



HAL
open science

Analysis on Lambert-Tsallis functions

Hideto Nakashima, Piotr Graczyk

► **To cite this version:**

| Hideto Nakashima, Piotr Graczyk. Analysis on Lambert-Tsallis functions. 2020. hal-03001260

HAL Id: hal-03001260

<https://hal.science/hal-03001260>

Preprint submitted on 12 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Analysis on Lambert-Tsallis functions

Hideto Nakashima and Piotr Graczyk

ABSTRACT. In this paper, we study the Lambert-Tsallis W function, which is a generalization of the Lambert W function with two real parameters. We give a condition on the parameters such that there exists a complex domain touching zero on boundary which is mapped bijectively to the upper half plane by the Lambert-Tsallis function.

INTRODUCTION

The Lambert W function, which is the multivalued inverse of the function $z \mapsto ze^z$ has many applications in many areas of mathematics. In a theory of random matrices, it appears in a formula of a eigenvalue distribution of a certain Wishart-type ensemble (cf. Cheliotis [4]). In the previous paper [8], we found that, for a certain class of Wishart-type ensembles, the corresponding eigenvalue distributions can be described by using the main branch of the inverse function of $z \mapsto \frac{z}{1+\gamma z} \left(1 + \frac{z}{\kappa}\right)^\kappa$, where γ, κ are real parameter. The aim of this paper is to give a complete analysis of this function, which we call the Lambert-Tsallis function.

1. PRELIMINARIES

For a non zero real number κ , we set

$$\exp_\kappa(z) := \left(1 + \frac{z}{\kappa}\right)^\kappa \quad \left(1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}\right),$$

where we take the main branch of the power function when κ is not integer. If $\kappa = \frac{1}{1-q}$, then it is exactly the so-called Tsallis q -exponential function (cf. [2, 9]). By virtue of $\lim_{\kappa \rightarrow \infty} \exp_\kappa(z) = e^z$, we regard $\exp_\infty(z) = e^z$.

For two real numbers κ, γ such that $\kappa \neq 0$, we introduce a holomorphic function $f_{\kappa, \gamma}(z)$, which we call *generalized Tsallis function*, by

$$f_{\kappa, \gamma}(z) := \frac{z}{1 + \gamma z} \exp_\kappa(z) \quad \left(1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}\right).$$

Analogously to Tsallis q -exponential, we also consider $f_{\infty, \gamma}(z) = \frac{ze^z}{1+\gamma z}$ ($z \in \mathbb{C}$). In particular, $f_{\infty, 0}(z) = ze^z$. Let $\mathcal{D}(f_{\kappa, \gamma})$ be the domain of $f_{\kappa, \gamma}$, that is, if κ is integer then $\mathcal{D}(f_{\kappa, \gamma}) = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$ if κ is not integer, then

$$\mathcal{D}(f_{\kappa, \gamma}) = \mathbb{C} \setminus \left\{x \in \mathbb{R}; 1 + \frac{x}{\kappa} \leq 0, \text{ or } x = -\frac{1}{\gamma}\right\}.$$

The purpose of this work is to study an inverse function to $f_{\kappa, \gamma}$ in detail. A multivariate inverse function of $f_{\infty, 0}(z) = ze^z$ is called the Lambert W function and studied in [6]. Hence, we call an inverse function to $f_{\kappa, \gamma}$ the *Lambert-Tsallis W function*. Since we have

$$f'_{\kappa, \gamma}(z) = \frac{\gamma z^2 + (1 + 1/\kappa)z + 1}{(1 + \gamma z)^2} \left(1 + \frac{z}{\kappa}\right)^{\kappa-1}, \quad (1)$$

the function $f_{\kappa, \gamma}(z)$ has the inverse function $w_{\kappa, \gamma}$ in a neighborhood of $z = 0$ by the fact $f'_{\kappa, \gamma}(0) = 1 \neq 0$.

Let $z = x + yi \in \mathbb{C}$ and we set for $\kappa \neq \infty$

$$\theta(x, y) := \text{Arg}\left(1 + \frac{z}{\kappa}\right),$$

where $\text{Arg}(w)$ stands for the principal argument of w ; $-\pi < \text{Arg}(w) \leq \pi$. Since we now take the main branch of power function, we have for $\kappa \neq \infty$

$$\begin{aligned} \left(1 + \frac{z}{\kappa}\right)^\kappa &= \exp\left(\kappa \left(\log \left|1 + \frac{z}{\kappa}\right| + i \text{Arg}\left(1 + \frac{z}{\kappa}\right)\right)\right) = \left(\left(1 + \frac{x}{\kappa}\right)^2 + \frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} e^{i\kappa\theta(x,y)} \\ &= \left(\left(1 + \frac{x}{\kappa}\right)^2 + \frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} \left(\cos(\kappa\theta(x,y)) + i \sin(\kappa\theta(x,y))\right). \end{aligned}$$

If $\kappa = \infty$, then we regard $\kappa\theta(x,y)$ as y because we have $\lim_{\kappa \rightarrow +\infty} \exp_\kappa(z) = e^z = e^x(\cos y + i \sin y)$. Since

$$\frac{z}{1 + \gamma z} = \frac{z(1 + \gamma \bar{z})}{|1 + \gamma z|^2} = \frac{(x + \gamma x^2 + \gamma y^2) + i(y + \gamma xy - \gamma xy)}{(1 + \gamma x)^2 + \gamma^2 y^2} = \frac{(x + \gamma x^2 + \gamma y^2) + iy}{(1 + \gamma x)^2 + \gamma^2 y^2},$$

we have

$$f_{\kappa,\gamma}(z) = \frac{\left(\left(1 + x/\kappa\right)^2 + (y/\kappa)^2\right)^{\frac{\kappa}{2}}}{(1 + \gamma x)^2 + \gamma^2 y^2} \begin{pmatrix} (x + \gamma x^2 + \gamma y^2) \cos(\kappa\theta(x,y)) - y \sin(\kappa\theta(x,y)) \\ + i\{(x + \gamma x^2 + \gamma y^2) \sin(\kappa\theta(x,y)) + y \cos(\kappa\theta(x,y))\} \end{pmatrix}. \quad (2)$$

Then, $f_{\kappa,\gamma}(z) \in \mathbb{R}$ implies

$$(x + \gamma x^2 + \gamma y^2) \sin(\kappa\theta(x,y)) + y \cos(\kappa\theta(x,y)) = 0. \quad (3)$$

If $\sin(\kappa\theta(x,y)) = 0$, then $\cos(\kappa\theta(x,y))$ does not vanish so that y needs to be zero. Thus, if $z = x + yi \in f_{\kappa,\gamma}(\mathbb{R})$ with $y \neq 0$, then we have $\sin(\kappa\theta(x,y)) \neq 0$. Thus, the equation (3) for $\sin(\kappa\theta(x,y)) \neq 0$ can be rewritten as

$$F(x,y) := (x + \gamma x^2 + \gamma y^2) + y \cot(\kappa\theta(x,y)) = 0. \quad (4)$$

For $y = 0$, we set $F(x,0) := \lim_{y \rightarrow 0} F(x,y)$. If $x = 0$, then we have $\theta(0,y) = \text{Arctan}(\frac{y}{\kappa})$ and hence $F(0,0) = \lim_{y \rightarrow 0} y \cot(\kappa \text{Arctan}(\frac{y}{\kappa})) = 1$. Let us introduce the connected set $\Omega = \Omega_{\kappa,\gamma}$ by

$$\Omega = \Omega_{\kappa,\gamma} := \{z = x + yi \in \mathcal{D}(f_{\kappa,\gamma}); F(x,y) > 0\}^\circ,$$

where A° is the connected component of an open set $A \subset \mathbb{C}$ containing $z = 0$. Note that since F is an even function on y , the domain Ω is symmetric with respect to the real axis. Set

$$\mathcal{S} := \mathbb{R} \setminus f_{\kappa,\gamma}(\Omega_{\kappa,\gamma} \cap \mathbb{R}). \quad (5)$$

Definition 1.1. If there exists a unique holomorphic extension $W_{\kappa,\gamma}$ of $w_{\kappa,\gamma}$ to $\mathbb{C} \setminus \bar{\mathcal{S}}$, then we call $W_{\kappa,\gamma}$ the main branch of Lambert-Tsallis W function. In this paper, we only study and use $W_{\kappa,\gamma}$ among other branches so that we call $W_{\kappa,\gamma}$ the Lambert-Tsallis function for short. Note that in our terminology the Lambert-Tsallis W function is multivalued and the Lambert-Tsallis function $W_{\kappa,\gamma}$ is single-valued.

Our goal is to prove the following theorem.

Theorem 1.2. Let $f_{\kappa,\gamma}$ be a generalized Tsallis function. Then, there exists the main branch of Lambert-Tsallis function if and only if (i) $0 < \kappa < 1$ and $\gamma \leq 0$, (ii) $\kappa \geq 1$ and $\gamma \leq \frac{1}{4}(1 + \frac{1}{\kappa})^2$, (iii) $-1 < \kappa < 0$ and $\gamma \leq \frac{1}{\kappa}$, (iv) $\kappa \leq -1$ and $\gamma \leq \frac{1}{4}(1 + \frac{1}{\kappa})^2$, and (v) $\kappa = \infty$ and $\gamma \leq \frac{1}{4}$.

Remark 1.3. In the case $\kappa > 1$ and $D(0) < 0$, the function $f_{\kappa,\gamma}$ maps $\Omega \cap \mathbb{C}^+$ to \mathbb{C}^+ two-to-one, and hence the extension exists on a smaller domain in Ω which $f_{\kappa,\gamma}$ maps to $\mathbb{C} \setminus \bar{\mathcal{S}}$, but it is not unique. In the case $0 < \kappa < 1$ and $\gamma > 0$, we need to extend the defining domain of $f_{\kappa,\gamma}$ so that there does not exist a complex domain such that $f_{\kappa,\gamma}$ maps to $\mathbb{C} \setminus \bar{\mathcal{S}}$.

The proof of this theorem is done in two steps, that is, we first give an explicit expression of $\Omega = \Omega_{\kappa,\gamma}$, and then show that $f_{\kappa,\gamma}$ maps Ω to $\mathbb{C} \setminus \bar{\mathcal{S}}$ bijectively. At first, we suppose that $0 < \kappa < +\infty$. Let us change variables in (4) by

$$r e^{i\theta} = 1 + \frac{z}{\kappa} \quad (r > 0, \theta \in (0, \pi)), \quad \text{or equivalently} \quad \begin{cases} x = \kappa(r \cos \theta - 1), \\ y = \kappa r \sin \theta, \end{cases} \quad (6)$$

and set $a := \gamma\kappa$ and

$$b(\theta) = (1 - 2a) \cos \theta + \sin \theta \cot(\kappa\theta). \quad (7)$$

Then, the equation (4) can be written as

$$ar^2 + b(\theta)r + a - 1 = 0. \quad (8)$$

In fact, we have

$$\begin{aligned} & \kappa(r \cos \theta - 1) + \gamma \{ (\kappa(r \cos \theta - 1))^2 + (\kappa r \sin \theta)^2 \} + \kappa r \sin \theta \cot(\kappa \theta) = 0 \\ \iff & \gamma \kappa^2 r^2 + \{ \kappa \cos \theta - 2\gamma \kappa^2 \cos \theta + \kappa \sin \theta \cot(\kappa \theta) \} r + (\gamma \kappa^2 - \kappa) = 0 \\ \iff & \gamma \kappa r^2 + \{ (1 - 2\gamma \kappa) \cos \theta + \sin \theta \cot(\kappa \theta) \} r + \gamma \kappa - 1 = 0. \end{aligned}$$

If $\sin(\kappa \theta) \neq 0$, then (8) has a solution

$$r = r_{\pm}(\theta) = \frac{-b(\theta) \pm \sqrt{b(\theta)^2 - 4a(a-1)}}{2a}.$$

Thus, for each angle θ , there exists at most two points on $f_{\kappa, \gamma}^{-1}(\mathbb{R})$. Since the change (6) of variables is the polar transformation, we need to know whether $r_{\pm}(\theta)$ is positive real or not. We note that

$$r'_{\varepsilon}(\theta) = \frac{1}{2a} \left(-b'(\theta) + \varepsilon \frac{2b(\theta)b'(\theta)}{2\sqrt{D(\theta)}} \right) = \frac{-\varepsilon b'(\theta)}{2a} \cdot \frac{-b(\theta) + \varepsilon \sqrt{D(\theta)}}{\sqrt{D(\theta)}} = -\varepsilon b'(\theta) \frac{r_{\varepsilon}(\theta)}{\sqrt{D(\theta)}} \quad (9)$$

for $\varepsilon = \pm 1$. Set $D(\theta) := b(\theta)^2 - 4a(a-1)$. Then, $r_{\pm}(\theta)$ are real if and only if $D(\theta) \geq 0$, and we have

$$D'(\theta) = 2b(\theta)b'(\theta).$$

Let α_1, α_2 be the two solutions of

$$q(z) := \gamma z^2 + (1 + 1/\kappa)z + 1 = 0. \quad (10)$$

If α_i are real, then we assume that $\alpha_1 \leq \alpha_2$, and if not, then we assume that $\text{Im } \alpha_1 > 0$ and $\text{Im } \alpha_2 < 0$. Then, $f'(z) = 0$ implies $z = \alpha_i$ ($i = 1, 2$) or $z = -\kappa$ if $\kappa > 1$. It is clear that α_1, α_2 are real numbers if and only if

$$\left(1 + \frac{1}{\kappa}\right)^2 - 4\gamma \geq 0 \iff \gamma \leq \frac{1}{4} \left(1 + \frac{1}{\kappa}\right)^2.$$

These two points α_i , $i = 1, 2$ correspond to the solutions of (8) with parameter $\theta = 0$. In fact, if $\theta = 0$, then $r = 1 + \frac{x}{\kappa}$ and

$$b(0) := \lim_{\theta \rightarrow 0} b(\theta) = \frac{\kappa + 1}{\kappa} - 2a, \quad (11)$$

and hence the equation (8) with $\theta = 0$ is described as

$$\begin{aligned} 0 &= ar^2 + b(0)r + a - 1 = a(r-1)^2 + (1 + \frac{1}{\kappa})(r-1) + \frac{1}{\kappa} \\ &= a \cdot \frac{x^2}{\kappa^2} + (1 + \frac{1}{\kappa})\frac{x}{\kappa} + \frac{1}{\kappa} = \frac{1}{\kappa} \left(\gamma x^2 + \left(1 + \frac{1}{\kappa}\right)x + 1 \right). \end{aligned}$$

We note that

$$q\left(-\frac{1}{\gamma}\right) = \frac{a-1}{a}, \quad q(-\kappa) = (a-1)\kappa, \quad (12)$$

and

$$D(0) = b(0)^2 - 4a(a-1) = \left(1 + \frac{1}{\kappa}\right)^2 - \frac{4a}{\kappa}.$$

In order to know whether the equation (8) has a positive solution, we need to study the signature of $D(\theta)$, and hence that of $b(\theta)$ and $b'(\theta)$. To do so, we introduce three functions H_{α} , F_{κ} and J_{κ} as below, and investigate them in detail.

For $\alpha > 0$, we set

$$H_{\alpha}(x) := \sin(\alpha x) - \alpha \sin(x) \quad (x \in \mathbb{R}).$$

Let us investigate the signature of $H_{\alpha}(x)$ on the interval $I_1 = (0, \min(\frac{2\pi}{\alpha}, 2\pi))$. Note that if $\alpha = 1$ then we have $H_1 \equiv 0$, and hence we exclude the case $\alpha = 1$. By differentiating, we have

$$H'_{\alpha}(x) = \alpha \cos(\alpha x) - \alpha \cos x = -2\alpha \sin\left(\frac{\alpha+1}{2}x\right) \sin\left(\frac{\alpha-1}{2}x\right).$$

$H'_{\alpha}(x) = 0$ implies that $x = \frac{2n\pi}{\alpha+1}$ ($n \in \mathbb{Z}$) or $x = \frac{2m\pi}{\alpha-1}$ ($m \in \mathbb{Z}$).

Lemma 1.4. (1) If $0 < \alpha \leq \frac{1}{2}$, then one has $H_{\alpha}(x) > 0$ for any $x \in I_1$.

(2) If $\frac{1}{2} < \alpha < 1$, then there exists a unique y_* such that $H_{\alpha}(y_*) = 0$, and one has $H_{\alpha}(x) > 0$ for $x \in (0, y_*)$ and $H_{\alpha}(x) < 0$ for $x \in (y_*, 2\pi)$.

(3) If $1 < \alpha < 2$, then there exists a unique $y_* \in I_1$ such that $H_{\alpha}(y_*) = 0$, and one has $H_{\alpha}(x) < 0$ for $x \in (0, y_*)$ and $H_{\alpha}(x) > 0$ for $x \in (y_*, \frac{2\pi}{\alpha})$.

(4) If $\alpha \geq 2$, then one has $H_\alpha(x) < 0$ for any $x \in I_1$.

Proof. It is obvious that $H_\alpha(0) = 0$. First, we assume that $0 < \alpha < 1$. In this case, since $\frac{1}{1-\alpha} \leq 1$ and $\frac{1}{2} < \frac{1}{\alpha+1} < 1$, we have

$$0 < \frac{2\pi}{\alpha+1} < 2\pi < \min\left(\frac{2\pi}{\alpha+1}, \frac{-2\pi}{\alpha-1}\right).$$

This means that $H'_\alpha(x) > 0$ when $x \in (0, \frac{2\pi}{\alpha+1})$, and $H'_\alpha(x) < 0$ when $x \in (\frac{2\pi}{\alpha+1}, 2\pi)$. On the other hand, we have $H_\alpha(2\pi) = \sin(2\pi\alpha) - \alpha \sin(2\pi) = \sin(2\pi\alpha)$. If $0 < \alpha \leq \frac{1}{2}$, then we have $0 < 2\alpha\pi \leq \pi$ and thus $H_\alpha(2\pi) \geq 0$, and if $\frac{1}{2} < \alpha < 1$ then we have $\pi < 2\pi\alpha < 2\pi$ which implies $H_\alpha(2\pi) < 0$. Thus, we obtain the assertions (1) and (2).

Next we assume that $\alpha > 1$. In this case, since $\frac{2}{\alpha+1} > \frac{1}{\alpha}$, we have

$$0 < \frac{2\pi}{\alpha+1} < \frac{2\pi}{\alpha} < \min\left(\frac{4\pi}{\alpha+1}, \frac{2\pi}{\alpha-1}\right).$$

This means that $H'_\alpha(x) < 0$ when $x \in (0, \frac{2\pi}{\alpha+1})$, and $H'_\alpha(x) > 0$ when $x \in (\frac{2\pi}{\alpha+1}, \frac{2\pi}{\alpha})$. On the other hand, we have $H_\alpha(\frac{2\pi}{\alpha}) = \sin(2\pi) - \alpha \sin(\frac{2\pi}{\alpha}) = -\alpha \sin(\frac{2\pi}{\alpha})$. If $1 < \alpha < 2$, then we have $\pi < \frac{2\pi}{\alpha} < 2\pi$ and hence $H_\alpha(\frac{2\pi}{\alpha}) > 0$. If $\alpha \geq 2$, then we have $0 < \frac{2\pi}{\alpha} \leq \pi$ so that $H_\alpha(\frac{2\pi}{\alpha}) \leq 0$. Therefore, we have proved the assertion (3) and (4). \square

For $\kappa > 0$, we set

$$F_\kappa(x) := \tan x \cdot \cot(\kappa x) \quad (x \in \mathbb{R}).$$

Let us investigate the behavior of $F_\kappa(x)$ on the interval $I_0 = (0, \min(\frac{\pi}{\kappa}, \pi))$. Since $F_1 \equiv 1$, we exclude the case $\kappa = 1$. Notice that if $\kappa < 2$, then $F_\kappa(x)$ has a pole at $x = \frac{\pi}{2}$ in the interval I_0 . At first, we see that

$$F_\kappa(0) := \lim_{x \rightarrow +0} F_\kappa(x) = \lim_{x \rightarrow +0} \frac{\sin x}{\sin(\kappa x)} \cdot \frac{\cos(\kappa x)}{\cos x} = \frac{1}{\kappa} > 0.$$

If $\kappa < 1$, then it is obvious that $F_\kappa(\pi) = 0$. On the other hand, if $1 < \kappa < 2$, then $\frac{\pi}{2} < \frac{\pi}{\kappa} < \pi$, and if $\kappa \geq 2$ then $0 < \frac{\pi}{\kappa} \leq \frac{\pi}{2}$, and hence we have

$$\lim_{x \rightarrow \frac{\pi}{\kappa} - 0} F_\kappa(x) = \begin{cases} +\infty & (\text{if } 1 < \kappa < 2), \\ -\infty & (\text{if } \kappa \geq 2). \end{cases}$$

By differentiating, we have

$$F'_\kappa(x) = \frac{\cot(\kappa x)}{\cos^2 x} - \frac{\kappa \tan x}{\sin^2(\kappa x)} = \frac{H_\kappa(2x)}{2(\cos x \sin(\kappa x))^2}.$$

For the case $\frac{1}{2} < \kappa < 2$ ($\kappa \neq 1$), Lemma 1.4 tells us that H_κ has a unique zero point y_* in the interval $(0, \min(2\pi, \frac{2\pi}{\kappa}))$. If we set $x_* = \frac{y_*}{2}$, then we have $x_* \in I_0$ and $F_\kappa(x_*) = 0$.

Lemma 1.5. *If $\frac{1}{2} < \kappa < 1$, then one has $F_\kappa(x_*) < 1$, and if $1 < \kappa < 2$, then one has $F_\kappa(x_*) > 1$.*

Proof. We first assume that $\frac{1}{2} < \kappa < 1$. In this case, since $0 < 1 - \kappa < \frac{1}{2}$ and since $\frac{\pi}{2} < x_* < \pi$, we have $\sin((1 - \kappa)x_*) > 0$. Thus, since $\cos x_* < 0$ and $\sin(\kappa x_*) > 0$, we obtain

$$\begin{aligned} \sin((1 - \kappa)x_*) > 0 &\iff \sin x_* \cos(\kappa x_*) - \cos x_* \sin(\kappa x_*) > 0 \\ &\iff \sin x_* \cos(\kappa x_*) > \cos x_* \sin(\kappa x_*) \\ &\iff F_\kappa(x_*) = \frac{\sin x_* \cos(\kappa x_*)}{\cos x_* \sin(\kappa x_*)} < 1. \end{aligned}$$

Next, we assume that $1 < \kappa < 2$. In this case, since $0 < \kappa - 1 < 1$ and since $\frac{\pi}{2} < x_* < \frac{\pi}{\kappa} < \pi$, we have $\sin((\kappa - 1)x_*) > 0$. Thus, since $\cos x_* < 0$ and $\sin(\kappa x_*) > 0$, we obtain

$$\begin{aligned} \sin((\kappa - 1)x_*) > 0 &\iff \cos x_* \sin(\kappa x_*) - \sin x_* \cos(\kappa x_*) > 0 \\ &\iff \cos x_* \sin(\kappa x_*) > \sin x_* \cos(\kappa x_*) \\ &\iff F_\kappa(x_*) = \frac{\sin x_* \cos(\kappa x_*)}{\cos x_* \sin(\kappa x_*)} > 1. \end{aligned}$$

We have proved the lemma. \square

Lemmas 1.4 and 1.5 yield the following table.

Lemma 1.6. *One has the following increasing/decreasing table of F_κ .*

(A) $0 < \kappa \leq \frac{1}{2}$	x	0	\dots	$\frac{\pi}{2}$	\dots	π			
	F'_κ		+	\times		+			
	F_κ	$\frac{1}{\kappa}$	$\nearrow^{+\infty}$	\times	$-\infty$	\nearrow	0		
(B) $\frac{1}{2} < \kappa < 1$	x	0	\dots	$\frac{\pi}{2}$	\dots	x_*	\dots	π	
	F'_κ		+	\times	+	0		-	
	F_κ	$\frac{1}{\kappa}$	$\nearrow^{+\infty}$	\times	$-\infty$	\nearrow	$F_\kappa(x_*)$	\searrow	0
(C) $1 < \kappa < 2$	x	0	\dots	$\frac{\pi}{2}$	\dots	x_*	\dots	$\frac{\pi}{\kappa}$	
	F'_κ		-	\times	-	0		+	
	F_κ	$\frac{1}{\kappa}$	$\searrow^{-\infty}$	\times	$+\infty$	\searrow	$F_\kappa(x_*)$	$\nearrow^{+\infty}$	\times
(D) $\kappa \geq 2$	x	0	\dots	$\frac{\pi}{\kappa}$					
	F'_κ			-	\times				
	F_κ	$\frac{1}{\kappa}$	$\searrow^{-\infty}$	\times					

For $\kappa > 0$, we set

$$J_\kappa(x) := \frac{2x - 2\kappa - 1}{4\kappa x - 2\kappa - 1} = \frac{1}{2\kappa} \cdot \frac{x - \frac{2\kappa+1}{2}}{x - \frac{2\kappa+1}{4\kappa}} = \frac{1}{2\kappa} - \frac{4\kappa^2 - 1}{8\kappa^2} \cdot \frac{1}{x - \frac{2\kappa+1}{4\kappa}}.$$

Then, we have

$$J_\kappa(0) = 1, \quad J_\kappa(1) = -1, \quad \lim_{x \rightarrow +\infty} J_\kappa(x) = \lim_{x \rightarrow -\infty} J_\kappa(x) = \frac{1}{2\kappa}. \quad (13)$$

Note that, if we set $\kappa = \frac{1}{2}$, then $J_{\frac{1}{2}} \equiv 1$. The following lemma is trivial.

Lemma 1.7. *Suppose that $\kappa \neq \frac{1}{2}$. Then, J_κ has a pole at $x = \frac{1}{2} + \frac{1}{4\kappa}$. If $0 < \kappa < \frac{1}{2}$, then it is monotonic decreasing on \mathbb{R} , and if $\kappa > \frac{1}{2}$, then it is monotonic increasing on \mathbb{R} .*

We now consider the function $b(\theta)$. If $\kappa = 1$, then we have $b(\theta) = 2(1 - a) \cos \theta$. Otherwise, since $b(\theta)$ can be described as

$$b(\theta) = ((1 - 2a) + F_\kappa(\theta)) \cos \theta \quad (\text{if } \cos \theta \neq 0),$$

the signature of $b(\theta)$ can be determined by using F_κ . Note that, $\cos \theta = 0$ occurs when $\kappa < 2$, and in this case, we have

$$b\left(\frac{\pi}{2}\right) = 0 + 1 \cdot \cot \frac{\kappa\pi}{2} = \cot \frac{\kappa\pi}{2}.$$

It is positive if $\kappa < 1$, and negative if $1 < \kappa < 2$. These observations together with Lemma 1.5 yield the following table.

Lemma 1.8. *The signature of $b(\theta)$ on the interval $I_0 = (0, \min(\pi, \frac{\pi}{\kappa}))$ is given as follows.*

$0 < \kappa \leq \frac{1}{2}$									
$2a - 1 > \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ	\cdots	π			$\varphi < \frac{\pi}{2}$
$0 \leq 2a - 1 \leq \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	\cdots	π					
$2a - 1 < 0$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ	\cdots	π			$\varphi > \frac{\pi}{2}$
$\frac{1}{2} < \kappa < 1$									
$2a - 1 > \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ	\cdots	π			$\varphi < \frac{\pi}{2}$
$F_\kappa(\theta_*) < 2a - 1 \leq \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	\cdots	π					
$0 \leq 2a - 1 \leq F_\kappa(\theta_*)$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ_1	\cdots	φ_2	\cdots	π	$\varphi_i > \frac{\pi}{2}$
$2a - 1 < 0$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ	\cdots	π			$\varphi > \frac{\pi}{2}$
$1 < \kappa < 2$									
$2a - 1 \geq F_\kappa(\theta_*)$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ_1	\cdots	φ_2	\cdots	$\frac{\pi}{\kappa}$	$\varphi_i > \frac{\pi}{2}$
$\frac{1}{\kappa} \leq 2a - 1 < F_\kappa(\theta_*)$	$\frac{\theta}{b(\theta)}$	0	\cdots	$\frac{\pi}{\kappa}$					
$2a - 1 < \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ	\cdots	$\frac{\pi}{\kappa}$			$\varphi < \frac{\pi}{2}$
$\kappa \geq 2$									
$2a - 1 \geq \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	\cdots	$\frac{\pi}{\kappa}$					
$2a - 1 < \frac{1}{\kappa}$	$\frac{\theta}{b(\theta)}$	0	\cdots	φ	\cdots	$\frac{\pi}{\kappa}$			

In this table, φ or φ_i ($i = 1, 2$) are solutions in I_0 of $b(\theta) = 0$. If $2a - 1 = F_\kappa(x_*)$, then $\varphi_1 = \varphi_2$.

Let us consider the function $b'(\theta)$. If $\kappa = \frac{1}{2}$, then we have

$$b'(\theta) = (1 - 2a) \cos \theta + 2 \sin \theta \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = (1 - 2a) \cos \theta + 2 \cos^2\left(\frac{\theta}{2}\right) = 2(a - 1) \sin \theta. \quad (14)$$

Let us assume that $\kappa \neq \frac{1}{2}$. Since the function $b(\theta)$ can be also written as

$$\begin{aligned} b(\theta) &= \cos \theta + \sin \theta \cot(\kappa\theta) - 2a \cos \theta = \frac{\cos \theta \sin(\kappa\theta) + \sin \theta \cos(\kappa\theta)}{\sin(\kappa\theta)} - 2a \cos \theta \\ &= \frac{\sin((\kappa + 1)\theta)}{\sin(\kappa\theta)} - 2a \cos \theta, \end{aligned}$$

its derivative can be written by using H_α as

$$b'(\theta) = \frac{H_{2\kappa+1}(\theta)}{2 \sin^2(\kappa\theta)} + 2a \sin \theta. \quad (15)$$

In fact,

$$\begin{aligned}
b'(\theta) &= \frac{(\kappa+1) \cos((\kappa+1)\theta) \sin \kappa\theta - \kappa \sin((\kappa+1)\theta) \cos \kappa\theta}{\sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{-\kappa \sin \theta + \cos((\kappa+1)\theta) \sin \kappa\theta}{\sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{-\kappa \sin \theta + \frac{1}{2}(\sin((2\kappa+1)\theta) - \sin \theta)}{\sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{\sin((2\kappa+1)\theta) - (2\kappa+1) \sin \theta}{2 \sin^2 \kappa\theta} + 2a \sin \theta \\
&= \frac{H_{2\kappa+1}(\theta)}{2 \sin^2(\kappa\theta)} + 2a \sin \theta.
\end{aligned}$$

Let us set

$$B(\theta) := 2 \sin^2(\kappa\theta) b'(\theta) = H_{2\kappa+1}(\theta) + 4a \sin \theta \sin^2(\kappa\theta), \quad \ell(\kappa, a) := 4a\kappa - 2\kappa - 1. \quad (16)$$

Then, the signatures of $b'(\theta)$ and $B(\theta)$ are the same, and the derivative of $B(\theta)$ is given as, if $\ell(\kappa, a) \neq 0$ then

$$B'(\theta) = 2\ell(\kappa, a) \cos \theta \sin^2(\kappa\theta) \left(F_\kappa(\theta) + J_\kappa(a) \right), \quad (17)$$

and if $\ell(\kappa, a) = 0$ then

$$B'(\theta) = \frac{1 - 4\kappa^2}{\kappa} \cos \theta \sin^2(\kappa\theta). \quad (18)$$

In fact, we have

$$\begin{aligned}
B'(\theta) &= -2(2\kappa+1) \sin((\kappa+1)\theta) \sin(\kappa\theta) + 4a(\sin \kappa\theta)(\cos \theta \sin(\kappa\theta) + 2\kappa \sin \theta \cos(\kappa\theta)) \\
&= (4a - 4\kappa - 2) \cos \theta \sin^2(\kappa\theta) + (8a\kappa - 4\kappa - 2) \sin \theta \sin(\kappa\theta) \cos(\kappa\theta) \\
&= 2 \cos \theta \sin^2(\kappa\theta) (2a - 2\kappa - 1 + (4a\kappa - 2\kappa - 1) \tan \theta \cot(\kappa\theta)) \\
&= 2(4a\kappa - 2\kappa - 1) \cos \theta \sin^2(\kappa\theta) \left(F_\kappa(\theta) + \frac{2a - 2\kappa - 1}{4a\kappa - 2\kappa - 1} \right).
\end{aligned}$$

Note that two conditions $\ell(\kappa, a) = 0$ and $2a - 2\kappa - 1 = 0$ occur simultaneously only when $\kappa = \frac{1}{2}$. Set

$$G(x) := \frac{2x^2 + 3x + 1}{6x}.$$

- Lemma 1.9.** (1) Suppose that $0 < \kappa < \frac{1}{2}$, or $\kappa > 1$. If $a > G(\kappa)$, then there exists a unique $\varphi_* \in I_0$ such that $b'(\varphi_*) = 0$, and one has $b'(\theta) > 0$ for $\theta \in (0, \varphi_*)$, and $b'(\theta) < 0$ for $\theta \in (\varphi_*, \pi)$. If $a \leq G(\kappa)$, then one has $b'(\theta)$ for any $\theta \in I_0$.
- (2) Suppose that $\frac{1}{2} < \kappa < 1$. If $a < G(\kappa)$, then there exists a unique $\varphi_* \in I_0$ such that $b'(\varphi_*) = 0$, and one has $b'(\theta) < 0$ for $\theta \in (0, \varphi_*)$, and $b'(\theta) > 0$ for $\theta \in (\varphi_*, \pi)$. If $a > G(\kappa)$, then one has $b'(\theta) > 0$ for any $\theta \in I_0$.

Proof. Since we have (17) or (18), we can use Lemmas 1.5 and 1.7. This lemma can be obtained by dividing cases and by elementary but tedious calculations. Note that $\frac{2\kappa^2+3\kappa+1}{6\kappa}$ comes from the equation $0 = F_\kappa(0) + J_\kappa(a)$. By differentiating $b(\theta)$ by using expression (7), we obtain

$$b'(\theta) = (2a - 1) \sin \theta + \cos \theta \cot(\kappa\theta) - \frac{\kappa \sin \theta}{\sin^2(\kappa\theta)},$$

and hence

$$\begin{aligned}
b'(\theta) \sin^2(\kappa\theta) &= (2a - 1) \sin \theta (1 - \cos^2(\kappa\theta)) + \cos \theta \sin(\kappa\theta) \cos(\kappa\theta) - \kappa \sin \theta \\
&= (2a - \kappa - 1) \sin \theta + (2a - 1) \sin \theta \cos^2(\kappa\theta) + \cos \theta \sin(\kappa\theta) \cos(\kappa\theta) \\
&= (2a - \kappa - 1) \sin \theta + \cos \theta \sin(\kappa\theta) \cos(\kappa\theta) ((2a - 1) F_\kappa(\theta) + 1).
\end{aligned}$$

If φ_i satisfies $B'(\varphi_i) = 0$, then we have $F_\kappa(\varphi_i) + J_\kappa(a) = 0$ and hence

$$(2a - 1) F_\kappa(\varphi_i) + 1 = \frac{-(2a - 1)(2a - 2\kappa - 1) + (4a\kappa - 2\kappa - 1)}{\ell(\kappa, a)} = \frac{4a(a - 1)}{\ell(\kappa, a)}. \quad (19)$$

The non-trivial cases are given as (i) $\frac{1}{2} < \kappa < 1$ and $0 \leq -J_\kappa(a) \leq F_\kappa(x_*)$, or (ii) $1 < \kappa < 2$ and $-J_\kappa(x) > F_\kappa(x_*)$.

first we consider the case (i). In this case, Lemma 1.7 tells us that we have $\ell(\kappa, a) > 0$. Since $-J_\kappa(a)$ is monotonic decreasing when $\ell(\kappa, a) > 0$ and since $-J_\kappa(1) = 1$ by (13), we see that a need satisfy $a \in (1, \frac{1}{2} + \kappa)$ because $F_\kappa(x_*) < 1$ by Lemma 1.5. Thus, we have $2a - \kappa - 1 > 0$. Again by

Lemma 1.5, the equation $B'(\theta) = F_\kappa(\theta) + J_\kappa(a) = 0$ has at most two solutions $\varphi_i \in (\frac{\pi}{2\kappa}, \pi)$, $i = 1, 2$. Set $\varphi_1 \leq \varphi_2$. Then, since

$$B(0) = 0, \quad B(\pi) = \sin(2\kappa\pi) > 0,$$

we have

θ	0	\dots	$\frac{\pi}{2}$	\dots	φ_1	\dots	φ_2	\dots	π
$\ell(\kappa, a)$		+		+		+		+	
$\cos \theta$		+	0	-		-		-	
$F_\kappa + J_\kappa$		+		-	0	+	0	-	
$B'(\theta)$		+		+	0	-	0	+	
$B(\theta)$	0	\nearrow		\nearrow		\searrow		\nearrow	$B(\pi) > 0$

We shall show that $B(\varphi_i) > 0$, ($i = 1, 2$), which implies $b'(\theta) > 0$. Since $a > 1$, we have $(2a - 1)F_\kappa(\varphi_i) + 1 > 0$ by (19). Since $\varphi_i \in (\frac{\pi}{2\kappa}, \pi)$, we see that $\cos \varphi_i \sin(\kappa\varphi_i) \cos(\kappa\varphi_i) > 0$ and hence we obtain $B(\varphi_i) > 0$.

Next, we consider the case (ii). Then, since $F_\kappa(x_*) > 1$ by Lemma 1.5 and since $J_\kappa(1) = -1$ by (13), we need to have $\ell(\kappa, a) > 0$ and $a < 1$, whence $2a - \kappa - 1 < 0$. Lemma 1.7 tells us that the function $-J_\kappa(a)$ is monotonic decreasing if $\ell(\kappa, a) > 0$. Again by Lemma 1.5, the equation $B'(\theta) = F_\kappa(\theta) + J_\kappa(a) = 0$ has at most two solutions $\varphi_i \in (\frac{\pi}{2\kappa}, \pi)$, $i = 1, 2$. Let $\varphi_1 \leq \varphi_2$. Then, since

$$B(0) = 0, \quad B\left(\frac{\pi}{\kappa}\right) = -2\kappa \sin \frac{\pi}{\kappa} < 0,$$

we have

θ	0	\dots	$\frac{\pi}{2}$	\dots	φ_1	\dots	φ_2	\dots	$\frac{\pi}{\kappa}$
$\ell(\kappa, a)$		+		+		+		+	
$\cos \theta$		+	0	-		-		-	
$F_\kappa + J_\kappa$		-		+	0	-	0	+	
$B'(\theta)$		-		-	0	+	0	-	
$B(\theta)$	0	\searrow		\searrow		\nearrow		\searrow	$B(\frac{\pi}{\kappa}) < 0$

Since $a < 1$, we have $(2a - 1)F_\kappa(\varphi_i) + 1 < 0$ by (19). Since $\varphi_i \in (\frac{\pi}{2}, \frac{\pi}{\kappa})$, we have $\cos \varphi_i \sin(\kappa\varphi_i) \cos(\kappa\varphi_i) > 0$ so that we obtain $B(\varphi_i) < 0$. \square

2. THE DOMAIN Ω FOR $\kappa < +\infty$

In this section, we shall determine Ω for finite $\kappa > 0$, which is the connected component of the set $\{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); F(x, y) > 0\}$ containing $z = 0$. Let $r_\pm(\theta)$ be the solutions of the equation (8) and let α_i , $i = 1, 2$ are the solutions of the equation (10). Since $F(x, y)$ is a continuous function, the boundary $\partial\Omega$ is included in the set $\{z = x + yi \in \mathbb{C}; F(x, y) = 0\} \subset f_{\kappa, \gamma}^{-1}(\mathbb{R})$. Since F is even function on y , the domain Ω is symmetric with respect to the real axis, and hence we consider $D := \Omega \cap \mathbb{C}^+$.

Proposition 2.1. *Suppose that $\kappa = 1$.*

- (1) *If $\gamma > 1$, then one has $\Omega = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$.*
- (2) *If $0 < \gamma \leq 1$, then one has $\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); (x + \frac{1}{\gamma})^2 + y^2 > \frac{1-\gamma}{\gamma^2}\}$.*
- (3) *If $\gamma = 0$, then one has $\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); 1 + 2x > 0\}$.*
- (4) *If $\gamma < 0$, then one has $\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); (x + \frac{1}{\gamma})^2 + y^2 < \frac{1-\gamma}{\gamma^2}\}$. In particular, Ω is bounded.*

Proof. For $z = x + yi$, we have

$$f_{1, \gamma}(z) = \frac{(x + \gamma x^2 + \gamma y^2 + yi)(x + 1 + yi)}{(1 + \gamma x)^2 + \gamma^2 y^2} = \frac{1}{(1 + \gamma x)^2 + \gamma^2 y^2} \begin{pmatrix} (x + \gamma x^2 + \gamma y^2)(1 + x) - y^2 \\ iy(1 + 2x + \gamma x^2 + \gamma y^2) \end{pmatrix}.$$

Thus, $f_{1, \gamma}(z) \in \mathbb{R}$ implies $y = 0$ or

$$1 + 2x + \gamma x^2 + \gamma y^2 = 0 \iff \begin{cases} (x + \frac{1}{2\gamma})^2 + y^2 = \frac{1-\gamma}{\gamma^2} & (\text{if } \gamma \neq 0), \\ x = -\frac{1}{2} & (\text{if } \gamma = 0). \end{cases}$$

Therefore, we have $\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); \operatorname{Re} z > -\frac{1}{2}\}$ when $\gamma = 0$ because it contains $z = 0$. Suppose that $\gamma \neq 0$ and set $C = \{z = x + yi \in \mathbb{C}; 1 + 2x + \gamma x^2 + \gamma y^2 = 0\}$. If $\gamma > 1$, then $C = \emptyset$, which implies $\Omega = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$. Assume that $0 < \gamma \leq 1$. Then, $C \neq \emptyset$ and $z = 0$ does

not contained in the interior of C and hence Ω is the out side of C , which is written as $\Omega = \{z = x + yi \in \mathbb{C}; 1 + 2x + \gamma x^2 + \gamma y^2 > 0\}$. Assume that $\gamma < 0$. Then, $C \neq \emptyset$ and $z = 0$ is contained in the interior of C , and thus Ω is the interior of C , which is written as

$$\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa,\gamma}); 1 + 2x + \gamma x^2 + \gamma y^2 > 0\}.$$

The proof is completed. \square

Recall that $a = \kappa\gamma$. Set $\theta_0 := \frac{\pi}{\kappa}$ and let I_0 be the interval $(0, \min(\pi, \theta_0))$.

Proposition 2.2. *Let $\kappa > 0$ with $\kappa \neq 1$. For $z \in \mathbb{C} \setminus \{x \leq -\kappa\}$, one sets $re^\theta = 1 + \frac{z}{\kappa}$. Then, Ω can be described as follows.*

- (1) *If $a < 0$, then there exists a unique $\theta_* \in (0, \frac{\pi}{\kappa+1})$ such that $r_+(\theta_*) = r_-(\theta_*)$ and that $r_\pm(\theta)$ are both positive with $r_-(\theta) \leq r_+(\theta)$ on $(0, \theta_*)$. Moreover, one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa,\gamma}); |\theta| < \theta_* \text{ and } r_-(\theta) < r < r_+(\theta)\}.$$

In particular, Ω is bounded. One has $\alpha_1, \alpha_2 \in \partial\Omega$ and $-\frac{1}{\gamma} \in \bar{\Omega}$, whereas $-\kappa \notin \bar{\Omega}$.

- (2) *If $a = 0$, then one has $r(\theta) = r_\pm(\theta) = \frac{\sin(\kappa\theta)}{\sin((\kappa+1)\theta)}$ which is positive on the interval $(0, \frac{\pi}{\kappa+1})$. Moreover, one has*

$$\Omega = \left\{ z \in \mathcal{D}(f_{\kappa,\gamma}); |\theta| < \frac{\pi}{\kappa+1} \text{ and } r > r(\theta) \right\}.$$

Ω has an asymptotic line $y = \pm(x + \frac{\kappa^2}{\kappa+1}) \tan \theta_1$. One has $\alpha_1 = \alpha_2 = -\frac{\kappa}{\kappa+1} \in \partial\Omega$ and $-\kappa \notin \bar{\Omega}$.

- (3) *If $0 < a < 1$, then $r_+(\theta)$ is the only positive solution of (8) on I_0 , and one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa,\gamma}); |\theta| < \min(\theta_0, \pi) \text{ and } r > r_+(\theta)\}.$$

If $\kappa > 1$, then Ω has an asymptotic line $y = \pm(\tan \theta_0)(x + \kappa - \frac{1}{a})$. One has $\alpha_2 \in \partial\Omega$, whereas $\alpha_1, -\frac{1}{\gamma}, -\kappa \notin \bar{\Omega}$.

- (4) *Suppose that $a = 1$ and $\kappa \neq 1$.*

(a) *If $0 < \kappa < 1$, then one has $r_+(\theta) = 0$ and $r_-(\theta) = -b(\theta) < 0$ for $\theta \in I_0$, and $\Omega = \mathcal{D}(f_{\kappa,\gamma})$. One has $\alpha_1 \in \partial\Omega$ and $\alpha_2 = -\kappa = -\frac{1}{\gamma} \in \partial\Omega$.*

(b) *If $\kappa > 1$, then one has $r_+(\theta) = -b(\theta) > 0$ and $r_-(\theta) = 0$ for $\theta \in I_0$, and one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa,\gamma}); |\theta| < \theta_0 \text{ and } r > r_+(\theta)\}.$$

One has $\alpha_2 = -1 \in \partial\Omega$, while $\alpha_1 = -\kappa = -\frac{1}{\gamma} \notin \bar{\Omega}$. Moreover, Ω has an asymptotic line $y = \pm(\tan \theta_0)(x + \kappa - \frac{1}{a})$.

- (5) *Suppose that $a > 1$.*

(a) *If $\kappa > 1$ with $D(0) \geq 0$, then $r_\pm(\theta)$ are both positive in I_0 with $r_-(\theta) \leq r_+(\theta)$, and one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa,\gamma}); |\theta| < \theta_0 \text{ and } r > r_+(\theta)\}.$$

Ω has an asymptotic line $y = \pm(\tan \theta_0)(x + \kappa - \frac{1}{a})$. One has $\alpha_2 \in \partial\Omega$, while $-\kappa, -\frac{1}{\gamma}, \alpha_1 \notin \bar{\Omega}$.

(b) *If $\kappa > 1$ and $D(0) < 0$, then there exists a unique $\theta_* \in (0, \theta_0)$ such that $D(\theta_*) = 0$, and $r_\pm(\theta)$ are both positive in the interval (θ_*, θ_0) . Moreover, one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa,\gamma}); |\theta| < \theta_0 \text{ and if } |\theta| \geq \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}.$$

In this case, $\alpha_i, i = 1, 2$ are both non-real and one has $\alpha_i, -\kappa \in \partial\Omega$ and $-\frac{1}{\gamma} \in \bar{\Omega}$. Moreover, Ω has an asymptotic line $y = \pm(\tan \theta_0)(x + \kappa - \frac{1}{a})$.

(c) *If $0 < \kappa < 1$, then there are no θ such that $D(\theta) > 0$, and one has $\Omega = \mathcal{D}(f_{\kappa,\gamma})$.*

Recall that, since Ω is symmetric with respect to the real axis, it is enough to determine the boundary $\partial\Omega$ of Ω in the upper half plane.

Proof. (1) Assume that $a < 0$. Since $r_+(\theta) \cdot r_-(\theta) = \frac{a-1}{a} > 0$, we see that $r_\pm(\theta)$ have the same signature if $r_\pm(\theta) \in \mathbb{R}$. Since $a < 0$, we have $b(0) = 1 + \frac{1}{\kappa} - 2a > 0$ by (11). Let $\theta_1 := \frac{\pi}{\kappa+1}$. Then, Lemmas 6.2 and 1.4 tells us that $b'(\theta) \leq 0$ for any $\theta \in (0, \theta_1)$. In fact, if κ does not satisfy $\frac{1}{2} < \kappa < 1$, then it is a direct consequence of Lemma 6.2, and if $\frac{1}{2} < \kappa < 1$ then we can verify it by Lemma 6.2 and by the fact that $b'(\theta_1) = H_{2\kappa+1}(\theta_1) + 2a \sin \theta_1 < 0$ by Lemma 1.4. Therefore, $b(\theta)$ is

monotonic decreasing on the interval $(0, \theta_1)$. If $b(\theta_1) < 0$, then there exists a unique $\varphi \in (0, \theta_1)$ such that $b(\varphi) = 0$, and if $b(\theta_1) \geq 0$ then we set $\varphi = \theta_1$. Since we have

$$D(0) = \left(1 + \frac{1}{\kappa}\right)^2 - \frac{4a}{\kappa} > 0 \quad \text{and} \quad D(\theta_1) = (-2a \cos \theta_1)^2 - 4a(a-1) = 4a^2 \cos^2 \theta_1 - 4a^2 + 4a \\ = 4a - 4a^2 \sin^2 \theta_1 = 4a(1 - a \sin^2 \theta_1) < 0$$

and $D'(\theta) = 2b(\theta)b'(\theta)$, the increasing/decreasing table of $D(\theta)$ is given as

θ	0	\dots	φ	\dots	θ_1
$b(\theta)$		+	0	-	
$b'(\theta)$		-	-	-	
$D'(\theta)$		-	0	+	
$D(\theta)$	$D(0) > 0$	\searrow	$D(\varphi)$	\nearrow	$D(\theta_1) < 0$

and hence there exists a unique $\theta_* \in (0, \varphi)$ such that $D(\theta_*) = 0$. In particular, D is monotonic decreasing in the interval $(0, \theta_*)$, and $D(\theta_* + \delta) < 0$ for $\delta > 0$ such that $\theta_* + \delta < \varphi$. Therefore, since $r_+(\theta) + r_-(\theta) = -b(\theta)/a$, the signatures of $r_{\pm}(\theta)$ is the same as that of $b(\theta)$ if r_{\pm} are real so that r_{\pm} are positive on $(0, \theta_*]$ and r_{\pm} are not real for $\theta \in (\theta_*, \varphi)$. Since $r_+(\theta_*) = r_-(\theta_*)$ by the fact $D(\theta_*) = 0$, the curves $r_+(\theta)$, $\theta \in (0, \theta_*]$ followed by $r_-(\theta_* - \theta)$, $\theta \in (0, \theta_*]$, form a continuous curve going from α_2 to α_1 in the upper half-plane. Let α_i , $i = 1, 2$ be the solutions of (10). Since $\alpha_1 \alpha_2 = \frac{1}{\gamma} < 0$ and $\alpha_1 \leq \alpha_2$, we have $\alpha_1 < 0 < \alpha_2$ so that

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_* \text{ and } r_-(\theta) < r < r_+(\theta)\}.$$

Since $q(-\frac{1}{\gamma}) > 0$ and $q(-\kappa) < 0$ by (12), where q is defined in (11), we see that $-\frac{1}{\gamma} \in \bar{\Omega}$ and $-\kappa \notin \bar{\Omega}$.

(2) Assume that $a = 0$. In this case, the equation (8) reduces to $b(\theta)r - 1 = 0$ so that

$$r_{\pm}(\theta) = r(\theta) = \frac{1}{b(\theta)} = \frac{1}{\cos \theta + \sin \theta \cot(\kappa \theta)} = \frac{\sin(\kappa \theta)}{\sin((\kappa + 1)\theta)}.$$

Let $\theta_1 = \frac{\pi}{\kappa + 1}$. Since $\sin(\kappa \theta)$ and $\sin((\kappa + 1)\theta)$ are both positive in the interval $(0, \theta_1)$, and since $\lim_{\theta \rightarrow \theta_1 - 0} \sin((\kappa + 1)\theta) = 0$, we see that

$$\lim_{\theta \rightarrow \theta_1 - 0} r(\theta) = +\infty.$$

Thus, it has an asymptotic line with slope $\tan \theta_1$, which is determined later. Since $\gamma = a/\kappa = 0$, the solutions α_i , $i = 1, 2$ are given as $\alpha_1 = \alpha_2 = -\frac{\kappa}{\kappa + 1}$. Since $q(0) = 1 > 0$ and $q(-\kappa) = -\kappa < 0$, we see that the domain Ω is given as

$$\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_1, r > r(\theta)\}$$

and $-\kappa \notin \bar{\Omega}$.

(3) Assume that $0 < a < 1$. In this case, we have

$$D(\theta) = b(\theta)^2 + 4a(1 - a) > 0$$

for any $\theta \in I_0$ so that the solutions of (8) are always real. Moreover, since $r_+(\theta) \cdot r_-(\theta) = -\frac{1-a}{a} < 0$, they have the different signatures, and $r_+(\theta)$ is the positive real solution of (8) by the fact $|b(\theta)| < \sqrt{D(\theta)}$. In particular, $r_+(\theta)$, $\theta \in I_0$ forms a continuous curve in \mathbb{C}^+ . The solutions α_i , $i = 1, 2$ of (10) are both negative because we have $\gamma = a/\kappa > 0$ and $\alpha_1 + \alpha_2 = -(1 + \frac{1}{\kappa})/\gamma < 0$. Since $q(-\frac{1}{\gamma}) < 0$ and $q(-\kappa) < 0$ by (12) and since $\frac{1}{\gamma} > \kappa$, we see that

$$\alpha_1 < -\frac{1}{\gamma} < -\kappa < \alpha_2 < 0. \quad (20)$$

(a) We first assume that $\kappa \geq 1$. Then, $b(\theta)$ is defined on the interval $I_0 = (0, \theta_0)$, and we have

$$\lim_{\theta \rightarrow \theta_0 - 0} b(\theta) = -\infty, \text{ which implies}$$

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty.$$

Therefore, the curve $r_+(\theta)$, $\theta \in (0, \theta_0)$ has the asymptotic line with gradient $\tan \theta_0$, which is determined later. By (20), we see that Ω is given as

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, r > r_+(\theta)\},$$

and $\alpha_1, -\frac{1}{\gamma}, -\kappa \notin \bar{\Omega}$.

(b) Next, we assume that $0 < \kappa < 1$. Then, $b(\theta)$ is defined for any $\theta \in I_0 = (0, \pi)$. If $\theta = 0$, then the positive solution of (8) corresponds to $\alpha_2 < 0$, and if $\theta = \pi$ then, since $b(\pi) = 2a - 1$, the positive

solution of (8) corresponds to $-\frac{1}{\gamma}$. Thus, the curve $r_+(\theta)$, $\theta \in (0, \pi)$ connects $z = \alpha_2$ and $z = -\frac{1}{\gamma}$ passing in the upper half plane. This means that $z = -\kappa$ is in the interior of the curve, whereas $z = 0$ is in its outside. Since Ω is the connected component including $z = 0$, Ω is the outside of the curve and hence

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \pi, r > r_+(\theta)\}.$$

(4) Assume that $a = 1$ and $\kappa \neq 1$. In this case, the equation (8) reduces to $r^2 + b(\theta)r = 0$, whose solutions are $r(\theta) = 0, -b(\theta)$. Recall that $I_0 = (0, \min(\pi, \frac{\pi}{\kappa}))$. By Lemmas 1.8 and 1.5, we see that if $0 < \kappa < 1$ then $b(\theta) > 0$ and if $\kappa > 1$ then $b(\theta) < 0$ for any $\theta \in I_0$. This means that if $0 < \kappa < 1$ then the equation (8) does not have a positive solution, and hence we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \pi, r > 0\} = \mathcal{D}(f_{\kappa, \gamma}),$$

which shows the assertion (4)-(a). If $\kappa > 1$, then since $b'(\theta) < 0$ for $\theta \in I_0$ and since $b(0) = 1 + \frac{1}{\kappa} - 2 < 0$ and $\lim_{\theta \rightarrow \theta_0 - 0} b(\theta) = -\infty$, the function $D(\theta)$ is monotonic increasing on I_0 and hence we have

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty.$$

Thus, the curve $r_+(\theta)$, $\theta \in I_0$ has an asymptotic line with gradient $\tan \theta_0$, which is determined later. Thus, we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0 \text{ and } r > r_+(\theta)\}.$$

(5) Suppose that $a > 1$. In this case, we have $b(0) = 1 + \frac{1}{\kappa} - 2a$ and

$$b\left(\frac{\pi}{\kappa}\right) = (1 - 2a) \cos \frac{\pi}{\kappa} \quad (\text{if } \kappa > 1), \quad b(\pi) = 2a - 1 > 0 \quad (0 < \kappa < 1).$$

Note that $b(0) > 0$ if $\kappa > 1$. Since $r_+(\theta) \cdot r_-(\theta) = \frac{a-1}{a} > 0$, two solutions $r_{\pm}(\theta)$ of (8) have the same signature if $r_{\pm}(\theta)$ are real.

(a) We first consider the case $\kappa > 1$ and $D(0) \geq 0$. Let us show that $D(\theta) > 0$ for $\theta \in I_0$. Set

$$K(x) := \frac{x}{4} \left(1 + \frac{1}{x}\right)^2, \quad G(x) = \frac{2x^2 + 3x + 1}{6x} \quad (x > 0).$$

Then, the condition $D(0) \geq 0$ is equivalent to $a \leq K(\kappa)$, and hence we have $a \leq G(\kappa)$ because

$$G(x) - K(x) = \frac{x^2 - 1}{12x} > 0 \quad \text{if } x > 1.$$

By the assumption $\kappa > 1$, we see that $b'(\theta) < 0$ for any $\theta \in I_0$ by Lemma 6.2 so that b is monotonic decreasing in this interval. Since $b(0) < 0$, the function b is negative in I_0 , and hence $D'(\theta) = 2b(\theta)b'(\theta) > 0$ so that $D(\theta)$ is monotonic increasing in the interval I_0 , and in particular, it is positive on I_0 . By (9), we have $r'_+(\theta) > 0$, and hence the function $r_+(\theta)$ is monotonic increasing, whereas $r_-(\theta)$ is monotonic decreasing because $r_-(\theta) = \frac{a-1}{ar_+(\theta)}$. Note that

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty, \quad \lim_{\theta \rightarrow \theta_0 - 0} r_-(\theta) = 0.$$

Thus, $r_+(\theta)$, $\theta \in I_0$ draws an bounded curve connecting $z = \alpha_2$ to ∞ , and $r_-(\theta)$, $\theta \in I_0$ draws a bounded curve connecting $z = \alpha_1$ to $z = -\kappa$. Since we have $-\kappa < -\frac{1}{\gamma} < \alpha_1 < \alpha_2 < 0$ and since Ω is the connected component including $z = 0$, we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, r > r_+(\theta)\}.$$

(b) Next, we assume that $\kappa > 1$ and $D(0) < 0$. According to Lemma 1.9 (1), we consider the function $b'(\theta)$ in two cases, that is, (i) $a \leq G(\kappa)$ and (ii) $a > G(\kappa)$.

(i) Assume that $a \leq G(\kappa)$. Then, $b(\theta)$ is monotonic decreasing. Since $b(0) < 0$, we see that $b(\theta) < 0$ for any $\theta \in I_0$ and therefore $D'(\theta) = 2b(\theta)b'(\theta) > 0$ for any $\theta \in I_0$. Thus, $D(\theta)$ is monotonic increasing with $D(0) < 0$. In particular, there exists a unique θ_* such that $D(\theta_*) = 0$, and $r_{\pm}(\theta)$ are real for $\theta \in (\theta_*, \theta_0)$. In this interval, since $r_+(\theta) + r_-(\theta) = -b(\theta)/a > 0$, we see that $r_{\pm}(\theta)$ are both positive. By (9), we have $r'_+(\theta) > 0$ and thus the function $r_+(\theta)$ is monotonic increasing, whereas $r_-(\theta)$ is monotonic decreasing because $r_-(\theta) = \frac{a-1}{ar_+(\theta)}$. By taking a limit $\theta \rightarrow \theta_0 - 0$, we have

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty, \quad \lim_{\theta \rightarrow \theta_0 - 0} r_-(\theta) = 0.$$

This means that $r_+(\theta)$ draws an unbounded curve connecting $z = \alpha_1$ and $z = \infty$, and $r_-(\theta)$ draws a bounded curve connecting $z = \alpha_1$ and $z = -\kappa$, where α_1 is the complex solution of (10) with positive imaginary part. Since we have $-\kappa < -\frac{1}{\gamma} < 0$, the domain Ω is given as

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, \text{ if } |\theta| \geq \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}.$$

(ii) Assume that $a > G(\kappa)$. Then, there are two possibilities on $b(\varphi_*)$. If $b(\varphi_*) \leq 0$, then we have

θ	0	\cdots	φ_*	\cdots	$\frac{\pi}{\kappa}$
$b'(\theta)$		+	0	-	
$b(\theta)$	-	-		-	
$D'(\theta)$		-	0	+	
$D(\theta)$	$D(0) < 0$	\searrow		$\nearrow^{+\infty}$	\times

and if $b(\varphi_*) > 0$, then there exist exactly two $\varphi_1 < \varphi_2$ such that $b(\varphi_i) = 0$ so that

θ	0	\cdots	φ_1	\cdots	φ_*	\cdots	φ_2	\cdots	$\frac{\pi}{\kappa}$
$b'(\theta)$			+		0		-		
$b(\theta)$	-	-	0		+		0	-	
$D'(\theta)$		-	0	+	0	-	0	+	
$D(\theta)$	$D(0) < 0$	\searrow		\nearrow		\searrow		$\nearrow^{+\infty}$	\times

We note that $D(\varphi_i) = b(\varphi_i)^2 - 4a(a-1) < 0$. Since $r_+(\theta) + r_-(\theta) = -b(\theta)/a$, if $D(\varphi_*) > 0$, then $r_{\pm}(\theta)$ are both negative so that we do not deal with this case. Thus, in both cases, there exists a unique $\theta_* \in I_0$ such that $D(\theta_*) = 0$ and $r_{\pm}(\theta) > 0$ for any $\theta \in (\theta_*, \theta_0)$. By (9), we see that $r_+(\theta)$ is monotonic increasing on (θ_*, θ_0) , whereas $r_-(\theta)$ is monotonic decreasing. Moreover, we have

$$\lim_{\theta \rightarrow \theta_0 - 0} r_+(\theta) = +\infty, \quad \lim_{\theta \rightarrow \theta_0 - 0} r_-(\theta) = 0,$$

and hence the curves $r_{\pm}(\theta)$, $\theta \in (\theta_*, \theta_0)$ form an unbounded curve connecting $z = \alpha_1$ and $z = \infty$. Since $-\kappa < -\frac{1}{\gamma} < 0$, we have

$$\Omega = \{z \in \mathcal{D}(f_{\kappa, \gamma}); |\theta| < \theta_0, \text{ and if } |\theta| \geq \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}.$$

(c) We finally assume that $0 < \kappa < 1$. In this case, we have $D(\pi) = (2a-1)^2 - 4a(a-1) = 1$. We note that $b(0) < 0$ implies $D(0) < 0$. In fact, $b(0) < 0$ means $1 + \frac{1}{\kappa} < 2a$ so that

$$D(0) = \left(1 + \frac{1}{\kappa}\right)^2 - \frac{4a}{\kappa} < 2a\left(1 + \frac{1}{\kappa}\right) - \frac{4a}{\kappa} = 2a\left(1 - \frac{1}{\kappa}\right) < 0. \quad (21)$$

Since $a > 1$, the signatures of $r_{\pm}(\theta)$ are the same, and they are the opposite to the signature of $b(\theta)$.

Let $I_D \subset I_0$ be the maximal interval such that D is positive on I_D . We shall show that there are no suitable solutions of (8), that is, $r_{\pm}(\theta) < 0$ for any $\theta \in I_D$, which yields that $\Omega = \mathcal{D}(f_{\kappa, \gamma})$. Let us recall Lemma 1.9.

(i) Assume that $0 < \kappa < \frac{1}{2}$ and $a > G(\kappa)$. In this case, we have

θ	0	\cdots	φ_*	\cdots	π
$b'(\theta)$		+	0	-	
$b(\theta)$	$b(0)$	\nearrow		\searrow	$b(\pi) > 0$

If $b(0) \geq 0$, then we see that $b(\theta) > 0$ for any $\theta \in I_0$, which implies $r_{\pm}(\theta) < 0$ for any $\theta \in I_D$. If $b(0) < 0$, then there exists a unique $0 < \varphi < \varphi_*$ such that $b(\varphi) = 0$, and we have $D(0) < 0$ by (21). Hence,

θ	0	\cdots	φ	\cdots	φ_*	\cdots	π
$b'(\theta)$		+	+	+	0	-	
$b(\theta)$	-	\nearrow	0	\nearrow	+	\searrow	+
$D'(\theta)$		-	0	+	0	-	
$D(\theta)$	$D(0) < 0$	\searrow	-	\nearrow	+	\searrow	1

This table yields that b is negative on I_D , whence $r_{\pm}(\theta) < 0$ for any $\theta \in I_D$.

(ii) Assume that $0 < \kappa < \frac{1}{2}$ and $a \leq G(\kappa)$. Then, Lemma 1.8 tells us that $b'(\theta) < 0$ for any $\theta \in I_0$. Thus, $b(\theta)$ is monotonic decreasing on the interval I_0 with $b(\pi) > 0$, and hence $b(\theta) > 0$ for any $\theta \in I_0$. This means that $D'(\theta) < 0$ and $D(\theta)$ is monotonic decreasing on I_0 . Since $D(\pi) = 1$, we see that $I_D = I_0$ and hence $r_{\pm}(\theta) < 0$ for any $\theta \in I_D$.

(iii) Assume that $\frac{1}{2} \leq \kappa < 1$. In this case, we have we have $b'(\theta) > 0$ for any $\theta \in I_0$ so that $b(\theta)$ is monotonic increasing. In fact, if $\kappa > \frac{1}{2}$, then since $\frac{(1+\sqrt{2})^2}{6} < G(\kappa) < 1$ for $\frac{1}{2} < \kappa < 1$ and since $a > 1$, we always have $a > G(\kappa)$ so that $b'(\theta) > 0$ by Lemma 1.8, and if $\kappa = \frac{1}{2}$ then we have $b'(\theta)2(a-1)\sin\theta$ so that $b'(\theta) > 0$ by (14). If $b(0) \geq 0$, then we have $b(\theta) \geq 0$ for any $\theta \in \theta$ and hence we see that $r_{\pm}(\theta) < 0$ for $\theta \in I_D$. If $b(0) < 0$ then there exists a unique φ such that $b(\varphi) = 0$ and we have

θ	0	...	φ	...	π
$b'(\theta)$		+	+	+	
$b(\theta)$	-	\nearrow	0	\nearrow	+
$D'(\theta)$		-	0	+	
$D(\theta)$	-	\searrow	-	\nearrow	1

This table indicates that for $\theta \in I_D$ we have $b(\theta) > 0$, which implies $r_{\pm}(\theta) < 0$.

We shall determine an asymptotic line with respect to Ω when $r_+(\theta) \rightarrow +\infty$ as $\theta \rightarrow \frac{\pi}{\kappa}$ or $\frac{\pi}{\kappa+1}$. To calculate them in a one scheme, we set $\vartheta = \frac{\pi}{\kappa}$ or $\frac{\pi}{\kappa+1}$, and denote its denominator by α . A line having gradient $\tan \vartheta$ can be written as $x \sin \vartheta - y \cos \vartheta = A$ with some constant A . Since $x = \kappa(r(\theta) \cos \theta - 1)$ and $y = \kappa r(\theta) \sin \theta$, we have

$$\begin{aligned} x \sin \vartheta - y \cos \vartheta &= \kappa \{ \sin \vartheta (r(\theta) \cos \theta - 1) - \cos \vartheta r(\theta) \sin \theta \} \\ &= \kappa \{ r(\theta) (\cos \theta \sin \vartheta - \sin \theta \cos \vartheta) - \sin \vartheta \} \\ &= -\kappa \{ r(\theta) \sin(\theta - \vartheta) + \sin \vartheta \}. \end{aligned}$$

When $\theta \rightarrow \vartheta$, we have $r_+(\theta) \rightarrow +\infty$ and $b(\theta) \rightarrow -\infty$, and

$$r_+(\theta) = \frac{-b(\theta) + \sqrt{b(\theta)^2 - 4a(a-1)}}{2a} = -\frac{b(\theta)}{2a} \left(1 + \sqrt{1 - \frac{4a(a-1)}{b(\theta)^2}} \right).$$

Since

$$\lim_{\theta \rightarrow \vartheta-0} \frac{\sin(\theta - \vartheta)}{\sin(\alpha\vartheta)} = \lim_{\theta \rightarrow \vartheta-0} -\frac{\sin(\theta - \vartheta)}{\sin(\alpha(\theta - \vartheta))} = \lim_{\theta \rightarrow \vartheta-0} -\frac{\theta - \vartheta}{\alpha(\theta - \vartheta)} = -\frac{1}{\alpha},$$

we have

$$\lim_{\theta \rightarrow \vartheta-0} b(\theta) \sin(\theta - \vartheta) = \lim_{\theta \rightarrow \vartheta-0} \left((1 - 2a) \cos \theta \sin(\theta - \vartheta) + \sin \theta \cos(\alpha\theta) \frac{\sin(\theta - \vartheta)}{\sin(\alpha\theta)} \right) = \frac{\sin \vartheta}{\alpha\alpha},$$

whence

$$\lim_{\theta \rightarrow \vartheta-0} r(\theta) \sin(\theta - \vartheta) = \lim_{\theta \rightarrow \vartheta-0} -\frac{b(\theta) \sin(\theta - \vartheta)}{2a} \left(1 + \sqrt{1 - \frac{4a(a-1)}{b(\theta)^2}} \right) = -\frac{\sin \vartheta}{\alpha\alpha}.$$

Thus, we have

$$A = \begin{cases} -\kappa \left(-\frac{\sin \theta_0}{\alpha\kappa} + \sin \theta_0 \right) = \left(\frac{1}{\alpha} - \kappa \right) \sin \theta_0 & (\text{if } \alpha = \kappa), \\ -\kappa \left(-\frac{\sin \theta_1}{\kappa+1} + \sin \theta_1 \right) = -\frac{\kappa^2}{\kappa+1} \sin \theta_1 & (\text{if } \alpha = \kappa + 1), \end{cases}$$

and therefore the proof is now complete. \square

3. THE DOMAIN Ω FOR $\kappa = +\infty$

In this section, we deal with the case $\kappa = \infty$. Since $\kappa\theta(x, y)$ is regarded as y in this case, the equation (4) can be written as

$$F(x, y) = x + \gamma x^2 + \gamma y^2 + y \cot y = 0. \quad (22)$$

Assume that $\gamma = 0$, that is, the case of the classical Lambert function. Then, the above equation reduces to $F(x, y) = x + y \cot y = 0$. Since the point $z = x + yi = 0$ satisfies $F(0, 0) = 1 > 0$, we have $\Omega = \{z = x + yi \in \mathbb{C}; x > -y \cot y, |y| < \pi\}$.

Assume that $\gamma \neq 0$. Then, the equation (22) can be calculated as

$$x^2 + \frac{x}{\gamma} + y^2 + \frac{y \cot y}{\gamma} = 0 \iff \left(x + \frac{1}{2\gamma} \right)^2 = \frac{1}{4\gamma^2} - y^2 - \frac{y \cot y}{\gamma}. \quad (23)$$

Let us consider the function

$$\begin{aligned} h(y) &:= \frac{1}{4\gamma^2} - y^2 - \frac{y \cot y}{\gamma} = \frac{1}{4\gamma^2} - \left(y + \frac{\cot y}{2\gamma}\right)^2 + \frac{\cot^2 y}{4\gamma^2} = \frac{1}{4\gamma^2 \sin^2 y} - \left(y + \frac{\cot y}{2\gamma}\right)^2 \\ &= \frac{1 - (2\gamma y \sin y + \cos y)^2}{4\gamma^2 \sin^2 y}. \end{aligned}$$

Note that

$$h(0) = \lim_{y \rightarrow 0} h(y) = \frac{1}{4\gamma^2} - \frac{1}{\gamma} \lim_{y \rightarrow 0} \frac{y}{\sin y} = \frac{1-4\gamma}{4\gamma^2} \quad \text{and} \quad \lim_{y \rightarrow \pi-0} |h(y)| = +\infty.$$

In order that the equation (22) has a real solution in x and y , the function $h(y)$ needs to be non-negative, and it is equivalent to the condition that the absolute value of the function $g(y) := \cos y + 2\gamma y \sin y$ is less than or equal to 1. At first, we observe that $g(0) = 1$ and $g(\pi) = -1$, and its derivative is

$$g'(y) = -\sin y + 2\gamma(\sin y + y \cos y) = -(1-2\gamma)\sin y + 2\gamma y \cos y = (2\gamma-1)\left(\frac{2\gamma}{2\gamma-1}y + \tan y\right) \cos y.$$

Set $c_\gamma = \frac{2\gamma}{2\gamma-1}$. Then, the signature of g' can be determined by the signatures of $2\gamma-1$, $\cos y$ and $c_\gamma y + \tan y$.

Assume that $\gamma > \frac{1}{4}$. If $\gamma > \frac{1}{2}$, then we have $c_\gamma > 1$ and hence there exists a unique $y_* \in (\frac{\pi}{2}, \pi)$ such that $c_\gamma y + \tan y = 0$ and we have

y	0	\cdots	$\frac{\pi}{2}$	\cdots	y_*	\cdots	π
g'		+		+	0	-	
g	1	\nearrow		\nearrow		\searrow	-1

If $\frac{1}{4} < \gamma < \frac{1}{2}$, then we have $c_\gamma < -1$ and hence there exists a unique $y_* \in (0, \frac{\pi}{2})$ such that $c_\gamma y_* + \tan y_* = 0$ and we have

y	0	\cdots	y_*	\cdots	$\frac{\pi}{2}$	\cdots	π
g'		+		-		-	
g	1	\nearrow		\searrow		\searrow	-1

If $\gamma = \frac{1}{2}$, then we have $g'(y) = y \cos y$ so that g is monotonic decreasing on the interval $(\frac{\pi}{2}, \pi)$ with $g(\frac{\pi}{2}) > 1$, and in this case we set $y_* = \frac{\pi}{2}$. These observation shows that, if $\gamma > \frac{1}{4}$, then there exists one and only one $y_0 \in (y_*, \pi)$ such that $g(y_0) = 1$ and $g(y_0 - \varepsilon) > 1$ for $\varepsilon \in (0, y_0 - y_*)$. In this case, $h(y)$ is non-negative in the interval $[y_0, \pi)$, and $h(y_0 - \varepsilon) < 0$ for $\varepsilon \in (0, y_0 - y_*)$. Let $x_i(y)$, $i = 1, 2$ be the real solutions of the equation (23) with $x_1(y) \leq x_2(y)$. Then, since we have $x_1(y_0) = x_2(y_0)$ and

$$\lim_{y \rightarrow \pi-0} x_1(y) = -\infty, \quad \lim_{y \rightarrow \pi-0} x_2(y) = +\infty,$$

the curves $x_i(y)$, $y \in (y_0, \pi)$ form a connected curve, and hence we have

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and if } |y| \geq y_0 \text{ then } x < x_1(y) \text{ or } x > x_2(y)\}.$$

Assume that $0 < \gamma \leq \frac{1}{4}$. Then, we have $-1 \leq c_\gamma < 1$ and hence there are no $y \in (0, \pi)$ such that $c_\gamma y + \tan y = 0$. Thus, we obtain $g'(y) < 0$ for any $y \in (0, \pi)$ so that g is monotonic decreasing from $g(0) = 1$ to $g(\pi) = -1$. This shows that $h(y)$ is non-negative in the interval $(0, \pi)$. Let $x_i(y)$, $i = 1, 2$ be the solutions of the equation (23) with $x_1(y) \leq x_2(y)$. Then, since we have

$$\lim_{y \rightarrow \pi-0} x_1(y) = -\infty, \quad \lim_{y \rightarrow \pi-0} x_2(y) = +\infty$$

and $x_1(0) < x_2(0) < 0$, we have

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and } x > x_2(y)\}.$$

Assume that $\gamma < 0$. Then, there exists a unique $y_* \in (\frac{\pi}{2}, \pi)$ such that $c_\gamma y_* + \tan y_* = 0$, and we have

y	0	\cdots	$\frac{\pi}{2}$	\cdots	y_*	\cdots	π
g'		-		-	0	+	
g	1	\searrow		\searrow		\nearrow	-1

This observation shows that there exists one and only one $y_0 \in (0, y_*)$ such that $g(y_0) = -1$ and $g(y_0 + \varepsilon) < -1$ for $\varepsilon \in (0, y_* - y_0)$. Thus, $h(y)$ is non-negative on $y \in [0, y_0]$, and $h(y_0 + \varepsilon) < 0$ for $\varepsilon \in (0, y_* - y_0)$. Let $x_i(y)$, $i = 1, 2$ be the solutions of the equation (23) with $x_1(y) \leq x_2(y)$. Then,

since $x_1(0) < 0 < x_2(0)$ and $x_1(y_0) = x_2(y_0)$, these two paths $(x_{\pm}(y), y)$ form a continuous curve connecting $x_+(0)$ and $x_-(0)$ and we have

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < y_0 \text{ and } x_1(y) < x < x_2(y)\}.$$

We summarize these calculations as a proposition.

Proposition 3.1. *Assume that $\kappa = +\infty$ and let $x_i(y)$, $i = 1, 2$ be solutions of (23) with $x_1(y) \leq x_2(y)$ if they are real.*

- (1) *If $\gamma = 0$, then one has $\Omega = \{z = x + yi; |y| < \pi \text{ and } x > -y \cot y\}$.*
- (2) *Suppose that $\gamma > \frac{1}{4}$. In this case, there exists a unique $y_0 \in (0, \pi)$ such that $x_1(y_0) = x_2(y_0)$ and if $y \geq y_0$ then $x_i(y)$, $i = 1, 2$ are real. Moreover, one has*

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and if } |y| \geq y_0 \text{ then } x < x_1(y) \text{ or } x > x_2(y)\}.$$

- (3) *If $0 < \gamma \leq \frac{1}{4}$, then $x_i(y)$, $i = 1, 2$ are both real for any $y \in (0, \pi)$, and one has*

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < \pi \text{ and } x > x_2(y)\}.$$

- (4) *Suppose that $\gamma < 0$. Then, there exists a unique $y_0 \in (0, \pi)$ such that $x_1(y_0) = x_2(y_0)$ and if $y \leq y_0$ then $x_i(y)$, $i = 1, 2$ are real for (23). Moreover, one has*

$$\Omega = \{z = x + yi \in \mathbb{C}; |y| < y_0 \text{ and } x_1(y) < x < x_2(y)\}.$$

In particular, Ω is bounded.

4. PROOF OF THEOREM 1.2

In this section, we shall show Theorem 1.2 for $\kappa > 0$ or $\kappa = \infty$. To do so, we consider the domain $D := \Omega \cap \mathbb{C}^+$, and show that $f_{\kappa, \gamma}$ maps D to \mathbb{C}^+ bijectively.

The key tool is the argument principle (see [1, Theorem 18, p.152], for example).

Theorem 4.1 (The argument principle). *If $f(z)$ is meromorphic in a domain Ω with the zeros a_j and the poles b_k , then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles. Here, $n(\gamma, a)$ is the winding number of γ with respect to a .

We also use the following elementary property of holomorphic functions.

Lemma 4.2. *Let $f(z) = u(x, y) + iv(x, y)$ be a holomorphic function. The implicit function $v(x, y) = 0$ has an intersection point at $z = x + yi$ only if $f'(z) = 0$.*

Proof. Let $p(t) = (x(t), y(t))$ be a continuous path in $\mathbb{C} \cong \mathbb{R}^2$ satisfying $v(p(t)) = 0$ for all $t \in [0, 1]$. We assume that $(x'(t), y'(t)) \neq (0, 0)$. Set

$$g(t) := u(p(t)) = u(x(t), y(t)), \quad h(t) := v(p(t)) = v(x(t), y(t)).$$

Obviously, we have $h'(t) \equiv 0$ for any t , and

$$h'(t) = v_x x'(t) + v_y y'(t) = (v_x, v_y) \cdot (x'(t), y'(t)).$$

Assume that $g'(t_0) = 0$ for some point $t_0 \in [0, 1]$. Then

$$\begin{aligned} g'(t) &= u_x x'(t) + u_y y'(t) = (u_x, u_y) \cdot (x'(t), y'(t)) \\ &= (v_x, v_y) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot (x'(t), y'(t)) = (v_x, v_y) \cdot (-y'(t), x'(t)), \end{aligned}$$

the condition $g'(t_0) = 0$ implies that the vector (v_x, v_y) is orthogonal both to $(x'(t_0), y'(t_0))$ and $(-y'(t_0), x'(t_0))$, which are non-zero vectors and mutually orthogonal. Such vector is only zero vector in \mathbb{R}^2 , that is, $(v_x, v_y) = (0, 0)$, and hence $(u_x, u_y) = (0, 0)$ by Cauchy-Riemann equations. Thus, if $g'(t_0) = 0$ then $p(t_0)$ needs to satisfy $f'(p(t_0)) = 0$. \square

4.1. **The case $\gamma < 0$.** Assume that $\gamma < 0$. By Propositions 2.1, 2.2 and 3.1, it is equivalent to the condition that Ω is bounded. We first consider the set \mathcal{S} (see (5) for definition). Let α_i , $i = 1, 2$ be solutions of (10). Since $\gamma < 0$, these are distinct real numbers. Set $\alpha_1 < \alpha_2$. Then, we have $\Omega \cap \mathbb{R} = (\alpha_1, \alpha_2)$.

Lemma 4.3. *One has $f(\alpha_2) < f(\alpha_1) < 0$ and $\mathcal{S} = (f_{\kappa, \gamma}(\alpha_2), f_{\kappa, \gamma}(\alpha_1))$.*

Proof. Assume that $\kappa < +\infty$. Since $\gamma < 0$, we have by (1)

x	$-\kappa$	\cdots	α_1	\cdots	0	\cdots	$-\frac{1}{\gamma}$	\cdots	α_2	\cdots	
$f'_{\kappa, \gamma}$		$-$	0	$+$	$+$	$+$	\times	$+$	0	$-$	
$f_{\kappa, \gamma}$		\searrow	$f_{\kappa, \gamma}(\alpha_1)$	\nearrow	0	$\nearrow^{+\infty}$	\times	$-\infty$	\nearrow	$f_{\kappa, \gamma}(\alpha_2)$	\searrow

The inequality $f(\alpha_1) < 0$ is obvious by the above table. We shall show $f(\alpha_1) > f(\alpha_2)$. By the fact that $\alpha_1 \alpha_2 = \frac{1}{\gamma}$, we have

$$\frac{f(\alpha_2)}{f(\alpha_1)} = \frac{\alpha_2(1 + \gamma\alpha_1)}{(1 + \gamma\alpha_2)\alpha_1} \cdot \left(\frac{1 + \alpha_2/\kappa}{1 + \alpha_1/\kappa} \right)^\kappa = \frac{\alpha_2 + 1}{\alpha_1 + 1} \cdot \left(\frac{1 + \alpha_2/\kappa}{1 + \alpha_1/\kappa} \right)^\kappa.$$

Since $1 + \gamma\alpha_2 < 0$ and $\alpha_1 < 0$, we have $\alpha_1 + 1 = (1 + \gamma\alpha_2)\alpha_1 > 0$. Moreover, the facts that $1 + \alpha_1/\kappa > 0$ and $\alpha_2 > \alpha_1$ yield that

$$\frac{\alpha_2 + 1}{\alpha_1 + 1} > 1 \quad \text{and} \quad \frac{1 + \alpha_2/\kappa}{1 + \alpha_1/\kappa} > 1,$$

whence we obtain $\frac{f(\alpha_2)}{f(\alpha_1)} > 1$. Since $f(\alpha_2) < 0$ because $\alpha_2 > -\frac{1}{\gamma}$ and $\gamma < 0$, we conclude that $0 > f(\alpha_1) > f(\alpha_2)$.

Assume that $\kappa = +\infty$. Since $\gamma < 0$ and $\gamma(-\frac{1}{\gamma})^2 + (-\frac{1}{\gamma}) + 1 = 1 > 0$, we have the following variation table of $f(x)$:

x	$-\infty$	\cdots	α_1	\cdots	0	\cdots	$-\frac{1}{\gamma}$	\cdots	α_2	\cdots	$+\infty$
f'		$-$	0		$+$		\times	$+$	0		$-$
f	0	\searrow	$f(\alpha_1)$	\nearrow	0	\nearrow	\times	\nearrow	$f(\alpha_2)$	\searrow	$-\infty$

Since $\gamma\alpha_i + 1 = -\frac{1}{\alpha_i}$, we see that $f(\alpha_i) = -\alpha_i^2 e^{\alpha_i} < 0$. By $\alpha_1 \alpha_2 = \frac{1}{\gamma}$, we have

$$\frac{f(\alpha_2)}{f(\alpha_1)} = \frac{\alpha_2(1 + \gamma\alpha_1)}{\alpha_1(1 + \gamma\alpha_2)} e^{\alpha_2 - \alpha_1} = \frac{\alpha_2 + 1}{\alpha_1 + 1} e^{\alpha_2 - \alpha_1} > 1,$$

whence $f(\alpha_2) < f(\alpha_1) < 0$. Thus, the proof is now completed. \square

Now we show that $f_{\kappa, \gamma}: D \rightarrow \mathbb{C}^+$ is bijective. Since the proof is completely analogous, we only prove the case $\kappa < +\infty$.

We take a path $C = C(t)$ ($t \in [0, 1]$) in such a way that by starting from $z = -\frac{1}{\gamma}$, it goes to $z = \alpha_2$ along the real axis, next goes to $z = \alpha_1$ along the curve r_{+-} defined by (4) and connecting α_2 and α_1 in the upper half plane, and then it goes to $z = -\frac{1}{\gamma}$ along the real axis (see Figure 1). Here, we can assume that $C'(t) \neq 0$ whenever $C(t) \neq \alpha_i$, $i = 1, 2$. Actually, the curve $v(x, y) = 0$ has a tangent line unless f' vanishes. If we take an arc-length parameter t , then $C'(t)$ represents the direction of the tangent line at $(x, y) = C(t)$. We note that $C(t)$ describes the boundary of D .

We show that $f_{\kappa, \gamma}$ maps the boundary of D to \mathbb{R} bijectively. We take t_i , $i = 1, 2$ as $C(t_i) = \alpha_i$. Note that the sub-curve $C(t)$, $t \in (t_2, t_1)$ describes the curve $r_{+-}(t)$, and $f_{\kappa, \gamma}$ does not have a pole or singular point on $C(t)$, $t \in (t_2, t_1)$. Set $f(z) = u(x, y) + iv(x, y)$. By Lemma 4.2, the implicit function $v(x, y) = 0$ may have an intersection point only if $f'(x + iy) = 0$, i.e. at $x + iy = \alpha_i$ ($i = 1, 2$) or at $x + iy = -\kappa$ if $\kappa > 1$. Then, the function $g(t) = u(C(t))$, $t \in [t_2, t_1]$ attains maximum and minimum in the interval because it is a continuous function on a compact set. Moreover, g' never vanishes in (t_2, t_1) by the above argument and by the fact that $f'(C(t)) \neq 0$ for $t \in (t_2, t_1)$. Therefore, g is monotone and hence it takes maximal and minimal values at the endpoints $t = t_2, t_1$. Now we have $f(\alpha_1) > f(\alpha_2)$ by the last claim so that the image of g is $[f(\alpha_2), f(\alpha_1)]$, and the function g is bijective.

We shall show that for any $w_0 \in \mathbb{C}^+$ there exists one and only one $z_0 \in D$ such that $f(z_0) = w_0$. Let us take an $R > 0$ such that $|w_0| < R$. For $\delta > 0$, let $C' = C_\delta$ be a path obtained from C in such a way that the pole $z = -1/\gamma$ is avoided by a semi-circle $-\frac{1}{\gamma} + \delta e^{i\theta}$, $\theta \in (0, \pi)$ of radius δ (see Figure 3). Denote by D' the domain surrounded by the curve C' .

Then, we can choose $\delta > 0$ such that

$$\left| f\left(-\frac{1}{\gamma} + \delta e^{i\theta}\right) \right| > R \quad (\text{for all } \theta \in (0, \pi)).$$

In fact, if $z = -\frac{1}{\gamma} + \delta e^{i\theta}$, then we have

$$|1 + \gamma z| = |\gamma|\delta, \quad |z| = \left| -\frac{1}{\gamma} + \delta e^{i\theta} \right| > \frac{1}{2|\gamma|} \quad (\text{if } \delta < \frac{1}{2|\gamma|}),$$

and

$$\left| 1 + \frac{z}{\kappa} \right| = \left| 1 - \frac{1}{\kappa\gamma} + \frac{\delta}{\kappa} e^{i\theta} \right| > \frac{\kappa\gamma - 1}{2\kappa\gamma} \quad (\text{if } \delta < \frac{\kappa}{2} \left| 1 - \frac{1}{\kappa\gamma} \right|),$$

so that

$$\left| f\left(-\frac{1}{\gamma} + \delta e^{i\theta}\right) \right| > \frac{1}{2|\gamma|^2} \left(\frac{\kappa\gamma - 1}{2\kappa\gamma} \right)^\kappa \cdot \frac{1}{\delta}.$$

Thus it is enough to take

$$\delta = \min\left(\frac{1}{2|\gamma|^2 R} \left(\frac{\kappa\gamma - 1}{2\kappa\gamma} \right)^\kappa, \frac{1}{2|\gamma|}, \frac{\kappa}{2} \left| 1 - \frac{1}{\kappa\gamma} \right| \right).$$

Since f is non-singular on the semi-circle $-\frac{1}{\gamma} + \delta e^{i\theta}$, $\theta \in [0, \pi]$, the curve $\theta \mapsto f(-\frac{1}{\gamma} + \delta e^{i\theta})$ does not have a singular angular point, so that it is homotopic to a large semicircle (with radius larger than R) in the upper half-plane (see Figure 5).

Note that

$$\operatorname{Im} f(x + yi) = \frac{\left((1 + x/\kappa)^2 + (y/\kappa)^2 \right)^{\frac{\kappa}{2}}}{(1 + \gamma x)^2 + \gamma^2 y^2} \left\{ (x + \gamma x^2 + \gamma y^2) \sin(\kappa\theta(x, y)) + y \cos(\kappa\theta(x, y)) \right\}.$$

By changing variables as in (6), we have

$$\begin{aligned} \operatorname{Im} f(re^{i\theta}) &= \text{positive factor} \times \sin(\kappa\theta) \cdot (ar^2 + b(\theta)r + a - 1) \\ &= \text{positive factor} \times \sin(\kappa\theta) \cdot a(r - r_-(\theta))(r - r_+(\theta)). \end{aligned}$$

Note that the inside of the path C can be written as $\{re^{i\theta}; \theta \in (0, \theta_*)\}$, $r \in (r_-(\theta), r_+(\theta))$ in (r, θ) coordinates. Since $a < 0$ and $\sin(\kappa\theta) > 0$ when $\theta \in (0, \theta_*)$, we see that $\operatorname{Im} f(z) > 0$ if z is inside of the path C . In particular, the inside set of the curve $f(C')$ is a bounded domain in \mathbb{C}^+ including w_0 .

Since the winding number of the path $f(C')$ with respect to $w = w_0$ is exactly one, we see that

$$\frac{1}{2\pi i} \int_{C'} \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi i} \int_{f(C')} \frac{dw}{w - w_0} = 1.$$

By definition of f , we see that $f(z) - w_0$ does not have a pole in D' . Therefore, by the argument principle, the function $f(z) - w_0$ has only one zero point, say $z_0 \in D' \subset D$. Thus, we obtain $f(z_0) = w_0$, and such $z_0 \in D$ is unique. We conclude that the map f is a bijection from D to the upper half-plane \mathbb{C}^+ .

4.2. The case $\gamma \geq 0$. Assume that $\gamma \geq 0$. By Propositions 2.2, 2.1 and 3.1, it is equivalent to the condition that Ω is unbounded. We first consider the behavior of $|f_{\kappa, \gamma}(z)|$ as $\partial\Omega \ni z \rightarrow \infty$. We have

$$|f_{\kappa, \gamma}(z)| = \frac{1}{\left| \gamma + \frac{1}{z} \right|} \cdot \left| 1 + \frac{z}{\kappa} \right|^\kappa.$$

We consider the change variables $1 + \frac{z}{\kappa} = re^{i\theta}$ for $z \in \Omega$. Then, we have

$$\left| \gamma + \frac{1}{z} \right| \leq |\gamma| + \frac{1}{|\kappa|} \cdot \frac{1}{|re^{i\theta} - 1|} \leq |\gamma| + \frac{1}{|\kappa| \cdot |r - 1|},$$

so that

$$|f_{\kappa, \gamma}(z)| \geq \frac{r^\kappa}{|\gamma| + \frac{1}{|\kappa| \cdot |r - 1|}} \rightarrow +\infty \quad (\text{as } r \rightarrow +\infty).$$

Propositions 2.2, 2.1 and 3.1 show that, if $\kappa > 1$ or $\gamma = 0$, then $\partial\Omega \cap \mathbb{C}^+$ can be described as a connected curve $C(t)$, $t \in [0, 1)$ with $C(0) = \alpha_2$ and $\lim_{t \rightarrow 1-0} C(t) = \infty$. In this case, if $z \in \partial\Omega \cap \mathbb{C}^+$, then we have $F(x, y) = 0$ and

$$f_{\kappa, \gamma}(z) = \frac{\left| 1 + \frac{z}{\kappa} \right|^\kappa}{\left| 1 + \gamma z \right|^2} \cdot \frac{-y}{\sin(\kappa\theta(x, y))} < 0. \quad (24)$$

This means that, if $z \in \mathbb{C}^+$ goes to ∞ along the path $\partial\Omega \cap \mathbb{C}^+$, then $f_{\kappa, \gamma}$ must tend to $-\infty$.

We next consider \mathcal{S} for cases $\kappa \geq 1$ or $\gamma = 0$.

Lemma 4.4. *Let α_i , $i = 1, 2$ be the solutions of (8).*

- (1) *Assume that $\kappa > 1$ or $\kappa = \infty$, and also $D(0) \geq 0$. Then, α_i are both real and $\mathcal{S} = (-\infty, f_{\kappa, \gamma}(\alpha_2))$ with $f_{\kappa, \gamma}(\alpha_2) < 0$.*
- (2) *Assume that $\gamma = 0$. Then, one has $\alpha_i = -\frac{\kappa}{\kappa+1}$ if $\kappa < +\infty$ and $\alpha_i = -1$ if $\kappa = \infty$ ($i = 1, 2$). Moreover, one has $\mathcal{S} = (-\infty, f_{\kappa, \gamma}(\alpha_2))$.*
- (3) *Assume that $\kappa = 1$ and $0 < \gamma < 1$. Then, α_i are both real, and one has $\mathcal{S} = (f_{\kappa, \gamma}(\alpha_1), f_{\kappa, \gamma}(\alpha_2))$ with $f_{\kappa, \gamma}(\alpha_2) < 0$.*
- (4) *Assume that $\kappa \geq 1$ or $\kappa = \infty$, and also $D(0) < 0$. Then, α_i are both non-real, and one has $\mathcal{S} = \emptyset$. On the other hand, one has $f_{\kappa, \gamma}(\partial\Omega \cap \mathbb{C}^+) = (-\infty, 0)$.*

Proof. (1) If $\kappa > 1$ and $D(0) \geq 0$, then Proposition 2.2 (5-a) tells us that $\Omega \cup \mathbb{R} = (\alpha_2, +\infty)$. On the other hand, if $\kappa = \infty$ and $D(0) \geq 0$, that is, $\gamma \leq \frac{1}{4}$, then we also have $\Omega \cup \mathbb{R} = (\alpha_2, +\infty)$ by Proposition 3.1 (1) and (3). Since it is easily verified that $f_{\kappa, \gamma}$ is monotonic increasing on this interval for both cases, we have $f_{\kappa, \gamma}(\alpha_2) < 0$ and $\lim_{x \rightarrow +\infty} f_{\kappa, \gamma}(x) = +\infty$. Thus, by definition (5), we obtain

$\mathcal{S} = (-\infty, f_{\kappa, \gamma}(\alpha_2))$ for both cases.

(2) Assume that $\gamma = 0$. Propositions 2.2 (2) and 3.1 (1) show that we have $\Omega \cap \mathbb{R} = (\alpha, +\infty)$ where $\alpha = \alpha_1 = \alpha_2 = -\frac{\kappa}{\kappa+1}$ if $\kappa < +\infty$, and if $\kappa = \infty$ then $\alpha = -1$. Since it is easily verified that $f_{\kappa, \gamma}$ is monotonic increasing on this interval, we have $f_{\kappa, \gamma}(\alpha_2) < 0$ and $\lim_{x \rightarrow +\infty} f_{\kappa, \gamma}(x) = +\infty$. Thus, by definition (5), we obtain $\mathcal{S} = (-\infty, f_{\kappa, \gamma}(\alpha))$.

(3) Assume that $\kappa = 1$ and $0 < \gamma < 1$. By Proposition 2.1, we have $\Omega \cap \mathbb{R} = (-\infty, \alpha_1) \cup (\alpha_2, +\infty)$. An elementary calculation yields that the image of this set by $f_{\kappa, \gamma}$ is $(-\infty, f_{\kappa, \gamma}(\alpha_1)) \cup (f_{\kappa, \gamma}(\alpha_2), +\infty)$ with $f_{\kappa, \gamma}(\alpha_1) < f_{\kappa, \gamma}(\alpha_2) < 0$, and therefore we obtain $\mathcal{S} = (f_{\kappa, \gamma}(\alpha_1), f_{\kappa, \gamma}(\alpha_2))$.

(4) Assume that $\kappa \geq 1$ or $\kappa = \infty$, and also assume that $D(0) < 0$. Then, Propositions 2.2 (5-b) and 3.1 (2) tell us that $\Omega \cap \mathbb{R} = (-\kappa, +\infty)$, where if $\kappa = \infty$ then we regard $-\kappa$ as $-\infty$. Since we have $-\kappa < -\frac{1}{\gamma} < 0$, we have the following increasing/decreasing table of $f_{\kappa, \gamma}$.

x	$-\kappa$	\cdots	$-\frac{1}{\gamma}$	\cdots	0	\cdots
$f'_{\kappa, \gamma}$		$+$	\times	$+$	$+$	$+$
$f_{\kappa, \gamma}$	0	$\nearrow +\infty$	\times	$-\infty \nearrow$	0	$\nearrow +\infty$

Thus, we obtain $\mathcal{S} = \emptyset$. Let $C(t)$, $t \in [0, 1)$ be a path describing $\partial\Omega \cap \mathbb{C}^+$ with $C(0) = \alpha_2$. Assume that $C(t_1) = \alpha_1$. Then, by (24) and by the discussion below of it, we see that $f_{\kappa, \gamma}(\alpha_1) < 0$ and $\lim_{t \rightarrow 1} f_{\kappa, \gamma}(C(t)) = -\infty$. Thus, by Lemma 4.2, we conclude that $f_{\kappa, \gamma}(\partial\Omega \cap \mathbb{C}^+) = (-\infty, 0)$. \square

We note that if $(\kappa, \gamma) = (1, 1)$, then we have $f_{\kappa, \gamma}(z) = z$, and hence we omit this case. Recall that $D = \Omega \cap \mathbb{C}^+$. We shall show that $f_{\kappa, \gamma}$ maps D to \mathbb{C}^+ bijectively. We divide cases according to the above lemma.

4.2.1. The case (1) and (2). Assume that $\kappa > 1$ or $\gamma = 0$. We also assume that $D(0) \geq 0$. Let us take a path $C = C(t)$, $t \in (0, 1)$ in such a way that by starting from $z = \infty$, it goes to $z = \alpha_2$ along the curve r_+ defined by (4) in the upper half plane, and then goes to $z = \infty$ along the real axis (see Figure 2). Here, we can assume that $C'(t) \neq 0$ whenever $C(t) \neq \alpha_i$, $i = 1, 2$. Actually, the curve $v(x, y) = 0$ has a tangent line unless f' vanishes. If we take an arc-length parameter t , then $C'(t)$ represents the direction of the tangent line at $(x, y) = C(t)$. We note that $C(t)$ describes the boundary of D . Lemma shows that $g(t) := f_{\kappa, \gamma}(C(t))$ is a monotonic increasing function on $(0, 1)$ such that $g(t) \rightarrow -\infty$ if $t \rightarrow 0$ and $g(t) \rightarrow +\infty$ if $t \rightarrow 1$.

We shall show that for any $w_0 \in \mathbb{C}^+$ there exists one and only one $z_0 \in D$ such that $f(z_0) = w_0$. Note that we have $\text{Im } f_{\kappa, \gamma}(z) > 0$ for any $z \in D$ by definition. Let us take an $R > 0$ such that $|w_0| < R$. For $L > 0$, let Γ_L be the circle $-\kappa + Le^{i\theta}$ of origin $z = -\kappa$ with radius L . Let $L - \kappa$ and z_L be two distinct intersection points of C and Γ_L . Let $C' := C_L$ be a closed path obtained from C by connecting $L - \kappa$ and z_L via the arc A of Γ_L included in the upper half plane, see Figure 4.

Since f is non-singular on the arc A , the curve $f(A)$ does not have a singular point so that it is homotopic to a large semi-circle (whose radius is larger than R) in the upper half plane (see Figure 6). In particular, the inside set $f(D')$ of the curve $f(C')$ is a bounded domain including $w_0 \in \mathbb{C}^+$. Since the winding number of the path $f(C')$ about $w = w_0$ is exactly one, we see that

$$\frac{1}{2\pi i} \int_{C'} \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi i} \int_{f(C')} \frac{dw}{w - w_0} = 1.$$

We know by definition of f that f does not have a pole on D' . Therefore, by the argument principle, the function $f(z) - w_0$ has the only one zero point, say $z_0 \in D'$. Then, we obtain $f(z_0) = w_0$, and such z_0 is unique. We conclude that the map f is bijection from the interior set D of C to the upper half plane.

4.2.2. *The case (3).* Assume that $\kappa = 1$ and $0 < \gamma < 1$. Then, Ω is given in Proposition 2.1. By a simple calculation, we see that α_i , $i = 1, 2$ are both real such that $\alpha_1 < \alpha_2 < 0$, and we have $\Omega \cap \mathbb{R} = (-\infty, \alpha_1) \cup (\alpha_2, +\infty)$. Moreover,

$$f_{\kappa, \gamma}([\alpha_1, +\infty)) = [f_{\kappa, \gamma}(\alpha_1), +\infty), \quad f_{\kappa, \gamma}((-\infty, \alpha_2]) = (-\infty, f_{\kappa, \gamma}(\alpha_2)],$$

and we have $\mathcal{S} = (f_{\kappa, \gamma}(\alpha_1), f_{\kappa, \gamma}(\alpha_2))$. Let $C = C(t)$, $t \in [0, 1]$ be a path in \mathbb{C}^+ connecting $z = \alpha_1$ to $z = \alpha_2$ along $\partial\Omega \cap \mathbb{C}^+$. Then, since $f'_{\kappa, \gamma}(C'(t)) \neq 0$ for any $t \in (0, 1)$ and since $f_{\kappa, \gamma}(C(t)) \in \mathbb{R}$, Lemma 4.2 tells us that $f_{\kappa, \gamma}(C(t))$ is monotonic increasing on the interval $(0, 1)$, and hence $f_{\kappa, \gamma}(C([0, 1])) = \overline{\mathcal{S}}$. This shows that $f_{\kappa, \gamma}$ maps ∂D to \mathbb{R} bijectively.

We can show the bijectivity of $f_{\kappa, \gamma}$ on D to \mathbb{C}^+ similarly to the case $\kappa > 1$ and $D(0) \geq 0$ by taking Γ_L to be the semi-circle $-1 + Le^{i\theta}$ of origin $z = -1$ with radius L , contained in the upper half plane. Thus, we omit the detail.

4.2.3. *The case (4).* Assume that $\kappa \geq 1$ and $D(0) < 0$. Then, the solutions α_i , $i = 1, 2$ of (10) are both non-real complex numbers by $D(0) < 0$. Let $\text{Im } \alpha_1 > 0$. In this case, we have $-\kappa < -\frac{1}{\gamma} < 0$ and

x		$-\kappa$		\cdots		$-\frac{1}{\gamma}$		\cdots		0		\cdots
$f'_{\kappa, \gamma}$		\times		$+$		\times		$+$		$+$		$+$
$f_{\kappa, \gamma}$		0		$\nearrow^{+\infty}$		\times		$-\infty \nearrow$		0		$\nearrow^{+\infty}$

Note that if $\kappa = \infty$, then we regard $-\kappa$ as $-\infty$. By the above table, we have $\mathcal{S} = \emptyset$, and set $\mathcal{S}' = (-\infty, 0)$. Since we have for $z = x + yi \in \partial\Omega$

$$f_{\kappa, \gamma}(z) = C((x + \gamma x^2 + \gamma y^2) \cos(\kappa\theta(x, y)) - y \sin(\kappa\theta(x, y))) = -C \frac{y}{\sin(\kappa\theta(x, y))},$$

we see that $f_{\kappa, \gamma}(\alpha_1) < 0$. This shows that the image of the function $f_{\kappa, \gamma}(z(\theta))$, $\theta \in (0, \theta_0)$ is $(-\infty, 0)$. Let us adopt a similar argument of the proof of bijectivity of the map $f_{\kappa, \gamma}$ as in unbounded cases. since the winding number of the path of the boundary ∂D is two, we see that the map $f_{\kappa, \gamma}$ maps D to \mathbb{C}^+ in two-to-one, and hence $f_{\kappa, \gamma}$ does not map D to \mathbb{C}^+ bijectively.

4.2.4. Assume that $0 < \kappa < 1$ and $\gamma > 0$. In this case, Proposition 2.2 tells us that if $0 < a = \kappa\gamma < 1$ then $\Omega \cap \mathbb{R} = (\alpha_2, +\infty)$, and if $a > 1$ then $\Omega \cap \mathbb{R} = \left\{ x \in \mathbb{R}; x > -\kappa \text{ and } x \neq -\frac{1}{\gamma} \right\}$. If $f_{\kappa, \gamma}$ maps $D = \Omega \cap \mathbb{C}^+$ to \mathbb{C}^+ bijectively, then it needs map $\partial D = \mathcal{I}$ to \mathbb{R} . However, if $x \in \mathbb{R}$ satisfies $x < \min(\alpha_1, -\kappa)$, then $\exp_{\kappa}(z)$ to $\left| 1 + \frac{x}{\kappa} \right|^{\kappa} e^{i\kappa\pi}$ as $z \rightarrow x$ via the arc $re^{i\theta}$ where $r = \left| 1 + \frac{x}{\kappa} \right|$. Since $0 < \kappa < 1$, then we see that $\lim_{z \rightarrow x} f_{\kappa, \gamma}(z)$ is not real and hence $f_{\kappa, \gamma}$ cannot map D to \mathbb{C}^+ bijectively.

Now we have completed the proof of Theorem 1.2 for the case $\kappa > 0$ or $\kappa = \infty$. \square

5. THE CASE OF $\kappa < 0$

We shall complete the proof of Theorem 1.2 by proving it for the case $\kappa < 0$. To do so, let us recall the homographic (linear fractional) action of $SL(2, \mathbb{R})$ on \mathbb{C} . For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ and $z \in \mathbb{C}^+$, we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

For each $g \in SL(2, \mathbb{R})$, the corresponding homographic action map \mathbb{C}^+ to \mathbb{C}^+ bijectively. Let $\kappa = -\kappa'$ with positive $\kappa' > 0$. Consider the transformation

$$1 + \frac{z'}{\kappa'} = \left(1 + \frac{z}{\kappa} \right)^{-1}.$$

Then, it can be written as

$$z' = \begin{pmatrix} 1 & 0 \\ 1/\kappa & 1 \end{pmatrix} \cdot z = \frac{z}{1 + z/\kappa} \iff z = \begin{pmatrix} 1 & 0 \\ -1/\kappa & 1 \end{pmatrix} \cdot z' = \frac{z'}{1 - z'/\kappa}.$$

Note that since $\begin{pmatrix} 1 & 0 \\ 1/\kappa & 1 \end{pmatrix} \in SL(2, \mathbb{R})$, it maps \mathbb{C}^+ to \mathbb{C}^+ bijectively. Then, since

$$\begin{aligned} \frac{z}{1+\gamma z} &= \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \cdot z = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/\kappa & 1 \end{pmatrix} \cdot z' = \begin{pmatrix} 1 & 0 \\ \gamma - 1/\kappa & 1 \end{pmatrix} \cdot z' \\ &= \frac{z'}{1 + (\gamma - 1/\kappa)z'} = \frac{z'}{1 + (\gamma + 1/\kappa')z'} \end{aligned}$$

and

$$\left(1 + \frac{z}{\kappa}\right)^\kappa = \left(\left(1 + \frac{z}{\kappa}\right)^{-1}\right)^{-\kappa} = \left(1 + \frac{z'}{\kappa'}\right)^{\kappa'}$$

(recall that we are taking the main branch so that $\log z = -\log(z^{-1})$), we obtain

$$f_{\gamma, \kappa}(z) = \frac{z}{1+\gamma z} \left(1 + \frac{z}{\kappa}\right)^\kappa = \frac{z'}{1 + (\gamma + 1/\kappa')z'} \left(1 + \frac{z'}{\kappa'}\right)^{\kappa'} = f_{\gamma+1/\kappa', \kappa'}(z').$$

Set $\gamma' = \gamma + 1/\kappa'$. Since homographic actions map \mathbb{C}^+ to \mathbb{C}^+ bijectively, there exists a domain Ω such that $f_{\kappa, \gamma}$ maps $D = \Omega \cap \mathbb{C}^+$ to \mathbb{C}^+ bijectively if and only if it holds for $f_{\gamma', \kappa'}$. Thus,

$$\gamma' \leq 0 \iff \gamma \leq \frac{1}{\kappa},$$

and $\kappa' > 1$ and $\gamma' > 0$ with $\gamma' \leq \frac{1}{4}(1 + \frac{1}{\kappa'})^2$ is equivalent to

$$\gamma > \frac{1}{\kappa} \quad \text{and} \quad \kappa < -1 \quad \text{with} \quad \gamma - \frac{1}{\kappa} \leq \frac{1}{4}\left(1 - \frac{1}{\kappa}\right)^2 \iff \gamma \leq \frac{1}{4}\left(1 + \frac{1}{\kappa}\right)^2.$$

This shows the case $\kappa < 0$ in Theorem 1.2, and hence we have completed the proof of Theorem 1.2.

REFERENCES

- [1] Ahlfors, L. (1979), *Complex Analysis, An introduction to the theory of analytic functions of one complex variable*. Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.
- [2] Amari, S., Ohara, A. (2011), Geometry of q -exponential family of probability distributions, *Entropy* **13**, no. 6, 1170–1185.
- [3] Borodin, A. (1999), Biorthogonal ensembles, *Nuclear Phys.* **B536**, no. 3, 704732.
- [4] Cheliotis, D. (2018), Triangular random matrices and biorthogonal ensembles, *Statist. Probab. Letter* **134**, 36–44.
- [5] Claeys, T., Romano, S. (2014), Biorthogonal ensembles with two-particle interactions, *Nonlinearity* **27**, no. 10, 2419–2443.
- [6] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., Knuth, D. E. (1996), On the Lambert W function, *Adv. Comput. Math.* **5**, no. 4, 329–359.
- [7] Dykema, K., Haagerup, U. (2004), DT-operator and decomposability of Voiculescu's circular operator, *Amer. J. Math.* **126**, 121–189.
- [8] Nakashima, H., Graczyk, P., Wigner and Wishart ensembles for graphical models, submitted, 30 pages. arXiv:2008.10446
- [9] Zhang, F. D., Ng, H. K. T., Shi, Y. M. (2018), Information geometry on the curved q -exponential family with application to survival data analysis, *Phys. A* **512**, 788–802.

6. APPENDIX

6.1. **Tables.** This subsection contains increasing/decreasing tables of H_α and $b'(\theta)$.

Lemma 6.1. *One has the following increasing/decreasing table of H_α .*

(A) $0 < \alpha \leq \frac{1}{2}$	x	0	\dots	$\frac{2\pi}{\alpha+1}$	\dots	2π	$H_\alpha(2\pi) \geq 0$
	H'_α	0	+	0		+	
	H_α	0	\nearrow	$H_\alpha(\frac{2\pi}{\alpha+1})$	\searrow	$H_\alpha(2\pi)$	
(B) $\frac{1}{2} < \alpha < 1$	x	0	\dots	$\frac{2\pi}{\alpha+1}$	\dots	2π	$H_\alpha(2\pi) < 0$
	H'_α	0	+	0		-	
	H_α	0	\nearrow	$H_\alpha(\frac{2\pi}{\alpha+1})$	\searrow	$H_\alpha(2\pi)$	
(C) $1 < \alpha < 2$	x	0	\dots	$\frac{2\pi}{\alpha+1}$	\dots	$\frac{2\pi}{\alpha}$	$H_\alpha(\frac{2\pi}{\alpha}) > 0$
	H'_α	0	-	0		+	
	H_α	0	\searrow	$H_\alpha(\frac{2\pi}{\alpha+1})$	\nearrow	$H_\alpha(\frac{2\pi}{\alpha})$	
(D) $\alpha \geq 2$	x	0	\dots	$\frac{2\pi}{\alpha+1}$	\dots	$\frac{2\pi}{\alpha}$	$H_\alpha(\frac{2\pi}{\alpha}) \leq 0$
	H'_α	0	-	0		+	
	H_α	0	\searrow	$H_\alpha(\frac{2\pi}{\alpha+1})$	\nearrow	$H_\alpha(\frac{2\pi}{\alpha})$	

Lemma 6.2. *The signature of $b'(\theta)$ on the interval I is given as follows.*

$0 < \kappa < \frac{1}{2}$								
(1)	θ	0	\dots	φ_*	\dots	π	(i) $\ell(\kappa, a) \geq 0$, (ii) $\ell(\kappa, a) < 0$, $-J_\kappa(a) > \frac{1}{\kappa}$	$a > \frac{2\kappa^2+3\kappa+1}{6\kappa}$
	$b'(\theta)$		+	0	-			
(2)	$b'(\theta) < 0$ for $\theta \in I$						(i) $\ell(\kappa, a) < 0$, $-J_\kappa(a) \leq \frac{1}{\kappa}$	$a \leq \frac{2\kappa^2+3\kappa+1}{6\kappa}$
$\kappa = \frac{1}{2}$								
(1)	$b'(\theta) = 0$ for $\theta \in I$						$\ell(\kappa, a) = 0$	$a = 1$
(2)	$b'(\theta) > 0$ for $\theta \in I$						$\ell(\kappa, a) > 0$	$a > 1$
(3)	$b'(\theta) < 0$ for $\theta \in I$						$\ell(\kappa, a) < 0$	$a < 1$
$\frac{1}{2} < \kappa < 1$								
(1)	θ	0	\dots	φ_*	\dots	π	(i) $\ell(\kappa, a) < 0$, (ii) $\ell(\kappa, a) = 0$, (iii) $\ell(\kappa, a) > 0$, $-J_\kappa(a) > \frac{1}{\kappa}$	$a < \frac{2\kappa^2+3\kappa+1}{6\kappa}$
	$b'(\theta)$		-	0	+			
(2)	$b'(\theta) > 0$ for $\theta \in I$						$-J_\kappa(a) \leq \frac{1}{\kappa}$, $\ell(\kappa, a) > 0$	$a \geq \frac{2\kappa^2+3\kappa+1}{6\kappa}$
$\kappa > 1$								
(1)	θ	0	\dots	φ_*	\dots	$\frac{\pi}{\kappa}$	$\ell(\kappa, a) > 0$, $-J_\kappa(a) < \frac{1}{\kappa}$	$a > \frac{2\kappa^2+3\kappa+1}{6\kappa}$
	$b'(\theta)$		+	0	-			
(2)	$b'(\theta) < 0$ for $\theta \in I$						(i) $\ell(\kappa, a) < 0$, (ii) $\ell(\kappa, a) = 0$, (iii) $\ell(\kappa, a) > 0$, $-J_\kappa(a) \geq \frac{1}{\kappa}$	$a \leq \frac{2\kappa^2+3\kappa+1}{6\kappa}$

If $\kappa = 1$, then one has

$$b'(\theta) = 2(a-1) \sin \theta.$$

6.2. **Figures.** This subsection collects figures of graphs of $f_{\kappa,\gamma}(x)$ (for real x), F_κ , and of shapes of Ω with some deformation.

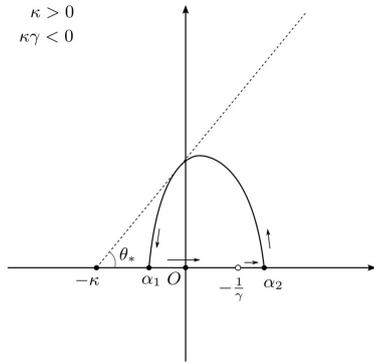


FIGURE 1. The case of (i)

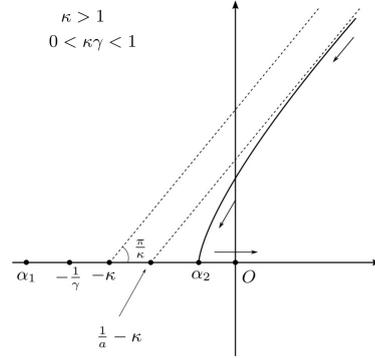


FIGURE 2. The case of (ii), when $\kappa > 2$

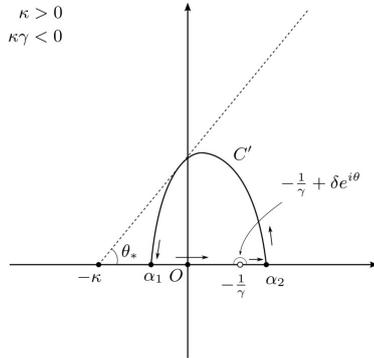


FIGURE 3. Curve C' in case (i)

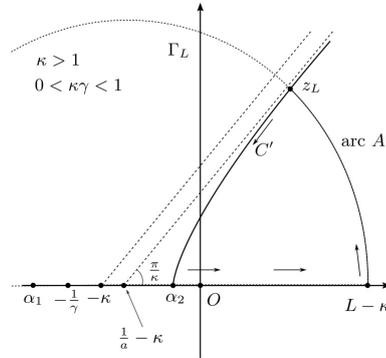


FIGURE 4. Curve C' in case (ii)

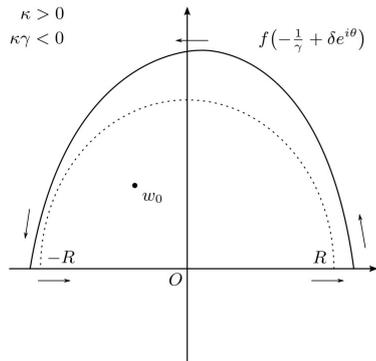


FIGURE 5. Curve $f(C')$ in case (i)

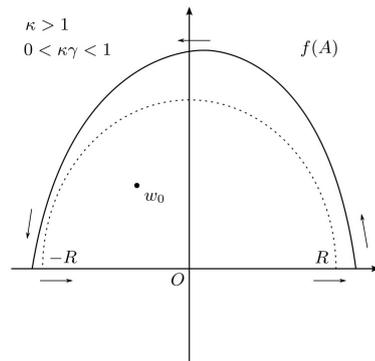


FIGURE 6. Curve $f(C')$ in case (ii)

- Graphs of $f_{\kappa,\gamma}(x)$ for real x .

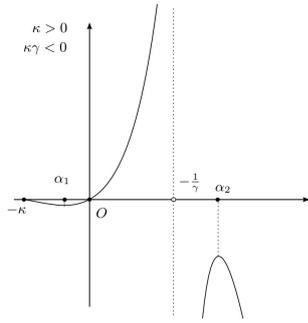


FIGURE 7. $f(x)$ for $x \geq -\kappa$, case (i)

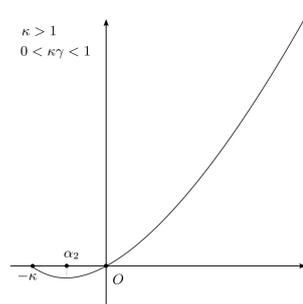
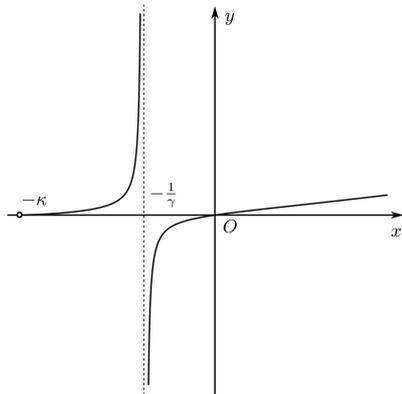
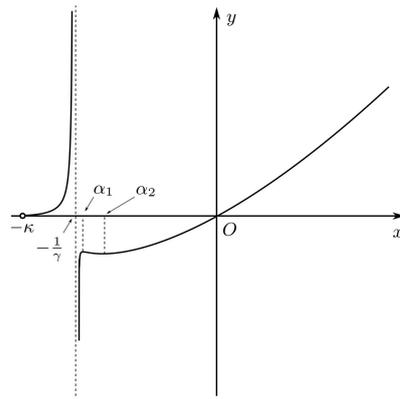
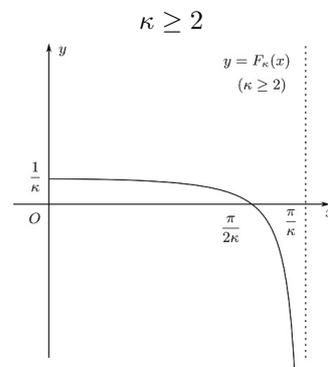
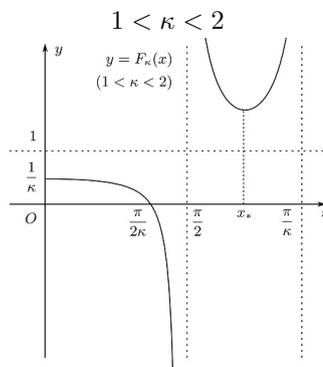
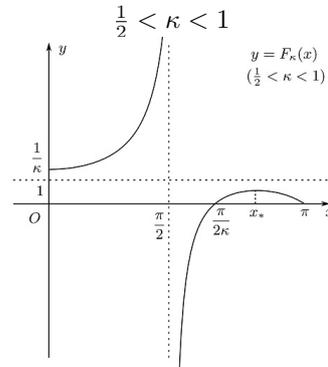
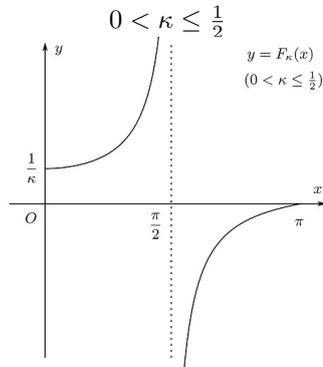
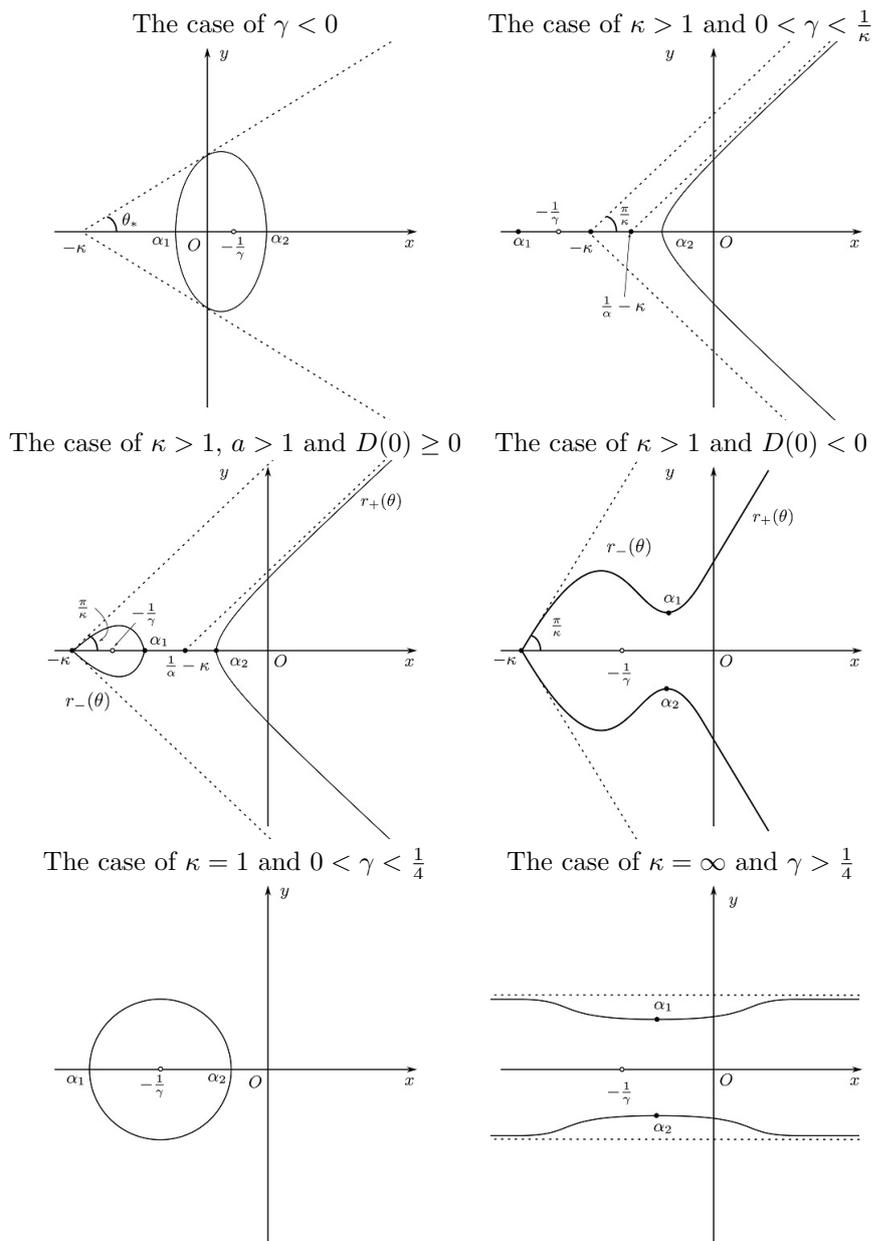


FIGURE 8. $f(x)$ for $x \geq -\kappa$, case (ii)

- Graphs of F_κ



- Shapes of Ω



(Hideto Nakashima) GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA 464-8602, JAPAN

E-mail address: h-nakashima@math.nagoya-u.ac.jp

(Piotr Graczyk) LABORATOIRE DE MATHÉMATIQUES LAREMA, UNIVERSITÉ D'ANGERS 2, BOULEVARD LAVOISIER, 49045 ANGERS CEDEX 01, FRANCE

E-mail address: piotr.graczyk@univ-angers.fr