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MEAN FIELD LIMIT AND QUANTITATIVE ESTIMATES WITH SINGULAR ATTRACTIVE KERNELS

DIDIER BRESCH, PIERRE-EMMANUEL JABIN, AND ZHENFU WANG

Abstract. This paper proves the mean field limit and quantitative estimates for many-particle systems with singular attractive interactions between particles. As an important example, a full rigorous derivation (with quantitative estimates) of the Patlak-Keller-Segel model in optimal subcritical regimes is obtained for the first time. To give an answer to this longstanding problem, we take advantage of a new modulated free energy and we prove some precise large deviation estimates encoding the competition between diffusion and attraction. Combined with the range of repulsive kernels, already treated in the séminaire Laurent Schwartz proceeding [https://slsedp.centre-mersenne.org/journals/SLSEDP/], we provide the full proof of results announced by the authors in [*C. R. Acad. Sciences*, Section Maths, (2019)].

Contents

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1. Introduction

The present paper coupled with the proceeding [Séminaire Laurent Schwartz, EDP et Applications, année 2019-2020, Exposé no II] published on website: https://slsedp.centre-mersenne.org/journals/SLSEDP/ corresponds to the extended version (*i.e.* with detailed proofs) of the announced results in the note *C.R. Acad. Sciences* [7].

Using a new weighted related entropy, we are able to derive for the first time the mean field limit for many-particle systems with singular attractive interactions of gradient-flow type. In particular we can positively answer the long standing open question of the mean field limit to the Patlak-Keller-Segel system.

More precisely, we consider the mean field limit for stochastic many-particle systems of the type

$$dX_i = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)dt + \sqrt{2\sigma}dB_i, \quad i = 1, 2, \cdots, N, \tag{1.1}$$

where the B_i are independent Brownian Motions or Wiener processes. For simplicity, we limit ourselves to the periodic domain Π^d.

We specifically focus on gradient flows with interaction kernels given by

$$K = -\nabla V \tag{1.2}$$

with general singular and *attractive* interaction potentials V.

A guiding example in this article (and the corresponding note [7]) is the *attractive* Poisson potential in dimension 2

$$V = \lambda \log|x| + V_e(x), \tag{1.3}$$

with $\lambda > 0$ and where V_e is a smooth correction so that V is periodic. Logarithmic potentials still play a critical role if the dimension $d > 2$ and for this reason we will still consider potentials like (1.3) in any dimension, even if there is no connection with the Poisson equation anymore.

Our main goal is to provide precise quantitative estimates for the convergence of (1.1) towards the limit McKean-Vlasov PDE

$$\begin{aligned} &\partial_t\bar\rho + \operatorname{div}_x(\bar\rho\, K \star_x \bar\rho) = \sigma\Delta_x\bar\rho, \\ &\text{with} \quad K = -\nabla V, \quad \bar\rho(t=0,x) = \bar\rho^0 \in \mathcal{P}(\Pi^d). \end{aligned} \tag{1.4}$$

In the case where V is given by (1.3) and $d = 2$, (1.4) is the famous Patlak-Keller-Segel model, which is one of the first models of chemotaxis for micro-organisms. The potential $-V\star\bar\rho$ can then be seen as the concentration of some chemical species (one has typically $V \leq 0$ here): From (1.3), one has that $\Delta V - V = 2\pi\,\lambda\,\delta_0$ so that the chemical species are produced by the population. Moreover (1.4) implies that the population follows the direction of higher chemical concentrations (more negative values of V).

It should be noted that the system (1.4) offers only a rough modeling of the biological processes involved in chemotaxis. For realistic applications, it is hence critical to be able to handle a *wide range* of potentials V that

may still share a singularity comparable to the one in (1.3). In that sense, the Patlak-Keller-Segel model is a good example of a typical setting where a singular attractive dynamics (all micro-organisms try to concentrate on a point) is competing with the spreading effect due to diffusion.

There exists a further advantage of considering (1.3): Since (1.4) has a simple structure, it is possible to fully characterize its behavior. Eq. (1.4) may indeed blow-up and form a Dirac mass in finite time and, one may exactly characterize that such a blow-up occurs simply by comparing λ and σ

- If $\lambda \leq 2\,d\,\sigma$ with d the dimension, then we always have a global solution to (1.4);
- If $\lambda > 2\,d\,\sigma$ then all solutions blow-up in final time (though it may be possible to extend the existence of some notion of solution past some blow-up as in [2]).

We refer for instance to [9, 13, 14] and the references therein. We note here that in our case $\bar{\rho}^0$ is normalized to be a probability density with total mass 1. The PDE literature typically instead normalizes V to be the Green kernel of the Poisson equation, so that the result above exactly corresponds to the classical $8\,\pi\,\sigma$ critical mass.

A key consequence of our main result is that $V \sim \log|x|$ is always critical for the mean field limit. As we will see in our main result, Theorem 2.1, we are able to prove the limit for essentially all $V \geq \gamma \log|x|$ for some $\gamma < 2\,d\,\sigma$ (with some reasonable assumptions on ∇V, see (2.6)-(2.8) below). This justifies the central role played by the Patlak-Keller-Segel case but it is remarkable that the exact same condition is found for the mean field limit as for the blow-up of the PDE system.

There are several ways to quantitatively compare (1.1) with the limit $\bar{\rho}$ given by (1.4), which one can very roughly separate into trajectorial and statistical approaches. We follow here [22, 23] by using the joint law $\rho_N(t, x_1, \ldots, x_N)$ of the process $(X_1, \ldots, X_N)$ which solves the Liouville or forward Kolmogorov equation

$$\begin{aligned} &\partial_t \rho_N + \sum_{i=1}^N \mathrm{div}_{x_i}\Big(\rho_N \frac{1}{N}\sum_{j\neq i}^N K(x_i - x_j)\Big) = \sigma \sum_{i=1}^N \Delta_{x_i}\rho_N, \\ &\rho_N|_{t=0} = \rho_N^0. \end{aligned} \tag{1.5}$$

Eq. (1.5) contains all the relevant statistical information about the position of the particles at any time. But it may fail to include some information on the trajectories: For example, it is not in general possible to identify time correlations in a given particle trajectory only from (1.5).

We also emphasize that, in addition to quantitative convergence estimates, Eq. (1.5) also offers a straightforward manner to understand solutions to system (1.1). We are actually not able to give a precise meaning to trajectorial solutions to the original SDE system (1.1). But instead we will

be working here with so-called *entropy solutions* to (1.5), which are more straightforward to define (see the appendix for an example). Of course any strong solution to (1.1) (in the probabilistic sense) would also yield an entropy solution to (1.5).

The joint law ρ_N is compared to the chaotic/tensorized law $\bar{\rho}_N := \bar{\rho}^{\otimes N} = \Pi_{i=1}^N \bar{\rho}(t, x_i)$, built from the limit $\bar{\rho}$. Of course $\bar{\rho}_N$ cannot be an exact solution to (1.5) so the method will have to account for the difference. As probability densities, both ρ_N and $\bar{\rho}_N$ are initially normalized by

$$\int_{\Pi^{dN}} \rho_N|_{t=0} = 1 = \int_{\Pi^d} \bar{\rho}_N|_{t=0}, \tag{1.6}$$

which is formally preserved by either (1.5) or (1.4).The method leads in particular to direct estimates between $\bar{\rho}^{\otimes k}$ and any observable or marginal of the system at a fixed rank k,

$$\rho_{N,k}(t, x_1, \dots, x_k) = \int_{\Pi^{(N-k)\,d}} \rho_N(t, x_1, \dots, x_N)\, dx_{k+1} \dots dx_N.$$

We postpone a full presentation of our main result till the main section as this requires a more technical discussion of the method. Still, a good example of corollary from our more complete Theorem 2.1 is a rigorous derivation of the Patlak-Keller-Segel system in the subcritical regime as given by

Theorem 1.1. *Assume that $\rho_N \in L^\infty(0,T;\ L^1(\Pi^{Nd}))$ is an entropy solution to Eq.* (1.5) *normalized by* (1.6) *in the sense of Definition 2.1, with initial condition $\rho_N(t=0) = \bar{\rho}^{\otimes N}(t=0)$, and for the potential V given by* (1.3)*. Assume that $\bar{\rho} \in L^\infty(0,T; W^{2,\infty}(\Pi^d))$ solves Eq.* (1.4) *with $\inf \bar{\rho} > 0$. Assume finally that $\lambda < 2d\,\sigma$. Then there exists a constant $C > 0$ and an exponent $\theta > 0$ independent of N s.t. for any fixed k*

$$\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^\infty(0,T;\ L^1(\Pi^{kd}))} \leq C\, k^{1/2}\, N^{-\theta}.$$

Theorem 1.1 follows directly from Theorem 2.1 will be stated below and the classical Csiszár-Kullback-Pinsker inequality. The exponent θ could be made fully explicit and actually depends only on $2d\,\sigma - \lambda$. We highlight that we obtain, in dimension 2, the optimal constant 4σ which corresponds to the critical mass $8\pi\sigma$ for which we have blow-up in finite time for the Patlak-Keller-Segel system.

Because of the singular behavior of the potential, a full rigorous derivation of the Patlak-Keller-Segel model from the stochastic equation (1.1) or the Liouville eq. (1.5) had remained elusive, in spite of recent progress in [12, 17] or [18, 20]. In particular, the results in [17] prove that any accumulation point as $N \to \infty$ of the random empirical measure associated to the system (1.1) is a weak solution in some sense to (1.4) provided that one is in the so-called very subcritical regime with $\lambda < \sigma$. While this provides the mean field limit, at least in some weak sense, it does not imply propagation of chaos. We also emphasize that [17] is also able to obtain well-posedness for

the original SDE system (1.1) in the same regime $\lambda < \sigma$. Of course the case of regularized Patlak-Keller-Segel interactions is much better understood with for example [25].

The paper is organized as follows: In Section 2, we introduce the new relative entropy with weights related to the Gibbs equilibrium G_N and the corresponding distribution $G_{\bar\rho_N}$ given in (2.2): modulated free energy. We also state our main quantitative Theorem 2.1 based on this modulated energy and we describe the main steps of the proof for reader's convenience. We finally present the explicit expression for the time evolution of such modulated free energy first in Proposition 2.2 for smooth solutions associated to smooth kernels and then Proposition 2.3 for entropy solutions associated to singular kernels. In Section 3, we present various large deviation type estimates which play crucial roles in the proof of Theorem 2.1 to control the non-negativity of the modulated energy. For the reader's convenience, we conclude with an appendix which recalls some previous large deviation estimates in [23] and proves the existence of entropy solutions for the Liouville equation (1.5) for the Patlak-Keller-Segel interaction kernel in two dimension.

2. New modulated free energy and main quantitative result

2.1. **Weighted relative entropy and the modulated free energy.** The method will revolve around the control of a rescaled entropy combining the relative entropy by Jabin-Wang [22] and the modulated energy by Serfaty [31] and Duerinckx [15]. This corresponds to a modulated free energy for the problem and reads

$$\begin{aligned} &E_N\Big(\frac{\rho_N}{G_N}\,|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\Big)\\ &\quad=\frac{1}{N}\int_{\Pi^{dN}}\rho_N(t,X^N)\log\Big(\frac{\rho_N(t,X^N)}{G_N(t,X^N)}\frac{G_{\bar\rho_N}(t,X^N)}{\bar\rho_N(t,X^N)}\Big)dX^N, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} G_N(t,X^N)&=\exp\Big(-\frac{1}{2N\sigma}\sum_{i\neq j}V(x_i-x_j)\Big),\\ G_{\bar\rho}(t,x)&=\exp\Big(-\frac{1}{\sigma}\,V\star\bar\rho(x)+\frac{1}{2\,\sigma}\int_{\Pi^d}V\star\bar\rho\,\bar\rho\Big),\\ G_{\bar\rho_N}(t,X^N)&=\exp\Big(-\frac{1}{\sigma}\sum_{i=1}^N V\star\bar\rho(x_i)+\frac{N}{2\sigma}\int_{\Pi^d}V\star\bar\rho\,\bar\rho\Big), \end{aligned} \tag{2.2}$$

and throughout this article $X^N=(x_1,x_2,\cdots,x_N)$. This free energy may be understood as a relative entropy with two weights (related to the Gibbs equilibrium) in the spirit of [6]. Note that

$$E_N\Big(\frac{\rho_N}{G_N}\,|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\Big)=\mathcal{H}_N(\rho_N|\bar\rho_N)+\mathcal{K}_N(G_N|G_{\bar\rho_N}),$$

where

$$\mathcal{H}_N(\rho_N|\bar\rho_N) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\Big(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\Big)\,dX^N \tag{2.3}$$

is exactly the relative entropy introduced in [22, 23] and

$$\mathcal{K}_N(G_N|G_{\bar\rho_N}) = -\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\big(\frac{G_N(t,X^N)}{G_{\bar\rho_N}(t,X^N)}\big)\,dX^N \tag{2.4}$$

with G_N and $G_{\bar\rho_N}$ given by Expressions (2.2) is the expectation of the modulated energy on which the method developed in [15, 31, 32] is based. Indeed, it is easy to check that

$$\mathcal{K}_N(G_N|G_{\bar\rho_N}) = \frac{1}{2\sigma}\int_{\Pi^{dN}} \mathrm{d}\rho_N \int_{\Pi^{2d}\cap\{x\neq y\}} V(x-y)(d\mu_N - d\bar\rho)^{\otimes 2}(x,y).$$

Note that E_N is not a priori a positive quantity. Since given any two measures f and g, not necessarily probability measures, by convexity one has $\int(f\log f/g + g - f)dx \geq 0$, which gives a lower bound for E_N,

$$E_N\Big(\frac{\rho_N}{G_N}\,|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\Big) \geq \frac{1}{N}\int_{\Pi^{dN}}\Big(\rho_N(t,X^N) - G_N(t,X^N)\frac{\bar\rho_N(t,X^N)}{G_{\bar\rho_N}(t,X^N)}\Big)dX^N,$$

or alternatively using Lemma 1 in [23]

$$E_N = \mathcal{H}_N - \frac{1}{N}\int \rho_N \log\frac{G_N}{G_{\bar\rho_N}} \geq -\frac{1}{N}\log\int \bar\rho_N\frac{G_N}{G_{\bar\rho_N}}.$$

As we mentioned already, the method in Theorem 2.1 combines the methods developed in [15, 31, 32] and [22, 23] (see also the summary in [30]). The modulated energy in [31, 32] proved effective for the mean field limit for Coulomb or Riesz gases, and was able to take advantage of the specific structure of the interaction to improve on previous results; though for less general interaction than [21] for example. On the other hand, the relative entropy in [22, 23] could not effectively handled gradient flows but performed well on interaction kernels that have bounded divergence with or without diffusion. In particular [22, 23] obtained quantitative mean field estimates from the 2d viscous model to the incompressible Navier-Stokes vs. previously only qualitative results in [16, 26, 27].

While relative entropy at the level of the Liouville equation have not been widely used for mean field limits, the relative entropy method initiated in [34] is maybe the closest. A different relative entropy approach at the level of the joint law of the full trajectories of the system was also developed in [4].

As mentioned in [7], combining the relative entropy with a modulated energy has already been very successfully used for various singular limits in kinetic theory. A first example concerns the so-called quasineutral limit for plasmas for which we refer for instance to [19] and [29]. Another example is the seminal derivation of the incompressible viscous Electro-magneto-hydrodynamics from the Vlasov-Maxwell-Boltzmann system in [1]; one issue

in that monograph is in particular to prove the asymptotic positivity of the combined free energy, which is a problem that we are facing as well as explained below. Specific tools are needed for the present (and different) context of the mean field limit for many-particle systems. Of course the role of the free energy for gradient flow systems has long been recognized, with [11, 28] being classical examples.

2.2. **The main quantitative theorem.** Because our method relies on propagating nonlinear quantities such as the relative free energy, some assumptions on the notion of solutions are required, namely

Definition 2.1. (Entropy solution) *Let $T > 0$ be fixed. A density $\rho_N \in L^\infty(0,T;L^1(\Pi^{dN})$ with $\rho_N \geq 0$ and $\int_{\Pi^{dN}} \rho_N dX^N = 1$, is an entropy solution to Eq.* (1.5) *on the time interval $[0,T]$ if it solves* (1.5) *in the sense of distributions, and for a.e. $t \leq T$*

$$\begin{aligned} &\int_{\Pi^{dN}} \rho_N(t,X^N) \log\Big(\frac{\rho_N(t,X^N)}{G_N}\Big)\, dX^N \\ &+ \sigma \sum_{i=1}^N \int_0^t \int_{\Pi^{dN}} \rho_N(s,X^N) \Big|\nabla_{x_i} \log\Big(\frac{\rho_N(s,X^N)}{G_N}\Big)\Big|^2 dX^N ds \\ &\leq \int_{\Pi^{dN}} \rho_N^0 \log\big(\frac{\rho_N^0}{G_N}\big)\, dX^N \end{aligned} \tag{2.5}$$

where for convenience we use in the article the notation $X^N = (x_1, \cdots, x_N)$.

Because of the singularity in the interaction, a weak solution to (1.5) may not be an entropy solution in the sense given above. We note however as well that entropy solutions need not be unique and Theorem 2.1 below applies to all entropy solutions if there exists more than one. While it is not the main purpose of this article, we include in the appendix a proof of the existence of entropy solutions for the Patlak-Keller-Segel setting.

Let us now specify the exact assumptions that are required on the potential V

$$V \in L^p(\Pi^d) \cap C^2(\Pi^d \setminus \{0\}) \quad \text{for some } p > 1, \tag{2.6}$$

$$V(x) \geq \lambda \log|x| + C \quad \text{for some } 0 \leq \lambda < 2\,d\,\sigma, \tag{2.7}$$

$$|\nabla V(x)| \leq \frac{C}{|x|}. \tag{2.8}$$

with $C > 0$ constant. Then the following theorem holds

Theorem 2.1. *Assume $K = -\nabla V$ with V a singular potential that satisfying* (2.6)–(2.8)*. Consider $\bar\rho \in L^\infty(0,T;W^{2,\infty}(\Pi^d))$ solves Eq.* (1.4) *with $\inf \bar\rho > 0$. Assume finally that $\lambda < 2d\,\sigma$. There exists constants $C > 0$ and $\theta > 0$ s.t. for $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t,x_i)$, and for the joint law ρ_N on Π^{dN} of any entropy solution to the SDE system* (1.1)*,*

$$\mathcal{H}_N(t) + |\mathcal{K}_N(t)| \leq e^{C_{\bar\rho}\,\|K\|\,t} \Big(\mathcal{H}_N(t=0) + |\mathcal{K}_N(t=0)| + \frac{C}{N^\theta}\Big),$$

where H_N and $\mathcal{K}_N$ are defined by (2.3) *and* (2.4)*, $C_{\bar\rho}$ and $\|K\|$ are constants depends on $\bar\rho$ and the assumptions* (2.6)–(2.8) *on V respectively. Hence if $H_N^0 + |\mathcal{K}_N^0| \leq C\, N^{-\theta}$, then for any fixed marginal $\rho_{N,k}$*

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar\rho(t,x_i)\|_{L^1(\Pi^{k\,d})} \leq C_{T,\bar\rho,k}\, N^{-\theta}.$$

Remark. Note that the same result may be obtained for attractive-repulsive kernels combining the present paper for attractive kernels to the method detailed in [8] for repulsive kernels. This corresponds to the announced results in the note *C.R. Acad. Sciences* [7]. More precisely we can get Theorem 2.1 for an even kernel V which may be decomposed as follows $V = V_a + V_r + V_s$ with V_a an attractive part satisfying (2.6)–(2.8), $V_s \in W^{2,\infty}(\Pi^d)$ a smooth part and V_r a repulsive part satisfying

$$V_r(-x) = V_r(x) \quad \text{and} \quad V_r \in L^p(\Pi^d) \text{ for some } p > 1 \tag{2.9}$$

with the following Fourier sign

$$\hat V_r(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^d. \tag{2.10}$$

One imposes the following pointwise controls for all $x \in \Pi^d$: There exists constants k and $C > 0$ such that

$$|\nabla V_r(\xi)| \leq \frac{C}{|x|^k}, \qquad |\nabla^2 V_r(x)| \leq \frac{C}{|x|^k}, \quad |\nabla V_r(x)| \leq C\frac{V_r(x)}{|x|}, \tag{2.11}$$

together with

$$\lim_{|x|\to 0} V_r(x) = +\infty, \qquad V_r(x) \leq C V_r(y) \text{ for all } |y| \leq 2|x|,$$

and

$$|\nabla_\xi \hat V_r(\xi) \leq \frac{C}{1+|\xi|}\Big(\hat V_r(\xi) + \frac{1}{1+|\xi|^{d-\alpha}}\Big) \text{ with } 0 < \alpha < d \text{ for all } \xi \in \mathbb{R}^d.$$

2.3. **Proof of Theorem 2.1: the main steps.** We describe here the various steps to prove Theorem 2.1. It follows the general strategy detailed in [8] which was dedicated to repulsive kernels but with some key differences due to the attractive singularity.

• *Step 1: The modulated free energy inequality.* The first step is of course to look at the time evolution of our modulated free energy. It is possible to show that it satisfies the following inequality

$$\begin{aligned} E_N\left(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t) &\leq E_N\left(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(0) \\ &- \frac{1}{2}\int_0^t \int_{\Pi^{dN}} \int_{\Pi^{2\,d}\cap\{x\neq y\}} \nabla V(x-y)\cdot \\ &\qquad \Big(\nabla \log \frac{\bar\rho}{G_{\bar\rho}}(x) - \nabla \log \frac{\bar\rho}{G_{\bar\rho}}(y)\Big)(d\mu_N - d\bar\rho)^{\otimes 2} d\rho_N. \end{aligned} \tag{2.12}$$

The inequality (2.12) exactly corresponds to the free energy inequality (2.25) that we prove later in Proposition 2.3 in subsection 2.5. This inequality

involves two difficulties: The formal calculations themselves are rather intricate and use in a critical manner the properties of the Gibbs equilibrium. The second issue is of course to justify those formal calculations for entropy solutions. For this reason we first explain the formal calculations in subsection 2.4 in Proposition 2.2 for smooth solutions, before presenting in Proposition 2.3 the argument to extend the inequality to entropy solutions.

• *Step 2: Control on the right-hand side.* Contrary to the repulsive case, the control on the right-hand side can immediately be obtained from [23]. Precisely since we assumed in (2.8) that $|\nabla V(x)| \leq C/|x|$, we have that

$$-\nabla V(x-y)\cdot(\phi(x)-\phi(y)) \in L^\infty,$$

with $\phi(x) = \nabla \log \frac{\bar\rho}{G_{\bar\rho}}(x)$, thus we may directly apply Lemma 1 as in [23]. We recall that this lemma reads

Lemma 2.1. *For any ρ_N, $\bar\rho_N$ in $\mathcal{P}(\Pi^{dN})$, any function $\psi \in L^\infty(\Pi^{d\,N})$ and any $\alpha > 0$*

$$\int_{\Pi^{dN}} \psi(X^N)\, d\rho_N \leq \frac{1}{\alpha\, N} \int_{\Pi^{dN}} d\rho_N \,\log \frac{\rho_N}{\bar\rho_N} + \frac{1}{\alpha\, N}\,\log \int_{\Pi^{dN}} e^{\alpha\, N\, \psi(X^N)}\, d\bar\rho_N.$$

For completeness, we give a simple proof of Lemma 2.1 in the appendix. Applying this lemma, we directly find that

$$-\int_{\Pi^{dN}} \int_{\{x\neq y\}} \nabla V(x-y)\cdot(\phi(x)-\phi(y))(d\mu_N - d\bar\rho)^{\otimes 2}\, d\rho_N \leq C\,\mathcal{H}_N(\rho|\,\bar\rho_N)$$
$$+\frac{C}{N}\int_{\Pi^{dN}} d\bar\rho_N \exp\Big(-\frac{N}{C}\int_{\{x\neq y\}} \nabla V(x-y)\cdot(\phi(x)-\phi(y))(d\mu_N - d\bar\rho)^{\otimes 2}\Big),$$

where we denote $\phi(x) = \nabla \log(\bar\rho/G_{\bar\rho})(x)$. We now apply a simplified version of Theorem 4 in [23] which reads

Theorem 2.2. (Theorem 4 in [23]). *Consider $\bar\rho \in \mathcal{P}(\Pi^d)$ and $f \in L^\infty(\Pi^{2d})$. Then there exists a constant $\alpha = \alpha(f) > 0$ small enough such that*

$$\sup_{N\geq 2} \int_{\Pi^{dN}} \bar\rho^{\otimes N} \exp\Big(\alpha N \int_{\Pi^{2d}} f(x,y)\,(d\mu_N - d\bar\rho)^{\otimes 2}\Big) dX^N \leq C < \infty.$$

Theorem 2.2 is a straightforward reformulation of Theorem 4 in [23] as we recall in the appendix (see Theorem 4.1 there and its following comments). This proves that

$$\int_{\Pi^{dN}} d\bar\rho_N \exp\Big(-\frac{N}{C}\int_{\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N - d\bar\rho)^{\otimes 2}\Big) \leq C_{\bar\rho}$$

for some constant $C_{\bar\rho}$ depending on the $W^{2,\infty}$ norm of $\log\bar\rho$ and it subsequently implies that

(2.13)
$$E_N\Big(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\Big)(t)\le E_N\Big(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\Big)(0)+C\int_0^t\mathcal{H}_N(\rho_N\mid\bar\rho_N)+\frac{C}{N}.$$

• *Step 3: A lower bound on $\mathcal{K}_N$ in E_N.* From (2.13), it remains to find a lower bound control on $E_N=\mathcal{H}_N+\mathcal{K}_N$ with

$$\mathcal{K}_N=\frac{1}{2\sigma}\int_{\Pi^{dN}}\int_{\Pi^{2d}\cap\{x\neq y\}}V(x-y)(d\mu_N-d\bar\rho)^{\otimes 2}(x,y)d\rho_N$$

a quantity which is non-necessarily positive or even asymptotically positive for attractive potentials V since $V\le 0$ and even $V(x)\to-\infty$ as $x\to 0$. A natural idea would be to try to control $\mathcal{K}_N$ from below by $\mathcal{H}_N$. Unfortunately direct inequalities between $\mathcal{K}_N$ and $\mathcal{H}_N$ do not appear to be true. However it is possible to compare $\mathcal{K}_N$ to $\mathcal{H}_N$ by splitting the study in two parts: short-range and long-range interactions. Therefore we introduce

$$V(x)=\lambda V_0(x)+W(x)=V(x)\,\chi(|x|/\eta)+V(x)\,(1-\chi(|x|/\eta)),$$

where χ is a smooth function with $\chi(x)=1$ if $x<1/2$ and $\operatorname{supp}\chi\in[0,1]$ with a parameter η which will be chosen later.

I) *The short-range interactions.* This case focuses on the main difficulty which is the singularity of V near 0. For this reason, we consider general truncated quantity (short-range interactions) of the type

$$\begin{aligned}&F(\mu)=-\int_{\Pi^{2d}\cap\{x\neq y\}}V_0(x-y)\,(\mu(dx)-\bar\rho(x)\,dx)\,(\mu(dy)-\bar\rho(y)\,dy),\\ &\text{with } V_0\in L^p(\Pi^d),\quad |\nabla V_0(x)|\le\frac{C}{|x|^k}\text{ with } k>1/2\\ &\text{satisfying the inequality: } V_0(x-y)\ge\log|x-y|\,\chi(|x-y|/\eta),\end{aligned}\tag{2.14}$$

for $p>1$ and where we emphasize that, from χ, $\operatorname{supp}V_0\in B(0,\eta)$. Of course we will use (2.14) for $\lambda V_0(x)=V(x)\chi(|x|/\eta)$ and we note that from assumptions (2.6)–(2.8), such a V_0 indeed satisfies the assumptions in (2.14). However (2.14) applies to V_0 with $|\nabla V_0(x)|\le C/|x|^k$ instead of the more restrictive (2.8) which is not required for the proposition below. Our goal is to prove the following

Proposition 2.1. *There exists η (depending only on $\|V_0\|_{L^p}$) s.t. for any $\gamma<d$, we have for F defined by (2.14) that for some $\theta>0$*

$$\begin{aligned}\gamma\int_{\Pi^{dN}}F(\mu_N)\,\rho_N\,dX^N&\le\mathcal{H}_N(\rho_N\mid\bar\rho_N)\\&+\frac{C}{N^\theta}\,(\log N+\|\log\bar\rho\|_{W^{1,\infty}}+\eta^{-1}),\end{aligned}$$

for all $N>\bar N$ with $\bar N$ depending only on the dimension and $d-\gamma$.

Prop. 2.1 is the main technical result of this article as it extends classical large deviation estimates to singular attractive potentials. Because of this, its proof is also rather intricate and is performed in Section 3. We also insist that Prop. 2.1 only holds for some η small enough which is the reason for the decomposition of the potential V into short and long ranges.

II) *The long-range interactions.* The second part consists in controlling the long-range interaction with $W(x) = V(x)\,(1-\chi(|x|/\eta))$. Because W is actually smooth, this can be done by rather straightforward contributions. Define G_N^W and $G_{\bar\rho_N}^W$ for W in the similar manner as for V,

$$
\begin{aligned}
&G_N^W(t,X^N) = \exp\Big(-\frac{1}{2N\sigma}\sum_{i,j=1}^N W(x_i-x_j)\Big),\\
&G_{\bar\rho}^W(x) = \exp\Big(-\frac{1}{\sigma}\,W\star\bar\rho(x) + \frac{1}{2\,\sigma}\int_{\Pi^d} W\star\bar\rho\,\bar\rho\Big),\\
&G_{\bar\rho_N}^W(t,X^N) = \exp\Big(-\frac{1}{\sigma}\sum_{i=1}^N W\star\bar\rho(x_i) + \frac{N}{2\sigma}\int_{\Pi^d} W\star\bar\rho\,\bar\rho\Big).
\end{aligned}
\tag{2.15}
$$

We calculate separately the evolution in time of the contribution of W in the modulated free energy through

Lemma 2.2. *For $W\in W^{2,\infty}_{\rm per}$ even and with $\Delta W(0)=0$, one has that*

$$
\begin{aligned}
-\frac{d}{dt}\mathcal{K}_N(G_N^W|G_{\bar\rho_N}^W) &= \frac{d}{dt}\frac{1}{N}\int_{\Pi^{dN}}\rho_N\log\frac{G_{\bar\rho_N}^W}{G_N^W}\,dX^N\\
&= \int_{\Pi^{dN}} d\rho_N\int_{\Pi^{2d}}\Delta W(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}\\
&-\frac{1}{\sigma}\int_{\Pi^{dN}} d\rho_N\int_{\Pi^{2d}}\int_{\Pi^d}\nabla W(z-x)\cdot\nabla V(z-y)\,\mathrm{d}\mu_N(z)\,(d\mu_N-d\bar\rho)^{\otimes 2}(x,y)\\
&-\frac{1}{2\,\sigma}\int_{\Pi^{dN}} d\rho_N\int_{\Pi^{2d}}\nabla W(x-y)\,(\nabla V\star\bar\rho(x)-\nabla V\star\bar\rho(y))\,(d\mu_N-d\bar\rho)^{\otimes 2}(x,y).
\end{aligned}
$$

Of course the right-hand side in this lemma involves more derivatives of W than what we observed in (2.12) because there is no particular structure in $\mathcal{K}_N(G_N^W|G_{\bar\rho_N}^W)$. However since W is smooth, this actually does not matter. In particular we observe that the 2nd term in the right-hand side can be rewritten as

$$
\begin{aligned}
&-\frac{1}{\sigma}\int_{\Pi^{dN}} d\rho_N\int_{\Pi^{2d}}\int_{\Pi^d}\nabla W(z-x)\cdot\nabla V(z-y)\,\bar\rho(z)\,\mathrm{d}z\,(d\mu_N-d\bar\rho)^{\otimes^2}\\
&-\frac{1}{2\sigma}\int_{\Pi^{dN}} d\rho_N\int_{\Pi^{2d}}\nabla V(x-y)(\nabla W\star(\mu_N-\bar\rho)(x)-\nabla W\star(\mu_N-\bar\rho)(y))\\
&\qquad\qquad (d\mu_N-d\bar\rho)^{\otimes^2}.
\end{aligned}
$$

We may then directly use Lemma 2.1 and Theorem 2.2 as in Step 2 to bound

$$
-\frac{d}{dt}\mathcal{K}_N(G_N^W|G_{\bar\rho_N}^W)\le C\mathcal{H}_N(\rho_N|\bar\rho_N)+\frac{C}{N}.
\tag{2.16}
$$

• *Step 4: Coercivity of* $\tilde{E}_N$ We write

$$\tilde{E}_N = E_N - \mathcal{K}_N(G_N^W|G_{\bar{\rho}_N}^W) = \mathcal{H}_N + \mathcal{K}_N(G_N|G_{\bar{\rho}_N}) - \mathcal{K}_N(G_N^W|G_{\bar{\rho}_N}^W),$$

where we remove the long range interaction from $\mathcal{K}_N$. From our choices and definitions, we have that

$$\mathcal{K}_N(G_N|G_{\bar{\rho}_N}) - \mathcal{K}_N(G_N^W|G_{\bar{\rho}_N}^W) = -\frac{\lambda}{2\sigma}\int_{\Pi^{dN}} F(\mu_N)\,\rho_N\,dX^N,$$

where F is defined through (2.14) with $\lambda V_0(x) = V(x)\chi(|x|/\eta)$. Applying Prop. 2.1 with $\gamma = \lambda/2\sigma$ we hence get that

$$\tilde{E}_N \geq \frac{1}{C}\,\mathcal{H}_N(\rho_N|\bar{\rho}_N) - \frac{C}{N^\theta}, \tag{2.17}$$

for some $C > 1$ and $\theta > 0$ assuming $\lambda < 2\sigma d$. See also for instance Eq. (26) and (27) in [7].

• *Step 5: Conclusion of the proof.* We are now ready to explain how to conclude the proof of the theorem. Combining (2.13) with (2.16) to obtain that

$$\tilde{E}_N \leq \tilde{E}_N(t=0) + C\int_0^t \mathcal{H}_N(\rho_N|\bar{\rho}_N) + \frac{C}{N}. \tag{2.18}$$

Inserting this into (2.17), we deduce by Gronwall's Lemma that

$$\frac{1}{C}\,\mathcal{H}_N(\rho_N|\bar{\rho}_N)(t) \leq e^{C\,t}\left(\tilde{E}_N(t=0) + \frac{C}{N^\theta}\right). \tag{2.19}$$

Finally since W is smooth, from Eq. (2.16) for instance, we trivially have that

$$|\mathcal{K}_N(G_N^W|G_{\bar{\rho}_N}^W)|(t) \leq |\mathcal{K}_N(G_N^W|G_{\bar{\rho}_N}^W)(0) + C\int_0^t \mathcal{H}_N(\rho_N|\bar{\rho}_N) + \frac{C}{N}. \tag{2.20}$$

Combining (2.17) with (2.18) and (2.19), we can control $|\mathcal{K}_N(G_N|G_{\bar{\rho}_N}) - \mathcal{K}_N(G_N^W|G_{\bar{\rho}_N}^W)|$ which finally allows to derive all estimates in the main theorem.

To fully complete the proof, it only remains to prove our modulated free energy inequality (2.12), Prop. 2.1 and Lemma 2.2. The inequality (2.12) is proved in the next two subsections in two steps: firstly Prop. 2.2 for the formal calculations and finally Prop. 2.3 for the rigorous derivation in subsection 2.5. Lemma 2.2 follows mostly the same calculations and is given just after in subsection 2.6. The proof of Prop. 2.1 is the main object of section 3.

2.4. **Modulated free energy control for smooth potential V.** Our first step is to prove the following explicit expression for the time evolution of the modulated free energy E_N where all interactions are smooth.

Proposition 2.2. *Assume that V is a $\mathcal{C}^2$ even function and that ρ_N is a classical solution to* (1.5) *and $\bar\rho$ solves* (1.4)*. Then the modulated free energy E_N defined by* (2.1) *satisfies*

$$\begin{aligned} &E_N\Big(\frac{\rho_N}{G_N}\,|\,\frac{\bar\rho_N}{G_{\rho_N}}\Big)(t)\\ &\le E_N\Big(\frac{\rho_N}{G_N}\,|\,\frac{\bar\rho_N}{G_{\rho_N}}\Big)(0) - \frac{\sigma}{N}\int_0^t\int_{\Pi^{d\,N}}\rho_N\left|\nabla\log\frac{\rho_N}{\bar\rho_N} - \nabla\log\frac{G_N}{G_{\bar\rho_N}}\right|^2\\ &-\frac{1}{2}\int_0^t\int_{\Pi^{dN}}\int_{\Pi^{2\,d}\cap\{x\neq y\}}\nabla V(x-y)\cdot\left(\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(x) - \nabla\log\frac{\bar\rho}{G_{\bar\rho}}(y)\right)\\ &\qquad\qquad (d\mu_N - d\bar\rho)^{\otimes 2}d\rho_N, \end{aligned}$$

where $\mu_N = \frac{1}{N}\sum_{i=1}^N\delta(x-x_i)$ is the empirical measure associated to the point configuration $X^N = (x_1,\cdots,x_N)$.

Remark. Note that

$$\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(x) = \nabla\log\bar\rho(x) + \frac{1}{\sigma}\nabla V\star\bar\rho(x).$$

Taking the right derivatives of this term will exactly cancel in the evolution of our modulated free energy the divergence term $\mathrm{div}K = -\Delta V$, that is otherwise present in the calculations in the time evolution of $\mathcal{H}_N(\rho_N|\bar\rho_N)$.

Proof. First of all we write ρ_N as a solution to the variant diffusion equation

$$\partial_t\frac{\rho_N}{G_N} - \frac{\sigma}{G_N}\,\mathrm{div}\left(G_N\nabla\frac{\rho_N}{G_N}\right) = 0.$$

We also try to put $\bar\rho_N$ under this form. Of course we have that

$$\partial_t\bar\rho = \sigma\,\mathrm{div}_x\left(G_{\bar\rho}\,\nabla\frac{\bar\rho}{G_{\bar\rho}}\right) = \sigma\,\mathrm{div}_x\left(\bar\rho\,\nabla\log\frac{\bar\rho}{G_{\bar\rho}}\right), \tag{2.21}$$

which we can tensorize trivially into

$$\partial_t\bar\rho_N - \sigma\,\mathrm{div}\left(G_{\bar\rho_N}\nabla\frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = 0.$$

So we just write

$$\partial_t\frac{\bar\rho_N}{G_{\bar\rho_N}} - \frac{\sigma}{G_N}\,\mathrm{div}\left(G_N\nabla\frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = R_N,$$

where

$$\begin{aligned} R_N = &-\frac{\sigma}{G_N}\sum_i\mathrm{div}_{x_i}\left(G_N\,\nabla_{x_i}\frac{\bar\rho_N}{G_{\bar\rho_N}}\right) + \frac{\sigma}{G_{\bar\rho_N}}\sum_i\mathrm{div}_{x_i}\left(G_{\bar\rho_N}\,\nabla_{x_i}\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)\\ &+\bar\rho_N\partial_t\frac{1}{G_{\bar\rho_N}}. \end{aligned}$$

We now recall the classical entropy-entropy dissipation inequality for self-adjoint diffusion equations: Consider two solutions u_i, $i = 1,\ 2$ to

$$\partial_t u_i - \frac{1}{M(x)} \operatorname{div}_x \left(M(x) \nabla_x u_i \right) = 0.$$

Then one has that by differentiating and integrating by parts

$$\begin{aligned} \frac{d}{dt} \int u_1 \log \frac{u_1}{u_2} M(x)\, dx &= \int \left(\left(1 + \log \frac{u_1}{u_2} \right) \partial_t u_1 - \frac{u_1}{u_2} \partial_t u_2 \right) M(x)\, dx \\ &= - \int \left(\nabla_x u_1 \cdot \nabla_x \log \frac{u_1}{u_2} - \nabla_x \frac{u_1}{u_2} \cdot \nabla_x u_2 \right) M(x)\, dx. \end{aligned}$$

By re-arranging the terms, we obtain the usual

$$\frac{d}{dt} \int u_1 \log \frac{u_1}{u_2} M(x)\, dx = - \int u_1 M(x) \left| \nabla_x \log \frac{u_1}{u_2} \right|^2 dx.$$

In our case, this gives immediately that

$$\begin{aligned} \frac{d}{dt} E_N = &- \frac{\sigma}{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \left| \nabla \log \frac{\rho_N}{\bar\rho_N} - \nabla \log \frac{G_N}{G_{\bar\rho_N}} \right|^2 \\ &- \frac{1}{N} \int \frac{\rho_N}{\bar\rho_N} G_{\bar\rho_N} R_N. \end{aligned} \tag{2.22}$$

So the whole point is to handle correctly the terms with R_N. Let us start with just expanding the divergence terms in R_N and getting the trivial cancellation

$$\begin{aligned} R_N = &\sigma \sum_i (\nabla_{x_i} \log G_{\bar\rho_N} - \nabla_{x_i} \log G_N) \cdot \nabla_{x_i} \frac{\bar\rho_N}{G_{\bar\rho_N}} \\ &+ \bar\rho_N \partial_t \frac{1}{G_{\bar\rho_N}}. \end{aligned} \tag{2.23}$$

Hence the remainder above just reads

$$\begin{aligned} r_N = &\frac{1}{N} \int \frac{\rho_N}{\bar\rho_N} G_{\bar\rho_N} R_N = \frac{\sigma}{N} \int \rho_N \sum_i \nabla_{x_i} \log \frac{\bar\rho_N}{G_{\bar\rho_N}} \cdot \nabla_{x_i} \log \frac{G_{\bar\rho_N}}{G_N} \\ &- \frac{1}{N} \int \rho_N \partial_t \log G_{\bar\rho_N}. \end{aligned}$$

This is of course directly

$$\begin{aligned} r_N = &\frac{1}{N} \int \rho_N \int_{\Pi^{2d}} \nabla V(x-y) \cdot \nabla_x \log \frac{\bar\rho_N}{G_{\bar\rho_N}}(x)\, \mu_N(dx)\, (\mu_N - \bar\rho)(dy) \\ &- \frac{1}{N} \int \rho_N \int_{\Pi^d} \mu_N(dx) \partial_t \log G_{\bar\rho}. \end{aligned}$$

Use now (2.21) to get that

$$\partial_t \log G_{\bar\rho} = -\frac{1}{\sigma} V \star \partial_t \bar\rho + \frac{1}{\sigma} \int V \star \bar\rho \partial_t \bar\rho = -\nabla V \star (\bar\rho\, \phi) - \int \nabla V \star \bar\rho\, \bar\rho \phi,$$

where we denote $\phi(x) = \nabla_x \log\big(\bar\rho_N / G_{\bar\rho_N}(x)\big)$.

If we insert this into r_N, we get that

$$r_N = \frac{1}{N}\int \rho_N \int_{\Pi^{2d}} \nabla V(x-y)\cdot\Big(\phi(x)\,\mu_N(dx)\,(\mu_N-\bar\rho)(dy) + \phi(y)\,\mu_N(dx)\,\bar\rho(dy) + \phi(x)\,\bar\rho(dx)\,\bar\rho(dy)\Big).$$

It just remains to symmetrize in x and y to obtain

$$\begin{aligned} r_N &= \frac{1}{N}\int_0^t \int \frac{\rho_N}{\bar\rho_N} G_{\bar\rho_N} R_N \\ &= \frac{1}{2N}\int \rho_N \int_{\Pi^{2d}} \nabla V(x-y) \\ &\quad\cdot (\phi(x)-\phi(y))(\mu_N(dx)\,\mu_N(dy) - 2\mu_N(dy)\,\bar\rho(dx) + \bar\rho(dx)\,\bar\rho(dy)), \end{aligned} \tag{2.24}$$

which concludes. □

2.5. **Modulated free energy control for non-smooth V.** We prove here an equivalent of Prop. 2.2 for realistic, singular potentials V, which finally implies (2.12).

Proposition 2.3. *Assume that V is an even function, and that ρ_N is an entropy solution to* (1.5) *in the sense of definition 2.1 below and $\bar\rho$ smooth solves* (1.4)*. Then the modulated free energy defined by* (2.1) *satisfies that*

$$\begin{aligned} &E_N\left(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t) \\ &\le E_N\left(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(0) - \frac{\sigma}{N}\int_0^t\int_{\Pi^{d\,N}} \rho_N\left|\nabla\log\frac{\rho_N}{\bar\rho_N} - \nabla\log\frac{G_N}{G_{\bar\rho_N}}\right|^2 \\ &- \frac{1}{2}\int_0^t\int_{\Pi^{dN}}\int_{\Pi^{2\,d}\cap\{x\neq y\}} \nabla V(x-y)\cdot\left(\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(x) - \nabla\log\frac{\bar\rho}{G_{\bar\rho}}(y)\right) \\ &\qquad\qquad (d\mu_N - d\bar\rho)^{\otimes 2} d\rho_N, \end{aligned} \tag{2.25}$$

where $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta(x-x_i)$ is the empirical measure.

Proof. We first note that the entropy estimate provides some useful *a priori* estimates. In particular from the entropy dissipation (2.5), we have that

$$\begin{aligned} &\sum_{i=1}^N \int_0^T \int_{\Pi^{dN}} \rho_N(s,X^N)\Big|\nabla_{x_i}\log\big(\frac{\rho_N(s,X^N)}{G_N}\big)\Big|^2 dX^N ds \\ &= \sum_{i=1}^N \int_0^T \int_{\Pi^{dN}} G_N\,|\nabla_{x_i}\frac{\rho_N(s,X^N)}{G_N}|^2 dX^N\,ds < +\infty \end{aligned} \tag{2.26}$$

where $X^N = (x_1,\cdots,x_N)$. This implies that $G_N\,\nabla_x\frac{\rho_N}{G_N} \in L^1([0,\ T]\times\Pi^{dN})$ and it allows to give meaning to Eq. (1.5) in the sense of distribution as it

can be rewritten as

$$\partial_t \rho_N - \sigma \operatorname{div}\Big(G_N \nabla \frac{\rho_N}{G_N} \Big) = 0. \tag{2.27}$$

The proof of Prop. 2.3 follows the same path in our previous formal derivation. We of course start with the entropy control (2.5) satisfied by the entropy solution which takes care of the terms in dE_N/dt that are non-linear in ρ_N.

Then we recall that both $\bar\rho$ and $G_{\bar\rho}$ are smooth and non-vanishing. As we have previously seen, they satisfy

$$\partial_t \frac{\bar\rho_N}{G_{\bar\rho_N}} - \frac{\sigma}{G_N} \operatorname{div}\Big(G_N \nabla \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) = R_N$$

with R_N given by (2.23) and therefore satisfies

$$\begin{aligned} \partial_t \Big(\rho_N \log \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) - \Big(\log \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) \partial_t \rho_N \\ - \sigma \frac{\rho_N G_{\bar\rho_N}}{G_N \bar\rho_N} \operatorname{div}\Big(G_N \nabla \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) = R_N \rho_N \frac{G_{\bar\rho_N}}{\bar\rho_N}. \end{aligned} \tag{2.28}$$

We note that all terms above are well defined in the sense of distribution with for example

$$\begin{aligned} \frac{\rho_N G_{\bar\rho_N}}{G_N \bar\rho_N} \operatorname{div}\Big(G_N \nabla \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) &= \operatorname{div}\Big(\frac{\rho_N G_{\bar\rho_N}}{\bar\rho_N} \nabla \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) - \rho_N \nabla \frac{\bar\rho_N}{G_{\bar\rho_N}} \cdot \nabla \frac{G_{\bar\rho_N}}{\bar\rho_N} \\ &\quad - G_N \nabla \frac{\rho_N}{G_N} \cdot \nabla \log \frac{\bar\rho_N}{G_{\bar\rho_N}}. \end{aligned}$$

Similarly the only non-smooth term in R_N is

$$-\sigma \sum_i \nabla_{x_i} \log G_N \cdot \nabla_{x_i} \frac{\bar\rho_N}{G_{\bar\rho_N}} = -\sigma \sum_i \frac{1}{G_N} \nabla_{x_i} G_N \cdot \nabla_{x_i} \frac{\bar\rho_N}{G_{\bar\rho_N}},$$

which, once multiplied against $\rho_N \frac{G_{\bar\rho_N}}{\bar\rho_N}$, can be rewritten as

$$-\sigma \sum_i \nabla_{x_i} \rho_N \cdot \nabla_{x_i} \log \frac{\bar\rho_N}{G_{\bar\rho_N}} + \sigma \sum_i G_N \nabla_{x_i} \frac{\rho_N}{G_N} \cdot \nabla_{x_i} \log \frac{\bar\rho_N}{G_{\bar\rho_N}}.$$

It remains to use (2.27) and test it with $\log\big(\bar\rho_N / G_{\bar\rho_N}\big)$ to get after commuting space derivatives

$$\begin{aligned} \int_0^t \int -\Big(\log \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) \partial_t \rho_N - \sigma \int_0^t \int \frac{\rho_N G_{\bar\rho_N}}{G_N \bar\rho_N} \operatorname{div}\Big(G_N \nabla \frac{\bar\rho_N}{G_{\bar\rho_N}} \Big) \\ = \sigma \Big(- \int_0^t \int \rho_N |\nabla_x \log \frac{G_{\bar\rho_N}}{\bar\rho_N}|^2 + 2 \int_0^t \int \frac{G_N G_{\bar\rho_N}}{\bar\rho_N} \nabla \frac{\bar\rho_N}{G_{\bar\rho_N}} \cdot \nabla \frac{\rho_N}{G_N} \Big). \end{aligned}$$

Note that everything is well defined due to the control $G_N \nabla_x \big(\rho_N / G_N \big) \in L^1$. It remains now to collect (2.5) with (2.28) integrated in space and time on $(0,t)$ using identity (2.24) to conclude the proposition. □

2.6. **Large range contribution: proof of Lemma 2.2.** We mostly perform a direct calculation, with any special need to use the structure of the dynamics as before. First of all

$$\frac{1}{N}\int_{\Pi^{dN}}\rho_N\log\frac{G^W_{\bar\rho_N}}{G^W_N}\,dX^N == \frac{1}{2\,\sigma}\int\rho_N\int_{\Pi^{2d}}W(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}\,dX^N$$
$$=\frac{1}{2\,N^2\,\sigma}\sum_{i,j}\int\rho_N\,\Big(W(x_i-x_j)-W\star\bar\rho(x_i)-W\star\bar\rho(x_j)+\int\bar\rho\,W\star\bar\rho\Big)$$

Therefore using the dynamics

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{N}\int_{\Pi^{dN}}\rho_N\log\frac{W_{\bar\rho_N}}{W_N}\,dX^N\\
&=\frac{1}{2\,N^2}\sum_{i,j}\int\rho_N\,(2\,\Delta W(x_i-x_j)-\Delta W\star\bar\rho(x_i)-\Delta W\star\bar\rho(x_j))\\
&\quad+\frac{1}{2\,N^2}\sum_{i,j}\int\rho_N\,\Big(2\int\bar\rho\,\Delta W\star\bar\rho-\Delta W\star\bar\rho(x_i)-\Delta W\star\bar\rho(x_j)\Big)\\
&\quad-\frac{1}{N^3\,\sigma}\sum_{i,j,k}\int\rho_N\,(\nabla W(x_i-x_j)-\nabla W\star\bar\rho(x_i))\cdot\nabla V(x_i-x_k)\\
&\quad-\frac{1}{2\,N^2\,\sigma}\sum_{i,j}\int\rho_N\,\Big(2\int\nabla V\star\bar\rho\,\bar\rho\,\nabla W\star\bar\rho+\nabla W\star(\bar\rho\,\nabla V\star\bar\rho)(x_i)\\
&\qquad\qquad-\nabla W\star(\bar\rho\,\nabla V\star\bar\rho)(x_j)\Big).
\end{aligned}$$

Now we just have to symmetrize as before, with first

$$\begin{aligned}
&\frac{1}{2\,N^2}\sum_{i,j}\int\rho_N\,(2\,\Delta W(x_i-x_j)-\Delta W\star\bar\rho(x_i)-\Delta W\star\bar\rho(x_j))\\
&\quad+\frac{1}{2\,N^2}\sum_{i,j}\int\rho_N\,\Big(2\int\bar\rho\,\Delta W\star\bar\rho-\Delta W\star\bar\rho(x_i)-\Delta W\star\bar\rho(x_j)\Big)\\
&\quad=\int\rho_N\int_{\Pi^{2d}}\Delta W(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}\,dX^N.
\end{aligned}$$

Similarly

$$\begin{aligned}
&-\frac{1}{N^3\,\sigma}\sum_{i,j,k}\int\rho_N\,(\nabla W(x_i-x_j)-\nabla W\star\bar\rho(x_i))\cdot\nabla V(x_i-x_k)\\
&\quad=-\frac{1}{\sigma}\int\rho_N\int_{\Pi^{2d}}\int_{\Pi^d}\nabla W(z-x)\cdot\nabla V(z-y)\;\mathrm{d}\mu(z)\,(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\qquad-\frac{1}{N^2\,\sigma}\sum_{i,j}\int\rho_N\,(\nabla W(x_i-x_j)-\nabla W\star\bar\rho(x_i))\cdot\nabla V\star\bar\rho(x_i).
\end{aligned}$$

Since ∇W is odd,

$$\begin{aligned}\frac{1}{N^2}\sum_{i,j}\int \rho_N\,\nabla W\star(\bar\rho\,\nabla V\star\bar\rho)(x_i)\\ &=\int\rho_N\int_{\Pi^d}\nabla W\star(\bar\rho\,\nabla V\star\bar\rho)(x)\,\mu_N(dx)\\ &=-\int\rho_N\int_{\Pi^d}\bar\rho\,\nabla V\star\bar\rho(z)\,\nabla W\star\mu_N(z)\,dz.\end{aligned}$$

Hence one also has

$$\begin{aligned}&-\frac{1}{N^2\,\sigma}\sum_{i,j}\int\rho_N\,(\nabla W(x_i-x_j)-\nabla W\star\bar\rho(x_i))\cdot\nabla V\star\bar\rho(x_i)\\ &-\frac{1}{2\,N^2\,\sigma}\sum_{i,j}\int\rho_N\Big(2\int\nabla V\star\bar\rho\,\bar\rho\,\nabla W\star\bar\rho+\nabla W\star(\bar\rho\,\nabla V\star\bar\rho)(x_i)\\ &\qquad+\nabla W\star(\bar\rho\,\nabla V\star\bar\rho)(x_j)\Big)\\ &=\frac{1}{\sigma}\int\rho_N\int_{\Pi^d}\nabla V\star\bar\rho(z)\Big((\nabla W\star\bar\rho-\nabla W\star\mu_N)\,\mu_N(dz)-\nabla W\star\bar\rho\,\bar\rho\,dz\\ &\qquad+\nabla W\star\mu_N\,\bar\rho\,dz\Big)\\ &=-\frac{1}{\sigma}\int\rho_N\int_{\Pi^d}\nabla V\star\bar\rho(z)\,(\nabla W\star\mu_N-\nabla W\star\bar\rho)\,(\mu_N(dz)-\bar\rho\,dz),\end{aligned}$$

which by an symmetrization equals to

$$-\frac{1}{2\,\sigma}\int\rho_N\int_{\Pi^{2d}}\nabla W(x-y)\,(\nabla V\star\bar\rho(x)-\nabla V\star\bar\rho(y))\,(\mu_N-\bar\rho)^{\otimes 2},$$

and summing up all terms concludes the proof.

3. Large deviation type estimates

The main goal of this section is to prove Prop 2.1, namely to derive a large deviation inequality on the functional

$$\gamma\int_{\Pi^{dN}}F(\mu_N)\,\rho_N\,dX^N,$$

where we recall that

$$F(\mu)=-\int_{\Pi^{2d}\cap\{x\neq y\}}V_0(x-y)\,(d\mu-d\bar\rho)^{\otimes 2}(x,y),$$

and for V_0 satisfying the assumptions of (2.14).

Classical large deviations approaches typically attempt at the limit of Gibbs equilibrium, see [3, 33] for example. In contrast, we only care here about bounds on corresponding quantities. The estimates are also made more delicate since $V_0(x)$ is allowed to be singular at $x=0$, (2.14) only imposes $V_0(x)\geq\log|x|\,\chi(|x|/\eta)$, and we want to treat the best possible constant γ: any $\gamma<d$.

The main strategy that we follow is hence to try to remove the singularity in V_0 by carving out those $|x-y|<\varepsilon$ for some ε. This process is obviously the most delicate part and is carried out in subsection 3.4.

This does not completely resolve the estimate though as it is still necessary to obtain a good quantitative control on the remaining functional in terms of ε. The derivation of such a control forces us to revisit more classical large deviation approaches in subsections 3.1 and 3.2.

3.1. **An explicit basic large deviation estimate.** For any $M>0$, define a decomposition of the torus Π^d into M^d disjoint hypercubes C_k^M, $k=1,\dots,M^d$, of size $1/M$. We then denote

$$L_M(x,y)=M^d\,\mathbb{I}_{x,\,y\in C_k^M},\quad L_M[f](x)=\int_{\Pi^d}L_M(x,y)\,f(dy).$$

Note that for $M\to\infty$, L_M is an approximation of the Dirac mass as for f Lipschitz

$$|f(x)-L_M[f](x)|\le\int_{\Pi^d}L_M(x,y)\,|f(x)-f(y)|\,dy\le C\|\nabla f\|_{L^\infty}\,\frac{1}{M}.$$

The kernel L_M makes it relatively straightforward to use elementary combinatorics for large deviation purposes. In particular we can derive the following estimate that we will make use of later.

Proposition 3.1. *There exists a constant C_d s.t. for any $\bar\rho\in L^1(\Pi^d)$, one has the exponential bound for $M^d\le N/2$,*

$$C^{-M^d}\,\frac{N^{M^d/2-1/2}}{M^{d\,M^d-d/2}}\le\int_{\Pi^{d\,N}}e^{N\,\int_{\Pi^d}\mu_N(dx)\,\log\frac{L_M[\mu_N](x)}{\bar\rho(x)}}\,\bar\rho_N\,dx_1\dots dx_N\le C\,N^{M^d+1/2}.$$

Remark. Note that the previous proposition will be used choosing M in terms of the number of particles N and the regularized parameter ε to provide the quantitative large deviation type estimate given by Proposition 3.2.

Proof of Prop. 3.1. The first step is to simply reduce to the case $\bar\rho=1$ by observing that

$$\begin{aligned}e^{N\,\int_{\Pi^d}\mu_N(dx)\,\log\frac{L_M[\mu_N](x)}{\bar\rho(x)}}\,\bar\rho_N&=\exp\left(\sum_i\log\frac{L_M[\mu_N](x_i)}{\bar\rho(x_i)}+\sum_i\log\bar\rho(x_i)\right)\\&=\exp\left(\sum_i\log L_M[\mu_N](x_i)\right),\end{aligned}$$

and therefore it is enough to bound

$$Z_{N,M}=\int_{\Pi^{d\,N}}e^{N\,\int_{\Pi^d}\mu_N(dx)\,\log L_M[\mu_N](x)}\,dx_1\dots dx_N.$$

Of course, one may simply write

$$Z_{N,M} = \int_{\Pi^{d\,N}} e^{\sum_i \log L_M[\mu_N](x_i)}\, dx_1 \dots dx_N$$
$$= \int_{\Pi^{d\,N}} \Pi_i L_M[\mu_N](x_i)\, dx_1 \dots dx_N.$$

On the other hand, if $x_i \in C_k^M$

$$L_M[\mu_N](x_i) = \frac{M^d}{N} \#\{j \mid x_j \in C_k^M\}.$$

This leads us to define, for any given i the unique index $k(x_i)$ s.t. $x_i \in C_{k(x_i)}^M$, and for a given k the number $n_k = \#\{j \mid x_j \in C_k^M\}$. This simply gives

$$Z_{N,M} = \frac{M^{d\,N}}{N^N} \int_{\Pi^{d\,N}} \Pi_i n_{k(x_i)}\, dx_1 \dots dx_N.$$

We can of course reverse the process and first choose any decomposition

$$n_1 + \ldots + n_{M^d} = N, \quad 0 \le n_k \le N,$$

and then denote by $\Omega = \Omega(n_1, \dots, n_{M^d})$ the subset of $\Pi^{d\,N}$ such that $n_k = \#\{i \mid x_i \in C_k^M\}$ for all $k = 1, \cdots, M^d$. Hence

$$Z_{N,M} = \frac{M^{d\,N}}{N^N} \sum_{n_1+\ldots+n_{M^d}=N} \Pi_k n_k^{n_k} \int_{\Omega(n_1,\ldots,n_{M^d})} dx_1 \dots dx_N$$
$$= \frac{M^{d\,N}}{N^N} \sum_{n_1+\ldots+n_{M^d}=N} \Pi_k n_k^{n_k}\, |\Omega(n_1, \dots, n_{M^d})|.$$

It is relatively straightforward to evaluate $|\Omega(n_1, \dots, n_{M^d})|$. We may first consider the reduced set $\Omega^r(n_1, \dots, n_{M^d})$ where we assign particles to hypercubes based on their rank: Simply put $x_1, \dots, x_{n_1}$ anywhere in C_1^M, $x_{n_1+1}, \dots, x_{n_1+n_2}$ anywhere in C_2^M and so on... Trivially

$$|\Omega^r(n_1, \dots, n_{M^d})| = M^{-d\,N}.$$

On the other hand, up to a permutation τ of the indices, if $(x_1, \dots, x_N) \in \Omega(n_1, \dots, n_{M^d})$ then $(x_{\tau(1)}, \dots, x_{\tau(N)}) \in \Omega^r(n_1, \dots, n_{M^d})$, implying that

$$|\Omega(n_1, \dots, n_{M^d})|$$
$$= M^{-d\,N} \left| \left\{ (k_1, \dots k_N) \in \{1, \dots, M^d\}^N \mid \forall l\ n_l = |\{i,\ k_i = l\}| \right\} \right|.$$

We now recall the classical combinatorics results (see [22] for instance) stating that

(3.1)
$$\left| \left\{ (k_1, \dots k_N) \in \{1, \dots, M^d\}^N \mid \forall l\ n_l = |\{i,\ k_i = l\}| \right\} \right| = \frac{N!}{n_1! \dots n_{M^d}!}.$$

This proves that

$$|\Omega(n_1,\dots,n_{M^d})| = \frac{M^{-d\,N}\,N!}{n_1!\dots n_{M^d}!},$$

and therefore

$$Z_{N,M} = \frac{N!}{N^N} \sum_{n_1+\dots+n_{M^d}=N} \Pi_k \frac{n_k^{n_k}}{n_k!}. \tag{3.2}$$

We may easily simplify further the expression by recalling as well that for any $n \geq 0$

$$\sqrt{n+1}\,\left(\frac{n}{e}\right)^n \leq n! \leq C\,\sqrt{n}\,\left(\frac{n}{e}\right)^n.$$

Hence

$$\frac{n_k^{n_k}}{n_k!} \leq \frac{1}{\sqrt{n_k+1}}\,e^{n_k}, \quad \frac{N!}{N^N} \leq C\,N^{1/2}\,e^{-N},$$

and

$$Z_{N,M} \leq C\,N^{1/2} \sum_{n_1+\dots+n_{M^d}=N} \Pi_k \frac{1}{\sqrt{n_k+1}}.$$

Observe that we still have not lost much and we can also derive the lower bound

$$Z_{N,M} \geq C^{-M^d}\,N^{1/2} \sum_{n_1+\dots+n_{M^d}=N} \Pi_k \frac{1}{\sqrt{n_k+1}}.$$

Here however for the upper bound, we proceed more roughly by simply bounding

$$Z_{N,M} \leq C\,N^{1/2}\,|\{(n_1,\dots,n_{M^d}) \in \mathbb{N}^{M^d} \mid \forall k\ n_k \geq 0,\ n_1+\dots+n_{M^d}=N\}|. \tag{3.3}$$

Similarly, for the lower bound, we simply use the trivial estimate $(n_k+1)^{-1/2} \geq N^{-1/2}$ to find

$$\begin{aligned} Z_{N,M} \geq\ & C^{-M^d}\,N^{1/2-M^d/2} \\ & |\{(n_1,\dots,n_{M^d}) \in \mathbb{N}^{M^d} \mid \forall k\ n_k \geq 0,\ n_1+\dots+n_{M^d}=N\}|. \end{aligned} \tag{3.4}$$

We finally recall that (Lemma 7 in [22])

$$|\{(b_1,\dots,b_p) \in \mathbb{N}^p \mid \forall k\ b_k \geq 1,\ b_1+\dots+b_p=q\}| = \binom{q-1}{p-1}. \tag{3.5}$$

Defining $b_l = n_l + 1$, $p = M^d$ and $q = N + M^d$, we obtain that

$$\begin{aligned} & |\{(n_1,\dots,n_{M^d}) \in \mathbb{N}^{M^d} \mid \forall k\ n_k \geq 0,\ \text{and } n_1+\dots+n_{M^d}=N\}| \\ & = \binom{N+M^d-1}{M^d-1}, \end{aligned}$$

or from (3.3) and (3.4)

$$C^{-M^d}\,N^{1/2-M^d/2}\binom{N+M^d-1}{M^d-1} \leq Z_{N,M} \leq C\,N^{1/2}\binom{N+M^d-1}{M^d-1}.$$

It now only remains to trivially bound again the binomial coefficient. Denoting $K = M^d - 1$ for simplicity and if $K \leq N/2$

$$\binom{N + M^d - 1}{M^d - 1} = \frac{(N+K)!}{K!\,N!} \leq C\,\frac{(N+K)^K}{K^K}\,\frac{(N+K)^{1/2}}{N^{1/2}\,K^{1/2}}\,\frac{(N+K)^N}{N^N}$$
$$\leq C\,N^K\,e^{N\,\log(1+K/N)}\,\frac{2^K}{K^K} \leq C\,N^K\,\frac{2^K\,e^K}{K^K} \leq C\,N^K.$$

as $\log(1 + K/N) \leq K/N$. We similarly have the lower bound

$$\binom{N + M^d - 1}{M^d - 1} = \frac{(N+K)!}{K!\,N!} \geq \frac{1}{C}\,\frac{(N+K)^K}{K^K}\,\frac{(N+K)^{1/2}}{N^{1/2}\,K^{1/2}}\,\frac{(N+K)^N}{N^N}$$
$$\geq \frac{1}{C}\,\frac{N^K}{K^{1/2}}\,e^{N\,\log(1+K/N)}\,\frac{1}{K^K} \geq \frac{1}{C}\,N^K\,\frac{1}{K^{K+1/2}} \geq \frac{1}{C}\,\frac{N^K}{K^{1/2+K}},$$

which concludes the estimate. □

3.2. **Quantitative large deviations and regularization.** We can now write a quantitative large deviations for abstract functional. Consider any $F : \mathcal{P}(\Pi^d) \to \mathbb{R}$, possibly unbounded. Our goal in this subsection is to derive an intermediary large deviation inequality where we have removed the singularity in F. More specifically, we will derive an estimate on

$$\frac{1}{N}\log\int_{\Pi^{dN}} e^{N\,F(L_\varepsilon\star\mu_N)}\,\bar\rho_N\,dX^N,$$

for some classical, smooth convolution kernel L, instead of the original

$$\frac{1}{N}\log\int_{\Pi^{dN}} e^{N\,F(\mu_N)}\,\bar\rho_N\,dX^N.$$

When F is given by (2.14), this will give the final bound once the singularity of V_0 at 0 has been removed. This bound is directly connected to the large deviation functional associated to F, of which we recall the definition

$$I(F) = \max_{\mu\in\mathcal{P}(\Pi^d)}\Big[F(\mu) - \int_{\Pi^d}\mu\,\log\frac{\mu}{\bar\rho}\,dx\Big]. \tag{3.6}$$

We may now state the quantitative large deviation type estimate for the regularized F

Proposition 3.2. *Assume that* $\log\bar\rho \in W^{1,\infty}$*, then there exists a constant* C *depending only on* d*,* L*, s.t. for any* F*, one has that*

$$\frac{1}{N}\log\int_{\Pi^{dN}} e^{N\,F(L_\varepsilon\star\mu_N)}\,\bar\rho_N\,dX^N \leq I(F)$$
$$+\,\frac{C}{N^{1/(d+1)}\,\varepsilon^{d/(d+1)}}\,(\log N + |\log\varepsilon| + \|\log\bar\rho\|_{L^\infty}) + C\,\varepsilon\,\|\log\bar\rho\|_{W^{1,\infty}}.$$

The main idea to prove Prop. 3.2 is to connect the left-hand side to an hypercubes averaging quantity for which it is possible to apply the large deviation estimate in Prop. 3.1 that we have proved previously. The first

step is hence to replace the general $L_\varepsilon \star \mu_N$ first by $L_\varepsilon \star L_M[\mu_N]$ and then by $L_M[\mu_N]$ for the kernel L_M previously defined. First we observe that one has the following lemma

Lemma 3.1. *There exists C depending only on d and L s.t. for any measure $\mu \in \mathcal{P}(\Pi^d)$ and for $M\,\varepsilon \geq 1$*

$$\|L_\varepsilon \star \mu - L_M[L_\varepsilon \star \mu]\|_{L^1(\Pi^d)} \leq \frac{C}{M\,\varepsilon}.$$

Proof of Lemma 3.1. The proof is rather straightforward and consists in noticing for example that for any x, if L has support in $B(0,r)$ then

$$\begin{aligned} L_\varepsilon \star \mu(x) &= \sum_k \int_{C_k^M} L_\varepsilon(x-y)\,\mu(dy) \\ &= \sum_k \frac{1}{|C_k^M|} \int_{C_k^M} L_\varepsilon(x-z)\,dz \int_{C_k^M} \mu(dy) \\ &\quad + \sum_k \int_{C_k^M} \frac{1}{|C_k^M|} \int_{C_k^M} (L_\varepsilon(x-y) - L_\varepsilon(x-z))\,dz\,\mu(dy), \end{aligned}$$

so that

$$|L_\varepsilon \star \mu(x) - L_\varepsilon \star L_M[\mu](x)| \leq C\,\|L\|_{W^{1,\infty}}\,\frac{1}{M\,\varepsilon} \sum_k \int_{C_k^M} \frac{\mathbb{I}_{|x-y|\leq r\,\varepsilon + 1/M}}{\varepsilon^d}\,\mu(dy),$$

with

$$\sum_k \int_{C_k^M} \frac{\mathbb{I}_{|x-y|\leq r\,\varepsilon+1/M}}{\varepsilon^d}\,\mu(dy) \leq 2,$$

since μ is a probability measure $\int d\mu = 1$. □

Turning now back to the main proof.

Proof of Prop. 3.2. We start by using the large deviation functional (3.6) to find

$$\begin{aligned} &\frac{1}{N} \log \int_{\Pi^{dN}} e^{N\,F(L_\varepsilon \star \mu_N)}\,\bar\rho_N\,dX^N \\ &\quad \leq \frac{1}{N} \log \int_{\Pi^{dN}} e^{N\,I(F) + N\int L_\varepsilon \star \mu_N\,\log \frac{L_\varepsilon \star \mu_N}{\bar\rho}}\,\bar\rho_N\,dX^N \\ &\quad \leq I_F + \frac{1}{N}\,\log \int_{\Pi^{dN}} e^{N\int L_\varepsilon \star \mu_N\,\log \frac{L_\varepsilon \star \mu_N}{\bar\rho}}\,\bar\rho_N\,dX^N = I_F + I_N^\varepsilon. \end{aligned}$$

One then finds from Lemma 3.1 that

$$\begin{aligned} \int L_\varepsilon \star \mu_N\,\log \frac{L_\varepsilon \star \mu_N}{\bar\rho} &\leq \int L_\varepsilon \star L_M[\mu_N]\,\log \frac{L_\varepsilon \star L_M[\mu_N]}{\bar\rho} \\ &+ \|L_\varepsilon \star \mu_N - L_\varepsilon \star L_M[\mu_N]\|_{L^1}\,(|\log \varepsilon| + \|\log \bar\rho\|_{L^\infty}). \end{aligned}$$

Therefore

$$I_N^\varepsilon \leq \frac{1}{N} \log \int_{\Pi^{dN}} e^{N \int L_\varepsilon \star L_M[\mu_N] \log \frac{L_\varepsilon \star L_M[\mu_N]}{\bar\rho}} \bar\rho_N \, dX^N + \frac{C}{M\,\varepsilon} (|\log \varepsilon| + \|\log \bar\rho\|_{L^\infty}).$$

Now we may just recall that $x \log x$ is a convex function so

$$\int L_\varepsilon \star L_M[\mu_N] \, \log L_\varepsilon \star L_M[\mu_N] \leq \int L_\varepsilon \star (L_M[\mu_N] \, \log L_M[\mu_N]) = \int L_M[\mu_N] \, \log L_M[\mu_N].$$

Moreover using the definition of L_M, it is straightforward to check that

$$\int L_M[\mu_N] \, \log L_M[\mu_N] = \int \mu_N \, \log L_M[\mu_N].$$

Using now that $\log \bar\rho \in W^{1,\infty}$ and since $1/M \leq \varepsilon$, we simply have that

$$- \int L_\varepsilon \star L_M[\mu_N] \, \log \bar\rho \leq - \int \mu_N \, \log \bar\rho + C\,\varepsilon \, \|\log \bar\rho\|_{W^{1,\infty}}.$$

This leads to

$$I_N^\varepsilon \leq \frac{1}{N} \log \int_{\Pi^{dN}} e^{N \int \mu_N \, \log \frac{L_M[\mu_N]}{\bar\rho}} \bar\rho_N \, dX^N + \frac{C}{M\,\varepsilon} (|\log \varepsilon| + \|\log \bar\rho\|_{L^\infty}) + C\,\varepsilon \, \|\log \bar\rho\|_{W^{1,\infty}}.$$

Using finally Prop. 3.1, we find

$$I_N^\varepsilon \leq C\,\frac{M^d}{N} \, \log N + \frac{C}{M\,\varepsilon} (|\log \varepsilon| + \|\log \bar\rho\|_{L^\infty}) + C\,\varepsilon \, \|\log \bar\rho\|_{W^{1,\infty}}. \tag{3.7}$$

It only remains to optimize in M by choosing for example $M^{d+1} = N/\varepsilon$ to conclude. □

3.3. **Estimating the large deviation functional.** Prop. 3.2 controls the regularized large deviation inequality in terms of the classical large deviation functional. Our next step is hence to estimate this functional and to prove that for some potential size $I(F) = 0$ namely

Lemma 3.2. *For any $\bar\rho \in L^\infty$ and any $c < d$, there exists a truncation δ s.t. for any $\tilde V$ with $\|\tilde V\|_{L^1} \leq \delta$ and $\tilde V(x) \geq c \log |x|$, and the functional*

$$F_{\tilde V}(\mu) = - \int_{\{x \neq y\}} \tilde V(x-y) \, (\mu(dx) - \bar\rho(x)\,dx) \, (\mu(dy) - \bar\rho(y)\,dy),$$

one then has that $I(F_{\tilde V}) = 0$.

This lemma precisely explains why we decompose the potential V into short-range V_0 and long range W in the main proof. We will indeed later apply the lemma to $\tilde V = \frac{\lambda}{2\sigma} V_0$ and choose η s.t. $\|\tilde V\|_{L^1} < \delta$. Otherwise the

case of optimality in the logarithmic Hardy-Littlewood-Sobolev shows that $I(F) \neq 0$.

Proof of Lemma 3.2. We start by recalling the classical logarithmic Hardy-Littlewood-Sobolev inequality (See Theorem 1 in [10] for instance),

$$-\int_{\mathbb{R}^d} \log|x-y|\,\mu(dx)\,\mu(dy) \leq \frac{1}{d}\int_{\mathbb{R}^d} \mu\,\log\mu\,dx + C_d,$$

for some constant C_d depending only on d and any probability measure μ. We refer for instance to Dolbeault-Campos [13] for a discussion of the importance of this inequality for the Patlak-Keller-Segel system.

This inequality shows that for $c < d$, using that $\tilde{V} \geq -c\,\log|x|$

$$\begin{aligned} F_{\tilde{V}}(\mu) - \int \mu\,\log\frac{\mu}{\bar{\rho}}\,dx &\leq C_d\,((1+\|\tilde{V}\|_{L^1})\,\|\bar{\rho}\|_{L^\infty} + \|\log\bar{\rho}\|_{L^\infty}) \\ &\quad - (1-\frac{c}{d})\int \mu\,\log\frac{\mu}{\bar{\rho}}\,dx, \end{aligned}$$

and is hence coercive. This implies that, if we consider a maximizing sequence μ_n, then μ_n is bounded in $L\,\log L$ and any weak limit is a maximum for $F_{\tilde{V}}(\mu) - \int \mu\,\log\frac{\mu}{\bar{\rho}}\,dx$.

The value of $I(F_{\tilde{V}})$ is hence given by a such a maximal measure $\bar{\mu}$. By standard arguments, a maximum $\bar{\mu}$ must satisfy that on the support of $\bar{\mu}$

$$1 + \log\frac{\bar{\mu}}{\bar{\rho}} + 2\,\tilde{V}\star(\bar{\mu}-\bar{\rho}) = \kappa.$$

The constant κ is chosen so that $\int\bar{\mu} = 1$. Note that this implies that $\bar{\mu}$ cannot vanish on the support of $\bar{\rho}$ and hence

$$\bar{\mu} = \frac{\bar{\rho}}{M}\,e^{-2\,\tilde{V}\star(\bar{\mu}-\bar{\rho})}, \quad M = \int \bar{\rho}\,e^{-2\,\tilde{V}\star(\bar{\mu}-\bar{\rho})}\,dx.$$

Let us denote $u = -\tilde{V}\star(\bar{\mu}-\bar{\rho})$ and to emphasize the dependence on u in M

$$M = M_u = \int \bar{\rho}\,e^{2\,u(x)}\,dx.$$

We observe that u is a solution to

$$u = -\tilde{V}\star\left(\bar{\rho}\,\left(\frac{e^{2\,u(x)}}{M_u} - 1\right)\right), \tag{3.8}$$

which is in fact a sort of non-linear elliptic equation. Our goal is simply to show that the unique solution to (3.8) is $u = 0$ provided that δ is chosen small enough.

This is straightforward enough: First note that since $\bar{\mu} \in L\,\log L$ then $u \in L^\infty$ (by Lemma 1 in [23] for instance) and in fact there exists C depending only on $1 - c/d$, $\|\tilde{V}\|_{L^1}$ and $\|\bar{\rho}\|_{L^\infty}$ s.t. $\|u\|_{L^\infty} \leq C$. Therefore

$$\left|e^{2\,u(x)} - e^{2\,u(y)}\right| \leq C\,|u(x)-u(y)| \leq C(|u(x)| + |u(y)|)$$

and

$$e^{-C} \leq M_u = \int \bar{\rho}(x) e^{2\,u(x)} dx \leq e^C.$$

Hence

$$\begin{aligned}\|\bar{\mu} - \bar{\rho}\|_{L^1} &= \left\| \bar{\rho} \left(\frac{e^{2\,u(x)}}{M_u} - 1 \right) \right\|_{L^1} \\ &\leq \frac{1}{M_u} \int \int \bar{\rho}(x)\bar{\rho}(y) |e^{2\,u(x)} - e^{2\,u(y)}| \,\mathrm{d}x \,\mathrm{d}y \leq C \,\|u\|_{L^1},\end{aligned}$$

for some constant C. To conclude, we note using (3.8) that

$$\|u\|_{L^1} \leq C \,\|\tilde{V}\|_{L^1} \,\|u\|_{L^1},$$

and it is enough to take δ small enough s.t. $C\,\|\tilde{V}\|_{L^1} < 1$ to have that $u = 0$ and finally $I(F_\eta) = 0$. □

3.4. **Proof of Prop 2.1.** We are finally ready to prove Prop. 2.1. We start by using again Lemma 2.1 (following again Lemma 1 in [23]) to obtain that

$$\gamma \int_{\Pi^{dN}} F(\mu_N)\, \rho_N \, dX^N \leq \mathcal{H}_N(\rho_N \mid \bar{\rho}_N) + \frac{1}{N} \log \int \bar{\rho}_N \, e^{N\,\gamma\, F(\mu_N)} \, dX^N,$$

so that the whole question resolves around estimating

$$Z_N(\gamma) = \int_{\Pi^{dN}} \bar{\rho}_N \, e^{N\,\gamma\, F(\mu_N)} \, dX^N. \tag{3.9}$$

Of course if one studies the maximization problem

$$\sup_{\rho_N} \left(\gamma \int_{\Pi^{dN}} F(\mu_N)\, \rho_N \, dX^N - \mathcal{H}_N(\rho_N \mid \bar{\rho}_N) \right),$$

then the maximum is actually given by

$$\rho_N = \frac{1}{Z_N} \, \bar{\rho}_N \, e^{N\,\gamma\, F(\mu_N)}, \qquad Z_N = \int_{\Pi^{dN}} \bar{\rho}_N \, e^{N\,\gamma\, F(\mu_N)},$$

and inserting this in the maximization problem exactly leads to $\frac{1}{N} \log Z_N$ with Z_N given by (3.9).

• *Step 1: Introducing a regularized F_ε.* To estimate (3.9), we first introduce the regularized quantity

$$F_\varepsilon(\mu) = - \int_{\Pi^{2d} \cap \{x \neq y\}} V_\varepsilon(x - y) \, (d\mu - d\bar{\rho})^{\otimes 2},$$

with V_ε some regularized V_ε. We denote

$$Z_{N,\varepsilon}(\gamma) = \int_{\Pi^{dN}} \bar{\rho}_N \, e^{N\,\gamma\, F_\varepsilon(\mu_N)} \, dX^N,$$

and the main point is to bound $Z_N(\gamma)$ in terms of the regularized $Z_{N,\varepsilon}(\gamma')$ for some $\gamma' > \gamma$. The control on $Z_{N,\varepsilon}$ will be performed at the end of the proof

and follows in a straightforward manner from Prop. 3.2 and Lemma 3.2. To define V_ε, we decompose V_0 by writing

$$V_0 = \tilde{V} + \bar{V},$$

where

$$\tilde{V} = V_0 - \bar{V} \text{ with } \bar{V}(x) = \log|x|\,\chi(|x|/\eta),$$

so that in particular $\tilde{V} \geq 0$ and still satisfies

$$|\nabla \tilde{V}(x)| \leq \frac{C}{|x|^k} \text{ for } k > 1/2. \tag{3.10}$$

We now choose $V_\varepsilon(x) = \tilde{V}_\varepsilon(x) + \bar{V}_\varepsilon(x)$ where

$$\tilde{V}_\varepsilon(x) = \tilde{V}(x)\,(1 - \chi(|x|/\varepsilon^{1/2k})) \text{ with } \bar{V}_\varepsilon(x) = \log(\max(|x|,\,\varepsilon))\,\chi(|x|/\eta).$$

Remark that we truncate $\tilde{V}_\varepsilon$ at a much larger scale than $\bar{V}_\varepsilon$: $\varepsilon^{1/2k}$ vs ε.

• *Step 2: Identifying the close and singular interactions.* Our next step is to relate Z_N with

$$I = \int_{\Pi^{dN}} \bar{\rho}_N\, e^{N\,\gamma\,F_\varepsilon(\mu_N)} e^{\gamma\,\sum_{j>1}\log\frac{\varepsilon}{|x_1-x_j|}\,\mathbb{I}_{|x_1-x_j|\leq\varepsilon}}\,\mathrm{d}X^N. \tag{3.11}$$

The integral in I clearly separates the regularized F_ε from the singularity in V_0. Moreover it identifies one test particle, which we choose as particle 1, and compare all singularities through this particle.

By developing in Z_N, we have that

$$\begin{aligned} F(\mu_N) &= -\int_{\Pi^{2d}\cap\{x\neq y\}} V(x-y)\,(d\mu_N - d\bar{\rho})^{\otimes 2} \\ &= -\frac{1}{N^2}\sum_{i\neq j} V(x_i - x_j) + 2\frac{1}{N}\sum_i V\star\bar{\rho}(x_i) \\ &\quad - \int_{\Pi^{2d}\cap\{x\neq y\}} V(x-y)\,\bar{\rho}(x)\,\bar{\rho}(y)\,dx\,dy, \end{aligned} \tag{3.12}$$

with a similar formula for $F_\varepsilon(\mu_N)$. Observe that for any x,

$$\begin{aligned} |V_\varepsilon\star\bar{\rho}(x) - V\star\bar{\rho}(x)| &\leq \|V_\varepsilon - V\|_{L^1}\,\|\bar{\rho}\|_{L^\infty} \leq \|V\,\mathbb{I}_{|z|\leq\varepsilon^{1/2k}}\|_{L^1}\,\|\bar{\rho}\|_{L^\infty} \\ &\leq C\,\varepsilon^{1/2kp^*}\,\|\bar{\rho}\|_{L^\infty}. \end{aligned}$$

Further note that since $\tilde{V} \geq 0$, we have that $\tilde{V}(x) \geq \tilde{V}_\varepsilon(x)$ so this directly implies that

$$F(\mu_N) \leq F_\varepsilon(\mu_N) + C\,\|\bar{\rho}\|_{L^\infty}\,\varepsilon^{1/2kp^*} + \frac{1}{N^2}\sum_{i\neq j}\log\frac{\varepsilon}{|x_i-x_j|}\,\mathbb{I}_{|x_i-x_j|\leq\varepsilon}.$$

Hence we obtain that

$$\begin{aligned} Z_N \leq\; & e^{C\,N\,\varepsilon^{1/2kp^*}\,\|\bar{\rho}\|_{L^\infty}} \\ & \int_{\Pi^{dN}} \bar{\rho}_N\, e^{N\,\gamma\,F_\varepsilon(\mu_N) + \frac{\gamma}{N}\,\sum_{i\neq j}\log\frac{\varepsilon}{|x_i-x_j|}\,\mathbb{I}_{|x_i-x_j|\leq\varepsilon}}\,\mathrm{d}X^N. \end{aligned} \tag{3.13}$$

By general Hölder inequality

$$\begin{aligned}
&\int_{\Pi^{dN}} \bar\rho_N \, e^{N\,\gamma\, F_\varepsilon(\mu_N)+\frac{\gamma}{N}\,\sum_{i\neq j}\log\frac{\varepsilon}{|x_i-x_j|}\,\mathbb{I}_{|x_i-x_j|\leq\varepsilon}}\,\mathrm{d}X^N\\
&=\int_{\Pi^{dN}} \bar\rho_N \, e^{N\,\gamma\, F_\varepsilon(\mu_N)}\Pi_{i=1}^N e^{\frac{\gamma}{N}\,\sum_{j\neq i}\log\frac{\varepsilon}{|x_i-x_j|}\,\mathbb{I}_{|x_i-x_j|\leq\varepsilon}}\,\mathrm{d}X^N\\
&\leq \Pi_{i=1}^N\left(\int_{\Pi^{dN}} \bar\rho_N \, e^{N\,\gamma\, F_\varepsilon(\mu_N)} e^{\gamma\,\sum_{j\neq i}\log\frac{\varepsilon}{|x_i-x_j|}\,\mathbb{I}_{|x_i-x_j|\leq\varepsilon}}\,\mathrm{d}X^N\right)^{1/N}.
\end{aligned}$$

Using the symmetry of ρ_N, we can simply keep one of the factors, for example with $i=1$ yielding

$$\begin{aligned}
&\int_{\Pi^{dN}} \bar\rho_N \, e^{N\,\gamma\, F_\varepsilon(\mu_N)+\gamma\,\sum_{i\neq j}\log\frac{\varepsilon}{|x_i-x_j|}\,\mathbb{I}_{|x_i-x_j|\leq\varepsilon}}\,\mathrm{d}X^N\\
&\leq\int_{\Pi^{dN}} \bar\rho_N \, e^{N\,\gamma\, F_\varepsilon(\mu_N)} e^{\gamma\,\sum_{j>1}\log\frac{\varepsilon}{|x_1-x_j|}\,\mathbb{I}_{|x_1-x_j|\leq\varepsilon}}\,\mathrm{d}X^N = I.
\end{aligned}$$

Combined with (3.13), this gives

$$Z_N \leq e^{C\,N\,\varepsilon^{1/2kp^*}\,\|\bar\rho\|_{L^\infty}}\,I. \tag{3.14}$$

• *Step 3: Introducing the functional with a "frozen" test particle.* Since we will take ε very small, it is natural to expect that there will only be a limited number of indices j s.t. $|x_1-x_j|\leq\varepsilon$. To make that precise, we introduce the number n of such indices j. Up to permutations, we may also assume that those are $j=2,\cdots,n+1$ and we decompose accordingly

$$\begin{aligned}
I&=\int_{\Pi^{dN}} \bar\rho_N \, e^{N\,\gamma\, F_\varepsilon(\mu_N)} e^{\gamma\,\sum_{j>1}\log\frac{\varepsilon}{|x_1-x_j|}\,\mathbb{I}_{|x_1-x_j|\leq\varepsilon}}\,\mathrm{d}X^N\\
&=\sum_{n=0}^{N-1}\binom{N-1}{n}\int_{\Pi^d}\mathrm{d}x_1\int_{|x_1-x_i|\leq\varepsilon,\,\forall i=2\ldots n+1}\mathrm{d}x_2\ldots\,\mathrm{d}x_{n+1}\\
&\int_{|x_1-x_j|>\varepsilon,\,\forall j>n+1}\mathrm{d}x_{n+2}\ldots\,\mathrm{d}x_N\,\bar\rho_N\,e^{N\,\gamma\,F_\varepsilon(\mu_N)}e^{\gamma\,\sum_{i=2}^{n+1}\log\frac{\varepsilon}{|x_1-x_i|}}.
\end{aligned} \tag{3.15}$$

In this expression, one should first observe that $F_\varepsilon(\mu_N)$ mostly do not depend on $x_2,\cdots,x_{n+1}$. For this, denote $\mu_N^{1,n}$ the empirical measure obtained by replacing all $x_2\ldots x_{n+1}$ by x_1

$$\mu_N^{1,n}=\frac{n+1}{N}\delta(x-x_1)+\frac{1}{N}\sum_{j>n+1}\delta(x-x_j),$$

and denote accordingly

$$F_\varepsilon^{1,n}(\mu_N)=F_\varepsilon(\mu_N^{1,n}).$$

Now $F_\varepsilon^{1,n}(\mu_N)$ does not depend on $x_2\ldots x_{n+1}$ but since x_i and x_1 are close if $i=2\ldots n+1$, we still expect it to be close to $F_\varepsilon(\mu_N)$. Our next steps aim at making this precise.

• *Step 4: Comparing the potential V_ε for close particles.* We derive here the following estimate for $i = 2 \dots n+1$

(3.16)
$$|\bar V_\varepsilon(x_i - x_j) - \bar V_\varepsilon(x_1 - x_j)| \le C_d\, \varepsilon^{1/2} + \begin{cases} \log 2 + \dfrac{C}{\eta}\, \varepsilon \text{ if } |x_1 - x_j| \le \varepsilon^{1/2}, \\ 2\, \varepsilon^{1/2} + \dfrac{C}{\eta}\, \varepsilon \quad \text{if } |x_1 - x_j| \ge \varepsilon^{1/2}. \end{cases}$$

We first recall that $k > 1/2$. Hence for $|x_i - x_1| \le \varepsilon$, we have that if $|x_i - x_j| \le (\varepsilon^{1/2k})/2$, then $|x_1 - x_j| \le \varepsilon^{1/2k}$ and then

$$\tilde V_\varepsilon(x_i - x_j) = \tilde V_\varepsilon(x_1 - x_j) = 0.$$

Otherwise we necessarily have $|x_i - x_j|,\ |x_1 - x_j| \ge (\varepsilon^{1/2k}/4)$ and recalling (3.10), we obtain

$$\begin{aligned} |\tilde V_\varepsilon(x_i - x_j) - \tilde V_\varepsilon(x_1 - x_j)| &\le C_d\, \varepsilon \max_{|z| \ge \frac{\varepsilon^{1/2k}}{2}} |\nabla \tilde V(z)| \\ &\le C_d\, \frac{\varepsilon}{\varepsilon^{1/2}} = C_d\, \varepsilon^{1/2}. \end{aligned} \tag{3.17}$$

On the other hand for $|x_1 - x_i| \le \varepsilon$, observe that we always have that

$$\frac{1}{2}\, \max(\varepsilon, |x_1 - x_j|) \le \max(\varepsilon, |x_i - x_j|) \le 2\, \max(\varepsilon, |x_1 - x_j|).$$

Indeed one has trivially

$$\max(\varepsilon, |x_i - x_j|) \le \max(\varepsilon, |x_1 - x_j|) + \varepsilon \le 2\, \max(\varepsilon, |x_1 - x_j|),$$

and of course

$$\max(\varepsilon, |x_1 - x_j|) \le \max(\varepsilon, |x_i - x_j|) + \varepsilon \le 2\, \max(\varepsilon, |x_i - x_j|).$$

This implies that

$$\left| \log \frac{1}{\max(\varepsilon, |x_i - x_j|)} - \log \frac{1}{\max(\varepsilon, |x_1 - x_j|)} \right| \le \log 2.$$

If $|x_1 - x_j| \ge \varepsilon^{1/2}$ then we can be more precise as

$$|\log a - \log b| = \int_{[a,\, b]} \frac{dx}{x} \le |a - b|\, \max(1/a, 1/b).$$

In this case of course we have that $|x_i - x_j| \ge |x_1 - x_j| - \varepsilon \ge \varepsilon^{1/2}/2$ so

$$\left| \log \frac{1}{\max(\varepsilon, |x_i - x_j|)} - \log \frac{1}{\max(\varepsilon, |x_1 - x_j|)} \right| \le 2\, \varepsilon^{-1/2}\, |x_1 - x_i| \le 2\, \varepsilon^{1/2}.$$

To summarize one has that

$$\begin{aligned} &\left| \log \frac{1}{\max(\varepsilon, |x_i - x_j|)} - \log \frac{1}{\max(\varepsilon, |x_1 - x_j|)} \right| \\ &\qquad \le \begin{cases} \log 2 & \text{if } |x_1 - x_j| \le \varepsilon^{1/2} \\ 2\, \varepsilon^{1/2} & \text{if } |x_1 - x_j| \ge \varepsilon^{1/2}. \end{cases} \end{aligned}$$

And since the truncation χ is smooth, this yields

$$|\bar V_\varepsilon(x_i-x_j)-\bar V_\varepsilon(x_1-x_j)|\le\begin{cases}\log 2+\dfrac{C}{\eta}\,\varepsilon \text{ if } |x_1-x_j|\le\varepsilon^{1/2},\\ 2\,\varepsilon^{1/2}+\dfrac{C}{\eta}\,\varepsilon \quad \text{if } |x_1-x_j|\ge\varepsilon^{1/2}.\end{cases}\tag{3.18}$$

Combining (3.18) with (3.17) proves (3.16).

• *Step 5: Introducing the intermediary scale $\varepsilon^{1/2}$.* The inequality (3.16) shows that those j s.t. $|x_1-x_j|\le\varepsilon^{1/2}$ will be playing a different role from those j s.t. $|x_1-x_j|>\varepsilon^{1/2}$. This leads us to introduce $n_{1/2}$ the number of j s.t. $\varepsilon\le|x_1-x_j|\le\varepsilon^{1/2}$. Using again developments such as (3.12), we have that

$$\begin{aligned}|F_\varepsilon^{1,n}(\mu_N)-F_\varepsilon(\mu_N)|&\le C\,\varepsilon\,\|\bar\rho\|_{L^\infty}\\&+\frac{1}{N^2}\sum_{i=2}^{n+1}\sum_{j\ge n+2}|V_\varepsilon(x_i-x_j)-V_\varepsilon(x_1-x_j)|\\&+\frac{1}{N^2}\sum_{i=2}^{n+1}\sum_{j=2,j\neq i}^{n+1}|V_\varepsilon(x_i-x_j)-V_\varepsilon(0)|.\end{aligned}$$

Therefore, one has

$$F_\varepsilon(\mu_N)\le F_\varepsilon^{1,n}(\mu_N)+C\,\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+C\,\varepsilon^{1/2}+\frac{n^2+n\,n_{1/2}}{N^2}\,\log 2.\tag{3.19}$$

Note that by symmetry between x_1 and x_i in the above bounds, we also have the symmetric

$$F_\varepsilon^{1,n}(\mu_N)\le F_\varepsilon(\mu_N)+C\,\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+C\,\varepsilon^{1/2}+\frac{n^2+n\,n_{1/2}}{N^2}\,\log 2,\tag{3.20}$$

which we will use later. Going back to (3.15) and using (3.19), we may again freely assume that the j s.t. $\varepsilon\le|x_1-x_j|\le\varepsilon^{1/2}$ are those indices from $n+2$ to $n+n_{1/2}+1$. We find that

(3.21)

$$\begin{aligned}I&=\int_{\Pi^{dN}}\bar\rho_N\,e^{N\,\gamma\,F_\varepsilon(\mu_N)}e^{\gamma\,\sum_{j>1}\log\frac{\varepsilon}{|x_1-x_j|}\,\mathbb{I}_{|x_1-x_j|\le\varepsilon}}\,\mathrm{d}X^N\\&\le e^{C\,N\,(\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+\varepsilon^{1/2})}\sum_{n=0}^{N-1}\binom{N-1}{n}\sum_{n_{1/2}=0}^{N-1-n}\binom{N-n-1}{n_{1/2}}2^{n+n_{1/2}}\,J_{n,n_{1/2}}\end{aligned}$$

with

$$J_{n,n_{1/2}} = \int_{\Pi^d} \mathrm{d}x_1 \int_{|x_1-x_i|\le\varepsilon,\ \forall i=2\dots n+1} \mathrm{d}x_2 \dots \mathrm{d}x_{n+1}$$
$$\int_{\varepsilon<|x_1-x_j|\le\varepsilon^{1/2},\ \forall j=n+2\dots n+1+n_{1/2}} \mathrm{d}x_{n+2} \dots \mathrm{d}x_{n+n_{1/2}+1}$$
$$\int_{|x_1-x_k|>\varepsilon^{1/2},\ \forall k>n+1+n_{1/2}} \mathrm{d}x_{n+n_{1/2}+2} \cdots \mathrm{d}x_N\, \bar\rho_N\, e^{N\gamma F_\varepsilon^{1,n}(\mu_N)} e^{\gamma \sum_{i=2}^{n+1} \log \frac{\varepsilon}{|x_1-x_i|}}.$$

• *Step 6: Bounding $J_{n,n_{1/2}}$ back in terms of F_ε.* We first recall that $F_\varepsilon^{1,n}(\mu_N)$ does not depend on $x_2, \cdots, x_{n+1}$ since it only depends on $\mu_N^{1,n}$. Hence the integrals in $J_{n,n_{1/2}}$ nicely separate. Moreover since $\gamma < d$

$$\begin{aligned}
&\int_{|x_1-x_i|\le\varepsilon,\ \forall i=2\dots n+1} \mathrm{d}x_2 \dots \mathrm{d}x_{n+1} e^{\gamma \sum_{i=2}^{n+1} \log \frac{\varepsilon}{|x_1-x_j|}} \\
&= \left(\int_{|x_1-y|\le\varepsilon} \frac{\varepsilon^\gamma}{|x_1-y|^\gamma}\, dy \right)^n \\
&= \bar C^n\, \varepsilon^{dn} = \bar C^n \int_{|x_1-x_i|\le\varepsilon,\ \forall i=2\dots n+1} \mathrm{d}x_2 \dots \mathrm{d}x_{n+1},
\end{aligned}$$

with $\bar C \sim \frac{1}{d-\gamma}$. Therefore, one may obtain that

$$\begin{aligned}
J_{n,n_{1/2}} \le \bar C^n \int_{\Pi^d} \mathrm{d}x_1 &\int_{|x_1-x_i|\le\varepsilon,\ \forall i=2\dots n+1} \mathrm{d}x_2 \dots \mathrm{d}x_{n+1} \\
&\int_{\varepsilon<|x_1-x_j|\le\varepsilon^{1/2},\ \forall j=n+2\dots n+1+n_{1/2}} \mathrm{d}x_{n+2} \dots \mathrm{d}x_{n+n_{1/2}+1} \\
&\int_{|x_1-x_k|>\varepsilon^{1/2},\ \forall k>n+1+n_{1/2}} \mathrm{d}x_{n+2} \cdots \mathrm{d}x_N\, \bar\rho_N\, e^{N\gamma F_\varepsilon^{1,n}(\mu_N)}.
\end{aligned}$$

Now that we have used the key property of $F_\varepsilon^{1,n}$, it is more convenient to change it back to $F_\varepsilon(\mu_N)$ by using the reverse inequality (3.20),

$$J_{n,n_{1/2}} \le e^{C\,N\,\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+C\,N\,\varepsilon^{1/2}}\, 2^{n+n_{1/2}}\, \bar C^n\, \bar J_{n,n_{1/2}}, \tag{3.22}$$

where

$$\begin{aligned}
\bar J_{n,n_{1/2}} \le \int_{\Pi^d} \mathrm{d}x_1 &\int_{|x_1-x_i|\le\varepsilon,\ \forall i=2\dots n+1} \mathrm{d}x_2 \dots \mathrm{d}x_{n+1} \\
&\int_{\varepsilon<|x_1-x_j|\le\varepsilon^{1/2},\ \forall j=n+2\dots n+1+n_{1/2}} \mathrm{d}x_{n+2} \dots \mathrm{d}x_{n+n_{1/2}+1} \\
&\int_{|x_1-x_k|>\varepsilon^{1/2},\ \forall k>n+1+n_{1/2}} \mathrm{d}x_{n+n_{1/2}+2} \cdots \mathrm{d}x_N\, \bar\rho_N\, e^{N\gamma F_\varepsilon(\mu_N)}.
\end{aligned}$$

• *Step 7: The final bound on I: Reconstructing the full integral.* We now wish to "undo" the decompositions performed at step 5 and earlier at step 3

where we introduced $n_{1/2}$ and n. In other words, using (3.21) and (3.22), we aim at expressing

$$I \leq e^{C\,N\,\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+C\,N\,\varepsilon^{1/2}} \sum_{n=0}^{N-1} \binom{N-1}{n} \sum_{n_{1/2}=0}^{N-1-n} \binom{N-n-1}{n_{1/2}} 4^{n+n_{1/2}}\,\bar C^n\,\bar J_{n,n_{1/2}},$$

in terms of the full integral over $\bar\rho_N\,e^{N\,\gamma\,F_\varepsilon(\mu_N)}$. Unfortunately we cannot directly reverse the decomposition because of the extra factor $\bar C^n\,4^{n+n_{1/2}}$ in the sums over n and $n_{1/2}$.

We do expect the probability of n or $n_{1/2}$ being of order N to be extremely small of course and this issue can be solved by performing a last Hölder estimate at exponent γ'/γ for some γ'.

$$I \leq e^{C\,N\,\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+C\,N\,\varepsilon^{1/2}}\,R_1^{1-\gamma/\gamma'}\,R_2^{\gamma/\gamma'},$$

where

$$R_1 = \sum_{n=0}^{N-1} \binom{N-1}{n} \sum_{n_{1/2}=0}^{N-1-n} \binom{N-n-1}{n_{1/2}} 4^{\Lambda(n+n_{1/2})} \bar C^{\Lambda\,n}\,\varepsilon^{d\,n}\,\varepsilon^{d\,n_{1/2}/2},$$

with $\Lambda = \left(1-\frac{\gamma}{\gamma'}\right)^{-1}$. On the other hand, we have

$$\begin{aligned} R_2 = \sum_{n=0}^{N-1} \binom{N-1}{n} \sum_{n_{1/2}=0}^{N-1-n} \binom{N-n-1}{n_{1/2}} \\ \int_{\Pi^d} \mathrm{d}x_1 \int_{|x_1-x_i|\leq\varepsilon,\ \forall i=2\ldots n+1} \mathrm{d}x_2\ldots\,\mathrm{d}x_{n+1} \\ \int_{\varepsilon<|x_1-x_j|\leq\varepsilon^{1/2},\ \forall j=n+2\ldots n+1+n_{1/2}} \mathrm{d}x_{n+2}\ldots\,\mathrm{d}x_{n+n_{1/2}+1} \\ \int_{|x_1-x_k|>\varepsilon^{1/2},\ \forall k>n+1} \mathrm{d}x_{n+n_{1/2}+2}\cdots\,\mathrm{d}x_N\,\bar\rho_N\,e^{N\,\gamma'\,F_\varepsilon(\mu_N)}. \end{aligned}$$

We easily have that

$$R_1 \leq (1+2\,\varepsilon)^N,$$

provided that $4^\Lambda\,\bar C^\Lambda\,\varepsilon^d \leq \varepsilon$ and $4^\Lambda\,\varepsilon^{d/2} \leq \varepsilon$. As for R_2, we may now easily reverse the decomposition implemented in steps 3 and 5 to find

$$R_2 = \int_{\Pi^{dN}} \bar\rho_N\,e^{N\,\gamma'\,F_\varepsilon(\mu_N)}\,\mathrm{d}X^N = Z_{N,\varepsilon}(\gamma').$$

Therefore we obtain that

$$\begin{aligned} I &= \int_{\Pi^{dN}} \bar\rho_N\,e^{N\,\gamma\,F_\varepsilon(\mu_N)} e^{\gamma\,\sum_{j>1}\log\frac{\varepsilon}{|x_1-x_j|}\,\mathbb{I}_{|x_1-x_j|\leq\varepsilon}}\,\mathrm{d}X^N \\ &\leq e^{C\,N\,\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+C\,\varepsilon^{1/2}}\,(1+2\,\varepsilon)^N\,(Z_{N,\varepsilon}(\gamma'))^{\frac{\gamma}{\gamma'}}. \end{aligned} \tag{3.23}$$

• Final Step: Using Prop. 3.2 and Lemma 3.2. Let us first gather all our estimates: By inserting (3.23) into (3.14), we have proved so far that

$$Z_N(\gamma) \leq e^{C\,N\,\varepsilon\,(\|\bar\rho\|_{L^\infty}+\eta^{-1})+C\,N\,\varepsilon^{1/2kp^*}+C\,\varepsilon^{1/2}}\,(1+2\,\varepsilon)^N\,(Z_{N,\varepsilon}(\gamma'))^{\frac{\gamma}{\gamma'}}. \tag{3.24}$$

It only remains to bound $Z_{N,\varepsilon}(\gamma')$. Note that since V_ε is smooth, for any convolution kernel L, we have that

$$|V_\varepsilon - L_{\varepsilon'} \star L_{\varepsilon'} \star V_\varepsilon| \leq C\,\frac{\varepsilon'}{\varepsilon},$$

and consequently

$$\frac{1}{N}\,\log Z_{N,\varepsilon}(\gamma') \leq C\,\frac{\varepsilon'}{\varepsilon} + \frac{1}{N}\,\log \int_{\Pi^{dN}} \bar\rho_N\, e^{N\,\gamma\,F_\varepsilon(L_{\varepsilon'}\star\mu_N)}\,dX^N.$$

The estimate is now straightforward thanks to the Prop. 3.2 which directly shows that

$$\begin{aligned}\frac{1}{N}\,\log Z_{N,\varepsilon}(\gamma') \leq I(\gamma'\,F_\varepsilon) + \frac{C}{N^{\frac{1}{d+1}}\,\varepsilon'^{\frac{d}{d+1}}}\,(\log N + |\log\varepsilon'| + \|\log\bar\rho\|_{L^\infty}) \\ + C\,\varepsilon'\,\|\log\bar\rho\|_{W^{1,\infty}} + C\,\frac{\varepsilon'}{\varepsilon}.\end{aligned}$$

From (3.24), and since we will take $\varepsilon' < \varepsilon$, this implies that

$$\begin{aligned}\frac{1}{N}\,\log Z_N(\gamma) \leq I(\gamma'\,F_\varepsilon) + \frac{C}{N^{1/(d+1)}\,\varepsilon'^{d/(d+1)}}\,(\log N + |\log\varepsilon'| + \|\log\bar\rho\|_{L^\infty}) \\ + C\,\varepsilon\,(\eta^{-1} + \|\log\bar\rho\|_{W^{1,\infty}} + \|\bar\rho\|_{L^\infty}) + C\,\varepsilon^{1/2} + C\,N\,\varepsilon^{1/2kp^*} + C\,\frac{\varepsilon'}{\varepsilon}.\end{aligned}$$

We may estimate $I(\gamma'\,F_\varepsilon)$ through Lemma 3.2. We observe that as long as $\gamma' < d$, we indeed have that $V_\varepsilon(x) \geq c\,\log|x|$ for some $c < d$. On the other hand, since $V \in L^p$ and $\operatorname{supp} V \in B(0,\eta)$, by choosing η small enough, we can guarantee that $\|V\|_{L^1} \leq \delta$. Now simply taking for example $\varepsilon' = N^{-1/(2d+1)}$ and $\varepsilon = \sqrt{\varepsilon}$, we deduce that provided $\gamma' < d$, there exists some $\theta > 0$ s.t.

$$\begin{aligned}\frac{1}{N}\,\log Z_{N,\varepsilon}(\gamma') \leq &\frac{C}{N^{1/(2d+1)}}\,(\log N + \|\log\bar\rho\|_{W^{1,\infty}} + \eta^{-1}) \\ &+ \frac{C}{N^\theta}.\end{aligned}$$

Note that of course $\varepsilon >> N^{-1/d}$. On the other hand we need N large enough so that $\varepsilon^{d-1} = N^{-(d-1)/2\,(2d+1)} \leq \bar C^{-\Lambda}$ which gives the condition on N in the proposition.

4. Appendix

4.1. **Large deviation type estimates in** [23]**.** We first recall and prove Lemma 2.1 which was Lemma 1 in [23]

Lemma 4.1. *For any $\rho_N,\ \bar\rho_N \in \mathcal{P}(\Pi^{dN})$, any test function $\psi \in L^\infty(\Pi^{dN})$, one has that for any $\alpha > 0$,*

$$\int_{\Pi^{dN}} \psi(X^N)\, d\rho_N \le \frac{1}{\alpha}\frac{1}{N}\int d\rho_N\, \log\frac{\rho_N}{\bar\rho_N} + \frac{1}{\alpha}\frac{1}{N}\log\int_{\Pi^{dN}} e^{\alpha N\,\psi(X^N)}\, d\bar\rho_N.$$

Proof. Without loss of generality, we assume that $\alpha = 1$. Define

$$f = \frac{1}{\lambda}\, e^{N\,\psi}\,\bar\rho_N, \quad \lambda = \int_{\Pi^{d\,N}} d\bar\rho_N\, e^{N\,\psi}.$$

Notice that f is a probability density as $f \ge 0$ and $\int f = 1$. Hence by the convexity of the entropy

$$\frac{1}{N}\int_{\Pi^{d\,N}} \rho_N\, \log f\, dX^N \le \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N\, \log\rho_N\, dX^N.$$

Expanding the left-hand side

$$\frac{1}{N}\int_{\Pi^{d\,N}} \rho_N\, \log f\, dX^N = \int_{\Pi^{d\,N}} \rho_N\, \Phi\, dX^N + \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N\, \log\bar\rho_N\, dX^N - \frac{\log\lambda}{N},$$

gives the desired inequality. □

Lemma 4.1 directly connects bounds on quantities like $\int \psi(X^N) d\rho_N$ to the relative entropy $\mathcal{H}_N$ and estimates on quantities that can be seen as partition functions

$$\int_{\Pi^{dN}} e^{N\int_{\Pi^{2d}} f(x,y)\,(d\mu_N - d\bar\rho)^{\otimes 2}}\, \bar\rho^{\otimes N}\, dX^N. \tag{4.1}$$

It is hence natural to try to use large deviation type of tools to bound (4.1). Note however that our goals here are different from classical large deviation approaches: We do not try to calculate the limit as $N \to \infty$ of (4.1) but instead to obtain bounds that are uniform in N.

We now recall the estimate from [23]

Theorem 4.1. (Theorem 4 in [23]). *Consider $\bar\rho \in L^1(\Pi^d)$ with $\bar\rho \ge 0$ and $\int_{\Pi^d} \bar\rho\, \mathrm{d}x = 1$. Consider further any $\phi(x,z) \in L^\infty$ with*

$$\gamma := C\left(\sup_{p\ge 1} \frac{\|\sup_z |\phi(.,z)|\|_{L^p(\bar\rho\,\mathrm{d}x)}}{p}\right)^2 < 1,$$

where C is a universal constant. Assume that ϕ satisfies the following cancellations

$$\int_{\Pi^d} \phi(x,z)\,\bar\rho(x)\, dx = 0 \quad \forall z, \qquad \int_{\Pi^d} \phi(x,z)\,\bar\rho(z)\, dz = 0 \quad \forall x. \tag{4.2}$$

Then

$$\int_{\Pi^{d\,N}} \bar\rho_N \exp\left(\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right) \mathrm{d}X^N \le \frac{2}{1-\gamma} < \infty, \tag{4.3}$$

where again $\bar\rho_N = \bar\rho^{\otimes N}$.

We may directly deduce Theorem 2.2 from this. Given a configuration $X^N = (x_1, \cdots, x_N)$ or $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, just write

$$N \int_{\Pi^{2d}} f(x,y)(d\mu_N - d\bar{\rho})^{\otimes 2} = \frac{1}{N}\sum_{i,j=1}^N \phi(x_i, x_j),$$

where

$$\phi(x,y) = f(x,y) - \int_{\Pi^d} f(x,w)\bar{\rho}(w)dw \\ - \int_{\Pi^d} f(z,y)\bar{\rho}(z)dz + \int_{\Pi^{2d}} f(z,w)\bar{\rho}(z)\bar{\rho}(w)dzdw.$$

This new ϕ is a symmetrization of f according to the reference measure $\bar{\rho}$ and ϕ indeed satisfies two cancellation rules in Theorem 4.1, i.e.

$$\int_{\Pi^d} \phi(x,y)\bar{\rho}(y)dy = 0, \forall x, \quad \int_{\Pi^d} \phi(x,y)\bar{\rho}(x)dx = 0, \forall y. \tag{4.4}$$

Finally $\|\phi\|_{L^\infty} \leq 4\|f\|_{L^\infty}$ so that we only need to take α small enough such that

$$\gamma = C\left(\sup_{p\geq 1} \frac{\|\sup_z |\alpha\,\phi(.,z)|\|_{L^p(\bar{\rho}\,\mathrm{d}x)}}{p}\right)^2 \leq 16\,C\,\alpha^2\,\|f\|_{L^\infty}^2 < 1.$$

We also want to emphasize here that in the case $\phi \in L^\infty$, a probabilistic proof of Theorem 4.1 was recently obtained in [24].

4.2. **Existence of entropy solution for the Liouville equation with the Patlak-Keller-Segel interaction kernel in 2D.** For the reader's convenience, we prove here the existence of an entropy solution to Eq. (1.5) in the case of the Patlak-Keller-Segel interaction kernel in dimension 2 namely

$$V(x) = \lambda \log|x| + V_e(x) \tag{4.5}$$

with $0 < \lambda < 2d\sigma$ and V_e a smooth correction so that V is periodic.

In general obtaining well-posedness to (1.5) may require a different set of assumptions than what we need to derive the mean field limit. In particular for existence as here, we need to be more specific than just asking $V(x) \geq \lambda \log|x|$ together with bounds on $|\nabla V|$. Moreover we emphasize that the argument below only shows existence of solutions to the Liouville equation (1.5). The existence of solutions to the original coupled SDE system (1.1) is much more difficult and essentially open out of the diffusion-dominated regime studied in [17]. Here we prove

Proposition 4.1. *Assume that the initial data ρ_N^0 satisfies that*

$$\int_{\Pi^{dN}} \rho_N^0 \log\left(\frac{\rho_N^0}{G_{N,\varepsilon}}\right) < \infty.$$

Then there exists a global in time entropy solution to (1.5) *with* V *given by* (4.5).

Proof. We consider the following regularization of V

$$\bar{V}_\varepsilon = \lambda \log \max(|x|, \varepsilon).$$

Since $\bar{V}_\varepsilon$ is now Lipschitz, we trivially have existence of a smooth solution $\rho_{N,\varepsilon}$ to (1.5) for this interaction kernel (it is a standard, linear advection-diffusion equation). The goal is of course to pass to the limit in $\rho_{N,\varepsilon}$ with two difficulties: Handle the singular interaction terms $\nabla V_\varepsilon(x_i - x_j)\,\rho_{N,\varepsilon}$ and obtain the non-linear entropy bound at the limit.

The first step is to use Prop. 2.2 for this corresponding solution $\rho_{N,\varepsilon}$ (again this is straightforward since $\bar{V}_\varepsilon$ is Lipschitz). This yields the entropy bound

$$\begin{aligned} &\int_{\Pi^{dN}} \rho_{N,\varepsilon}(t, X^N) \log\big(\frac{\rho_{N,\varepsilon}(t, X^N)}{G_{N,\varepsilon}}\big)\, dX^N \\ &+ \sigma \sum_{i=1}^N \int_0^t \int_{\Pi^{dN}} \rho_{N,\varepsilon}(s, X^N) \Big|\nabla_{x_i} \log\big(\frac{\rho_{N,\varepsilon}(s, X^N)}{G_{N,\varepsilon}}\big)\Big|^2 dX^N ds \\ &\leq \int_{\Pi^{dN}} \rho_N^0 \log\big(\frac{\rho_N^0}{G_{N,\varepsilon}}\big)\, dX^N. \end{aligned} \tag{4.6}$$

The next step is to use our large deviation estimates, namely Prop. 2.1: Since $\lambda < 2d\sigma$, we have that for some constant C independent of ε

$$\int_{\Pi^{dN}} \rho_{N,\varepsilon} \log G_{N,\varepsilon} \leq C, \quad \int_{\Pi^{dN}} \rho_{N,\varepsilon} \log \rho_{N,\varepsilon} \leq C. \tag{4.7}$$

This implies that $\rho_{N,\varepsilon}$ is equi-integrable in X^N and we may now extract a converging subsequence (still denoted by $\rho_{N,\varepsilon}$) s.t. $\rho_{N,\varepsilon} \to \rho_N$ weakly in $L^\infty([0,\ T],\ L^1(\Pi^{dN}))$.

Next we start to use the specific structure of V_ε. Since $V \leq 0$, $G_{N,\varepsilon}$ is increasing in ε (when $\varepsilon \to 0$). Moreover $G_{N,\varepsilon} \geq 1$ so in particular $1/G_{N,\varepsilon}$ is bounded in L^∞ by 1 and converges pointwise to $1/G_N$. It hence converges *strongly* in every space between L^1 and L^∞ strictly. Let us now denote

$$X_\varepsilon = \frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}}.$$

We next observe that X_ε converges weakly to $X = \rho_N/G_N$ as $\varepsilon \to 0$. This is a consequence of the equi-integrability of $\rho_{N,\varepsilon}$ given by (4.7) and the above strong convergence of $G_{N,\varepsilon}$. To be more specific, fix any ε_0, then for $\varepsilon \leq \varepsilon_0$, and any smooth test function ϕ,

$$\int \frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}}\,\phi = \int_{G_{N,\varepsilon_0} \leq M} \frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}}\,\phi + \int_{G_{N,\varepsilon_0} \geq M} \frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}}\,\phi,$$

where we choose any M s.t. $M < \exp(\frac{1}{N\sigma}\log\frac{1}{\varepsilon_0})$. As $G_{N,\varepsilon} \geq 1$, we have that

$$\int_{G_{N,\varepsilon_0}\geq M} \frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}}\,\phi \leq \int_{G_{N,\varepsilon_0}\geq M} \rho_{N,\varepsilon}\,|\phi| \leq C\,\frac{\|\phi\|_{L^\infty}}{\log M}$$

by the first point of (4.7) since $\{G_{N,\varepsilon_0} \geq M\} \subset \{G_{N,\varepsilon} \geq M\}$. By doing the same estimate at the limit, we have that

$$\left|\int \phi\left(\frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}} - \frac{\rho_N}{G_N}\right)\right| \leq C\,\frac{\|\phi\|_{L^\infty}}{\log M} + \left|\int_{G_{N,\varepsilon_0}\leq M} \phi\left(\frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}} - \frac{\rho_N}{G_N}\right)\right|.$$

But now we note that $G_{N,\varepsilon}$ is in fact uniformly smooth in ε on $G_{N,\varepsilon_0} \leq M$. Indeed denoting by $\delta = \min_{i\neq j}|x_i - x_j|$, we have the trivial bound $G_{N,\varepsilon_0} \geq \exp(-\frac{1}{N\sigma}\log\max(\delta,\varepsilon_0))$. We recall that $M < \exp(\frac{1}{N\sigma}\log\frac{1}{\varepsilon_0})$ which implies that $\exp(\frac{1}{N\sigma}\log\frac{1}{\delta}) \leq M$ thus providing an explicit control on δ in terms of M. Hence on $G_{N,\varepsilon_0} \leq M$, $G_{N,\varepsilon}$ convergence smoothly to G_ε. By the weak convergence of $\rho_{N,\varepsilon}$, we have that

$$\limsup_{\varepsilon\to 0}\left|\int \phi\left(\frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}} - \frac{\rho_N}{G_N}\right)\right| \leq C\,\frac{\|\phi\|_{L^\infty}}{\log M}.$$

Taking now $\varepsilon_0 \to 0$ and hence $M \to \infty$, we conclude that $X_\varepsilon \to X$.

Now the entropy inequality (2.5) at ε implies that for any $\varepsilon_0 \geq \varepsilon$

$$\int_{\Pi^{dN}} \rho_{N,\varepsilon}\,\log\frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}} + \sigma\int_0^t\int_{\Pi^{dN}} \frac{|\nabla X_\varepsilon|^2}{X_\varepsilon}\,G_{N,\varepsilon_0} \leq \int_{\Pi^{dN}} \rho_N^0\,\log\frac{\rho_N^0}{G_{N,\varepsilon}}, \tag{4.8}$$

again because $G_{N,\varepsilon_0} \leq G_N$.

Convexity and the large deviation estimates from Prop. 2.1 show that

$$\int_{\Pi^{dN}} \rho_N\,\log\frac{\rho_N}{G_N} \leq \liminf_{\varepsilon\to 0}\int_{\Pi^{dN}} \rho_{N,\varepsilon}\,\log\frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}}.$$

Since $G_{N,\varepsilon}$ is increasing in ε, we also have directly that

$$\int_{\Pi^{dN}} \rho_N^0\,\log\frac{\rho_N^0}{G_{N,\varepsilon}} \leq \int_{\Pi^{dN}} \rho_N^0\,\log\frac{\rho_N^0}{G_N}.$$

It only remains to treat the dissipation term is in (4.8). Since ε_0 is fixed then G_{N,ε_0} is now smooth, ∇X_ε and X_ε converge both in the sense of distribution and, by convexity, we have that

$$\int_0^t\int_{\Pi^{dN}} \frac{|\nabla X|^2}{X}\,G_{N,\varepsilon_0} \leq \liminf_{\varepsilon\to 0}\int_0^t\int_{\Pi^{dN}} \frac{|\nabla X_\varepsilon|^2}{X_\varepsilon}\,G_{N,\varepsilon_0}.$$

This gives that for any ε_0,

$$\int_{\Pi^{dN}} \rho_N\,\log\frac{\rho_N}{G_N} + \sigma\int_0^t\int_{\Pi^{dN}} \frac{|\nabla X|^2}{X}\,G_{N,\varepsilon_0} \leq \int_{\Pi^{dN}} \rho_N^0\,\log\frac{\rho_N^0}{G_N}.$$

We recall again that $G_{N,\varepsilon}$ is increasing in ε so, by the monotone convergence theorem, this yields the desired entropy bound

$$\int_{\Pi^{dN}} \rho_N \, \log \frac{\rho_N}{G_N} + \sigma \int_0^t \int_{\Pi^{dN}} \frac{|\nabla X|^2}{X} \, G_N \leq \int_{\Pi^{dN}} \rho_N^0 \, \log \frac{\rho_N^0}{G_N}.$$

The last question is to show that Eq. (1.5) is satisfied in the sense of distribution at the limit. The only difficulty is to pass to the the limit in the advection term, which we may rewrite in a non-linear form as

$$\operatorname{div}\Big(\rho_{N,\varepsilon} \nabla \log \frac{\rho_{N,\varepsilon}}{G_{N,\varepsilon}}\Big)$$

To do so let us introduce the following quantity

$$I = \int_{\Pi^{dN}} \varphi \rho_{N,\varepsilon} (\nabla \log G_{N,\varepsilon} - \nabla \log \rho_{N,\varepsilon})$$

where φ is a $\mathcal{C}^\infty$ test function. Let us define a cut-off function χ such that $\chi(|x|) = 1$ is $|x| \leq 1$ and $\chi(|x|) = 0$ if $|x| > 2$ and choose ε and M such that again

$$2\,M < \exp\big(\frac{C}{N} \log 1/\varepsilon_0\big)$$

then

$$\begin{aligned} I = \quad & \int_{\Pi^{dN}} \varphi \rho_{N,\varepsilon} (\nabla \log G_{N,\varepsilon} - \nabla \log \rho_{N,\varepsilon}) \chi\Big(\frac{G_{N,\varepsilon_0}}{M}\Big) \\ & + \int_{\Pi^{dN}} \varphi \rho_{N,\varepsilon} (\nabla \log G_{N,\varepsilon} - \nabla \log \rho_{N,\varepsilon}) \Big(1 - \chi\Big(\frac{G_{N,\varepsilon_0}}{M}\Big)\Big) \\ = \quad & I_{1,\varepsilon} + I_{2,\varepsilon}. \end{aligned}$$

We now have as before the convergence of $I_{1,\varepsilon}$ to

$$I_1 = \int_{\Pi^{dN}} \varphi \rho_N (\nabla \log G_N - \nabla \log \rho_N) \chi\Big(\frac{G_{N,\varepsilon_0}}{M}\Big).$$

More precisely $I_{1,\varepsilon}$ converges to I_1 since as before $G_{\varepsilon_0,M} \leq 2M$ provides a uniform lower bound on $\inf_{i \neq j} |x_i - x_j|$.

Concerning $I_{2,\varepsilon}$, we remark that

$$\begin{aligned} |I_{2,\varepsilon}| \quad & \leq \int_{G_{N,\varepsilon_0} \geq M} \rho_{N,\varepsilon} \big|\nabla \log G_{N,\varepsilon} - \nabla \log \rho_{N,\varepsilon}\big| \\ & \leq \Big(\int_{\Pi^{dN}} \rho_{N,\varepsilon} \big|\nabla \log \frac{G_{N,\varepsilon}}{\rho_{N,\varepsilon}}|^2\Big)^{1/2} \Big(\int_{\Pi^{dN}} \rho_{N,\varepsilon} \, \mathbb{I}_{G_{N,\varepsilon} \geq M}\Big)^{1/2} \\ & \leq C \frac{1}{(\log M)^{1/2}} \Big(\int_{\Pi^{dN}} \rho_{N,\varepsilon} \log \rho_{N,\varepsilon}\Big)^{1/2}. \\ & \leq \frac{C}{(\log M)^{1/2}}, \end{aligned} \tag{4.9}$$

with C independent on ε. Now letting $\varepsilon_0 \to 0$ and hence $M \to \infty$, we conclude that

$$\int_{\Pi^{dN}} \varphi \rho_{N,\varepsilon} \big(\nabla \log G_{N,\varepsilon} - \nabla \log \rho_{N,\varepsilon} \big) \to \int_{\Pi^{dN}} \varphi \rho_N \big(\nabla \log G_N - \nabla \log \rho_N \big),$$

and hence that ρ_N solves (1.5) in the sense of distribution. □

Conclusion

We have been able to derive for the first time the mean field limit for attractive singular interaction of gradient flow type. Our approach relies critically on the use of the right physics through the structure of the free energy of the system which allows to combine the two methods in [23] and [31]. An important application is the answer to the longstanding open problem of the full rigorous derivation with quantitative estimates of the Patlak–Keller-Segel model in the optimal subcritical regime.

Controlling the time evolution of our modulated free energy leads to the development of new large deviation estimates that encode the competition between diffusion and attraction or concentration of the particles. Those large deviation estimates only require simple one-sided bound on the potential near its singularity at 0 without strong structural assumptions.

By adding to the proofs in the present paper for attractive kernels the estimates in the proceeding [8] for repulsive kernels, our method provides quantitative mean field estimates for a large class of attractive-repulsive interactions. Furthermore, our approach is compatible with vanishing or degenerate diffusion systems where the diffusion coefficient vanishes as the number of particles increases and which are especially relevant for some Coulomb gases related to the complex Ginibre Ensemble in random matrix theory, see for example [5].

Finally, we wish to highlight the following open questions for which this new method could be helpful:

- It is unclear what the optimal rate of convergence should be in Theorem 2.1 and additional work is still needed. In particular the proof in section 3 is somewhat careless in that regard for the sake of simplicity. In particular the use of different regularizations in Prop. 3.1, Prop. 3.2 and the proof of Prop. 2.1 likely leads to an artificially lower rate. However this proof still suggests that a polynomial rate in N cannot be uniformly maintained as the potential V approaches the critical case $\lambda \log |x|$ with $\lambda = 2\, d\, \sigma$.
- Is it possible to obtain uniform in time convergence or in general to work on the infinite time interval $[0, +\infty)$? This would both provide the mean field limit and the large time asymptotics of the dynamics. In that regards, we point out that our proof does not use the dissipation term in the time evolution of the modulated free energy in Inequality (2.25). Of course this dissipation term is an equivalent of a weighted and modulated Fisher information and it is

technically challenging to use because it involves the singular Gibbs equilibrium. But a good entropy-entropy dissipation estimate for this term could lead to uniform in time estimates.

- Does the mean field limit hold in the supercritical cases? In that case the limiting Patlak-Keller-Segel system blows-up in finite time but it would not be unreasonable to conjecture that the limit holds on the time interval before the blow-up. There are again significant technical issues (including at the level of the existence of our entropy solutions) but we believe that it is possible to develop localized relative entropies that would allow such a result.
- Can our method provide some insights for the existence of strong stochastic solutions to the trajectorial many-particle system (1.1)? The entropy solution that we derive in the appendix is so far the only example of some sort of existence outside of the diffusion dominant result $\lambda < \sigma$ as in [17]. Such entropy solutions rely on a (simplified) version of the large deviation inequalities which could provide further insights into whether particles can actually collide and how they collide.
- Can the modulated free energy be extended to different type of interactions than gradient flows such as Hamiltonian systems? Another example is given by interactions between particles that solve evolution in time equation, with in particular the Patlak–Keller–Segel parabolic-parabolic equations.

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