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# Analysis on Lambert-Tsallis functions

Hideto Nakashima and Piotr Graczyk

ABSTRACT. In this paper, we study the Lambert-Tsallis  $W$  function, which is a generalization of the Lambert  $W$  function with two real parameters. We give a condition on the parameters such that there exists a complex domain touching zero on boundary which is mapped bijectively to the upper half plane by the Lambert-Tsallis function.

## INTRODUCTION

The Lambert  $W$  function, which is the multivalued inverse of the function  $z \mapsto ze^z$  has many applications in many areas of mathematics. In a theory of random matrices, it appears in a formula of a eigenvalue distribution of a certain Wishart-type ensemble (cf. Cheliotis [4]). In the previous paper [8], we found that, for a certain class of Wishart-type ensembles, the corresponding eigenvalue distributions can be described by using the main branch of the inverse function of  $z \mapsto \frac{z}{1+\gamma z} \left(1 + \frac{z}{\kappa}\right)^\kappa$ , where  $\gamma, \kappa$  are real parameter. The aim of this paper is to give a complete analysis of this function, which we call the Lambert-Tsallis function.

## 1. PRELIMINARIES

For a non zero real number  $\kappa$ , we set

$$\exp_\kappa(z) := \left(1 + \frac{z}{\kappa}\right)^\kappa \quad \left(1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}\right),$$

where we take the main branch of the power function when  $\kappa$  is not integer. If  $\kappa = \frac{1}{1-q}$ , then it is exactly the so-called Tsallis  $q$ -exponential function (cf. [2, 9]). By virtue of  $\lim_{\kappa \rightarrow \infty} \exp_\kappa(z) = e^z$ , we regard  $\exp_\infty(z) = e^z$ .

For two real numbers  $\kappa, \gamma$  such that  $\kappa \neq 0$ , we introduce a holomorphic function  $f_{\kappa, \gamma}(z)$ , which we call *generalized Tsallis function*, by

$$f_{\kappa, \gamma}(z) := \frac{z}{1 + \gamma z} \exp_\kappa(z) \quad \left(1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}\right).$$

Analogously to Tsallis  $q$ -exponential, we also consider  $f_{\infty, \gamma}(z) = \frac{ze^z}{1+\gamma z}$  ( $z \in \mathbb{C}$ ). In particular,  $f_{\infty, 0}(z) = ze^z$ . Let  $\mathcal{D}(f_{\kappa, \gamma})$  be the domain of  $f_{\kappa, \gamma}$ , that is, if  $\kappa$  is integer then  $\mathcal{D}(f_{\kappa, \gamma}) = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$  if  $\kappa$  is not integer, then

$$\mathcal{D}(f_{\kappa, \gamma}) = \mathbb{C} \setminus \left\{x \in \mathbb{R}; 1 + \frac{x}{\kappa} \leq 0, \text{ or } x = -\frac{1}{\gamma}\right\}.$$

The purpose of this work is to study an inverse function to  $f_{\kappa, \gamma}$  in detail. A multivariate inverse function of  $f_{\infty, 0}(z) = ze^z$  is called the Lambert  $W$  function and studied in [6]. Hence, we call an inverse function to  $f_{\kappa, \gamma}$  the *Lambert-Tsallis  $W$  function*. Since we have

$$f'_{\kappa, \gamma}(z) = \frac{\gamma z^2 + (1 + 1/\kappa)z + 1}{(1 + \gamma z)^2} \left(1 + \frac{z}{\kappa}\right)^{\kappa-1}, \quad (1)$$

the function  $f_{\kappa, \gamma}(z)$  has the inverse function  $w_{\kappa, \gamma}$  in a neighborhood of  $z = 0$  by the fact  $f'_{\kappa, \gamma}(0) = 1 \neq 0$ .

Let  $z = x + yi \in \mathbb{C}$  and we set for  $\kappa \neq \infty$

$$\theta(x, y) := \text{Arg}\left(1 + \frac{z}{\kappa}\right),$$

where  $\text{Arg}(w)$  stands for the principal argument of  $w$ ;  $-\pi < \text{Arg}(w) \leq \pi$ . Since we now take the main branch of power function, we have for  $\kappa \neq \infty$

$$\begin{aligned} \left(1 + \frac{z}{\kappa}\right)^\kappa &= \exp\left(\kappa \left(\log\left|1 + \frac{z}{\kappa}\right| + i \text{Arg}\left(1 + \frac{z}{\kappa}\right)\right)\right) = \left(\left(1 + \frac{x}{\kappa}\right)^2 + \frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} e^{i\kappa\theta(x,y)} \\ &= \left(\left(1 + \frac{x}{\kappa}\right)^2 + \frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} \left(\cos(\kappa\theta(x,y)) + i \sin(\kappa\theta(x,y))\right). \end{aligned}$$

If  $\kappa = \infty$ , then we regard  $\kappa\theta(x,y)$  as  $y$  because we have  $\lim_{\kappa \rightarrow +\infty} \exp_\kappa(z) = e^z = e^x(\cos y + i \sin y)$ . Since

$$\frac{z}{1 + \gamma z} = \frac{z(1 + \gamma \bar{z})}{|1 + \gamma z|^2} = \frac{(x + \gamma x^2 + \gamma y^2) + i(y + \gamma xy - \gamma xy)}{(1 + \gamma x)^2 + \gamma^2 y^2} = \frac{(x + \gamma x^2 + \gamma y^2) + iy}{(1 + \gamma x)^2 + \gamma^2 y^2},$$

we have

$$f_{\kappa,\gamma}(z) = \frac{\left((1 + x/\kappa)^2 + (y/\kappa)^2\right)^{\frac{\kappa}{2}}}{(1 + \gamma x)^2 + \gamma^2 y^2} \begin{pmatrix} (x + \gamma x^2 + \gamma y^2) \cos(\kappa\theta(x,y)) - y \sin(\kappa\theta(x,y)) \\ + i\{(x + \gamma x^2 + \gamma y^2) \sin(\kappa\theta(x,y)) + y \cos(\kappa\theta(x,y))\} \end{pmatrix}. \quad (2)$$

Then,  $f_{\kappa,\gamma}(z) \in \mathbb{R}$  implies

$$(x + \gamma x^2 + \gamma y^2) \sin(\kappa\theta(x,y)) + y \cos(\kappa\theta(x,y)) = 0. \quad (3)$$

If  $\sin(\kappa\theta(x,y)) = 0$ , then  $\cos(\kappa\theta(x,y))$  does not vanish so that  $y$  needs to be zero. Thus, if  $z = x + yi \in f_{\kappa,\gamma}(\mathbb{R})$  with  $y \neq 0$ , then we have  $\sin(\kappa\theta(x,y)) \neq 0$ . Thus, the equation (3) for  $\sin(\kappa\theta(x,y)) \neq 0$  can be rewritten as

$$F(x,y) := (x + \gamma x^2 + \gamma y^2) + y \cot(\kappa\theta(x,y)) = 0. \quad (4)$$

For  $y = 0$ , we set  $F(x,0) := \lim_{y \rightarrow 0} F(x,y)$ . If  $x = 0$ , then we have  $\theta(0,y) = \text{Arctan}(\frac{y}{\kappa})$  and hence  $F(0,0) = \lim_{y \rightarrow 0} y \cot(\kappa \text{Arctan}(\frac{y}{\kappa})) = 1$ . Let us introduce the connected set  $\Omega = \Omega_{\kappa,\gamma}$  by

$$\Omega = \Omega_{\kappa,\gamma} := \{z = x + yi \in \mathcal{D}(f_{\kappa,\gamma}); F(x,y) > 0\}^\circ,$$

where  $A^\circ$  is the connected component of an open set  $A \subset \mathbb{C}$  containing  $z = 0$ . Note that since  $F$  is an even function on  $y$ , the domain  $\Omega$  is symmetric with respect to the real axis. Set

$$\mathcal{S} := \mathbb{R} \setminus f_{\kappa,\gamma}(\Omega_{\kappa,\gamma} \cap \mathbb{R}). \quad (5)$$

**Definition 1.1.** If there exists a unique holomorphic extension  $W_{\kappa,\gamma}$  of  $w_{\kappa,\gamma}$  to  $\mathbb{C} \setminus \bar{\mathcal{S}}$ , then we call  $W_{\kappa,\gamma}$  the main branch of Lambert-Tsallis  $W$  function. In this paper, we only study and use  $W_{\kappa,\gamma}$  among other branches so that we call  $W_{\kappa,\gamma}$  the Lambert-Tsallis function for short. Note that in our terminology the Lambert-Tsallis  $W$  function is multivalued and the Lambert-Tsallis function  $W_{\kappa,\gamma}$  is single-valued.

Our goal is to prove the following theorem.

**Theorem 1.2.** Let  $f_{\kappa,\gamma}$  be a generalized Tsallis function. Then, there exists the main branch of Lambert-Tsallis function if and only if (i)  $0 < \kappa < 1$  and  $\gamma \leq 0$ , (ii)  $\kappa \geq 1$  and  $\gamma \leq \frac{1}{4}(1 + \frac{1}{\kappa})^2$ , (iii)  $-1 < \kappa < 0$  and  $\gamma \leq \frac{1}{\kappa}$ , (iv)  $\kappa \leq -1$  and  $\gamma \leq \frac{1}{4}(1 + \frac{1}{\kappa})^2$ , and (v)  $\kappa = \infty$  and  $\gamma \leq \frac{1}{4}$ .

**Remark 1.3.** In the case  $\kappa > 1$  and  $D(0) < 0$ , the function  $f_{\kappa,\gamma}$  maps  $\Omega \cap \mathbb{C}^+$  to  $\mathbb{C}^+$  two-to-one, and hence the extension exists on a smaller domain in  $\Omega$  which  $f_{\kappa,\gamma}$  maps to  $\mathbb{C} \setminus \bar{\mathcal{S}}$ , but it is not unique. In the case  $0 < \kappa < 1$  and  $\gamma > 0$ , we need to extend the defining domain of  $f_{\kappa,\gamma}$  so that there does not exist a complex domain such that  $f_{\kappa,\gamma}$  maps to  $\mathbb{C} \setminus \bar{\mathcal{S}}$ .

The proof of this theorem is done in two steps, that is, we first give an explicit expression of  $\Omega = \Omega_{\kappa,\gamma}$ , and then show that  $f_{\kappa,\gamma}$  maps  $\Omega$  to  $\mathbb{C} \setminus \bar{\mathcal{S}}$  bijectively. At first, we suppose that  $0 < \kappa < +\infty$ . Let us change variables in (4) by

$$re^{i\theta} = 1 + \frac{z}{\kappa} \quad (r > 0, \theta \in (0, \pi)), \quad \text{or equivalently} \quad \begin{cases} x = \kappa(r \cos \theta - 1), \\ y = \kappa r \sin \theta, \end{cases} \quad (6)$$

and set  $a := \gamma\kappa$  and

$$b(\theta) = (1 - 2a) \cos \theta + \sin \theta \cot(\kappa\theta). \quad (7)$$

Then, the equation (4) can be written as

$$ar^2 + b(\theta)r + a - 1 = 0. \quad (8)$$

In fact, we have

$$\begin{aligned} & \kappa(r \cos \theta - 1) + \gamma \{ (\kappa(r \cos \theta - 1))^2 + (\kappa r \sin \theta)^2 \} + \kappa r \sin \theta \cot(\kappa \theta) = 0 \\ \iff & \gamma \kappa^2 r^2 + \{ \kappa \cos \theta - 2\gamma \kappa^2 \cos \theta + \kappa \sin \theta \cot(\kappa \theta) \} r + (\gamma \kappa^2 - \kappa) = 0 \\ \iff & \gamma \kappa r^2 + \{ (1 - 2\gamma \kappa) \cos \theta + \sin \theta \cot(\kappa \theta) \} r + \gamma \kappa - 1 = 0. \end{aligned}$$

If  $\sin(\kappa \theta) \neq 0$ , then (8) has a solution

$$r = r_{\pm}(\theta) = \frac{-b(\theta) \pm \sqrt{b(\theta)^2 - 4a(a-1)}}{2a}.$$

Thus, for each angle  $\theta$ , there exists at most two points on  $f_{\kappa, \gamma}^{-1}(\mathbb{R})$ . Since the change (6) of variables is the polar transformation, we need to know whether  $r_{\pm}(\theta)$  is positive real or not. We note that

$$r'_{\varepsilon}(\theta) = \frac{1}{2a} \left( -b'(\theta) + \varepsilon \frac{2b(\theta)b'(\theta)}{2\sqrt{D(\theta)}} \right) = \frac{-\varepsilon b'(\theta)}{2a} \cdot \frac{-b(\theta) + \varepsilon \sqrt{D(\theta)}}{\sqrt{D(\theta)}} = -\varepsilon b'(\theta) \frac{r_{\varepsilon}(\theta)}{\sqrt{D(\theta)}} \quad (9)$$

for  $\varepsilon = \pm 1$ . Set  $D(\theta) := b(\theta)^2 - 4a(a-1)$ . Then,  $r_{\pm}(\theta)$  are real if and only if  $D(\theta) \geq 0$ , and we have

$$D'(\theta) = 2b(\theta)b'(\theta).$$

Let  $\alpha_1, \alpha_2$  be the two solutions of

$$q(z) := \gamma z^2 + (1 + 1/\kappa)z + 1 = 0. \quad (10)$$

If  $\alpha_i$  are real, then we assume that  $\alpha_1 \leq \alpha_2$ , and if not, then we assume that  $\text{Im } \alpha_1 > 0$  and  $\text{Im } \alpha_2 < 0$ . Then,  $f'(z) = 0$  implies  $z = \alpha_i$  ( $i = 1, 2$ ) or  $z = -\kappa$  if  $\kappa > 1$ . It is clear that  $\alpha_1, \alpha_2$  are real numbers if and only if

$$\left(1 + \frac{1}{\kappa}\right)^2 - 4\gamma \geq 0 \iff \gamma \leq \frac{1}{4} \left(1 + \frac{1}{\kappa}\right)^2.$$

These two points  $\alpha_i$ ,  $i = 1, 2$  correspond to the solutions of (8) with parameter  $\theta = 0$ . In fact, if  $\theta = 0$ , then  $r = 1 + \frac{x}{\kappa}$  and

$$b(0) := \lim_{\theta \rightarrow 0} b(\theta) = \frac{\kappa + 1}{\kappa} - 2a, \quad (11)$$

and hence the equation (8) with  $\theta = 0$  is described as

$$\begin{aligned} 0 &= ar^2 + b(0)r + a - 1 = a(r-1)^2 + (1 + \frac{1}{\kappa})(r-1) + \frac{1}{\kappa} \\ &= a \cdot \frac{x^2}{\kappa^2} + (1 + \frac{1}{\kappa})\frac{x}{\kappa} + \frac{1}{\kappa} = \frac{1}{\kappa} \left( \gamma x^2 + \left(1 + \frac{1}{\kappa}\right)x + 1 \right). \end{aligned}$$

We note that

$$q\left(-\frac{1}{\gamma}\right) = \frac{a-1}{a}, \quad q(-\kappa) = (a-1)\kappa, \quad (12)$$

and

$$D(0) = b(0)^2 - 4a(a-1) = \left(1 + \frac{1}{\kappa}\right)^2 - \frac{4a}{\kappa}.$$

In order to know whether the equation (8) has a positive solution, we need to study the signature of  $D(\theta)$ , and hence that of  $b(\theta)$  and  $b'(\theta)$ . To do so, we introduce three functions  $H_{\alpha}$ ,  $F_{\kappa}$  and  $J_{\kappa}$  as below, and investigate them in detail.

For  $\alpha > 0$ , we set

$$H_{\alpha}(x) := \sin(\alpha x) - \alpha \sin(x) \quad (x \in \mathbb{R}).$$

Let us investigate the signature of  $H_{\alpha}(x)$  on the interval  $I_1 = (0, \min(\frac{2\pi}{\alpha}, 2\pi))$ . Note that if  $\alpha = 1$  then we have  $H_1 \equiv 0$ , and hence we exclude the case  $\alpha = 1$ . By differentiating, we have

$$H'_{\alpha}(x) = \alpha \cos(\alpha x) - \alpha \cos x = -2\alpha \sin\left(\frac{\alpha+1}{2}x\right) \sin\left(\frac{\alpha-1}{2}x\right).$$

$H'_{\alpha}(x) = 0$  implies that  $x = \frac{2n\pi}{\alpha+1}$  ( $n \in \mathbb{Z}$ ) or  $x = \frac{2m\pi}{\alpha-1}$  ( $m \in \mathbb{Z}$ ).

**Lemma 1.4.** (1) If  $0 < \alpha \leq \frac{1}{2}$ , then one has  $H_{\alpha}(x) > 0$  for any  $x \in I_1$ .

(2) If  $\frac{1}{2} < \alpha < 1$ , then there exists a unique  $y_*$  such that  $H_{\alpha}(y_*) = 0$ , and one has  $H_{\alpha}(x) > 0$  for  $x \in (0, y_*)$  and  $H_{\alpha}(x) < 0$  for  $x \in (y_*, 2\pi)$ .

(3) If  $1 < \alpha < 2$ , then there exists a unique  $y_* \in I_1$  such that  $H_{\alpha}(y_*) = 0$ , and one has  $H_{\alpha}(x) < 0$  for  $x \in (0, y_*)$  and  $H_{\alpha}(x) > 0$  for  $x \in (y_*, \frac{2\pi}{\alpha})$ .

(4) If  $\alpha \geq 2$ , then one has  $H_\alpha(x) < 0$  for any  $x \in I_1$ .

*Proof.* It is obvious that  $H_\alpha(0) = 0$ . First, we assume that  $0 < \alpha < 1$ . In this case, since  $\frac{1}{1-\alpha} \leq 1$  and  $\frac{1}{2} < \frac{1}{\alpha+1} < 1$ , we have

$$0 < \frac{2\pi}{\alpha+1} < 2\pi < \min\left(\frac{2\pi}{\alpha+1}, \frac{-2\pi}{\alpha-1}\right).$$

This means that  $H'_\alpha(x) > 0$  when  $x \in (0, \frac{2\pi}{\alpha+1})$ , and  $H'_\alpha(x) < 0$  when  $x \in (\frac{2\pi}{\alpha+1}, 2\pi)$ . On the other hand, we have  $H_\alpha(2\pi) = \sin(2\pi\alpha) - \alpha \sin(2\pi) = \sin(2\pi\alpha)$ . If  $0 < \alpha \leq \frac{1}{2}$ , then we have  $0 < 2\alpha\pi \leq \pi$  and thus  $H_\alpha(2\pi) \geq 0$ , and if  $\frac{1}{2} < \alpha < 1$  then we have  $\pi < 2\pi\alpha < 2\pi$  which implies  $H_\alpha(2\pi) < 0$ . Thus, we obtain the assertions (1) and (2).

Next we assume that  $\alpha > 1$ . In this case, since  $\frac{2}{\alpha+1} > \frac{1}{\alpha}$ , we have

$$0 < \frac{2\pi}{\alpha+1} < \frac{2\pi}{\alpha} < \min\left(\frac{4\pi}{\alpha+1}, \frac{2\pi}{\alpha-1}\right).$$

This means that  $H'_\alpha(x) < 0$  when  $x \in (0, \frac{2\pi}{\alpha+1})$ , and  $H'_\alpha(x) > 0$  when  $x \in (\frac{2\pi}{\alpha+1}, \frac{2\pi}{\alpha})$ . On the other hand, we have  $H_\alpha(\frac{2\pi}{\alpha}) = \sin(2\pi) - \alpha \sin(\frac{2\pi}{\alpha}) = -\alpha \sin(\frac{2\pi}{\alpha})$ . If  $1 < \alpha < 2$ , then we have  $\pi < \frac{2\pi}{\alpha} < 2\pi$  and hence  $H_\alpha(\frac{2\pi}{\alpha}) > 0$ . If  $\alpha \geq 2$ , then we have  $0 < \frac{2\pi}{\alpha} \leq \pi$  so that  $H_\alpha(\frac{2\pi}{\alpha}) \leq 0$ . Therefore, we have proved the assertion (3) and (4).  $\square$

For  $\kappa > 0$ , we set

$$F_\kappa(x) := \tan x \cdot \cot(\kappa x) \quad (x \in \mathbb{R}).$$

Let us investigate the behavior of  $F_\kappa(x)$  on the interval  $I_0 = (0, \min(\frac{\pi}{\kappa}, \pi))$ . Since  $F_1 \equiv 1$ , we exclude the case  $\kappa = 1$ . Notice that if  $\kappa < 2$ , then  $F_\kappa(x)$  has a pole at  $x = \frac{\pi}{2}$  in the interval  $I_0$ . At first, we see that

$$F_\kappa(0) := \lim_{x \rightarrow +0} F_\kappa(x) = \lim_{x \rightarrow +0} \frac{\sin x}{\sin(\kappa x)} \cdot \frac{\cos(\kappa x)}{\cos x} = \frac{1}{\kappa} > 0.$$

If  $\kappa < 1$ , then it is obvious that  $F_\kappa(\pi) = 0$ . On the other hand, if  $1 < \kappa < 2$ , then  $\frac{\pi}{2} < \frac{\pi}{\kappa} < \pi$ , and if  $\kappa \geq 2$  then  $0 < \frac{\pi}{\kappa} \leq \frac{\pi}{2}$ , and hence we have

$$\lim_{x \rightarrow \frac{\pi}{\kappa} - 0} F_\kappa(x) = \begin{cases} +\infty & (\text{if } 1 < \kappa < 2), \\ -\infty & (\text{if } \kappa \geq 2). \end{cases}$$

By differentiating, we have

$$F'_\kappa(x) = \frac{\cot(\kappa x)}{\cos^2 x} - \frac{\kappa \tan x}{\sin^2(\kappa x)} = \frac{H_\kappa(2x)}{2(\cos x \sin(\kappa x))^2}.$$

For the case  $\frac{1}{2} < \kappa < 2$  ( $\kappa \neq 1$ ), Lemma 1.4 tells us that  $H_\kappa$  has a unique zero point  $y_*$  in the interval  $(0, \min(2\pi, \frac{2\pi}{\kappa}))$ . If we set  $x_* = \frac{y_*}{2}$ , then we have  $x_* \in I_0$  and  $F_\kappa(x_*) = 0$ .

**Lemma 1.5.** *If  $\frac{1}{2} < \kappa < 1$ , then one has  $F_\kappa(x_*) < 1$ , and if  $1 < \kappa < 2$ , then one has  $F_\kappa(x_*) > 1$ .*

*Proof.* We first assume that  $\frac{1}{2} < \kappa < 1$ . In this case, since  $0 < 1 - \kappa < \frac{1}{2}$  and since  $\frac{\pi}{2} < x_* < \pi$ , we have  $\sin((1 - \kappa)x_*) > 0$ . Thus, since  $\cos x_* < 0$  and  $\sin(\kappa x_*) > 0$ , we obtain

$$\begin{aligned} \sin((1 - \kappa)x_*) > 0 &\iff \sin x_* \cos(\kappa x_*) - \cos x_* \sin(\kappa x_*) > 0 \\ &\iff \sin x_* \cos(\kappa x_*) > \cos x_* \sin(\kappa x_*) \\ &\iff F_\kappa(x_*) = \frac{\sin x_* \cos(\kappa x_*)}{\cos x_* \sin(\kappa x_*)} < 1. \end{aligned}$$

Next, we assume that  $1 < \kappa < 2$ . In this case, since  $0 < \kappa - 1 < 1$  and since  $\frac{\pi}{2} < x_* < \frac{\pi}{\kappa} < \pi$ , we have  $\sin((\kappa - 1)x_*) > 0$ . Thus, since  $\cos x_* < 0$  and  $\sin(\kappa x_*) > 0$ , we obtain

$$\begin{aligned} \sin((\kappa - 1)x_*) > 0 &\iff \cos x_* \sin(\kappa x_*) - \sin x_* \cos(\kappa x_*) > 0 \\ &\iff \cos x_* \sin(\kappa x_*) > \sin x_* \cos(\kappa x_*) \\ &\iff F_\kappa(x_*) = \frac{\sin x_* \cos(\kappa x_*)}{\cos x_* \sin(\kappa x_*)} > 1. \end{aligned}$$

We have proved the lemma.  $\square$

Lemmas 1.4 and 1.5 yield the following table.

**Lemma 1.6.** *One has the following increasing/decreasing table of  $F_\kappa$ .*

(A) $0 < \kappa \leq \frac{1}{2}$	$x$	0	$\dots$	$\frac{\pi}{2}$	$\dots$	$\pi$	
	$\frac{F'_\kappa}{F_\kappa}$		+	$\times$	+		
	$\frac{1}{F_\kappa}$	$\frac{1}{\kappa}$	$\nearrow +\infty$	$\times$	$-\infty \nearrow$	0	
(B) $\frac{1}{2} < \kappa < 1$	$x$	0	$\dots$	$\frac{\pi}{2}$	$\dots$	$x_*$	$\dots$ $\pi$
	$\frac{F'_\kappa}{F_\kappa}$		+	$\times$	+	0	—
	$\frac{1}{F_\kappa}$	$\frac{1}{\kappa}$	$\nearrow +\infty$	$\times$	$-\infty \nearrow$	$F_\kappa(x_*)$	$\searrow$ 0
(C) $1 < \kappa < 2$	$x$	0	$\dots$	$\frac{\pi}{2}$	$\dots$	$x_*$	$\dots$ $\frac{\pi}{\kappa}$
	$\frac{F'_\kappa}{F_\kappa}$		—	$\times$	—	0	+
	$\frac{1}{F_\kappa}$	$\frac{1}{\kappa}$	$\searrow -\infty$	$\times$	$+\infty \searrow$	$F_\kappa(x_*)$	$\nearrow +\infty$ $\times$
(D) $\kappa \geq 2$	$x$	0	$\dots$	$\frac{\pi}{\kappa}$			
	$\frac{F'_\kappa}{F_\kappa}$		—	$\times$			
	$\frac{1}{F_\kappa}$	$\frac{1}{\kappa}$	$\searrow -\infty$	$\times$			

For  $\kappa > 0$ , we set

$$J_\kappa(x) := \frac{2x - 2\kappa - 1}{4\kappa x - 2\kappa - 1} = \frac{1}{2\kappa} \cdot \frac{x - \frac{2\kappa+1}{2}}{x - \frac{2\kappa+1}{4\kappa}} = \frac{1}{2\kappa} - \frac{4\kappa^2 - 1}{8\kappa^2} \cdot \frac{1}{x - \frac{2\kappa+1}{4\kappa}}.$$

Then, we have

$$J_\kappa(0) = 1, \quad J_\kappa(1) = -1, \quad \lim_{x \rightarrow +\infty} J_\kappa(x) = \lim_{x \rightarrow -\infty} J_\kappa(x) = \frac{1}{2\kappa}. \quad (13)$$

Note that, if we set  $\kappa = \frac{1}{2}$ , then  $J_{\frac{1}{2}} \equiv 1$ . The following lemma is trivial.

**Lemma 1.7.** *Suppose that  $\kappa \neq \frac{1}{2}$ . Then,  $J_\kappa$  has a pole at  $x = \frac{1}{2} + \frac{1}{4\kappa}$ . If  $0 < \kappa < \frac{1}{2}$ , then it is monotonic decreasing on  $\mathbb{R}$ , and if  $\kappa > \frac{1}{2}$ , then it is monotonic increasing on  $\mathbb{R}$ .*

We now consider the function  $b(\theta)$ . If  $\kappa = 1$ , then we have  $b(\theta) = 2(1 - a) \cos \theta$ . Otherwise, since  $b(\theta)$  can be described as

$$b(\theta) = ((1 - 2a) + F_\kappa(\theta)) \cos \theta \quad (\text{if } \cos \theta \neq 0),$$

the signature of  $b(\theta)$  can be determined by using  $F_\kappa$ . Note that,  $\cos \theta = 0$  occurs when  $\kappa < 2$ , and in this case, we have

$$b\left(\frac{\pi}{2}\right) = 0 + 1 \cdot \cot \frac{\kappa\pi}{2} = \cot \frac{\kappa\pi}{2}.$$

It is positive if  $\kappa < 1$ , and negative if  $1 < \kappa < 2$ . These observations together with Lemma 1.5 yield the following table.

**Lemma 1.8.** *The signature of  $b(\theta)$  on the interval  $I_0 = (0, \min(\pi, \frac{\pi}{\kappa}))$  is given as follows.*

$0 < \kappa \leq \frac{1}{2}$									
$2a - 1 > \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi$	$\dots$	$\pi$		$\varphi < \frac{\pi}{2}$	
$0 \leq 2a - 1 \leq \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\pi$					
$2a - 1 < 0$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi$	$\dots$	$\pi$		$\varphi > \frac{\pi}{2}$	
$\frac{1}{2} < \kappa < 1$									
$2a - 1 > \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi$	$\dots$	$\pi$		$\varphi < \frac{\pi}{2}$	
$F_\kappa(\theta_*) < 2a - 1 \leq \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\pi$					
$0 \leq 2a - 1 \leq F_\kappa(\theta_*)$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi_1$	$\dots$	$\varphi_2$	$\dots$	$\pi$	$\varphi_i > \frac{\pi}{2}$
$2a - 1 < 0$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi$	$\dots$	$\pi$		$\varphi > \frac{\pi}{2}$	
$1 < \kappa < 2$									
$2a - 1 \geq F_\kappa(\theta_*)$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi_1$	$\dots$	$\varphi_2$	$\dots$	$\frac{\pi}{\kappa}$	$\varphi_i > \frac{\pi}{2}$
$\frac{1}{\kappa} \leq 2a - 1 < F_\kappa(\theta_*)$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\frac{\pi}{\kappa}$					
$2a - 1 < \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi$	$\dots$	$\frac{\pi}{\kappa}$		$\varphi < \frac{\pi}{2}$	
$\kappa \geq 2$									
$2a - 1 \geq \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\frac{\pi}{\kappa}$					
$2a - 1 < \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	$\dots$	$\varphi$	$\dots$	$\frac{\pi}{\kappa}$			

In this table,  $\varphi$  or  $\varphi_i$  ( $i = 1, 2$ ) are solutions in  $I_0$  of  $b(\theta) = 0$ . If  $2a - 1 = F_\kappa(x_*)$ , then  $\varphi_1 = \varphi_2$ .

Let us consider the function  $b'(\theta)$ . If  $\kappa = \frac{1}{2}$ , then we have

$$b'(\theta) = (1 - 2a) \cos \theta + 2 \sin \theta \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = (1 - 2a) \cos \theta + 2 \cos^2 \left( \frac{\theta}{2} \right) = 2(a - 1) \sin \theta. \quad (14)$$

Let us assume that  $\kappa \neq \frac{1}{2}$ . Since the function  $b(\theta)$  can be also written as

$$\begin{aligned} b(\theta) &= \cos \theta + \sin \theta \cot(\kappa \theta) - 2a \cos \theta = \frac{\cos \theta \sin(\kappa \theta) + \sin \theta \cos(\kappa \theta)}{\sin(\kappa \theta)} - 2a \cos \theta \\ &= \frac{\sin((\kappa + 1)\theta)}{\sin(\kappa \theta)} - 2a \cos \theta, \end{aligned}$$

its derivative can be written by using  $H_\alpha$  as

$$b'(\theta) = \frac{H_{2\kappa+1}(\theta)}{2 \sin^2(\kappa \theta)} + 2a \sin \theta. \quad (15)$$

In fact,

$$\begin{aligned}
b'(\theta) &= \frac{(\kappa+1) \cos((\kappa+1)\theta) \sin \kappa\theta - \kappa \sin((\kappa+1)\theta) \cos \kappa\theta}{\sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{-\kappa \sin \theta + \cos((\kappa+1)\theta) \sin \kappa\theta}{\sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{-\kappa \sin \theta + \frac{1}{2}(\sin((2\kappa+1)\theta) - \sin \theta)}{\sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{\sin((2\kappa+1)\theta) - (2\kappa+1) \sin \theta}{2 \sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{H_{2\kappa+1}(\theta)}{2 \sin^2(\kappa\theta)} + 2a \sin \theta.
\end{aligned}$$

Let us set

$$B(\theta) := 2 \sin^2(\kappa\theta) b'(\theta) = H_{2\kappa+1}(\theta) + 4a \sin \theta \sin^2(\kappa\theta), \quad \ell(\kappa, a) := 4a\kappa - 2\kappa - 1. \quad (16)$$

Then, the signatures of  $b'(\theta)$  and  $B(\theta)$  are the same, and the derivative of  $B(\theta)$  is given as, if  $\ell(\kappa, a) \neq 0$  then

$$B'(\theta) = 2\ell(\kappa, a) \cos \theta \sin^2(\kappa\theta) \left( F_\kappa(\theta) + J_\kappa(a) \right), \quad (17)$$

and if  $\ell(\kappa, a) = 0$  then

$$B'(\theta) = \frac{1 - 4\kappa^2}{\kappa} \cos \theta \sin^2(\kappa\theta). \quad (18)$$

In fact, we have

$$\begin{aligned}
B'(\theta) &= -2(2\kappa+1) \sin((\kappa+1)\theta) \sin(\kappa\theta) + 4a(\sin \kappa\theta)(\cos \theta \sin(\kappa\theta) + 2\kappa \sin \theta \cos(\kappa\theta)) \\
&= (4a - 4\kappa - 2) \cos \theta \sin^2(\kappa\theta) + (8a\kappa - 4\kappa - 2) \sin \theta \sin(\kappa\theta) \cos(\kappa\theta) \\
&= 2 \cos \theta \sin^2(\kappa\theta) (2a - 2\kappa - 1 + (4a\kappa - 2\kappa - 1) \tan \theta \cot(\kappa\theta)) \\
&= 2(4a\kappa - 2\kappa - 1) \cos \theta \sin^2(\kappa\theta) \left( F_\kappa(\theta) + \frac{2a - 2\kappa - 1}{4a\kappa - 2\kappa - 1} \right).
\end{aligned}$$

Note that two conditions  $\ell(\kappa, a) = 0$  and  $2a - 2\kappa - 1 = 0$  occur simultaneously only when  $\kappa = \frac{1}{2}$ . Set

$$G(x) := \frac{2x^2 + 3x + 1}{6x}.$$

- Lemma 1.9.** (1) Suppose that  $0 < \kappa < \frac{1}{2}$ , or  $\kappa > 1$ . If  $a > G(\kappa)$ , then there exists a unique  $\varphi_* \in I_0$  such that  $b'(\varphi_*) = 0$ , and one has  $b'(\theta) > 0$  for  $\theta \in (0, \varphi_*)$ , and  $b'(\theta) < 0$  for  $\theta \in (\varphi_*, \pi)$ . If  $a \leq G(\kappa)$ , then one has  $b'(\theta)$  for any  $\theta \in I_0$ .
- (2) Suppose that  $\frac{1}{2} < \kappa < 1$ . If  $a < G(\kappa)$ , then there exists a unique  $\varphi_* \in I_0$  such that  $b'(\varphi_*) = 0$ , and one has  $b'(\theta) < 0$  for  $\theta \in (0, \varphi_*)$ , and  $b'(\theta) > 0$  for  $\theta \in (\varphi_*, \pi)$ . If  $a \geq G(\kappa)$ , then one has  $b'(\theta) > 0$  for any  $\theta \in I_0$ .

*Proof.* Since we have (17) or (18), we can use Lemmas 1.5 and 1.7. This lemma can be obtained by dividing cases and by elementary but tedious calculations. Note that  $\frac{2\kappa^2+3\kappa+1}{6\kappa}$  comes from the equation  $0 = F_\kappa(0) + J_\kappa(a)$ . By differentiating  $b(\theta)$  by using expression (7), we obtain

$$b'(\theta) = (2a - 1) \sin \theta + \cos \theta \cot(\kappa\theta) - \frac{\kappa \sin \theta}{\sin^2(\kappa\theta)},$$

and hence

$$\begin{aligned}
b'(\theta) \sin^2(\kappa\theta) &= (2a - 1) \sin \theta (1 - \cos^2(\kappa\theta)) + \cos \theta \sin(\kappa\theta) \cos(\kappa\theta) - \kappa \sin \theta \\
&= (2a - \kappa - 1) \sin \theta + (2a - 1) \sin \theta \cos^2(\kappa\theta) + \cos \theta \sin(\kappa\theta) \cos(\kappa\theta) \\
&= (2a - \kappa - 1) \sin \theta + \cos \theta \sin(\kappa\theta) \cos(\kappa\theta) ((2a - 1) F_\kappa(\theta) + 1).
\end{aligned}$$

If  $\varphi_i$  satisfies  $B'(\varphi_i) = 0$ , then we have  $F_\kappa(\varphi_i) + J_\kappa(a) = 0$  and hence

$$(2a - 1) F_\kappa(\varphi_i) + 1 = \frac{-(2a - 1)(2a - 2\kappa - 1) + (4a\kappa - 2\kappa - 1)}{\ell(\kappa, a)} = \frac{4a(a - 1)}{\ell(\kappa, a)}. \quad (19)$$

The non-trivial cases are given as (i)  $\frac{1}{2} < \kappa < 1$  and  $0 \leq -J_\kappa(a) \leq F_\kappa(x_*)$ , or (ii)  $1 < \kappa < 2$  and  $-J_\kappa(x) > F_\kappa(x_*)$ .

first we consider the case (i). In this case, Lemma 1.7 tells us that we have  $\ell(\kappa, a) > 0$ . Since  $-J_\kappa(a)$  is monotonic decreasing when  $\ell(\kappa, a) > 0$  and since  $-J_\kappa(1) = 1$  by (13), we see that  $a$  need satisfy  $a \in (1, \frac{1}{2} + \kappa)$  because  $F_\kappa(x_*) < 1$  by Lemma 1.5. Thus, we have  $2a - \kappa - 1 > 0$ . Again by

Lemma 1.5, the equation  $B'(\theta) = F_\kappa(\theta) + J_\kappa(a) = 0$  has at most two solutions  $\varphi_i \in (\frac{\pi}{2\kappa}, \pi)$ ,  $i = 1, 2$ . Set  $\varphi_1 \leq \varphi_2$ . Then, since

$$B(0) = 0, \quad B(\pi) = \sin(2\kappa\pi) > 0,$$

we have

$\theta$	0	$\dots$	$\frac{\pi}{2}$	$\dots$	$\varphi_1$	$\dots$	$\varphi_2$	$\dots$	$\pi$
$\ell(\kappa, a)$		+		+		+		+	
$\cos \theta$		+	0	-		-		-	
$F_\kappa + J_\kappa$		+		-	0	+	0	-	
$B'(\theta)$		+		+	0	-	0	+	
$B(\theta)$	0	$\nearrow$		$\nearrow$		$\searrow$		$\nearrow$	$B(\pi) > 0$

We shall show that  $B(\varphi_i) > 0$ , ( $i = 1, 2$ ), which implies  $b'(\theta) > 0$ . Since  $a > 1$ , we have  $(2a - 1)F_\kappa(\varphi_i) + 1 > 0$  by (19). Since  $\varphi_i \in (\frac{\pi}{2\kappa}, \pi)$ , we see that  $\cos \varphi_i \sin(\kappa\varphi_i) \cos(\kappa\varphi_i) > 0$  and hence we obtain  $B(\varphi_i) > 0$ .

Next, we consider the case (ii). Then, since  $F_\kappa(x_*) > 1$  by Lemma 1.5 and since  $J_\kappa(1) = -1$  by (13), we need to have  $\ell(\kappa, a) > 0$  and  $a < 1$ , whence  $2a - \kappa - 1 < 0$ . Lemma 1.7 tells us that the function  $-J_\kappa(a)$  is monotonic decreasing if  $\ell(\kappa, a) > 0$ . Again by Lemma 1.5, the equation  $B'(\theta) = F_\kappa(\theta) + J_\kappa(a) = 0$  has at most two solutions  $\varphi_i \in (\frac{\pi}{2\kappa}, \pi)$ ,  $i = 1, 2$ . Let  $\varphi_1 \leq \varphi_2$ . Then, since

$$B(0) = 0, \quad B\left(\frac{\pi}{\kappa}\right) = -2\kappa \sin \frac{\pi}{\kappa} < 0,$$

we have

$\theta$	0	$\dots$	$\frac{\pi}{2}$	$\dots$	$\varphi_1$	$\dots$	$\varphi_2$	$\dots$	$\frac{\pi}{\kappa}$
$\ell(\kappa, a)$		+		+		+		+	
$\cos \theta$		+	0	-		-		-	
$F_\kappa + J_\kappa$		-		+	0	-	0	+	
$B'(\theta)$		-		-	0	+	0	-	
$B(\theta)$	0	$\searrow$		$\searrow$		$\nearrow$		$\searrow$	$B(\frac{\pi}{\kappa}) < 0$

Since  $a < 1$ , we have  $(2a - 1)F_\kappa(\varphi_i) + 1 < 0$  by (19). Since  $\varphi_i \in (\frac{\pi}{2}, \frac{\pi}{\kappa})$ , we have  $\cos \varphi_i \sin(\kappa\varphi_i) \cos(\kappa\varphi_i) > 0$  so that we obtain  $B(\varphi_i) < 0$ .  $\square$

## 2. THE DOMAIN $\Omega$ FOR $\kappa < +\infty$

In this section, we shall determine  $\Omega$  for finite  $\kappa > 0$ , which is the connected component of the set  $\{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); F(x, y) > 0\}$  containing  $z = 0$ . Let  $r_\pm(\theta)$  be the solutions of the equation (8) and let  $\alpha_i$ ,  $i = 1, 2$  are the solutions of the equation (10). Since  $F(x, y)$  is a continuous function, the boundary  $\partial\Omega$  is included in the set  $\{z = x + yi \in \mathbb{C}; F(x, y) = 0\} \subset f_{\kappa, \gamma}^{-1}(\mathbb{R})$ . Since  $F$  is even function on  $y$ , the domain  $\Omega$  is symmetric with respect to the real axis, and hence we consider  $D := \Omega \cap \mathbb{C}^+$ .

**Proposition 2.1.** *Suppose that  $\kappa = 1$ .*

- (1) *If  $\gamma > 1$ , then one has  $\Omega = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$ .*
- (2) *If  $0 < \gamma \leq 1$ , then one has  $\Omega = \left\{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); \left(x + \frac{1}{\gamma}\right)^2 + y^2 > \frac{1-\gamma}{\gamma^2}\right\}$ .*
- (3) *If  $\gamma = 0$ , then one has  $\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); 1 + 2x > 0\}$ .*
- (4) *If  $\gamma < 0$ , then one has  $\Omega = \left\{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); \left(x + \frac{1}{\gamma}\right)^2 + y^2 < \frac{1-\gamma}{\gamma^2}\right\}$ . In particular,  $\Omega$  is bounded.*

*Proof.* For  $z = x + yi$ , we have

$$f_{1, \gamma}(z) = \frac{(x + \gamma x^2 + \gamma y^2 + yi)(x + 1 + yi)}{(1 + \gamma x)^2 + \gamma^2 y^2} = \frac{1}{(1 + \gamma x)^2 + \gamma^2 y^2} \begin{pmatrix} (x + \gamma x^2 + \gamma y^2)(1 + x) - y^2 \\ iy(1 + 2x + \gamma x^2 + \gamma y^2) \end{pmatrix}.$$

Thus,  $f_{1, \gamma}(z) \in \mathbb{R}$  implies  $y = 0$  or

$$1 + 2x + \gamma x^2 + \gamma y^2 = 0 \iff \begin{cases} \left(x + \frac{1}{2\gamma}\right)^2 + y^2 = \frac{1-\gamma}{\gamma^2} & (\text{if } \gamma \neq 0), \\ x = -\frac{1}{2} & (\text{if } \gamma = 0). \end{cases}$$

Therefore, we have  $\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); \operatorname{Re} z > -\frac{1}{2}\}$  when  $\gamma = 0$  because it contains  $z = 0$ . Suppose that  $\gamma \neq 0$  and set  $C = \{z = x + yi \in \mathbb{C}; 1 + 2x + \gamma x^2 + \gamma y^2 = 0\}$ . If  $\gamma > 1$ , then  $C = \emptyset$ , which implies  $\Omega = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$ . Assume that  $0 < \gamma \leq 1$ . Then,  $C \neq \emptyset$  and  $z = 0$  does

not contained in the interior of  $C$  and hence  $\Omega$  is the out side of  $C$ , which is written as  $\Omega = \{z = x + yi \in \mathbb{C}; 1 + 2x + \gamma x^2 + \gamma y^2 > 0\}$ . Assume that  $\gamma < 0$ . Then,  $C \neq \emptyset$  and  $z = 0$  is contained in the interior of  $C$ , and thus  $\Omega$  is the interior of  $C$ , which is written as

$$\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); 1 + 2x + \gamma x^2 + \gamma y^2 > 0\}.$$

The proof is completed.  $\square$

Recall that  $a = \kappa\gamma$ . Set  $\theta_0 := \frac{\pi}{\kappa}$  and let  $I_0$  be the interval  $(0, \min(\pi, \theta_0))$ .

**Proposition 2.2.** *Let  $\kappa > 0$  with  $\kappa \neq 1$ . For  $z \in \mathbb{C} \setminus \{x \leq -\kappa\}$ , one sets  $re^\theta = 1 + \frac{z}{\kappa}$ . Then,  $\Omega$  can be described as follows.*

- (1) *If  $a < 0$ , then there exists a unique  $\theta_* \in (0, \frac{\pi}{\kappa+1})$  such that  $r_+(\theta_*) = r_-(\theta_-)$  and that  $r_\pm(\theta)$  are both positive with  $r_-(\theta) \leq r_+(\theta)$  on  $(0, \theta_*)$ . Moreover, one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_* \text{ and } r_-(\theta) < r < r_+(\theta)\}.$$

*In particular,  $\Omega$  is bounded. One has  $\alpha_1, \alpha_2 \in \partial\Omega$  and  $-\frac{1}{\gamma} \in \overline{\Omega}$ , whereas  $-\kappa \notin \overline{\Omega}$ .*

- (2) *If  $a = 0$ , then one has  $r(\theta) = r_\pm(\theta) = \frac{\sin(\kappa\theta)}{\sin((\kappa+1)\theta)}$  which is positive on the interval  $(0, \frac{\pi}{\kappa+1})$ . Moreover, one has*

$$\Omega = \left\{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \frac{\pi}{\kappa+1} \text{ and } r > r(\theta)\right\}.$$

*$\Omega$  has an asymptotic line  $y = \pm(x + \frac{\kappa^2}{\kappa+1})\tan\theta_1$ . One has  $\alpha_1 = \alpha_2 = -\frac{\kappa}{\kappa+1} \in \partial\Omega$  and  $-\kappa \notin \overline{\Omega}$ .*

- (3) *If  $0 < a < 1$ , then  $r_+(\theta)$  is the only positive solution of (8) on  $I_0$ , and one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \min(\theta_0, \pi) \text{ and } r > r_+(\theta)\}.$$

*If  $\kappa > 1$ , then  $\Omega$  has an asymptotic line  $y = \pm(\tan\theta_0)(x + \kappa - \frac{1}{a})$ . One has  $\alpha_2 \in \partial\Omega$ , whereas  $\alpha_1, -\frac{1}{\gamma}, -\kappa \notin \overline{\Omega}$ .*

- (4) *Suppose that  $a = 1$  and  $\kappa \neq 1$ .*

(a) *If  $0 < \kappa < 1$ , then one has  $r_+(\theta) = 0$  and  $r_-(\theta) = -b(\theta) < 0$  for  $\theta \in I_0$ , and  $\Omega = \mathcal{D}(f_{\kappa, \gamma})$ . One has  $\alpha_1 \in \partial\Omega$  and  $\alpha_2 = -\kappa = -\frac{1}{\gamma} \in \partial\Omega$ .*

(b) *If  $\kappa > 1$ , then one has  $r_+(\theta) = -b(\theta) > 0$  and  $r_-(\theta) = 0$  for  $\theta \in I_0$ , and one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0 \text{ and } r > r_+(\theta)\}.$$

*One has  $\alpha_2 = -1 \in \partial\Omega$ , while  $\alpha_1 = -\kappa = -\frac{1}{\gamma} \notin \overline{\Omega}$ . Moreover,  $\Omega$  has an asymptotic line  $y = \pm(\tan\theta_0)(x + \kappa - \frac{1}{a})$ .*

- (5) *Suppose that  $a > 1$ .*

(a) *If  $\kappa > 1$  with  $D(0) \geq 0$ , then  $r_\pm(\theta)$  are both positive in  $I_0$  with  $r_-(\theta) \leq r_+(\theta)$ , and one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0 \text{ and } r > r_+(\theta)\}.$$

*$\Omega$  has an asymptotic line  $y = \pm(\tan\theta_0)(x + \kappa - \frac{1}{a})$ . One has  $\alpha_2 \in \partial\Omega$ , while  $-\kappa, -\frac{1}{\gamma}, \alpha_1 \notin \overline{\Omega}$ .*

- (b) *If  $\kappa > 1$  and  $D(0) < 0$ , then there exists a unique  $\theta_* \in (0, \theta_0)$  such that  $D(\theta_*) = 0$ , and  $r_\pm(\theta)$  are both positive in the interval  $(\theta_*, \theta_0)$ . Moreover, one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0 \text{ and if } |\theta| \geq \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}.$$

*In this case,  $\alpha_i, i = 1, 2$  are both non-real and one has  $\alpha_i, -\kappa \in \partial\Omega$  and  $-\frac{1}{\gamma} \in \overline{\Omega}$ . Moreover,  $\Omega$  has an asymptotic line  $y = \pm(\tan\theta_0)(x + \kappa - \frac{1}{a})$ .*

- (c) *If  $0 < \kappa < 1$ , then there are no  $\theta$  such that  $D(\theta) > 0$ , and one has  $\Omega = \mathcal{D}(f_{\kappa, \gamma})$ .*

Recall that, since  $\Omega$  is symmetric with respect to the real axis, it is enough to determine the boundary  $\partial\Omega$  of  $\Omega$  in the upper half plane.

*Proof.* (1) Assume that  $a < 0$ . Since  $r_+(\theta) \cdot r_-(\theta) = \frac{a-1}{a} > 0$ , we see that  $r_\pm(\theta)$  have the same signature if  $r_\pm(\theta) \in \mathbb{R}$ . Since  $a < 0$ , we have  $b(0) = 1 + \frac{1}{\kappa} - 2a > 0$  by (11). Let  $\theta_1 := \frac{\pi}{\kappa+1}$ . Then, Lemmas 6.2 and 1.4 tells us that  $b'(\theta) \leq 0$  for any  $\theta \in (0, \theta_1)$ . In fact, if  $\kappa$  does not satisfy  $\frac{1}{2} < \kappa < 1$ , then it is a direct consequence of Lemma 6.2, and if  $\frac{1}{2} < \kappa < 1$  then we can verify it by Lemma 6.2 and by the fact that  $b'(\theta_1) = H_{2\kappa+1}(\theta_1) + 2a \sin\theta_1 < 0$  by Lemma 1.4. Therefore,  $b(\theta)$  is

monotonic decreasing on the interval  $(0, \theta_1)$ . If  $b(\theta_1) < 0$ , then there exists a unique  $\varphi \in (0, \theta_1)$  such that  $b(\varphi) = 0$ , and if  $b(\theta_1) \geq 0$  then we set  $\varphi = \theta_1$ . Since we have

$$D(0) = \left(1 + \frac{1}{\kappa}\right)^2 - \frac{4a}{\kappa} > 0 \quad \text{and} \quad D(\theta_1) = (-2a \cos \theta_1)^2 - 4a(a-1) = 4a^2 \cos^2 \theta_1 - 4a^2 + 4a \\ = 4a - 4a^2 \sin^2 \theta_1 = 4a(1 - a \sin^2 \theta_1) < 0$$

and  $D'(\theta) = 2b(\theta)b'(\theta)$ , the increasing/decreasing table of  $D(\theta)$  is given as

$\theta$	0	$\dots$	$\varphi$	$\dots$	$\theta_1$
$b(\theta)$		+	0	-	
$b'(\theta)$		-	-	-	
$D'(\theta)$		-	0	+	
$D(\theta)$	$D(0) > 0$	$\searrow$	$D(\varphi)$	$\nearrow$	$D(\theta_1) < 0$

and hence there exists a unique  $\theta_* \in (0, \varphi)$  such that  $D(\theta_*) = 0$ . In particular,  $D$  is monotonic decreasing in the interval  $(0, \theta_*)$ , and  $D(\theta_* + \delta) < 0$  for  $\delta > 0$  such that  $\theta_* + \delta < \varphi$ . Therefore, since  $r_+(\theta) + r_-(\theta) = -b(\theta)/a$ , the signatures of  $r_{\pm}(\theta)$  is the same as that of  $b(\theta)$  if  $r_{\pm}$  are real so that  $r_{\pm}$  are positive on  $(0, \theta_*]$  and  $r_{\pm}$  are not real for  $\theta \in (\theta_*, \varphi)$ . Since  $r_+(\theta_*) = r_-(\theta_*)$  by the fact  $D(\theta_*) = 0$ , the curves  $r_+(\theta)$ ,  $\theta \in (0, \theta_*]$  followed by  $r_-(\theta_* - \theta)$ ,  $\theta \in (0, \theta_*]$ , form a continuous curve going from  $\alpha_2$  to  $\alpha_1$  in the upper half-plane. Let  $\alpha_i$ ,  $i = 1, 2$  be the solutions of (10). Since  $\alpha_1 \alpha_2 = \frac{1}{\gamma} < 0$  and  $\alpha_1 \leq \alpha_2$ , we have  $\alpha_1 < 0 < \alpha_2$  so that

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_* \text{ and } r_-(\theta) < r < r_+(\theta)\}.$$

Since  $q(-\frac{1}{\gamma}) > 0$  and  $q(-\kappa) < 0$  by (12), where  $q$  is defined in (11), we see that  $-\frac{1}{\gamma} \in \overline{\Omega}$  and  $-\kappa \notin \overline{\Omega}$ .

(2) Assume that  $a = 0$ . In this case, the equation (8) reduces to  $b(\theta)r - 1 = 0$  so that

$$r_{\pm}(\theta) = r(\theta) = \frac{1}{b(\theta)} = \frac{1}{\cos \theta + \sin \theta \cot(\kappa \theta)} = \frac{\sin(\kappa \theta)}{\sin((\kappa + 1)\theta)}.$$

Let  $\theta_1 = \frac{\pi}{\kappa + 1}$ . Since  $\sin(\kappa \theta)$  and  $\sin((\kappa + 1)\theta)$  are both positive in the interval  $(0, \theta_1)$ , and since  $\lim_{\theta \rightarrow \theta_1 - 0} \sin((\kappa + 1)\theta) = 0$ , we see that

$$\lim_{\theta \rightarrow \theta_1 - 0} r(\theta) = +\infty.$$

Thus, it has an asymptotic line with slope  $\tan \theta_1$ , which is determined later. Since  $\gamma = a/\kappa = 0$ , the solutions  $\alpha_i$ ,  $i = 1, 2$  are given as  $\alpha_1 = \alpha_2 = -\frac{\kappa}{\kappa + 1}$ . Since  $q(0) = 1 > 0$  and  $q(-\kappa) = -\kappa < 0$ , we see that the domain  $\Omega$  is given as

$$\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_1, r > r(\theta)\}$$

and  $-\kappa \notin \overline{\Omega}$ .

(3) Assume that  $0 < a < 1$ . In this case, we have

$$D(\theta) = b(\theta)^2 + 4a(1 - a) > 0$$

for any  $\theta \in I_0$  so that the solutions of (8) are always real. Moreover, since  $r_+(\theta) \cdot r_-(\theta) = -\frac{1-a}{a} < 0$ , they have the different signatures, and  $r_+(\theta)$  is the positive real solution of (8) by the fact  $|b(\theta)| < \sqrt{D(\theta)}$ . In particular,  $r_+(\theta)$ ,  $\theta \in I_0$  forms a continuous curve in  $\mathbb{C}^+$ . The solutions  $\alpha_i$ ,  $i = 1, 2$  of (10) are both negative because we have  $\gamma = a/\kappa > 0$  and  $\alpha_1 + \alpha_2 = -(1 + \frac{1}{\kappa})/\gamma < 0$ . Since  $q(-\frac{1}{\gamma}) < 0$  and  $q(-\kappa) < 0$  by (12) and since  $\frac{1}{\gamma} > \kappa$ , we see that

$$\alpha_1 < -\frac{1}{\gamma} < -\kappa < \alpha_2 < 0. \quad (20)$$

(a) We first assume that  $\kappa \geq 1$ . Then,  $b(\theta)$  is defined on the interval  $I_0 = (0, \theta_0)$ , and we have  $\lim_{\theta \rightarrow \theta_0 - 0} b(\theta) = -\infty$ , which implies

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty.$$

Therefore, the curve  $r_+(\theta)$ ,  $\theta \in (0, \theta_0)$  has the asymptotic line with gradient  $\tan \theta_0$ , which is determined later. By (20), we see that  $\Omega$  is given as

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, r > r_+(\theta)\},$$

and  $\alpha_1, -\frac{1}{\gamma}, -\kappa \notin \overline{\Omega}$ .

(b) Next, we assume that  $0 < \kappa < 1$ . Then,  $b(\theta)$  is defined for any  $\theta \in I_0 = (0, \pi)$ . If  $\theta = 0$ , then the positive solution of (8) corresponds to  $\alpha_2 < 0$ , and if  $\theta = \pi$  then, since  $b(\pi) = 2a - 1$ , the positive

solution of (8) corresponds to  $-\frac{1}{\gamma}$ . Thus, the curve  $r_+(\theta)$ ,  $\theta \in (0, \pi)$  connects  $z = \alpha_2$  and  $z = -\frac{1}{\gamma}$  passing in the upper half plane. This means that  $z = -\kappa$  is in the interior of the curve, whereas  $z = 0$  is in its outside. Since  $\Omega$  is the connected component including  $z = 0$ ,  $\Omega$  is the outside of the curve and hence

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \pi, r > r_+(\theta)\}.$$

(4) Assume that  $a = 1$  and  $\kappa \neq 1$ . In this case, the equation (8) reduces to  $r^2 + b(\theta)r = 0$ , whose solutions are  $r(\theta) = 0, -b(\theta)$ . Recall that  $I_0 = (0, \min(\pi, \frac{\pi}{\kappa}))$ . By Lemmas 1.8 and 1.5, we see that if  $0 < \kappa < 1$  then  $b(\theta) > 0$  and if  $\kappa > 1$  then  $b(\theta) < 0$  for any  $\theta \in I_0$ . This means that if  $0 < \kappa < 1$  then the equation (8) does not have a positive solution, and hence we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \pi, r > 0\} = \mathcal{D}(f_{\kappa, \gamma}),$$

which shows the assertion (4)-(a). If  $\kappa > 1$ , then since  $b'(\theta) < 0$  for  $\theta \in I_0$  and since  $b(0) = 1 + \frac{1}{\kappa} - 2 < 0$  and  $\lim_{\theta \rightarrow \theta_0 - 0} b(\theta) = -\infty$ , the function  $D(\theta)$  is monotonic increasing on  $I_0$  and hence we have

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty.$$

Thus, the curve  $r_+(\theta)$ ,  $\theta \in I_0$  has an asymptotic line with gradient  $\tan \theta_0$ , which is determined later. Thus, we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0 \text{ and } r > r_+(\theta)\}.$$

(5) Suppose that  $a > 1$ . In this case, we have  $b(0) = 1 + \frac{1}{\kappa} - 2a$  and

$$b\left(\frac{\pi}{\kappa}\right) = (1 - 2a) \cos \frac{\pi}{\kappa} \quad (\text{if } \kappa > 1), \quad b(\pi) = 2a - 1 > 0 \quad (0 < \kappa < 1).$$

Note that  $b(0) > 0$  if  $\kappa > 1$ . Since  $r_+(\theta) \cdot r_-(\theta) = \frac{a-1}{a} > 0$ , two solutions  $r_{\pm}(\theta)$  of (8) have the same signature if  $r_{\pm}(\theta)$  are real.

(a) We first consider the case  $\kappa > 1$  and  $D(0) \geq 0$ . Let us show that  $D(\theta) > 0$  for  $\theta \in I_0$ . Set

$$K(x) := \frac{x}{4} \left(1 + \frac{1}{x}\right)^2, \quad G(x) = \frac{2x^2 + 3x + 1}{6x} \quad (x > 0).$$

Then, the condition  $D(0) \geq 0$  is equivalent to  $a \leq K(\kappa)$ , and hence we have  $a \leq G(\kappa)$  because

$$G(x) - K(x) = \frac{x^2 - 1}{12x} > 0 \quad \text{if } x > 1.$$

By the assumption  $\kappa > 1$ , we see that  $b'(\theta) < 0$  for any  $\theta \in I_0$  by Lemma 6.2 so that  $b$  is monotonic decreasing in this interval. Since  $b(0) < 0$ , the function  $b$  is negative in  $I_0$ , and hence  $D'(\theta) = 2b(\theta)b'(\theta) > 0$  so that  $D(\theta)$  is monotonic increasing in the interval  $I_0$ , and in particular, it is positive on  $I_0$ . By (9), we have  $r'_+(\theta) > 0$ , and hence the function  $r_+(\theta)$  is monotonic increasing, whereas  $r_-(\theta)$  is monotonic decreasing because  $r_-(\theta) = \frac{a-1}{ar_+(\theta)}$ . Note that

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty, \quad \lim_{\theta \rightarrow \theta_0 - 0} r_-(\theta) = 0.$$

Thus,  $r_+(\theta)$ ,  $\theta \in I_0$  draws an bounded curve connecting  $z = \alpha_2$  to  $\infty$ , and  $r_-(\theta)$ ,  $\theta \in I_0$  draws a bounded curve connecting  $z = \alpha_1$  to  $z = -\kappa$ . Since we have  $-\kappa < -\frac{1}{\gamma} < \alpha_1 < \alpha_2 < 0$  and since  $\Omega$  is the connected component including  $z = 0$ , we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, r > r_+(\theta)\}.$$

(b) Next, we assume that  $\kappa > 1$  and  $D(0) < 0$ . According to Lemma 1.9 (1), we consider the function  $b'(\theta)$  in two cases, that is, (i)  $a \leq G(\kappa)$  and (ii)  $a > G(\kappa)$ .

(i) Assume that  $a \leq G(\kappa)$ . Then,  $b(\theta)$  is monotonic decreasing. Since  $b(0) < 0$ , we see that  $b(\theta) < 0$  for any  $\theta \in I_0$  and therefore  $D'(\theta) = 2b(\theta)b'(\theta) > 0$  for any  $\theta \in I_0$ . Thus,  $D(\theta)$  is monotonic increasing with  $D(0) < 0$ . In particular, there exists a unique  $\theta_*$  such that  $D(\theta_*) = 0$ , and  $r_{\pm}(\theta)$  are real for  $\theta \in (\theta_*, \theta_0)$ . In this interval, since  $r_+(\theta) + r_-(\theta) = -b(\theta)/a > 0$ , we see that  $r_{\pm}(\theta)$  are both positive. By (9), we have  $r'_+(\theta) > 0$  and thus the function  $r_+(\theta)$  is monotonic increasing, whereas  $r_-(\theta)$  is monotonic decreasing because  $r_-(\theta) = \frac{a-1}{ar_+(\theta)}$ . By taking a limit  $\theta \rightarrow \theta_0 - 0$ , we have

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty, \quad \lim_{\theta \rightarrow \theta_0 - 0} r_-(\theta) = 0.$$

This means that  $r_+(\theta)$  draws an unbounded curve connecting  $z = \alpha_1$  and  $z = \infty$ , and  $r_-(\theta)$  draws a bounded curve connecting  $z = \alpha_1$  and  $z = -\kappa$ , where  $\alpha_1$  is the complex solution of (10) with positive imaginary part. Since we have  $-\kappa < -\frac{1}{\gamma} < 0$ , the domain  $\Omega$  is given as

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, \text{ if } |\theta| \geq \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}.$$

(ii) Assume that  $a > G(\kappa)$ . Then, there are two possibilities on  $b(\varphi_*)$ . If  $b(\varphi_*) \leq 0$ , then we have

$\theta$	0	$\cdots$	$\varphi_*$	$\cdots$	$\frac{\pi}{\kappa}$
$b'(\theta)$		+	0	−	
$b(\theta)$	−	−		−	
$D'(\theta)$		−	0	+	
$D(\theta)$	$D(0) < 0$	$\searrow$		$\nearrow^{+\infty}$	$\times$

and if  $b(\varphi_*) > 0$ , then there exist exactly two  $\varphi_1 < \varphi_2$  such that  $b(\varphi_i) = 0$  so that

$\theta$	0	$\cdots$	$\varphi_1$	$\cdots$	$\varphi_*$	$\cdots$	$\varphi_2$	$\cdots$	$\frac{\pi}{\kappa}$
$b'(\theta)$			+		0		−		
$b(\theta)$	−	−	0		+		0	−	
$D'(\theta)$		−	0	+	0	−	0	+	
$D(\theta)$	$D(0) < 0$	$\searrow$		$\nearrow$		$\searrow$		$\nearrow^{+\infty}$	$\times$

We note that  $D(\varphi_i) = b(\varphi_i)^2 - 4a(a-1) < 0$ . Since  $r_+(\theta) + r_-(\theta) = -b(\theta)/a$ , if  $D(\varphi_*) > 0$ , then  $r_{\pm}(\theta)$  are both negative so that we do not deal with this case. Thus, in both cases, there exists a unique  $\theta_* \in I_0$  such that  $D(\theta_*) = 0$  and  $r_{\pm}(\theta) > 0$  for any  $\theta \in (\theta_*, \theta_0)$ . By (9), we see that  $r_+(\theta)$  is monotonic increasing on  $(\theta_*, \theta_0)$ , whereas  $r_-(\theta)$  is monotonic decreasing. Moreover, we have

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty, \quad \lim_{\theta \rightarrow \theta_0 - 0} r_-(\theta) = 0,$$

and hence the curves  $r_{\pm}(\theta)$ ,  $\theta \in (\theta_*, \theta_0)$  form an unbounded curve connecting  $z = \alpha_1$  and  $z = \infty$ . Since  $-\kappa < -\frac{1}{\gamma} < 0$ , we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, \text{ and if } |\theta| \geq \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}.$$

(c) We finally assume that  $0 < \kappa < 1$ . In this case, we have  $D(\pi) = (2a-1)^2 - 4a(a-1) = 1$ . We note that  $b(0) < 0$  implies  $D(0) < 0$ . In fact,  $b(0) < 0$  means  $1 + \frac{1}{\kappa} < 2a$  so that

$$D(0) = \left(1 + \frac{1}{\kappa}\right)^2 - \frac{4a}{\kappa} < 2a\left(1 + \frac{1}{\kappa}\right) - \frac{4a}{\kappa} = 2a\left(1 - \frac{1}{\kappa}\right) < 0. \quad (21)$$

Since  $a > 1$ , the signatures of  $r_{\pm}(\theta)$  are the same, and they are the opposite to the signature of  $b(\theta)$ .

Let  $I_D \subset I_0$  be the maximal interval such that  $D$  is positive on  $I_D$ . We shall show that there are no suitable solutions of (8), that is,  $r_{\pm}(\theta) < 0$  for any  $\theta \in I_D$ , which yields that  $\Omega = \mathcal{D}(f_{\kappa, \gamma})$ . Let us recall Lemma 1.9.

(i) Assume that  $0 < \kappa < \frac{1}{2}$  and  $a > G(\kappa)$ . In this case, we have

$\theta$	0	$\cdots$	$\varphi_*$	$\cdots$	$\pi$
$b'(\theta)$		+	0	−	
$b(\theta)$	$b(0)$	$\nearrow$		$\searrow$	$b(\pi) > 0$

If  $b(0) \geq 0$ , then we see that  $b(\theta) > 0$  for any  $\theta \in I_0$ , which implies  $r_{\pm}(\theta) < 0$  for any  $\theta \in I_D$ . If  $b(0) < 0$ , then there exists a unique  $0 < \varphi < \varphi_*$  such that  $b(\varphi) = 0$ , and we have  $D(0) < 0$  by (21). Hence,

$\theta$	0	$\cdots$	$\varphi$	$\cdots$	$\varphi_*$	$\cdots$	$\pi$
$b'(\theta)$		+	+	+	0	−	
$b(\theta)$	−	$\nearrow$	0	$\nearrow$	+	$\searrow$	+
$D'(\theta)$		−	0	+	0	−	
$D(\theta)$	$D(0) < 0$	$\searrow$	−	$\nearrow$	+	$\searrow$	1

This table yields that  $b$  is negative on  $I_D$ , whence  $r_{\pm}(\theta) < 0$  for any  $\theta \in I_D$ .

(ii) Assume that  $0 < \kappa < \frac{1}{2}$  and  $a \leq G(\kappa)$ . Then, Lemma 1.8 tells us that  $b'(\theta) < 0$  for any  $\theta \in I_0$ . Thus,  $b(\theta)$  is monotonic decreasing on the interval  $I_0$  with  $b(\pi) > 0$ , and hence  $b(\theta) > 0$  for any  $\theta \in I_0$ . This means that  $D'(\theta) < 0$  and  $D(\theta)$  is monotonic decreasing on  $I_0$ . Since  $D(\pi) = 1$ , we see that  $I_D = I_0$  and hence  $r_{\pm}(\theta) < 0$  for any  $\theta \in I_D$ .

(iii) Assume that  $\frac{1}{2} \leq \kappa < 1$ . In this case, we have  $b'(\theta) > 0$  for any  $\theta \in I_0$  so that  $b(\theta)$  is monotonic increasing. In fact, if  $\kappa > \frac{1}{2}$ , then since  $\frac{(1+\sqrt{2})^2}{6} < G(\kappa) < 1$  for  $\frac{1}{2} < \kappa < 1$  and since  $a > 1$ , we always have  $a > G(\kappa)$  so that  $b'(\theta) > 0$  by Lemma 1.8, and if  $\kappa = \frac{1}{2}$  then we have  $b'(\theta)2(a-1)\sin\theta$  so that  $b'(\theta) > 0$  by (14). If  $b(0) \geq 0$ , then we have  $b(\theta) \geq 0$  for any  $\theta \in \theta$  and hence we see that  $r_{\pm}(\theta) < 0$  for  $\theta \in I_D$ . If  $b(0) < 0$  then there exists a unique  $\varphi$  such that  $b(\varphi) = 0$  and we have

$\theta$	0	$\dots$	$\varphi$	$\dots$	$\pi$
$b'(\theta)$		+	+	+	
$b(\theta)$	-	$\nearrow$	0	$\nearrow$	+
$D'(\theta)$		-	0	+	
$D(\theta)$	-	$\searrow$	-	$\nearrow$	1

This table indicates that for  $\theta \in I_D$  we have  $b(\theta) > 0$ , which implies  $r_{\pm}(\theta) < 0$ .

We shall determine an asymptotic line with respect to  $\Omega$  when  $r_+(\theta) \rightarrow +\infty$  as  $\theta \rightarrow \frac{\pi}{\kappa}$  or  $\frac{\pi}{\kappa+1}$ . To calculate them in a one scheme, we set  $\vartheta = \frac{\pi}{\kappa}$  or  $\frac{\pi}{\kappa+1}$ , and denote its denominator by  $\alpha$ . A line having gradient  $\tan \vartheta$  can be written as  $x \sin \vartheta - y \cos \vartheta = A$  with some constant  $A$ . Since  $x = \kappa(r(\theta) \cos \theta - 1)$  and  $y = \kappa r(\theta) \sin \theta$ , we have

$$\begin{aligned} x \sin \vartheta - y \cos \vartheta &= \kappa \{ \sin \vartheta (r(\theta) \cos \theta - 1) - \cos \vartheta r(\theta) \sin \theta \} \\ &= \kappa \{ r(\theta) (\cos \theta \sin \vartheta - \sin \theta \cos \vartheta) - \sin \vartheta \} \\ &= -\kappa \{ r(\theta) \sin(\theta - \vartheta) + \sin \vartheta \}. \end{aligned}$$

When  $\theta \rightarrow \vartheta$ , we have  $r_+(\theta) \rightarrow +\infty$  and  $b(\theta) \rightarrow -\infty$ , and

$$r_+(\theta) = \frac{-b(\theta) + \sqrt{b(\theta)^2 - 4a(a-1)}}{2a} = -\frac{b(\theta)}{2a} \left( 1 + \sqrt{1 - \frac{4a(a-1)}{b(\theta)^2}} \right).$$

Since

$$\lim_{\theta \rightarrow \vartheta-0} \frac{\sin(\theta - \vartheta)}{\sin(\alpha \vartheta)} = \lim_{\theta \rightarrow \vartheta-0} -\frac{\sin(\theta - \vartheta)}{\sin(\alpha(\theta - \vartheta))} = \lim_{\theta \rightarrow \vartheta-0} -\frac{\theta - \vartheta}{\alpha(\theta - \vartheta)} = -\frac{1}{\alpha},$$

we have

$$\lim_{\theta \rightarrow \vartheta-0} b(\theta) \sin(\theta - \vartheta) = \lim_{\theta \rightarrow \vartheta-0} \left( (1 - 2a) \cos \theta \sin(\theta - \vartheta) + \sin \theta \cos(\alpha \theta) \frac{\sin(\theta - \vartheta)}{\sin(\alpha \theta)} \right) = \frac{\sin \vartheta}{a\alpha},$$

whence

$$\lim_{\theta \rightarrow \vartheta-0} r(\theta) \sin(\theta - \vartheta) = \lim_{\theta \rightarrow \vartheta-0} -\frac{b(\theta) \sin(\theta - \vartheta)}{2a} \left( 1 + \sqrt{1 - \frac{4a(a-1)}{b(\theta)^2}} \right) = -\frac{\sin \vartheta}{a\alpha}.$$

Thus, we have

$$A = \begin{cases} -\kappa(-\frac{\sin \theta_0}{a\kappa} + \sin \theta_0) = \left(\frac{1}{a} - \kappa\right) \sin \theta_0 & (\text{if } \alpha = \kappa), \\ -\kappa(-\frac{\sin \theta_1}{\kappa+1} + \sin \theta_1) = -\frac{\kappa^2}{\kappa+1} \sin \theta_1 & (\text{if } \alpha = \kappa + 1), \end{cases}$$

and therefore the proof is now complete.  $\square$

### 3. THE DOMAIN $\Omega$ FOR $\kappa = +\infty$

In this section, we deal with the case  $\kappa = \infty$ . Since  $\kappa\theta(x, y)$  is regarded as  $y$  in this case, the equation (4) can be written as

$$F(x, y) = x + \gamma x^2 + \gamma y^2 + y \cot y = 0. \quad (22)$$

Assume that  $\gamma = 0$ , that is, the case of the classical Lambert function. Then, the above equation reduces to  $F(x, y) = x + y \cot y = 0$ . Since the point  $z = x + yi = 0$  satisfies  $F(0, 0) = 1 > 0$ , we have  $\Omega = \{z = x + yi \in \mathbb{C}; x > -y \cot y, |y| < \pi\}$ .

Assume that  $\gamma \neq 0$ . Then, the equation (22) can be calculated as

$$x^2 + \frac{x}{\gamma} + y^2 + \frac{y \cot y}{\gamma} = 0 \iff \left(x + \frac{1}{2\gamma}\right)^2 = \frac{1}{4\gamma^2} - y^2 - \frac{y \cot y}{\gamma}. \quad (23)$$

Let us consider the function

$$\begin{aligned} h(y) &:= \frac{1}{4\gamma^2} - y^2 - \frac{y \cot y}{\gamma} = \frac{1}{4\gamma^2} - \left(y + \frac{\cot y}{2\gamma}\right)^2 + \frac{\cot^2 y}{4\gamma^2} = \frac{1}{4\gamma^2 \sin^2 y} - \left(y + \frac{\cot y}{2\gamma}\right)^2 \\ &= \frac{1 - (2\gamma y \sin y + \cos y)^2}{4\gamma^2 \sin^2 y}. \end{aligned}$$

Note that

$$h(0) = \lim_{y \rightarrow 0} h(y) = \frac{1}{4\gamma^2} - \frac{1}{\gamma} \lim_{y \rightarrow 0} \frac{y}{\sin y} = \frac{1-4\gamma}{4\gamma^2} \quad \text{and} \quad \lim_{y \rightarrow \pi-0} |h(y)| = +\infty.$$

In order that the equation (22) has a real solution in  $x$  and  $y$ , the function  $h(y)$  needs to be non-negative, and it is equivalent to the condition that the absolute value of the function  $g(y) := \cos y + 2\gamma y \sin y$  is less than or equal to 1. At first, we observe that  $g(0) = 1$  and  $g(\pi) = -1$ , and its derivative is

$$g'(y) = -\sin y + 2\gamma(\sin y + y \cos y) = -(1-2\gamma)\sin y + 2\gamma y \cos y = (2\gamma-1)\left(\frac{2\gamma}{2\gamma-1}y + \tan y\right) \cos y.$$

Set  $c_\gamma = \frac{2\gamma}{2\gamma-1}$ . Then, the signature of  $g'$  can be determined by the signatures of  $2\gamma-1$ ,  $\cos y$  and  $c_\gamma y + \tan y$ .

Assume that  $\gamma > \frac{1}{4}$ . If  $\gamma > \frac{1}{2}$ , then we have  $c_\gamma > 1$  and hence there exists a unique  $y_* \in (\frac{\pi}{2}, \pi)$  such that  $c_\gamma y + \tan y = 0$  and we have

$y$	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$y_*$	$\cdots$	$\pi$
$g'$		+		+	0	-	
$g$	1	$\nearrow$		$\nearrow$		$\searrow$	-1

If  $\frac{1}{4} < \gamma < \frac{1}{2}$ , then we have  $c_\gamma < -1$  and hence there exists a unique  $y_* \in (0, \frac{\pi}{2})$  such that  $c_\gamma y_* + \tan y_* = 0$  and we have

$y$	0	$\cdots$	$y_*$	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$\pi$
$g'$		+		-		-	
$g$	1	$\nearrow$		$\searrow$		$\searrow$	-1

If  $\gamma = \frac{1}{2}$ , then we have  $g'(y) = y \cos y$  so that  $g$  is monotonic decreasing on the interval  $(\frac{\pi}{2}, \pi)$  with  $g(\frac{\pi}{2}) > 1$ , and in this case we set  $y_* = \frac{\pi}{2}$ . These observation shows that, if  $\gamma > \frac{1}{4}$ , then there exists one and only one  $y_0 \in (y_*, \pi)$  such that  $g(y_0) = 1$  and  $g(y_0 - \varepsilon) > 1$  for  $\varepsilon \in (0, y_0 - y_*)$ . In this case,  $h(y)$  is non-negative in the interval  $[y_0, \pi)$ , and  $h(y_0 - \varepsilon) < 0$  for  $\varepsilon \in (0, y_0 - y_*)$ . Let  $x_i(y)$ ,  $i = 1, 2$  be the real solutions of the equation (23) with  $x_1(y) \leq x_2(y)$ . Then, since we have  $x_1(y_0) = x_2(y_0)$  and

$$\lim_{y \rightarrow \pi-0} x_1(y) = -\infty, \quad \lim_{y \rightarrow \pi-0} x_2(y) = +\infty,$$

the curves  $x_i(y)$ ,  $y \in (y_0, \pi)$  form a connected curve, and hence we have

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and if } |y| \geq y_0 \text{ then } x < x_1(y) \text{ or } x > x_2(y)\}.$$

Assume that  $0 < \gamma \leq \frac{1}{4}$ . Then, we have  $-1 \leq c_\gamma < 1$  and hence there are no  $y \in (0, \pi)$  such that  $c_\gamma y + \tan y = 0$ . Thus, we obtain  $g'(y) < 0$  for any  $y \in (0, \pi)$  so that  $g$  is monotonic decreasing from  $g(0) = 1$  to  $g(\pi) = -1$ . This shows that  $h(y)$  is non-negative in the interval  $(0, \pi)$ . Let  $x_i(y)$ ,  $i = 1, 2$  be the solutions of the equation (23) with  $x_1(y) \leq x_2(y)$ . Then, since we have

$$\lim_{y \rightarrow \pi-0} x_1(y) = -\infty, \quad \lim_{y \rightarrow \pi-0} x_2(y) = +\infty$$

and  $x_1(0) < x_2(0) < 0$ , we have

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and } x > x_2(y)\}.$$

Assume that  $\gamma < 0$ . Then, there exists a unique  $y_* \in (\frac{\pi}{2}, \pi)$  such that  $c_\gamma y_* + \tan y_* = 0$ , and we have

$y$	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$y_*$	$\cdots$	$\pi$
$g'$		-		-	0	+	
$g$	1	$\searrow$		$\searrow$		$\nearrow$	-1

This observation shows that there exists one and only one  $y_0 \in (0, y_*)$  such that  $g(y_0) = -1$  and  $g(y_0 + \varepsilon) < -1$  for  $\varepsilon \in (0, y_* - y_0)$ . Thus,  $h(y)$  is non-negative on  $y \in [0, y_0]$ , and  $h(y_0 + \varepsilon) < 0$  for  $\varepsilon \in (0, y_* - y_0)$ . Let  $x_i(y)$ ,  $i = 1, 2$  be the solutions of the equation (23) with  $x_1(y) \leq x_2(y)$ . Then,

since  $x_1(0) < 0 < x_2(0)$  and  $x_1(y_0) = x_2(y_0)$ , these two paths  $(x_{\pm}(y), y)$  form a continuous curve connecting  $x_+(0)$  and  $x_-(0)$  and we have

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < y_0 \text{ and } x_1(y) < x < x_2(y)\}.$$

We summarize these calculations as a proposition.

**Proposition 3.1.** *Assume that  $\kappa = +\infty$  and let  $x_i(y)$ ,  $i = 1, 2$  be solutions of (23) with  $x_1(y) \leq x_2(y)$  if they are real.*

- (1) *If  $\gamma = 0$ , then one has  $\Omega = \{z = x + yi; |y| < \pi \text{ and } x > -y \cot y\}$ .*
- (2) *Suppose that  $\gamma > \frac{1}{4}$ . In this case, there exists a unique  $y_0 \in (0, \pi)$  such that  $x_1(y_0) = x_2(y_0)$  and if  $y \geq y_0$  then  $x_i(y)$ ,  $i = 1, 2$  are real. Moreover, one has*

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and if } |y| \geq y_0 \text{ then } x < x_1(y) \text{ or } x > x_2(y)\}.$$

- (3) *If  $0 < \gamma \leq \frac{1}{4}$ , then  $x_i(y)$ ,  $i = 1, 2$  are both real for any  $y \in (0, \pi)$ , and one has*

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and } x > x_2(y)\}.$$

- (4) *Suppose that  $\gamma < 0$ . Then, there exists a unique  $y_0 \in (0, \pi)$  such that  $x_1(y_0) = x_2(y_0)$  and if  $y \leq y_0$  then  $x_i(y)$ ,  $i = 1, 2$  are real for (23). Moreover, one has*

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < y_0 \text{ and } x_1(y) < x < x_2(y)\}.$$

*In particular,  $\Omega$  is bounded.*

#### 4. PROOF OF THEOREM 1.2

In this section, we shall show Theorem 1.2 for  $\kappa > 0$  or  $\kappa = \infty$ . To do so, we consider the domain  $D := \Omega \cap \mathbb{C}^+$ , and show that  $f_{\kappa, \gamma}$  maps  $D$  to  $\mathbb{C}^+$  bijectively.

The key tool is the argument principle (see [1, Theorem 18, p.152], for example).

**Theorem 4.1 (The argument principle).** *If  $f(z)$  is meromorphic in a domain  $\Omega$  with the zeros  $a_j$  and the poles  $b_k$ , then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

*for every cycle  $\gamma$  which is homologous to zero in  $\Omega$  and does not pass through any of the zeros or poles. Here,  $n(\gamma, a)$  is the winding number of  $\gamma$  with respect to  $a$ .*

We also use the following elementary property of holomorphic functions.

**Lemma 4.2.** *Let  $f(z) = u(x, y) + iv(x, y)$  be a holomorphic function. The implicit function  $v(x, y) = 0$  has an intersection point at  $z = x + yi$  only if  $f'(z) = 0$ .*

*Proof.* Let  $p(t) = (x(t), y(t))$  be a continuous path in  $\mathbb{C} \cong \mathbb{R}^2$  satisfying  $v(p(t)) = 0$  for all  $t \in [0, 1]$ . We assume that  $(x'(t), y'(t)) \neq (0, 0)$ . Set

$$g(t) := u(p(t)) = u(x(t), y(t)), \quad h(t) := v(p(t)) = v(x(t), y(t)).$$

Obviously, we have  $h'(t) \equiv 0$  for any  $t$ , and

$$h'(t) = v_x x'(t) + v_y y'(t) = (v_x, v_y) \cdot (x'(t), y'(t)).$$

Assume that  $g'(t_0) = 0$  for some point  $t_0 \in [0, 1]$ . Then

$$\begin{aligned} g'(t) &= u_x x'(t) + u_y y'(t) = (u_x, u_y) \cdot (x'(t), y'(t)) \\ &= (v_x, v_y) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot (x'(t), y'(t)) = (v_x, v_y) \cdot (-y'(t), x'(t)), \end{aligned}$$

the condition  $g'(t_0) = 0$  implies that the vector  $(v_x, v_y)$  is orthogonal both to  $(x'(t_0), y'(t_0))$  and  $(-y'(t_0), x'(t_0))$ , which are non-zero vectors and mutually orthogonal. Such vector is only zero vector in  $\mathbb{R}^2$ , that is,  $(v_x, v_y) = (0, 0)$ , and hence  $(u_x, u_y) = (0, 0)$  by Cauchy-Riemann equations. Thus, if  $g'(t_0) = 0$  then  $p(t_0)$  needs to satisfy  $f'(p(t_0)) = 0$ .  $\square$

4.1. **The case  $\gamma < 0$ .** Assume that  $\gamma < 0$ . By Propositions 2.1, 2.2 and 3.1, it is equivalent to the condition that  $\Omega$  is bounded. We first consider the set  $\mathcal{S}$  (see (5) for definition). Let  $\alpha_i$ ,  $i = 1, 2$  be solutions of (10). Since  $\gamma < 0$ , these are distinct real numbers. Set  $\alpha_1 < \alpha_2$ . Then, we have  $\Omega \cap \mathbb{R} = (\alpha_1, \alpha_2)$ .

**Lemma 4.3.** *One has  $f(\alpha_2) < f(\alpha_1) < 0$  and  $\mathcal{S} = (f_{\kappa, \gamma}(\alpha_2), f_{\kappa, \gamma}(\alpha_1))$ .*

*Proof.* Assume that  $\kappa < +\infty$ . Since  $\gamma < 0$ , we have by (1)

$x$	$-\kappa$	$\cdots$	$\alpha_1$	$\cdots$	$0$	$\cdots$	$-\frac{1}{\gamma}$	$\cdots$	$\alpha_2$	$\cdots$
$f'_{\kappa, \gamma}$		$-$	$0$	$+$	$+$	$+$	$\times$	$+$	$0$	$-$
$f_{\kappa, \gamma}$		$\searrow$	$f_{\kappa, \gamma}(\alpha_1)$	$\nearrow$	$0$	$\nearrow^{+\infty}$	$\times$	$-\infty \nearrow$	$f_{\kappa, \gamma}(\alpha_2)$	$\searrow$

The inequality  $f(\alpha_1) < 0$  is obvious by the above table. We shall show  $f(\alpha_1) > f(\alpha_2)$ . By the fact that  $\alpha_1 \alpha_2 = \frac{1}{\gamma}$ , we have

$$\frac{f(\alpha_2)}{f(\alpha_1)} = \frac{\alpha_2(1 + \gamma\alpha_1)}{(1 + \gamma\alpha_2)\alpha_1} \cdot \left( \frac{1 + \alpha_2/\kappa}{1 + \alpha_1/\kappa} \right)^\kappa = \frac{\alpha_2 + 1}{\alpha_1 + 1} \cdot \left( \frac{1 + \alpha_2/\kappa}{1 + \alpha_1/\kappa} \right)^\kappa.$$

Since  $1 + \gamma\alpha_2 < 0$  and  $\alpha_1 < 0$ , we have  $\alpha_1 + 1 = (1 + \gamma\alpha_2)\alpha_1 > 0$ . Moreover, the facts that  $1 + \alpha_1/\kappa > 0$  and  $\alpha_2 > \alpha_1$  yield that

$$\frac{\alpha_2 + 1}{\alpha_1 + 1} > 1 \quad \text{and} \quad \frac{1 + \alpha_2/\kappa}{1 + \alpha_1/\kappa} > 1,$$

whence we obtain  $\frac{f(\alpha_2)}{f(\alpha_1)} > 1$ . Since  $f(\alpha_2) < 0$  because  $\alpha_2 > -\frac{1}{\gamma}$  and  $\gamma < 0$ , we conclude that  $0 > f(\alpha_1) > f(\alpha_2)$ .

Assume that  $\kappa = +\infty$ . Since  $\gamma < 0$  and  $\gamma(-\frac{1}{\gamma})^2 + (-\frac{1}{\gamma}) + 1 = 1 > 0$ , we have the following variation table of  $f(x)$ :

$x$	$-\infty$	$\cdots$	$\alpha_1$	$\cdots$	$0$	$\cdots$	$-\frac{1}{\gamma}$	$\cdots$	$\alpha_2$	$\cdots$	$+\infty$
$f'$		$-$	$0$		$+$		$\times$	$+$	$0$		$-$
$f$	$0$	$\searrow$	$f(\alpha_1)$	$\nearrow$	$0$	$\nearrow$	$\times$	$\nearrow$	$f(\alpha_2)$	$\searrow$	$-\infty$

Since  $\gamma\alpha_i + 1 = -\frac{1}{\alpha_i}$ , we see that  $f(\alpha_i) = -\alpha_i^2 e^{\alpha_i} < 0$ . By  $\alpha_1 \alpha_2 = \frac{1}{\gamma}$ , we have

$$\frac{f(\alpha_2)}{f(\alpha_1)} = \frac{\alpha_2(1 + \gamma\alpha_1)}{\alpha_1(1 + \gamma\alpha_2)} e^{\alpha_2 - \alpha_1} = \frac{\alpha_2 + 1}{\alpha_1 + 1} e^{\alpha_2 - \alpha_1} > 1,$$

whence  $f(\alpha_2) < f(\alpha_1) < 0$ . Thus, the proof is now completed.  $\square$

Now we show that  $f_{\kappa, \gamma}: D \rightarrow \mathbb{C}^+$  is bijective. Since the proof is completely analogous, we only prove the case  $\kappa < +\infty$ .

We take a path  $C = C(t)$  ( $t \in [0, 1]$ ) in such a way that by starting from  $z = -\frac{1}{\gamma}$ , it goes to  $z = \alpha_2$  along the real axis, next goes to  $z = \alpha_1$  along the curve  $r_{+-}$  defined by (4) and connecting  $\alpha_2$  and  $\alpha_1$  in the upper half plane, and then it goes to  $z = -\frac{1}{\gamma}$  along the real axis (see Figure 1). Here, we can assume that  $C'(t) \neq 0$  whenever  $C(t) \neq \alpha_i$ ,  $i = 1, 2$ . Actually, the curve  $v(x, y) = 0$  has a tangent line unless  $f'$  vanishes. If we take an arc-length parameter  $t$ , then  $C'(t)$  represents the direction of the tangent line at  $(x, y) = C(t)$ . We note that  $C(t)$  describes the boundary of  $D$ .

We show that  $f_{\kappa, \gamma}$  maps the boundary of  $D$  to  $\mathbb{R}$  bijectively. We take  $t_i$ ,  $i = 1, 2$  as  $C(t_i) = \alpha_i$ . Note that the sub-curve  $C(t)$ ,  $t \in (t_2, t_1)$  describes the curve  $r_{+-}(t)$ , and  $f_{\kappa, \gamma}$  does not have a pole or singular point on  $C(t)$ ,  $t \in (t_2, t_1)$ . Set  $f(z) = u(x, y) + iv(x, y)$ . By Lemma 4.2, the implicit function  $v(x, y) = 0$  may have an intersection point only if  $f'(x + iy) = 0$ , i.e. at  $x + iy = \alpha_i$  ( $i = 1, 2$ ) or at  $x + iy = -\kappa$  if  $\kappa > 1$ . Then, the function  $g(t) = u(C(t))$ ,  $t \in [t_2, t_1]$  attains maximum and minimum in the interval because it is a continuous function on a compact set. Moreover,  $g'$  never vanishes in  $(t_2, t_1)$  by the above argument and by the fact that  $f'(C(t)) \neq 0$  for  $t \in (t_2, t_1)$ . Therefore,  $g$  is monotone and hence it takes maximal and minimal values at the endpoints  $t = t_2, t_1$ . Now we have  $f(\alpha_1) > f(\alpha_2)$  by the last claim so that the image of  $g$  is  $[f(\alpha_2), f(\alpha_1)]$ , and the function  $g$  is bijective.

We shall show that for any  $w_0 \in \mathbb{C}^+$  there exists one and only one  $z_0 \in D$  such that  $f(z_0) = w_0$ . Let us take an  $R > 0$  such that  $|w_0| < R$ . For  $\delta > 0$ , let  $C' = C_\delta$  be a path obtained from  $C$  in such a way that the pole  $z = -1/\gamma$  is avoided by a semi-circle  $-\frac{1}{\gamma} + \delta e^{i\theta}$ ,  $\theta \in (0, \pi)$  of radius  $\delta$  (see Figure 3). Denote by  $D'$  the domain surrounded by the curve  $C'$ .

Then, we can choose  $\delta > 0$  such that

$$\left| f\left(-\frac{1}{\gamma} + \delta e^{i\theta}\right) \right| > R \quad (\text{for all } \theta \in (0, \pi)).$$

In fact, if  $z = -\frac{1}{\gamma} + \delta e^{i\theta}$ , then we have

$$|1 + \gamma z| = |\gamma| \delta, \quad |z| = \left| -\frac{1}{\gamma} + \delta e^{i\theta} \right| > \frac{1}{2|\gamma|} \quad (\text{if } \delta < \frac{1}{2|\gamma|}),$$

and

$$\left| 1 + \frac{z}{\kappa} \right| = \left| 1 - \frac{1}{\kappa\gamma} + \frac{\delta}{\kappa} e^{i\theta} \right| > \frac{\kappa\gamma - 1}{2\kappa\gamma} \quad (\text{if } \delta < \frac{\kappa}{2} \left| 1 - \frac{1}{\kappa\gamma} \right|),$$

so that

$$\left| f\left(-\frac{1}{\gamma} + \delta e^{i\theta}\right) \right| > \frac{1}{2|\gamma|^2} \left( \frac{\kappa\gamma - 1}{2\kappa\gamma} \right)^\kappa \cdot \frac{1}{\delta}.$$

Thus it is enough to take

$$\delta = \min \left( \frac{1}{2|\gamma|^2 R} \left( \frac{\kappa\gamma - 1}{2\kappa\gamma} \right)^\kappa, \frac{1}{2|\gamma|}, \frac{\kappa}{2} \left| 1 - \frac{1}{\kappa\gamma} \right| \right).$$

Since  $f$  is non-singular on the semi-circle  $-\frac{1}{\gamma} + \delta e^{i\theta}$ ,  $\theta \in [0, \pi]$ , the curve  $\theta \mapsto f(-\frac{1}{\gamma} + \delta e^{i\theta})$  does not have a singular angular point, so that it is homotopic to a large semicircle (with radius larger than  $R$ ) in the upper half-plane (see Figure 5).

Note that

$$\operatorname{Im} f(x + yi) = \frac{\left( (1 + x/\kappa)^2 + (y/\kappa)^2 \right)^{\frac{\kappa}{2}}}{(1 + \gamma x)^2 + \gamma^2 y^2} \left\{ (x + \gamma x^2 + \gamma y^2) \sin(\kappa\theta(x, y)) + y \cos(\kappa\theta(x, y)) \right\}.$$

By changing variables as in (6), we have

$$\begin{aligned} \operatorname{Im} f(re^{i\theta}) &= \text{positive factor} \times \sin(\kappa\theta) \cdot (ar^2 + b(\theta)r + a - 1) \\ &= \text{positive factor} \times \sin(\kappa\theta) \cdot a(r - r_-(\theta))(r - r_+(\theta)). \end{aligned}$$

Note that the inside of the path  $C$  can be written as  $\{re^{i\theta}; \theta \in (0, \theta_*) , r \in (r_-(\theta), r_+(\theta))\}$  in  $(r, \theta)$  coordinates. Since  $a < 0$  and  $\sin(\kappa\theta) > 0$  when  $\theta \in (0, \theta_*)$ , we see that  $\operatorname{Im} f(z) > 0$  if  $z$  is inside of the path  $C$ . In particular, the inside set of the curve  $f(C')$  is a bounded domain in  $\mathbb{C}^+$  including  $w_0$ .

Since the winding number of the path  $f(C')$  with respect to  $w = w_0$  is exactly one, we see that

$$\frac{1}{2\pi i} \int_{C'} \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi i} \int_{f(C')} \frac{dw}{w - w_0} = 1.$$

By definition of  $f$ , we see that  $f(z) - w_0$  does not have a pole in  $D'$ . Therefore, by the argument principle, the function  $f(z) - w_0$  has only one zero point, say  $z_0 \in D' \subset D$ . Thus, we obtain  $f(z_0) = w_0$ , and such  $z_0 \in D$  is unique. We conclude that the map  $f$  is a bijection from  $D$  to the upper half-plane  $\mathbb{C}^+$ .

**4.2. The case  $\gamma \geq 0$ .** Assume that  $\gamma \geq 0$ . By Propositions 2.2, 2.1 and 3.1, it is equivalent to the condition that  $\Omega$  is unbounded. We first consider the behavior of  $|f_{\kappa, \gamma}(z)|$  as  $\partial\Omega \ni z \rightarrow \infty$ . We have

$$|f_{\kappa, \gamma}(z)| = \frac{1}{\left| \gamma + \frac{1}{z} \right|} \cdot \left| 1 + \frac{z}{\kappa} \right|^\kappa.$$

We consider the change variables  $1 + \frac{z}{\kappa} = re^{i\theta}$  for  $z \in \Omega$ . Then, we have

$$\left| \gamma + \frac{1}{z} \right| \leq |\gamma| + \frac{1}{|\kappa|} \cdot \frac{1}{|re^{i\theta} - 1|} \leq |\gamma| + \frac{1}{|\kappa| \cdot |r - 1|},$$

so that

$$|f_{\kappa, \gamma}(z)| \geq \frac{r^\kappa}{|\gamma| + \frac{1}{|\kappa| \cdot |r - 1|}} \rightarrow +\infty \quad (\text{as } r \rightarrow +\infty).$$

Propositions 2.2, 2.1 and 3.1 show that, if  $\kappa > 1$  or  $\gamma = 0$ , then  $\partial\Omega \cap \mathbb{C}^+$  can be described as a connected curve  $C(t)$ ,  $t \in [0, 1)$  with  $C(0) = \alpha_2$  and  $\lim_{t \rightarrow 1-0} C(t) = \infty$ . In this case, if  $z \in \partial\Omega \cap \mathbb{C}^+$ , then we have  $F(x, y) = 0$  and

$$f_{\kappa, \gamma}(z) = \frac{\left| 1 + \frac{z}{\kappa} \right|^\kappa}{|1 + \gamma z|^2} \cdot \frac{-y}{\sin(\kappa\theta(x, y))} < 0. \quad (24)$$

This means that, if  $z \in \mathbb{C}^+$  goes to  $\infty$  along the path  $\partial\Omega \cap \mathbb{C}^+$ , then  $f_{\kappa, \gamma}$  must tend to  $-\infty$ .

We next consider  $\mathcal{S}$  for cases  $\kappa \geq 1$  or  $\gamma = 0$ .

**Lemma 4.4.** *Let  $\alpha_i$ ,  $i = 1, 2$  be the solutions of (8).*

- (1) *Assume that  $\kappa > 1$  or  $\kappa = \infty$ , and also  $D(0) \geq 0$ . Then,  $\alpha_i$  are both real and  $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_2))$  with  $f_{\kappa,\gamma}(\alpha_2) < 0$ .*
- (2) *Assume that  $\gamma = 0$ . Then, one has  $\alpha_i = -\frac{\kappa}{\kappa+1}$  if  $\kappa < +\infty$  and  $\alpha_i = -1$  if  $\kappa = \infty$  ( $i = 1, 2$ ). Moreover, one has  $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_2))$ .*
- (3) *Assume that  $\kappa = 1$  and  $0 < \gamma < 1$ . Then,  $\alpha_i$  are both real, and one has  $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_1), f_{\kappa,\gamma}(\alpha_2))$  with  $f_{\kappa,\gamma}(\alpha_2) < 0$ .*
- (4) *Assume that  $\kappa \geq 1$  or  $\kappa = \infty$ , and also  $D(0) < 0$ . Then,  $\alpha_i$  are both non-real, and one has  $\mathcal{S} = \emptyset$ . On the other hand, one has  $f_{\kappa,\gamma}(\partial\Omega \cap \mathbb{C}^+) = (-\infty, 0)$ .*

*Proof.* (1) If  $\kappa > 1$  and  $D(0) \geq 0$ , then Proposition 2.2 (5-a) tells us that  $\Omega \cup \mathbb{R} = (\alpha_2, +\infty)$ . On the other hand, if  $\kappa = \infty$  and  $D(0) \geq 0$ , that is,  $\gamma \leq \frac{1}{4}$ , then we also have  $\Omega \cup \mathbb{R} = (\alpha_2, +\infty)$  by Proposition 3.1 (1) and (3). Since it is easily verified that  $f_{\kappa,\gamma}$  is monotonic increasing on this interval for both cases, we have  $f_{\kappa,\gamma}(\alpha_2) < 0$  and  $\lim_{x \rightarrow +\infty} f_{\kappa,\gamma}(x) = +\infty$ . Thus, by definition (5), we obtain

$\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_2))$  for both cases.

(2) Assume that  $\gamma = 0$ . Propositions 2.2 (2) and 3.1 (1) show that we have  $\Omega \cap \mathbb{R} = (\alpha, +\infty)$  where  $\alpha = \alpha_1 = \alpha_2 = -\frac{\kappa}{\kappa+1}$  if  $\kappa < +\infty$ , and if  $\kappa = \infty$  then  $\alpha = -1$ . Since it is easily verified that  $f_{\kappa,\gamma}$  is monotonic increasing on this interval, we have  $f_{\kappa,\gamma}(\alpha_2) < 0$  and  $\lim_{x \rightarrow +\infty} f_{\kappa,\gamma}(x) = +\infty$ . Thus, by definition (5), we obtain  $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha))$ .

(3) Assume that  $\kappa = 1$  and  $0 < \gamma < 1$ . By Proposition 2.1, we have  $\Omega \cap \mathbb{R} = (-\infty, \alpha_1) \cup (\alpha_2, +\infty)$ . An elementary calculation yields that the image of this set by  $f_{\kappa,\gamma}$  is  $(-\infty, f_{\kappa,\gamma}) \cup (f_{\kappa,\gamma}(\alpha_2), +\infty)$  with  $f_{\kappa,\gamma}(\alpha_1) < f_{\kappa,\gamma}(\alpha_2) < 0$ , and therefore we obtain  $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_1), f_{\kappa,\gamma}(\alpha_2))$ .

(4) Assume that  $\kappa \geq 1$  or  $\kappa = \infty$ , and also assume that  $D(0) < 0$ . Then, Propositions 2.2 (5-b) and 3.1 (2) tell us that  $\Omega \cap \mathbb{R} = (-\kappa, +\infty)$ , where if  $\kappa = \infty$  then we regard  $-\kappa$  as  $-\infty$ . Since we have  $-\kappa < -\frac{1}{\gamma} < 0$ , we have the following increasing/decreasing table of  $f_{\kappa,\gamma}$ .

$x$	$-\kappa$	$\cdots$	$-\frac{1}{\gamma}$	$\cdots$	$0$	$\cdots$
$f'_{\kappa,\gamma}$		$+$	$\times$	$+$	$+$	$+$
$f_{\kappa,\gamma}$	$0$	$\nearrow +\infty$	$\times$	$-\infty \nearrow$	$0$	$\nearrow +\infty$

Thus, we obtain  $\mathcal{S} = \emptyset$ . Let  $C(t)$ ,  $t \in [0, 1]$  be a path describing  $\partial\Omega \cap \mathbb{C}^+$  with  $C(0) = \alpha_2$ . Assume that  $C(t_1) = \alpha_1$ . Then, by (24) and by the discussion below of it, we see that  $f_{\kappa,\gamma}(\alpha_1) < 0$  and  $\lim_{t \rightarrow 1} f_{\kappa,\gamma}(C(t)) = -\infty$ . Thus, by Lemma 4.2, we conclude that  $f_{\kappa,\gamma}(\partial\Omega \cap \mathbb{C}^+) = (-\infty, 0)$ .  $\square$

We note that if  $(\kappa, \gamma) = (1, 1)$ , then we have  $f_{\kappa,\gamma}(z) = z$ , and hence we omit this case. Recall that  $D = \Omega \cap \mathbb{C}^+$ . We shall show that  $f_{\kappa,\gamma}$  maps  $D$  to  $\mathbb{C}^+$  bijectively. We divide cases according to the above lemma.

**4.2.1. The case (1) and (2).** Assume that  $\kappa > 1$  or  $\gamma = 0$ . We also assume that  $D(0) \geq 0$ . Let us take a path  $C = C(t)$ ,  $t \in (0, 1)$  in such a way that by starting from  $z = \infty$ , it goes to  $z = \alpha_2$  along the curve  $r_+$  defined by (4) in the upper half plane, and then goes to  $z = \infty$  along the real axis (see Figure 2). Here, we can assume that  $C'(t) \neq 0$  whenever  $C(t) \neq \alpha_i$ ,  $i = 1, 2$ . Actually, the curve  $v(x, y) = 0$  has a tangent line unless  $f'$  vanishes. If we take an arc-length parameter  $t$ , then  $C'(t)$  represents the direction of the tangent line at  $(x, y) = C(t)$ . We note that  $C(t)$  describes the boundary of  $D$ . Lemma shows that  $g(t) := f_{\kappa,\gamma}(C(t))$  is a monotonic increasing function on  $(0, 1)$  such that  $g(t) \rightarrow -\infty$  if  $t \rightarrow 0$  and  $g(t) \rightarrow +\infty$  if  $t \rightarrow 1$ .

We shall show that for any  $w_0 \in \mathbb{C}^+$  there exists one and only one  $z_0 \in D$  such that  $f(z_0) = w_0$ . Note that we have  $\text{Im } f_{\kappa,\gamma}(z) > 0$  for any  $z \in D$  by definition. Let us take an  $R > 0$  such that  $|w_0| < R$ . For  $L > 0$ , let  $\Gamma_L$  be the circle  $-\kappa + Le^{i\theta}$  of origin  $z = -\kappa$  with radius  $L$ . Let  $L - \kappa$  and  $z_L$  be two distinct intersection points of  $C$  and  $\Gamma_L$ . Let  $C' := C_L$  be a closed path obtained from  $C$  by connecting  $L - \kappa$  and  $z_L$  via the arc  $A$  of  $\Gamma_L$  included in the upper half plane, see Figure 4.

Since  $f$  is non-singular on the arc  $A$ , the curve  $f(A)$  does not have a singular point so that it is homotopic to a large semi-circle (whose radius is larger than  $R$ ) in the upper half plane (see Figure 6). In particular, the inside set  $f(D')$  of the curve  $f(C')$  is a bounded domain including  $w_0 \in \mathbb{C}^+$ . Since the winding number of the path  $f(C')$  about  $w = w_0$  is exactly one, we see that

$$\frac{1}{2\pi i} \int_{C'} \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi i} \int_{f(C')} \frac{dw}{w - w_0} = 1.$$

We know by definition of  $f$  that  $f$  does not have a pole on  $D'$ . Therefore, by the argument principle, the function  $f(z) - w_0$  has the only one zero point, say  $z_0 \in D'$ . Then, we obtain  $f(z_0) = w_0$ , and such  $z_0$  is unique. We conclude that the map  $f$  is bijection from the interior set  $D$  of  $C$  to the upper half plane.

4.2.2. *The case (3).* Assume that  $\kappa = 1$  and  $0 < \gamma < 1$ . Then,  $\Omega$  is given in Proposition 2.1. By a simple calculation, we see that  $\alpha_i$ ,  $i = 1, 2$  are both real such that  $\alpha_1 < \alpha_2 < 0$ , and we have  $\Omega \cap \mathbb{R} = (-\infty, \alpha_1) \cup (\alpha_2, +\infty)$ . Moreover,

$$f_{\kappa,\gamma}([\alpha_1, +\infty)) = [f_{\kappa,\gamma}(\alpha_1), +\infty), \quad f_{\kappa,\gamma}((-\infty, \alpha_2]) = (-\infty, f_{\kappa,\gamma}(\alpha_2)],$$

and we have  $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_1), f_{\kappa,\gamma}(\alpha_2))$ . Let  $C = C(t)$ ,  $t \in [0, 1]$  be a path in  $\mathbb{C}^+$  connecting  $z = \alpha_1$  to  $z = \alpha_2$  along  $\partial\Omega \cap \mathbb{C}^+$ . Then, since  $f'_{\kappa,\gamma}(C'(t)) \neq 0$  for any  $t \in (0, 1)$  and since  $f_{\kappa,\gamma}(C(t)) \in \mathbb{R}$ , Lemma 4.2 tells us that  $f_{\kappa,\gamma}(C(t))$  is monotonic increasing on the interval  $(0, 1)$ , and hence  $f_{\kappa,\gamma}(C([0, 1])) = \mathcal{S}$ . This shows that  $f_{\kappa,\gamma}$  maps  $\partial D$  to  $\mathbb{R}$  bijectively.

We can show the bijectivity of  $f_{\kappa,\gamma}$  on  $D$  to  $\mathbb{C}^+$  similarly to the case  $\kappa > 1$  and  $D(0) \geq 0$  by taking  $\Gamma_L$  to be the semi-circle  $-1 + Le^{i\theta}$  of origin  $z = -1$  with radius  $L$ , contained in the upper half plane. Thus, we omit the detail.

4.2.3. *The case (4).* Assume that  $\kappa \geq 1$  and  $D(0) < 0$ . Then, the solutions  $\alpha_i$ ,  $i = 1, 2$  of (10) are both non-real complex numbers by  $D(0) < 0$ . Let  $\text{Im } \alpha_1 > 0$ . In this case, we have  $-\kappa < -\frac{1}{\gamma} < 0$  and

$x$	$-\kappa$	$\cdots$	$-\frac{1}{\gamma}$	$\cdots$	$0$	$\cdots$
$f'_{\kappa,\gamma}$		$+$	$\times$	$+$	$+$	$+$
$f_{\kappa,\gamma}$	$0$	$\nearrow +\infty$	$\times$	$-\infty \nearrow$	$0$	$\nearrow +\infty$

Note that if  $\kappa = \infty$ , then we regard  $-\kappa$  as  $-\infty$ . By the above table, we have  $\mathcal{S} = \emptyset$ , and set  $\mathcal{S}' = (-\infty, 0)$ . Since we have for  $z = x + yi \in \partial\Omega$

$$f_{\kappa,\gamma}(z) = C((x + \gamma x^2 + \gamma y^2) \cos(\kappa\theta(x, y)) - y \sin(\kappa\theta(x, y))) = -C \frac{y}{\sin(\kappa\theta(x, y))},$$

we see that  $f_{\kappa,\gamma}(\alpha_1) < 0$ . This shows that the image of the function  $f_{\kappa,\gamma}(z(\theta))$ ,  $\theta \in (0, \theta_0)$  is  $(-\infty, 0)$ . Let us adopt a similar argument of the proof of bijectivity of the map  $f_{\kappa,\gamma}$  as in unbounded cases. since the winding number of the path of the boundary  $\partial D$  is two, we see that the map  $f_{\kappa,\gamma}$  maps  $D$  to  $\mathbb{C}^+$  in two-to-one, and hence  $f_{\kappa,\gamma}$  does not map  $D$  to  $\mathbb{C}^+$  bijectively.

4.2.4. Assume that  $0 < \kappa < 1$  and  $\gamma > 0$ . In this case, Proposition 2.2 tells us that if  $0 < a = \kappa\gamma < 1$  then  $\Omega \cap \mathbb{R} = (\alpha_2, +\infty)$ , and if  $a > 1$  then  $\Omega \cap \mathbb{R} = \left\{x \in \mathbb{R}; x > -\kappa \text{ and } x \neq -\frac{1}{\gamma}\right\}$ . If  $f_{\kappa,\gamma}$  maps  $D = \Omega \cap \mathbb{C}^+$  to  $\mathbb{C}^+$  bijectively, then it needs map  $\partial D = \mathcal{I}$  to  $\mathbb{R}$ . However, if  $x \in \mathbb{R}$  satisfies  $x < \min(\alpha_1, -\kappa)$ , then  $\exp_\kappa(z)$  to  $\left|1 + \frac{x}{\kappa}\right|^\kappa e^{i\kappa\pi}$  as  $z \rightarrow x$  via the arc  $re^{i\theta}$  where  $r = \left|1 + \frac{x}{\kappa}\right|$ . Since  $0 < \kappa < 1$ , then we see that  $\lim_{z \rightarrow x} f_{\kappa,\gamma}(z)$  is not real and hence  $f_{\kappa,\gamma}$  cannot map  $D$  to  $\mathbb{C}^+$  bijectively.

Now we have completed the proof of Theorem 1.2 for the case  $\kappa > 0$  or  $\kappa = \infty$ .  $\square$

## 5. THE CASE OF $\kappa < 0$

We shall complete the proof of Theorem 1.2 by proving it for the case  $\kappa < 0$ . To do so, let us recall the homographic (linear fractional) action of  $SL(2, \mathbb{R})$  on  $\mathbb{C}$ . For  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$  and  $z \in \mathbb{C}^+$ , we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

For each  $g \in SL(2, \mathbb{R})$ , the corresponding homographic action map  $\mathbb{C}^+$  to  $\mathbb{C}^+$  bijectively. Let  $\kappa = -\kappa'$  with positive  $\kappa' > 0$ . Consider the transformation

$$1 + \frac{z'}{\kappa'} = \left(1 + \frac{z}{\kappa}\right)^{-1}.$$

Then, it can be written as

$$z' = \begin{pmatrix} 1 & 0 \\ 1/\kappa & 1 \end{pmatrix} \cdot z = \frac{z}{1 + z/\kappa} \iff z = \begin{pmatrix} 1 & 0 \\ -1/\kappa & 1 \end{pmatrix} \cdot z' = \frac{z'}{1 - z'/\kappa}.$$

Note that since  $\begin{pmatrix} 1 & 0 \\ 1/\kappa & 1 \end{pmatrix} \in SL(2, \mathbb{R})$ , it maps  $\mathbb{C}^+$  to  $\mathbb{C}^+$  bijectively. Then, since

$$\begin{aligned} \frac{z}{1+\gamma z} &= \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \cdot z = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/\kappa & 1 \end{pmatrix} \cdot z' = \begin{pmatrix} 1 & 0 \\ \gamma - 1/\kappa & 1 \end{pmatrix} \cdot z' \\ &= \frac{z'}{1 + (\gamma - 1/\kappa)z'} = \frac{z'}{1 + (\gamma + 1/\kappa')z'} \end{aligned}$$

and

$$\left(1 + \frac{z}{\kappa}\right)^\kappa = \left(\left(1 + \frac{z}{\kappa}\right)^{-1}\right)^{-\kappa} = \left(1 + \frac{z'}{\kappa'}\right)^{\kappa'}$$

(recall that we are taking the main branch so that  $\log z = -\log(z^{-1})$ ), we obtain

$$f_{\gamma, \kappa}(z) = \frac{z}{1+\gamma z} \left(1 + \frac{z}{\kappa}\right)^\kappa = \frac{z'}{1 + (\gamma + 1/\kappa')z'} \left(1 + \frac{z'}{\kappa'}\right)^{\kappa'} = f_{\gamma+1/\kappa', \kappa'}(z').$$

Set  $\gamma' = \gamma + 1/\kappa'$ . Since homographic actions map  $\mathbb{C}^+$  to  $\mathbb{C}^+$  bijectively, there exists a domain  $\Omega$  such that  $f_{\kappa, \gamma}$  maps  $D = \Omega \cap \mathbb{C}^+$  to  $\mathbb{C}^+$  bijectively if and only if it holds for  $f_{\gamma', \kappa'}$ . Thus,

$$\gamma' \leq 0 \iff \gamma \leq \frac{1}{\kappa},$$

and  $\kappa' > 1$  and  $\gamma' > 0$  with  $\gamma' \leq \frac{1}{4}(1 + \frac{1}{\kappa'})^2$  is equivalent to

$$\gamma > \frac{1}{\kappa} \quad \text{and} \quad \kappa < -1 \quad \text{with} \quad \gamma - \frac{1}{\kappa} \leq \frac{1}{4}\left(1 - \frac{1}{\kappa}\right)^2 \iff \gamma \leq \frac{1}{4}\left(1 + \frac{1}{\kappa}\right)^2.$$

This shows the case  $\kappa < 0$  in Theorem 1.2, and hence we have completed the proof of Theorem 1.2.

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## 6. APPENDIX

6.1. **Tables.** This subsection contains increasing/decreasing tables of  $H_\alpha$  and  $b'(\theta)$ .

**Lemma 6.1.** *One has the following increasing/decreasing table of  $H_\alpha$ .*

(A) $0 < \alpha \leq \frac{1}{2}$	$x$	0	$\dots$	$\frac{2\pi}{\alpha+1}$	$\dots$	$2\pi$	$H_\alpha(2\pi) \geq 0$
	$H'_\alpha$	0	+	0		+	
	$H_\alpha$	0	$\nearrow$	$H_\alpha(\frac{2\pi}{\alpha+1})$	$\searrow$	$H_\alpha(2\pi)$	
(B) $\frac{1}{2} < \alpha < 1$	$x$	0	$\dots$	$\frac{2\pi}{\alpha+1}$	$\dots$	$2\pi$	$H_\alpha(2\pi) < 0$
	$H'_\alpha$	0	+	0		-	
	$H_\alpha$	0	$\nearrow$	$H_\alpha(\frac{2\pi}{\alpha+1})$	$\searrow$	$H_\alpha(2\pi)$	
(C) $1 < \alpha < 2$	$x$	0	$\dots$	$\frac{2\pi}{\alpha+1}$	$\dots$	$\frac{2\pi}{\alpha}$	$H_\alpha(\frac{2\pi}{\alpha}) > 0$
	$H'_\alpha$	0	-	0		+	
	$H_\alpha$	0	$\searrow$	$H_\alpha(\frac{2\pi}{\alpha+1})$	$\nearrow$	$H_\alpha(\frac{2\pi}{\alpha})$	
(D) $\alpha \geq 2$	$x$	0	$\dots$	$\frac{2\pi}{\alpha+1}$	$\dots$	$\frac{2\pi}{\alpha}$	$H_\alpha(\frac{2\pi}{\alpha}) \leq 0$
	$H'_\alpha$	0	-	0		+	
	$H_\alpha$	0	$\searrow$	$H_\alpha(\frac{2\pi}{\alpha+1})$	$\nearrow$	$H_\alpha(\frac{2\pi}{\alpha})$	

**Lemma 6.2.** *The signature of  $b'(\theta)$  on the interval  $I$  is given as follows.*

$0 < \kappa < \frac{1}{2}$								
(1)	$\frac{\theta}{b'(\theta)}$	0	$\cdots$	$\varphi_*$	$\cdots$	$\pi$	(i) $\ell(\kappa, a) \geq 0$ , (ii) $\ell(\kappa, a) < 0, -J_\kappa(a) > \frac{1}{\kappa}$	$a > \frac{2\kappa^2+3\kappa+1}{6\kappa}$
(2)	$b'(\theta) < 0$ for $\theta \in I$						(i) $\ell(\kappa, a) < 0, -J_\kappa(a) \leq \frac{1}{\kappa}$	$a \leq \frac{2\kappa^2+3\kappa+1}{6\kappa}$
$\kappa = \frac{1}{2}$								
(1)	$b'(\theta) = 0$ for $\theta \in I$						$\ell(\kappa, a) = 0$	$a = 1$
(2)	$b'(\theta) > 0$ for $\theta \in I$						$\ell(\kappa, a) > 0$	$a > 1$
(3)	$b'(\theta) < 0$ for $\theta \in I$						$\ell(\kappa, a) < 0$	$a < 1$
$\frac{1}{2} < \kappa < 1$								
(1)	$\frac{\theta}{b'(\theta)}$	0	$\cdots$	$\varphi_*$	$\cdots$	$\pi$	(i) $\ell(\kappa, a) < 0$ , (ii) $\ell(\kappa, a) = 0$ , (iii) $\ell(\kappa, a) > 0, -J_\kappa(a) > \frac{1}{\kappa}$	$a < \frac{2\kappa^2+3\kappa+1}{6\kappa}$
(2)	$b'(\theta) > 0$ for $\theta \in I$						$-J_\kappa(a) \leq \frac{1}{\kappa}, \ell(\kappa, a) > 0$	$a \geq \frac{2\kappa^2+3\kappa+1}{6\kappa}$
$\kappa > 1$								
(1)	$\frac{\theta}{b'(\theta)}$	0	$\cdots$	$\varphi_*$	$\cdots$	$\frac{\pi}{\kappa}$	$\ell(\kappa, a) > 0, -J_\kappa(a) < \frac{1}{\kappa}$	$a > \frac{2\kappa^2+3\kappa+1}{6\kappa}$
(2)	$b'(\theta) < 0$ for $\theta \in I$						(i) $\ell(\kappa, a) < 0$ , (ii) $\ell(\kappa, a) = 0$ , (iii) $\ell(\kappa, a) > 0, -J_\kappa(a) \geq \frac{1}{\kappa}$	$a \leq \frac{2\kappa^2+3\kappa+1}{6\kappa}$

If  $\kappa = 1$ , then one has

$$b'(\theta) = 2(a-1) \sin \theta.$$

**6.2. Figures.** This subsection collects figures of graphs of  $f_{\kappa,\gamma}(x)$  (for real  $x$ ),  $F_\kappa$ , and of shapes of  $\Omega$  with some deformation.

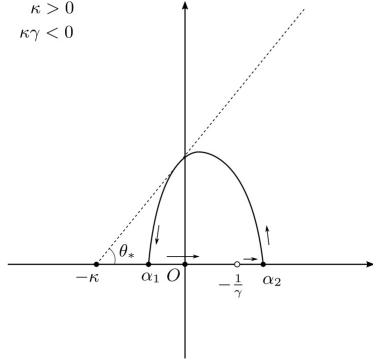
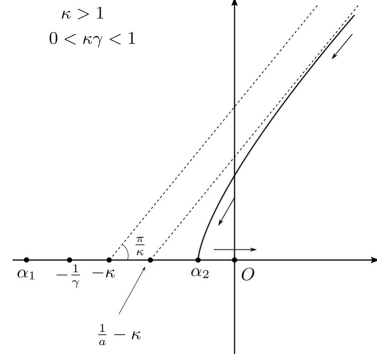
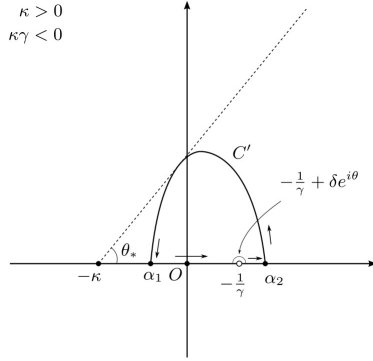
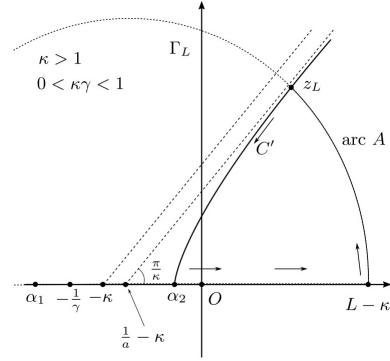
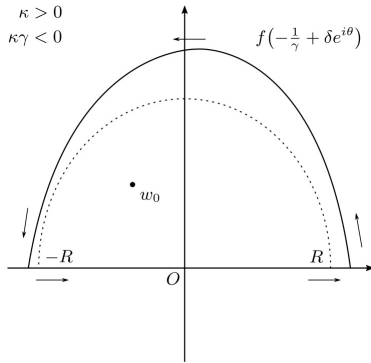
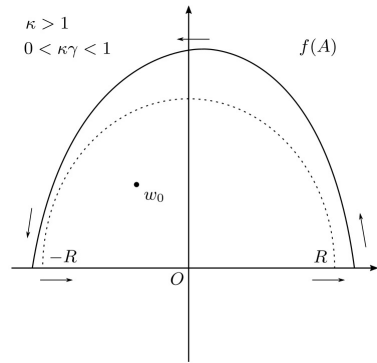
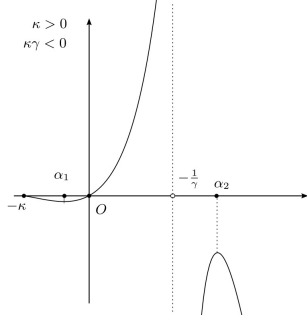
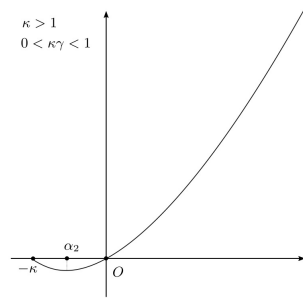
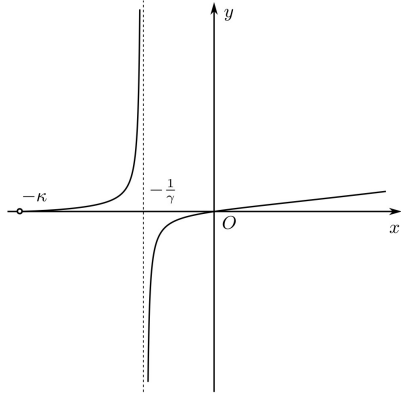
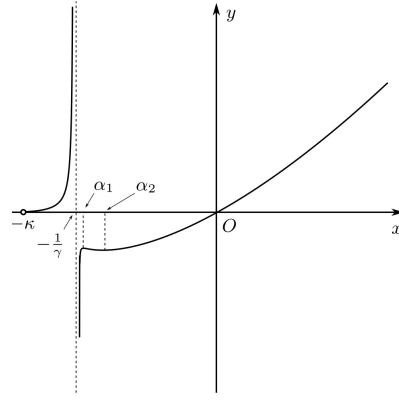


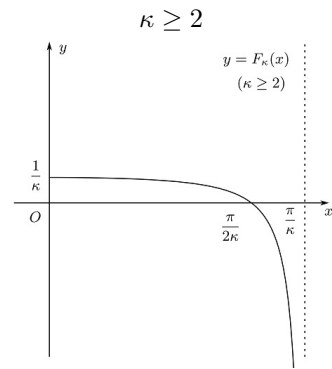
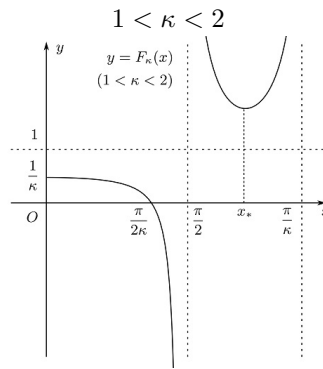
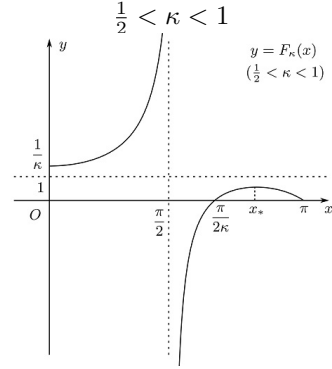
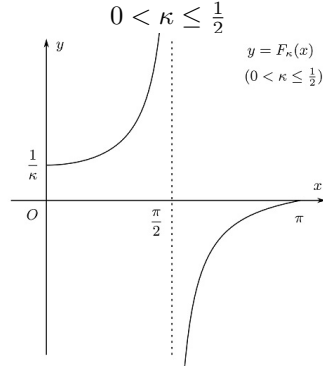
FIGURE 1. The case of (i)

FIGURE 2. The case of (ii),  
when  $\kappa > 2$ FIGURE 3. Curve  $C'$  in  
case (i)FIGURE 4. Curve  $C'$  in  
case (ii)FIGURE 5. Curve  $f(C')$  in  
case (i)FIGURE 6. Curve  $f(C')$  in  
case (ii)

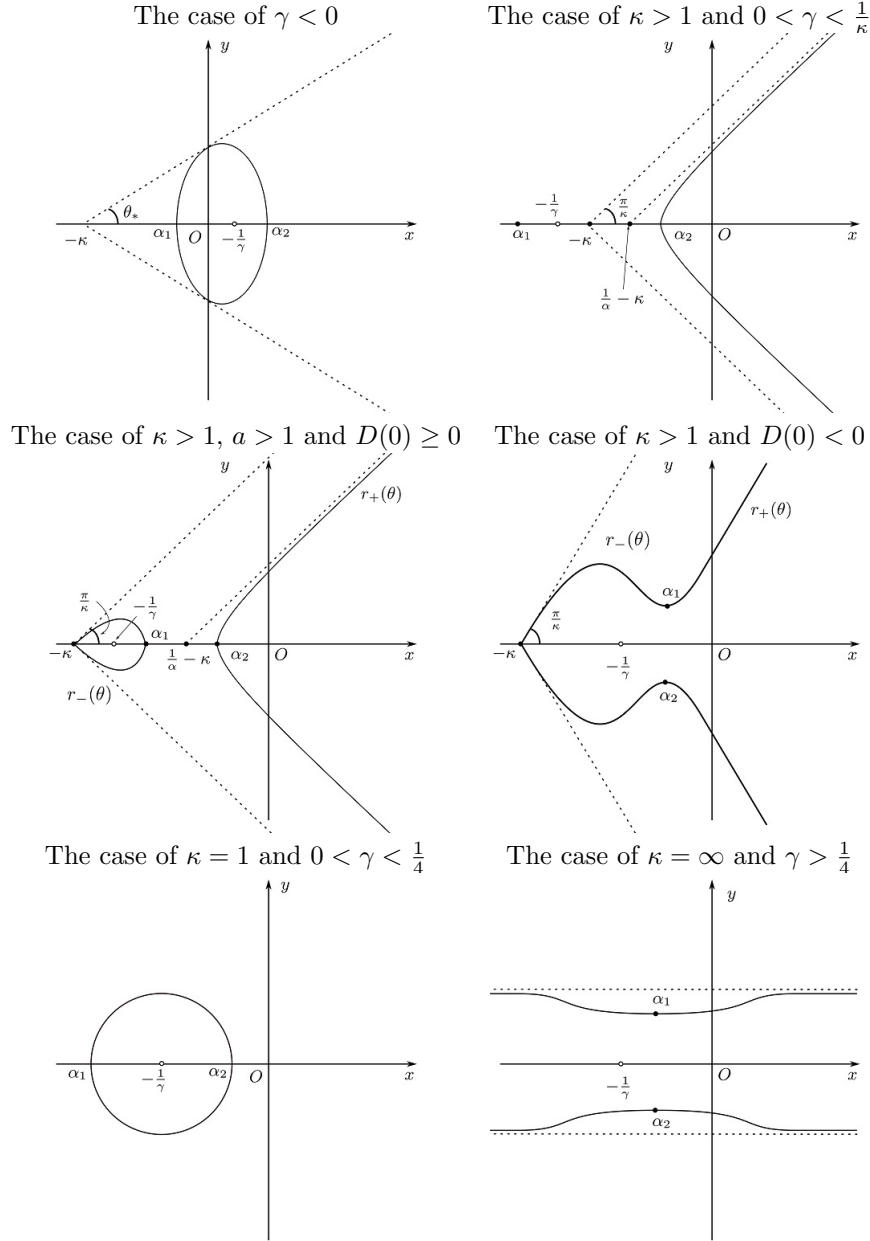
- Graphs of  $f_{\kappa,\gamma}(x)$  for real  $x$ .


 FIGURE 7.  $f(x)$  for  $x \geq -\kappa$ , case (i)

 FIGURE 8.  $f(x)$  for  $x \geq -\kappa$ , case (ii)

- Graphs of  $F_\kappa$



- Shapes of  $\Omega$



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