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Abstract. In 2010, Kerckhoff and Storm discovered a path of hyperbolic 4-polytopes eventually collapsing to an ideal right-angled cuboctahedron. This is expressed by a deformation of the inclusion of a discrete reflection group (a right-angled Coxeter group) in the isometry group of hyperbolic 4-space. More recently, we have shown that the path of polytopes can be extended to Anti-de Sitter geometry so as to have geometric transition on a naturally associated 4-orbifold, via a transitional half-pipe structure.

In this paper, we study the hyperbolic, Anti-de Sitter, and half-pipe character varieties of Kerckhoff and Storm's right-angled Coxeter group near each of the found holonomy representations, including a description of the singularity that appears at the collapse. An essential tool is the study of some rigidity properties of right-angled cusp groups in dimension four.

1. Introduction

In the Seventies, Thurston [Thu79] introduced the notion *degeneration* of (G, X)-structures, later widely studied and used in dimension three [Hod86, Por98, CHK00, HPS01, Por02, BLP05, Ser05, PW07, Por13, Koz13, LMA15a, LMA15b, Koz16]. Typical instances of this phenomenon are paths of hyperbolic cone structures on a 3-manifold eventually collapsing to some lower-dimensional orbifold, whose geometric structure is said to *regenerate* to 3-dimensional hyperbolic structures.

In his thesis [Dan11], Danciger showed that when the limit is 2-dimensional and hyperbolic, it often regenerates to Anti-de Sitter (AdS) structures as well, so as to have *geometric transition* from hyperbolic to AdS structures (see also [Dan13, Dan14, AP15, FS19, Tre19]). To that purpose, he introduced half-pipe (HP) geometry, which is a *limit geometry* [CDW18] of both hyperbolic and AdS geometries inside projective geometry, and encodes the behaviour of such a collapse "at the first order". One can indeed suitably "rescale" the structures inside the "ambient" projective geometry along the direction of collapse, so as to get at the limit a 3-dimensional "transitional" HP structure.

Concerning dimension four, Kerckhoff and Storm [KS10] described a path $t \mapsto \mathcal{P}_t$, $t \in (0, 1]$, of hyperbolic 4-polytopes which collapse as $t \to 0$ to a 3-dimensional ideal right-angled cuboctahedron. This induces a path of incomplete hyperbolic structures on a naturally associated 4-orbifold $\mathcal{O}$. The orbifold fundamental group of $\mathcal{O}$ is a right-angled Coxeter group Γ_{22}, which embeds in $\mathrm{Isom}(\mathbb{H}^4)$ as a discrete reflection group when $t = 1$. In [RS] (see also [Sep]), we found a similar path of AdS 4-polytopes such that the two paths, suitably

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rescaled, can be joined so as to give geometric transition on the orbifold $\mathcal{O}$. In particular, there is a transitional HP orbifold structure on $\mathcal{O}$ joining the two paths.

Keckhoff and Storm's deformation has been studied and used in [MR18] to show, among other things, the first examples of collapse of 4-dimensional hyperbolic cone structures to 3-dimensional ones. Similarly, thanks to the found AdS deformation and HP transitional structure, in [RS] the authors provided the first examples of geometric transition from hyperbolic to AdS cone structures in dimension four.

The goal of this paper is to describe the hyperbolic, AdS, and HP character varieties of the right-angled Coxeter group Γ_{22}, including a study of the behaviour at the collapse. The results are summarised in Theorems 1.1 and 1.2 below.

The three character varieties of Γ_{22}. Let G be $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, or the group G_{HP^4} of transformations of half-pipe geometry, and let $G^+ < G$ be the subgroup of orientation-preserving transformations. We call *character variety* of Γ_{22} the GIT quotient

$$X(\Gamma_{22}, G) = \mathrm{Hom}(\Gamma_{22}, G) /\!\!/ G^+$$

by the action of G^+ by conjugation, with its structure of real algebraic affine set. (In general, the GIT quotient by G is a semialgebraic affine set [RS90].)

The holonomy representations of the geometric structures on the orbifold $\mathcal{O}$ constructed in [KS10, RS] provide a smooth path $t \mapsto [\rho_t^G]$ in $X(\Gamma_{22}, G)$. This path was originally defined in [KS10] when $G = \mathrm{Isom}(\mathbb{H}^4)$ only for $t \in (0, 1]$, and is easily continued analytically also for non-positive times. The Anti-de Sitter path, introduced in [RS], is only defined for $t \in (-1, 1)$ and diverges as $|t| \to 1^-$, while for $G = G_{\mathsf{HP}^4}$ there is a "trivial" path of non-equivalent HP representations (defined for $t \in \mathbb{R}$, and diverging as $|t| \to +\infty$) differing from one another by "stretching" in the ambient real projective space (see below).

The representations obtained at $t = 0$ correspond geometrically to a "collapse" and play a special role in two ways. First, they correspond to a "symmetry" in the character varieties, since the representations ρ_t^G and ρ_{-t}^G are conjugated in G but not in G^+, and therefore are holonomies of geometric structures which admit an orientation-reversing isometry. Second, interpreting $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ and G_{HP^4} as subgroups of $\mathrm{PGL}(5, \mathbb{R})$, the three representations ρ_0^G coincide. They correspond to a representation (we omit the superscript G here)

$$\rho_0 \colon \Gamma_{22} \to \mathrm{Stab}(\mathbb{H}^3) < G ,$$

for a fixed copy of $\mathbb{H}^3$ in $\mathbb{H}^4$, $\mathbb{A}\mathrm{d}\mathbb{S}^4$, or HP^4, respectively. Projecting the image of ρ_0 in $\mathrm{Stab}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3) \times \mathbb{Z}/2\mathbb{Z}$ to $\mathrm{Isom}(\mathbb{H}^3)$ gives the reflection group of an ideal right-angled cuboctahedron (see Figure 5).

We show (see Figure 1):

Theorem 1.1. *Let G be $\mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, or G_{HP^4}. Then $[\rho_0]$ has a neighbourhood $\mathcal{U} = \mathcal{V} \cup \mathcal{H}$ in $X(\Gamma_{22}, G)$ homeomorphic to $\mathcal{S} = \{(x_1^2 + \ldots + x_{12}^2) \cdot x_{13} = 0\} \subset \mathbb{R}^{13}$, so that:*

- *$[\rho_0]$ corresponds to the origin;*
- *$\mathcal{V}$ corresponds to the x_{13}-axis, and consists of the conjugacy classes of the holonomy representations ρ_t^G;*
- *$\mathcal{H}$ corresponds to $\{x_{13} = 0\}$, identified to a neighbourhood of the complete hyperbolic orbifold structure of the ideal right-angled cuboctahedron in its deformation space.*

The group $G/G^+ \cong \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{S}$ by changing sign to the last coordinate x_{13}.

The proof will actually show that, near each $[\rho_t]$, the GIT quotient $X(\Gamma_{22}, G)$ is homeomorphic to the topological quotient $\mathrm{Hom}(\Gamma_{22}, G)/G^+$ (see Remark 4.5). In other words, the

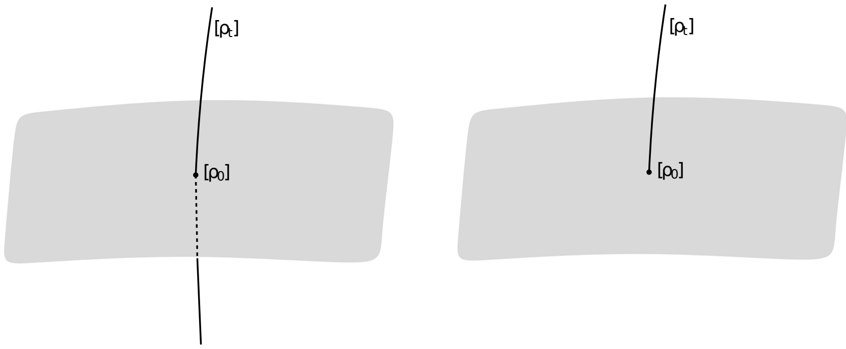

FIGURE 1. On the left, a topological picture of $X(\Gamma_{22}, G)$ near the collapse, which corresponds to the point $[\rho_0]$. The vertical component $\mathcal{V}$ is $\{[\rho_t^G]\}_t$. The horizontal component $\mathcal{H}$ is 12-dimensional and corresponds to the deformations of the complete hyperbolic structure of the ideal right-angled cuboctahedron. On the right, the corresponding neighbourhood in the semialgebraic affine set $\mathrm{Hom}(\Gamma_{22}, G)/\!\!/G$, i.e. in the further quotient of $X(\Gamma_{22}, G)$ by $G/G^+ \cong \mathbb{Z}/2\mathbb{Z}$.

latter is Hausdorff near $[\rho_t]$. We remark however that the language of GIT and schemes is not needed here, the adopted techniques being rather elementary.

The statement of Theorem 1.1 is purely topological, only dealing with the structure of the character variety *up to homeomorphism*. Nevertheless, $\mathcal{U} \subset X(\Gamma, G)$ is an affine algebraic set. Although we decided not to enter much into technical details from this point of view, we provide also some results in this direction.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ be the adjoint representation. The Zariski tangent space of $X(\Gamma, G)$ at $[\rho_0]$ is identified to the cohomology group $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$. Since ρ_0 preserves a totally geodesic copy of $\mathbb{H}^3$, we have a natural decomposition:

$$H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g}) \cong H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{o}(1,3)) \oplus H^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) \ . \tag{1}$$

By Theorem 1.1, the neighbourhood $\mathcal{U}$ of $[\rho_0]$ has two components: a "vertical" curve $\mathcal{V}$ and a "horizontal" 12-dimensional component $\mathcal{H}$. We show:

Theorem 1.2. *The Zariski tangent space of $X(\Gamma_{22}, G)$ at $[\rho_0]$ is 13-dimensional. In the decomposition (1), the first factor is 12-dimensional and tangent to $\mathcal{H}$, while the second factor is 1-dimensional and tangent to $\mathcal{V}$. Integrable vectors are precisely those lying in one of the two factors.*

In order to further discuss our results, we first need to describe the three paths of geometric representations of Theorem 1.1.

The three deformations. For $t = 1$, the hyperbolic polytope $\mathcal{P}_1 \subset \mathbb{H}^4$ is obtained in [KS10] from the ideal right-angled 24-cell by removing two opposite bounding hyperplanes. So $\mathcal{P}_1$ has two "Fuchsian ends", and in particular its volume is infinite. The reflection group

$$\Gamma_{22} < \mathrm{Isom}(\mathbb{H}^4)$$

associated to $\mathcal{P}_1$ is thus a right-angled Coxeter group obtained by removing from the reflection group Γ_{24} of the ideal right-angled 24-cell two generators (reflections at two opposite facets).

As a sort of "reflective hyperbolic Dehn filling", Kerckhoff and Storm show that the inclusion $\Gamma_{22} < \mathrm{Isom}(\mathbb{H}^4)$ is not locally rigid. This is done by moving the bounding hyperplanes of $\mathcal{P}_1$ in such a way that the orthogonality conditions given by the relations of Γ_{22} are maintained, and thus obtaining a path $\rho_t^{\mathbb{H}^4}$ of geometric representations of Γ_{22}.

As t decreases from 1, the combinatorics of $\mathcal{P}_t$ changes a few times, until the volume of $\mathcal{P}_t$ becomes finite. Most of the dihedral angles of $\mathcal{P}_t$ are constantly right, while the varying ones are all equal and tend to π as $t \to 0$, when $\mathcal{P}_t$ collapses to the cuboctahedron. As an abstract group, Γ_{22} can be identified to the orbifold fundamental group of an orbifold $\mathcal{O}$ supported on the complement in $\mathcal{P}_t$ of the ridges with non-constant dihedral angle.

Kerckhoff and Storm show moreover that the space of conjugacy classes of representations $\Gamma_{22} \to \mathrm{Isom}(\mathbb{H}^4)$ deforming the inclusion is a smooth curve outside of the collapse. In other words, the only non-trivial deformation (up to conjugacy) is given by the found holonomies $\rho_t^{\mathbb{H}^4}$.

In [RS], we produced a path of AdS 4-polytopes with the same combinatorics of the hyperbolic polytope $\mathcal{P}_t$ for $t \in (0, \varepsilon)$, such that the same orthogonality conditions between the bounding hyperplanes are satisfied, and again collapsing to an ideal right-angled cuboctahedron in a spacelike hyperplane $\mathbb{H}^3$ of $\mathbb{A}\mathrm{d}\mathbb{S}^4$. Some bounding hyperplanes are spacelike, and some others are timelike. We have in particular a path of AdS orbifold structures on $\mathcal{O}$, with holonomy representation $\rho_t^{\mathbb{A}\mathrm{d}\mathbb{S}^4} \colon \Gamma_{22} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ given by sending each generator to the corresponding AdS reflection.

We moreover find in [RS] a one-parameter family of transitional HP structures on $\mathcal{O}$, with holonomy $\rho_t^{\mathsf{HP}^4}$. To interpret distinct elements in this family, recall that in half-pipe space there is a preferred direction under which the HP metric is degenerate. An HP structure is never rigid, because one can always conjugate with a transformation which "stretches" the degenerate direction, and obtain a new structure equivalent to the initial one *as a real projective structure*, but inequivalent *as a half-pipe structure*. We discover here (Theorem 1.1) that such stretchings are the only possible deformations, so that the found HP structures are essentially unique.

Finally, we remark that the geometric transition described in [RS] induces a continuous deformation connecting in the $\mathrm{PGL}(5, \mathbb{R})$-character variety "half" of the path in $\mathcal{V} \subset X(\Gamma_{22}, \mathrm{Isom}(\mathbb{H}^4))$ (which is exactly the path of hyperbolic representations exhibited by Kerckhoff and Storm, for $t \in (0, 1]$) and "half" of the analogous path in $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$, going through a single half-pipe representations $\rho_{t_0}^{\mathsf{HP}^4}$ with $t_0 \neq 0$ (this value of t_0 can be chosen arbitrarily, up to reparameterising the entire deformation).

About the result. Theorems 1.1 and 1.2 contain several novelties. First, while the smoothness of the $\mathrm{Isom}(\mathbb{H}^4)$-character variety for $t > 0$ was proved in [KS10], the smoothness on the AdS and HP sides is completely new. Second, the study of the character variety at the "collapsed" point $[\rho_0]$ is a new result in all three settings. Some motivations follow.

First of all, we found worthwhile analysing the behaviour of the deformation space of the AdS orbifold $\mathcal{O}$ — equivalently, the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-character variety of Γ_{22} near $[\rho_t]$ — and compare it with the hyperbolic counterpart. In fact, the literature seems to miss a study of deformations of AdS polytopes in this spirit. With respect to hyperbolic geometry, one may expect more flexibility in AdS geometry, but we find that the behaviour on the AdS side is the same as the hyperbolic counterpart. (In regard, see however [BBD+12, Question 9.3] and the related discussion.)

Regarding the collapse, in [KS10, Section 14] Kerckhoff and Storm mention that the family of hyperbolic polytopes $\mathcal{P}_t \subset \mathbb{H}^4$, $t > 0$, is expected to have interesting geometric limits by rescaling in the direction transverse to the collapse. On the other hand, they assert that "providing the details of this geometric construction would require more space than perhaps is merited here".

The work [RS] provides a complete description of such a geometric limit in half-pipe geometry, which, after the work of Danciger, seems the best suited in order to analyse this kind of collapse. One can in fact consider the limiting HP structure as an object which encodes the collapse *at the first order*, essentially keeping track of the derivatives of all the associated geometric quantities. Thus, if [RS] describes the collapse *at level of geometric structures*, Theorems 1.1 and 1.2 (in the hyperbolic and AdS setting) give a precise description of the collapse *at level of the character variety*.

Finally, our study of the HP character variety shows that the only deformations of the found half-pipe orbifold structure are obtained by "stretching" in the degenerate direction. The presence of many commutation relations forces the rigidity of the HP structures.

Together with the hyperbolic and AdS picture, this shows that "nearby" there is no collapsing path of hyperbolic or AdS orbifold structures other than the ones we found (up to reparameterisation). This should be compared with some 3-dimensional examples found by Danciger [Dan13, Section 6], where the transitional HP structure deforms non-trivially to nearby HP structures that regenerate to non-equivalent AdS structures, despite not regenerating to hyperbolic structures.

All in all, Theorems 1.1 and 1.2 exhibit a strong lack of flexibility around this example. Its proof, explained in the next section, suggests that this could be more generally due to dimension issues, confirming the usual feeling that "the rigidity increases with the dimension".

Cusp rigidity in dimension four. Let us give an overview of the ideas behind the proof of Theorem 1.1, focusing first on the hyperbolic and AdS case.

The holonomy representations ρ_t have the property that each generator in the standard presentation of Γ_{22} is sent by ρ_t to a (hyperbolic or AdS) reflection, and this property is preserved by small deformations. As in [KS10], we thus reduce to studying the configurations of hyperplanes of reflections satisfying certain orthogonality conditions. Once this set-up is established, there are two main facts to prove: the smoothness of the character variety outside the collapse, and the description of the collapse itself.

For the first fact, the proof on the AdS side follows the general lines of the proof given in [KS10] for the hyperbolic case. However, different arguments are required for one point of fundamental importance concerning a property of rigidity of cusp representations in dimension four.

In fact, in [KS10] a preliminary lemma is proved, which can be summarised by saying that in dimension 4 "cusp groups stay cusp groups". More precisely, if we consider the orbifold fundamental group of a Euclidean cube Γ_{cube}, this property states that any representation of Γ_{cube} into $\mathrm{Isom}(\mathbb{H}^4)$ sending the six standard generators to reflections in six distinct hyperplanes sharing the same point at infinity (a "cusp group") can only be deformed by preserving these tangencies at infinity. Note that the analogue fact is false in dimension three, where the situation is more flexible.

We do prove the analogous property for Anti-de Sitter geometry in dimension 4 (Proposition 3.10), where a cusp group is defined analogously. There are however remarkable differences due to the different nature of hyperbolic and AdS geometries, for instance a cusp group in $\mathbb{A}\mathrm{d}\mathbb{S}^4$ will be generated by 4 reflections in timelike hyperplanes and 2 reflections in spacelike hyperplanes. The proof of this rigidity property in AdS uses therefore ad hoc arguments and is somehow more surprising than its hyperbolic counterpart, as in general a little more flexibility might be expected for AdS geometry.

Once this fundamental property is established, the proof of the smoothness of the curve is based on a careful analysis of the structure of the group Γ_{22} and the possible deformations of the polytope $\mathcal{P}_t$, relying on the application of the above rigidity property to each peripheral subgroup (there is a cusp group $\Gamma_{\mathrm{cube}} < G$ associated to each ideal vertex of $\mathcal{P}_t$). The methods are rather elementary, although some intricate computation is necessary, and the general strategy is similar to that of the hyperbolic analogue provided in [KS10].

Let us now explain our arguments to analyse the collapse in both the $\mathbb{H}^4$ and $\mathbb{A}\mathrm{d}\mathbb{S}^4$ character variety. The proof is essentially the same for both cases, so let us focus on the hyperbolic case (that is, $G = \mathrm{Isom}(\mathbb{H}^4)$) in this introduction for definiteness.

It is not difficult to describe the two components $\mathcal{V}$ and $\mathcal{H}$ of the neighbourhood $\mathcal{U}$ from a geometric point of view: the "vertical" curve $\mathcal{V}$ consists of the conjugacy classes of the holonomy representations ρ_t^G of $\mathcal{O}$, $t > 0$, plus the natural extension of the path for $t < 0$ given by $r \circ \rho_{-t}^G \circ r$. Here r is the reflection in the totally geodesic copy of $\mathbb{H}^3$ to which the polytope collapses as $t = 0$. On the other hand, the "horizontal" 12-dimensional component $\mathcal{H}$ consists of representations which fix setwise this copy of $\mathbb{H}^3$, and deform the reflection group of the ideal right-angled cuboctahedron in $\mathbb{H}^3$.

One then has to show that there exists a neighborhood of $[\rho_0]$ such that every point in this neighborhood belongs to one of these two components — namely, there are no other conjugacy classes of representations nearby $[\rho_0]$. To prove this, we refine the study of the rigidity properties of the cusps. We introduce a notion of *collapsed cusp group*: a representation of Γ_{cube} defined similarly to cusp groups, but allowing that two generators are sent to reflections in the same hyperplane. The restriction of ρ_0 to each peripheral subgroup is in fact a collapsed cusp group. Then we prove a more general version of the aforementioned rigidity property "cusp groups stay cusp groups", by showing that "collapsed cusp groups either stay collapsed, or deform to cusp groups". More precisely, representations nearby a collapsed cusp group either keep the property that two opposite generators are sent to the same reflection, or they become cusp groups in the usual sense.

By an analysis of the character variety in the spirit of Kerckhoff and Storm, we show that the "vertical" curve $\mathcal{V}$ is smooth also at $t = 0$ if we impose that the tangency conditions at infinity are preserved. Applying the more general property of rigidity which includes the "collapsed" case is then the fundamental step to conclude the proof.

Concerning the proof for the half-pipe case, it follows a similar line, but many steps are dramatically simpler. The key point is again a rigidity property for four-dimensional (collapsed) cusp groups, which is showed rather easily by using the isomorphism between G_{HP^4} and the group of isometries of Minkowski space $\mathbb{R}^{1,3}$, which is a semidirect product $\mathrm{O}(1,3) \ltimes \mathbb{R}^{1,3}$. The proof then parallels the steps for the hyperbolic and AdS case, except that the smoothness of the vertical component $\mathcal{V}$ is granted by the fact that — thanks to this semidirect product structure of G_{HP^4} — $\mathcal{V}$ identifies with the first cohomology vector space $H^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$. The proof that this vector space is 1-dimensional (see (2) below) requires a certain amount of technicality and relies on a precise study of the group-theoretical structure of Γ_{22}.

The Zariski tangent space and the first cohomology group. Such cohomological analysis finds another application, besides the proof of the half-pipe version of Theorem 1.1, in the study of the Zariski tangent space. Let us briefly explain how it is involved in the proof of Theorem 1.2.

As mentioned above, we prove (in Proposition 6.5) that the second factor in the decomposition (1) has dimension 1, namely

$$H^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) \cong \mathbb{R} \ . \tag{2}$$

The non-trivial elements in this vector space are obtained geometrically from the half-pipe holonomy representations that we constructed in [RS], and are easily shown to be "first-order deformations" of paths in the "vertical" component $\mathcal{V}$ of Theorem 1.1. On the other hand, the first factor in the decomposition (1) is 12-dimensional by general reasons (namely, by an orbifold version of hyperbolic Dehn filling), and its elements are again integrable and tangent to deformations in the "horizontal" component $\mathcal{H}$. Together with Theorem 1.1, this cohomological computation is therefore the essential step in the proof of Theorem 1.2.

As another noteworthy comment on the consequences of (2), recall Danciger's result [Dan13, Theorem 1.2]: the existence of geometric transition on a compact half-pipe 3-manifold $\mathcal{X}$, with singular locus a knot Σ, is proved under the sole condition

$$H^1_{\mathrm{Ad}\,\rho_0}(\pi_1(\mathcal{X} \smallsetminus \Sigma), \mathfrak{so}(1,2)) \cong \mathbb{R}\ . \tag{3}$$

This sufficient condition for the regeneration is certainly not necessary, but it is rather reasonable at least when the singular locus is connected.

Now, for any representation $\rho\colon \Gamma \to \mathrm{O}(1,2)$ with Γ finitely generated there is a natural identification

$$H^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{so}(1,2)) \cong H^1_{\rho}(\Gamma, \mathbb{R}^{1,2})\ .$$

In presence of a collapse of hyperbolic or AdS structures of dimension n, the holonomy representations of a rescaled HP structure naturally provide elements of the cohomology group $H^1_{\rho_0}(\pi_1(\mathcal{X} \smallsetminus \Sigma), \mathbb{R}^{1,n-1})$. In particular, the correct generalisation of Danciger's condition (3) to any dimension n would be:

$$H^1_{\rho_0}(\pi_1(\mathcal{X} \smallsetminus \Sigma), \mathbb{R}^{1,n-1}) \cong \mathbb{R}\ , \tag{4}$$

in agreement with what we found for Γ_{22} (the orbifold fundamental group of $\mathcal{O}$) — compare with (2). In Section 7.2 we relate more concretely Danciger's condition (3) with the viewpoint of this paper, which is suited to higher dimension.

In conclusion, this suggests that a higher-dimensional regeneration result in the spirit of Danciger, although far from reach at the present time, might be reasonable.

Organisation of the paper. In Section 2 we establish the set-up for the proof of Theorem 1.1 in the hyperbolic and AdS cases. In Section 3, which may be of independent interest, we study deformations of some right-angled Coxeter groups represented as "cusp groups" in $\mathrm{Isom}(\mathbb{H}^n)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, and G_{HP^n} for $n = 3, 4$. In Section 4, we study the $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ character varieties of Γ_{22}, concluding the proof of Theorem 1.1 for the hyperbolic and AdS case. Section 5 is the analogue of Section 3 for half-pipe geometry. In Section 6 we provide the explicit computation of (2), and use it to prove the HP version of Theorem 1.1. In Section 7 we prove Theorem 1.2, and finally explain in details how to relate (4) with Danciger's condition (3).

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2. Hyperbolic and AdS character varieties

In this section we establish the set-up for the study of hyperbolic and AdS character varieties of right-angled Coxeter groups.

2.1. **Hyperbolic and AdS geometry.**

We begin with the necessary definitions and notation. Let $q_{\pm 1}$ be the quadratic form on $\mathbb{R}^{n+1}$ of signature $(-,+,\ldots,+,\pm)$ defined by:

$$q_{\pm 1}(x) = -x_0^2 + x_1^2 + \ldots + x_{n-1}^2 \pm x_n^2 ,$$

and let $b_{\pm 1}$ be the associated bilinear form.

The n-dimensional *hyperbolic space* $\mathbb{H}^n$ is defined as the projective domain:

$$\mathbb{H}^n = \mathrm{P}\{x \in \mathbb{R}^{n+1} \,|\, q_1(x) < 0\} \subset \mathbb{RP}^n ,$$

It is endowed with a complete Riemannian metric of constant negative sectional curvature, whose isometry group $\mathrm{Isom}(\mathbb{H}^n)$ is identified to $\mathrm{PO}(q_1) \cong \mathrm{PO}(1,n)$. The *boundary at infinity* of $\mathbb{H}^n$ is the projectivisation of the cone of null directions for the quadratic form q_1:

$$\partial\mathbb{H}^n = \mathrm{P}\{x \in \mathbb{R}^{n+1} \,|\, q_1(x) = 0\} .$$

Similarly, *Anti-de Sitter space* of dimension n is defined as:

$$\mathbb{A}\mathrm{d}\mathbb{S}^n = \mathrm{P}\{x \in \mathbb{R}^{n+1} \,|\, q_{-1}(x) < 0\} ,$$

and is endowed with a Lorentzian metric of constant negative sectional curvature. Its isometry group $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ is identified to $\mathrm{PO}(q_{-1}) \cong \mathrm{PO}(2,n-1)$. The *boundary at infinity* of $\mathbb{A}\mathrm{d}\mathbb{S}^n$ is defined analogously:

$$\partial\mathbb{A}\mathrm{d}\mathbb{S}^n = \mathrm{P}\{x \in \mathbb{R}^{n+1} \,|\, q_{-1}(x) = 0\} .$$

2.2. **Hyperplanes and reflections.**

A *hyperplane* of $\mathbb{H}^n$ is the intersection, when non-empty, of a projective hyperplane of $\mathbb{RP}^n$ with $\mathbb{H}^n$.

Let us denote by $\perp_1$ the orthogonal complement with respect to the bilinear form b_1, and let $X \in \mathbb{R}^{n+1}$ be a vector. The projectivisation of the linear hyperplane $X^{\perp_1}$ of $\mathbb{R}^{n+1}$ intersects $\mathbb{H}^n$ if and only if $q_1(X) > 0$, i.e. if X is *spacelike* for q_1. Hence to every q_1-spacelike vector X is associated a hyperplane

$$H_X = \mathrm{P}(X^{\perp_1}) \cap \mathbb{H}^n .$$

It is clearly harmless to assume that $q_1(X) = 1$, so that the vector X is uniquely determined up to changing the sign.

A *reflection* in hyperbolic, resp. Anti-de Sitter, geometry is a non-trivial involution $r \in \mathrm{Isom}(\mathbb{H}^n)$, resp. $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, that fixes point-wise a totally geodesic hyperplane.

Given a hyperplane H_X in $\mathbb{H}^n$, there is a unique reflection r_X fixing the given hyperplane. Indeed, the reflection r_X is (the projective class of) the linear transformation in $\mathrm{O}(q_1)$ which fixes $X^{\perp_1}$ and acts on the subspace generated by X as minus the identity. Two spacelike unit vectors X and Y give the same reflection if and only if $X = \pm Y$. Finally, it is a simple exercise to show that two reflections r_X and r_Y commute if and only if either $X = \pm Y$ or X and Y are orthogonal for the bilinear form b_1.

We summarise the above considerations in the following statement:

Lemma 2.1. *There is a two-sheeted covering map*

$$\{X \in \mathbb{R}^{n+1} \,:\, q_1(X) = +1\} \to \{r \in \mathrm{Isom}(\mathbb{H}^n) : r \text{ is a reflection}\} ,$$

which maps a spacelike unit vector X to the unique reflection r_X fixing H_X point-wise. Moroever, two distinct reflections r_X and r_Y commute if and only if $b_1(X,Y)=0$.

The subset of vectors in $\mathbb{R}^{n+1}$ such that $q_1(X)=+1$ is usually called *de Sitter space*.

Let us now move to Anti-de Sitter geometry. Again, totally geodesic subspaces are the intersections of $\mathbb{A}\mathrm{d}\mathbb{S}^n$ with linear subspaces of $\mathbb{R}^{n+1}$. Such a subspace is called *spacelike*, *timelike* or *lightlike* if the induced bilinear form, obtained as the restriction of the Lorentzian metric of $\mathbb{A}\mathrm{d}\mathbb{S}^n$, is positive definite, indefinite or degenerate, respectively. Spacelike hyperplanes are isometrically embedded copies of $\mathbb{H}^{n-1}$, while timelike hyperplanes are isometrically embedded copies of $\mathbb{A}\mathrm{d}\mathbb{S}^{n-1}$.

Let us denote by $\perp_{-1}$ the orthogonality relation with respect to b_{-1}. We have:

Lemma 2.2. *Given a vector $X\in\mathbb{R}^{n+1}$, the intersection $X^{\perp_{-1}}\cap\mathbb{A}\mathrm{d}\mathbb{S}^n$ is non-empty, and is:*

- *a spacelike hyperplane if $q_{-1}(X)<0$,*
- *a timelike hyperplane if $q_{-1}(X)>0$,*
- *a lightlike hyperplane if $q_{-1}(X)=0$.*

The hyperplane of fixed points of an AdS reflection is either spacelike or timelike. Similarly to the hyperbolic case, given a vector X such that $q_{-1}(X)=\pm1$, the unique reflection fixing

$$H_X=\mathrm{P}(X^{\perp_{-1}})\cap\mathbb{A}\mathrm{d}\mathbb{S}^n$$

is induced by the linear transformation in $\mathrm{O}(q_{-1})$ acting on $X^{\perp_{-1}}$ as the identity and on $\mathrm{Span}(X)$ (which is in direct sum with $X^{\perp_{-1}}$ since $q_{-1}(X)\neq 0$) as minus the identity.

In conclusion, we have another summarising statement:

Lemma 2.3. *There is a two-sheeted covering map*

$$\{X\in\mathbb{R}^{n+1}\,:\,q_{-1}(X)=\pm1\}\to\{r\in\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)\,:\,r\text{ is a reflection}\}\ ,$$

which maps a spacelike or timelike unit vector X to the unique reflection r_X fixing H_X point-wise. Moroever, two distinct reflections r_X and r_Y commute if and only if $b_{-1}(X,Y)=0$.

The space $\{X\in\mathbb{R}^{n+1}\,:\,q_{-1}(X)=\pm1\}$ has two connected components, as well as the space of reflections. The component defined by $q_{-1}(X)=-1$ is a double cover of Anti-de Sitter space itself.

2.3. **Relative position of hyperplanes.** It will be useful to discuss the relative position of hyperplanes. For hyperbolic geometry, this is easily summarised:

Lemma 2.4. *Let H_X and H_Y be two distinct hyperplanes in $\mathbb{H}^n$, for $q_1(X)=q_1(Y)=1$. Then,*

- *H_X and H_Y intersect in $\mathbb{H}^n$ if and only if $|b_1(X,Y)|<1$;*
- *The closures of H_X and H_Y intersect in $\partial\mathbb{H}^n\smallsetminus\mathbb{H}^n$ if and only if $|b_1(X,Y)|=1$;*
- *The closures of H_X and H_Y are disjoint in $\mathbb{H}^n\cup\partial\mathbb{H}^n$ if and only if $|b_1(X,Y)|>1$.*

For Anti-de Sitter hyperplanes, it is necessary to distinguish several cases. Here we will only consider the cases of interest for the proofs of our main results.

For spacelike hyperplanes, we have (see Figure 2):

Lemma 2.5. *Let H_X and H_Y be two distinct spacelike hyperplanes in $\mathbb{A}\mathrm{d}\mathbb{S}^n$, for $q_{-1}(X)=q_{-1}(Y)=-1$. Then,*

- *H_X and H_Y intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ if and only if $|b_{-1}(X,Y)|>1$;*

- *The closures of H_X and H_Y intersect in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^n \smallsetminus \mathbb{A}\mathrm{d}\mathbb{S}^n$ if and only if $|b_{-1}(X,Y)| = 1$;*
- *The closures of H_X and H_Y are disjoint in $\mathbb{A}\mathrm{d}\mathbb{S}^n \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^n$ if and only if $|b_{-1}(X,Y)| < 1$.*

For two AdS timelike hyperplanes the situation is different, as explained in the following lemma. See also Figure 3.

Lemma 2.6. *Let H_X and H_Y be two distinct timelike hyperplanes in $\mathbb{A}\mathrm{d}\mathbb{S}^n$, for $q_{-1}(X) = q_{-1}(Y) = 1$. Then H_X and H_Y intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ and the intersection $H_X \cap H_Y$ is:*

- *spacelike if and only if $|b_{-1}(X,Y)| > 1$;*
- *lightlike if and only if $|b_{-1}(X,Y)| = 1$;*
- *timelike if and only if $|b_{-1}(X,Y)| < 1$.*

In the first case of Lemma 2.6, the intersection $H_X \cap H_Y$ is a totally geodesic copy of $\mathbb{H}^{n-2}$; in the third case it is a totally geodesic copy of $\mathbb{A}\mathrm{d}\mathbb{S}^{n-2}$.

2.4. **Coxeter groups and representation varieties.** Given a finitely presented group Γ and an algebraic Lie group G, let us denote by $\mathrm{Hom}(\Gamma, G)$ the space of representations $\rho\colon \Gamma \to G$. Since $\mathrm{Hom}(\Gamma, G)$ is naturally an affine algebraic set (see also Section 7.1), it is called *representation variety*.

In the remainder of the paper, we will restrict the attention to the case when Γ is a right-angled Coxeter group, which we now define.

Definition 2.7 (RACG)**.** Given a finite set S and a subset R of (unordered) pairs of distinct elements of S, the associated *right-angled Coxeter group* has presentation:

$$\langle\, S \;\big|\; \boldsymbol{s}^2 = 1 \;\forall\, \boldsymbol{s} \in S,\; \boldsymbol{s}_1\boldsymbol{s}_2 = \boldsymbol{s}_1\boldsymbol{s}_2 \;\forall\, (\boldsymbol{s}_1, \boldsymbol{s}_2) \in R \,\rangle \;.$$

For instance, the group generated by the reflections in the sides of a right-angled Euclidean or hyperbolic polytope is a right-angled Coxeter group.

We will only be interested in representations of a right-angled Coxeter group Γ which send every generator to a reflection. Let us introduce more formally this space.

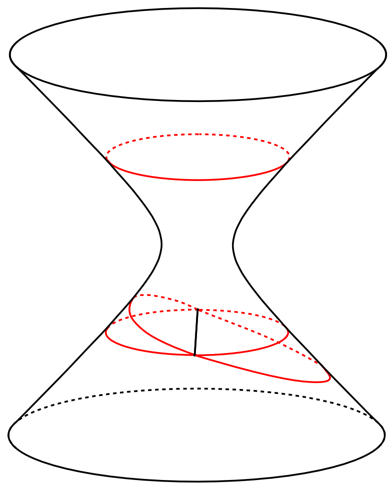

FIGURE 2. Two spacelike hyperplanes in $\mathbb{A}\mathrm{d}\mathbb{S}^n$ can intersect in a totally geodesic spacelike hyperplane, at a point at infinity, or be disjoint. The picture ($n = 3$) is in an affine chart for real projective space, where Anti-de Sitter space is the interior of a one-sheeted hyperboloid.

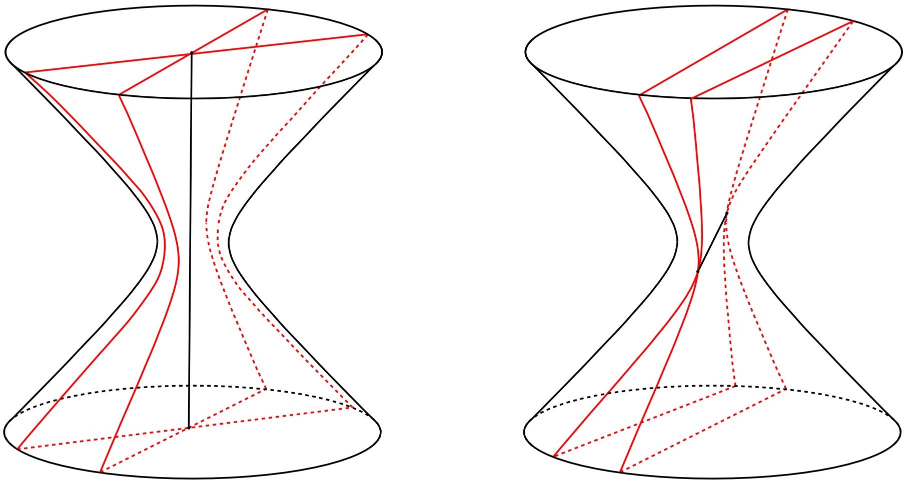

FIGURE 3. Two timelike planes in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ intersecting in a timelike (left) or spacelike (right) line.

Definition 2.8 (The space $\mathrm{Hom}_{\mathrm{refl}}$)**.** Let G be $\mathrm{Isom}(\mathbb{H}^n)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ and Γ a right-angled Coxeter group as above. We define $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ as the subset of $\mathrm{Hom}(\Gamma, G)$ of representations ρ such that:

- for every $\boldsymbol{s} \in S$, the isometry $\rho(\boldsymbol{s})$ is a reflection, and
- for every $(\boldsymbol{s}_1, \boldsymbol{s}_2) \in R$, the reflections $\rho(\boldsymbol{s}_1)$ and $\rho(\boldsymbol{s}_2)$ are distinct.

Reflections constitute a connected component in the space of order-two isometries in $\mathrm{Isom}(\mathbb{H}^n)$, while in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ they constitute two connected components, given by reflections in spacelike and timelike hyperplanes. Moreover, by Lemmas 2.1 and 2.3, two distinct reflections commute if and only if their fixed hyperplanes are orthogonal. Hence we immediately get:

Lemma 2.9. *The subset* $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ *is clopen in* $\mathrm{Hom}(\Gamma, G)$.

To simplify the computations, we will follow [KS10] and adopt a model for the character variety which is well-adapted to our setting. Roughly speaking, we only consider the deformations of the hyperplanes fixed by the reflection associated to each generator. This will reduce significantly the complexity of the problem, since (in dimension n) for each generator we have a vector of $n+1$ entries (giving the hyperplane of reflection) in place of a $(n+1) \times (n+1)$ matrix (giving the reflection itself).

More precisely, the following lemma gives a local parametrisation of the space $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$:

Lemma 2.10. *The space* $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ *is finitely covered by a disjoint union of subsets of* $\mathbb{R}^{(n+1)|S|}$ *defined by the vanishing of* $|S| + |R|$ *quadratic conditions.*

Proof. Let us first give the proof for $G = \mathrm{Isom}(\mathbb{H}^n)$. Let us identify $\mathbb{R}^{(n+1)|S|}$ to the vector space of functions $f \colon S \to \mathbb{R}^{n+1}$. For every representation $\rho \colon \Gamma \to G$ in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$, we can choose a function f such that $q_1(f(\boldsymbol{s})) = 1$ and $\rho(\boldsymbol{s}) = r_{f(\boldsymbol{s})}$ for every generator $\boldsymbol{s}$. In fact there are $2^{|S|}$ possible choices of such an f, differing by changing sign to the image of each generator, and they all satisfy the following conditions:

(1) The vector $f(\boldsymbol{s})$ is unit, meaning that $q_1(f(\boldsymbol{s})) = 1$, hence giving $|S|$ quadratic conditions.

(2) For each of the commutation relations $\boldsymbol{s}_i\boldsymbol{s}_j = \boldsymbol{s}_j\boldsymbol{s}_i$ in Γ, by Lemma 2.1 the corresponding vectors $f(\boldsymbol{s}_i)$ and $f(\boldsymbol{s}_j)$ are orthogonal with respect to b_1.

Conversely, every f satisfying these conditions induces the representation ρ in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ defined by $\rho(\boldsymbol{s}) = r_{f(\boldsymbol{s})}$. Define a function

$$g\colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$$

by

$$g(f) = \left((q_1(f(\boldsymbol{s})) - 1)_{\boldsymbol{s}\in S}, (b_1(f(\boldsymbol{s}_i), f(\boldsymbol{s}_j)))_{(\boldsymbol{s}_i,\boldsymbol{s}_j)\in R}\right) .$$

We have shown that $g^{-1}(0)$ is a $2^{|S|}$-sheeted covering of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, \mathrm{Isom}(\mathbb{H}^n))$, with deck transformations given by the group $(\mathbb{Z}/2\mathbb{Z})^{|S|}$.

The proof for the Anti-de Sitter case is completely analogous, except that we have to distinguish several cases, depending whether $\rho(\boldsymbol{s})$ is a reflection in a spacelike or timelike hyperplane. In the former case, we must impose $q_{-1}(f(\boldsymbol{s})) = -1$, and in the latter $q_{-1}(f(\boldsymbol{s})) = 1$ (see Lemma 2.2). The orthogonality conditions are exactly the same, now using the bilinear form b_{-1}, by Lemma 2.3. In conclusion we have that $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n))$ is finitely covered by a disjoint union of $|S|$ subsets each defined by the vanishing of a quadratic function $g\colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$. □

Remark 2.11. The covering map of Lemma 2.10 is equivariant with respect to the following two actions of $\mathrm{O}(q_{\pm 1})$:

- The action on the space of functions $f\colon S \to \mathbb{R}^{n+1}$ by post-composition $(A \cdot f)(\boldsymbol{s}) = A(f(\boldsymbol{s}))$. This action preserves the zero locus of the defining functions $g\colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$ introduced in the proof of Lemma 2.10;
- The action on $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ given by composition of the projection $\mathrm{O}(q_{\pm 1}) \to G$ and the G-action by conjugation. This action preserves the component $\mathrm{Hom}_{\mathrm{refl}}(\Gamma, G)$ of $\mathrm{Hom}(\Gamma, G)$.

Remark 2.12. In this paper we are considering representations in the groups $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, which are double quotients of $\mathrm{O}(1, n)$ and $\mathrm{O}(2, n-1)$, respectively. However, all the representations that we consider (that is, those as in Definition 2.8) do actually lift to representations with values in $\mathrm{O}(q_1) \cong \mathrm{O}(1, n)$ and $\mathrm{O}(q_{-1}) \cong \mathrm{O}(2, n-1)$ respectively.

Indeed, if $\rho\colon \Gamma \to G$ maps a generator $\boldsymbol{s}$ to the reflection $r_{f(\boldsymbol{s})}$, then a lift of ρ can be simply defined by sending $\boldsymbol{s}$ to a matrix representing $r_{f(\boldsymbol{s})}$, and the relations between generators of Γ are automatically sent to the identity. Therefore, the local pictures of the $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ character varieties, provided by Theorem 1.1, will actually coincide with the local pictures for the $\mathrm{O}(1, 4)$ and $\mathrm{O}(2, 3)$ character varieties of Γ_{22}.

3. Cusp flexibility and rigidity in AdS geometry

In this section, which may be of independent interest, we study deformations of some right-angled Coxeter groups represented as "cusp groups" in $\mathrm{Isom}(\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, for $n = 3, 4$.

3.1. **Flexibility in dimension three.**

Let Γ_{rect} denote the right-angled Coxeter group generated by the reflections along the sides of a Euclidean rectangle. The standard presentation of Γ_{rect} has 4 generators (one for each side of the rectangle), and relations such that each generator has order two and reflections in adjacent sides commute.

Definition 3.1 (Cusp group in dimension 3)**.** The image of a representation of Γ_{rect} into $\text{Isom}(\mathbb{H}^3)$ or $\text{Isom}(\mathbb{A}\text{d}\mathbb{S}^3)$ is called a *cusp group* if the four generators are sent to reflections in four distinct planes which share the same point at infinity.

We will also consider other similar representations of Γ_{rect}, which occur in correspondence to a collapse, when two non-commuting generators are sent to the same reflection. Let us begin with the hyperbolic case:

Definition 3.2 (Collapsed cusp group for $\mathbb{H}^3$)**.** The image of a representation of Γ_{rect} into $\text{Isom}(\mathbb{H}^3)$ is called a *collapsed cusp group* if the four generators are sent to reflections along three distinct planes which share the same point at infinity.

Let ρ' be a representation near a given $\rho \in \text{Hom}_{\text{refl}}(\Gamma_{\text{rect}}, \mathbb{H}^3)$, and $\boldsymbol{s}$ be a generator of Γ_{rect}. In virtue of Lemma 2.9 and the discussion below, we refer to the fixed-point set of $\rho'(\boldsymbol{s})$ as a *plane* of ρ'.

In [KS10, Lemma 5.1], the following property of cusp groups is proved:

Proposition 3.3. *Let $\rho\colon \Gamma_{\text{rect}} \to \text{Isom}(\mathbb{H}^3)$ be a representation whose image is a cusp group. For all nearby representations whose image is not a cusp group, a pair of opposite planes intersect in $\mathbb{H}^3$, while the other pair of opposite planes have disjoint closures in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$.*

In fact, a simple adaptation of the proof shows:

Proposition 3.4. *Let $\rho\colon \Gamma_{\text{rect}} \to \text{Isom}(\mathbb{H}^3)$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

1. *If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$.*
2. *The image of ρ' is a cusp group.*
3. *A pair of opposite planes intersect in $\mathbb{H}^3$, while the other pair of opposite planes have disjoint closures in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$.*

The first case may hold only if ρ is a collapsed cusp group. Under this hypothesis, Proposition 3.4 can be rephrased by saying that a deformation of a collapsed cusp group either preserves the property that two planes corresponding to non-adjacent sides of the rectangle coincide (which is the case when the collapsed cusp group remains a collapsed cusp group, for instance), or it falls in the class of representations described in Proposition 3.3, namely, deformations of non-collapsed cusp groups. If ρ is a cusp group, then the content of Proposition 3.4 is the same as Proposition 3.3.

We now move to the AdS version of Propositions 3.3 and 3.4, for which we will give a complete proof. Proofs for the hyperbolic case are easier and can be repeated by mimicking the AdS case.

Note that for an AdS cusp group the four planes necessarily satisfy the orthogonality conditions as in a rectangle, and therefore two of them are spacelike and two timelike. We will show the following proposition, which is the AdS version of Proposition 3.3.

Proposition 3.5. *Let $\rho\colon \Gamma_{\text{rect}} \to \text{Isom}(\mathbb{A}\text{d}\mathbb{S}^3)$ be a representation whose image is a cusp group. For all nearby representations whose image is not a cusp group, exactly one of the following possibilities holds:*

1. *The two spacelike planes are disjoint in $\mathbb{A}\text{d}\mathbb{S}^3 \cup \partial\mathbb{A}\text{d}\mathbb{S}^3$, whereas the two timelike planes intersect in a timelike geodesic of $\mathbb{A}\text{d}\mathbb{S}^3$;*
2. *The two spacelike planes intersect in $\mathbb{A}\text{d}\mathbb{S}^3$, whereas the two timelike planes intersect in a spacelike geodesic of $\mathbb{A}\text{d}\mathbb{S}^3$.*

Proposition 3.5 follows from the more general Proposition 3.7 below, which also includes the collapsed case. We will consider only the degeneration of cusp groups to a collapsed cusp group when the two planes which coincide are spacelike, as in the following definition:

Definition 3.6 (Collapsed cusp group for $\mathbb{A}\mathrm{d}\mathbb{S}^3$)**.** The image of a representation of Γ_{rect} into $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ is called a *collapsed cusp group* if the four generators are sent to reflections along three distinct planes, two timelike and one spacelike, sharing the same point at infinity.

Proposition 3.7. *Let $\rho\colon \Gamma_{\mathrm{rect}} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

(1) If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$ is a reflection in a spacelike hyperplane, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$.
(2) The image of ρ' is a cusp group.
(3) The two spacelike planes are disjoint in $\mathbb{A}\mathrm{d}\mathbb{S}^3 \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$, whereas the two timelike planes intersect in a timelike geodesic of $\mathbb{A}\mathrm{d}\mathbb{S}^3$.
(4) The two spacelike planes intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, whereas the two timelike planes intersect in a spacelike geodesic of $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Proof. By Lemmas 2.9 and 2.10, it is sufficient to analyse a neighborhood of a lift $f\colon S \to \mathbb{R}^4$ of ρ, where S is the standard generating set of Γ_{rect}. Let us denote $\boldsymbol{s}_1, \boldsymbol{s}_2$ the generators which are sent by ρ to a reflection in a spacelike plane, and $\boldsymbol{t}_1, \boldsymbol{t}_2$ those sent to a reflection in a timelike plane. The same will occur for representations nearby ρ.

Let us fix a nearby representation ρ' and a lift $f'\colon S \to \mathbb{R}^4$. Let us denote $X_i = f'(\boldsymbol{s}_i)$ and $Y_i = f'(\boldsymbol{t}_i)$. Recall that X_i is orthogonal to Y_j for $i, j = 1, 2$.

Suppose that $X_1 \neq \pm X_2$, for otherwise we are in the case of item (1). Up to the action of $\mathrm{O}(q_{-1})$ (see Remark 2.11) and up to changing signs, we can assume once and forever that

$$X_1 = (1, 0, 0, 0) \quad \text{and} \quad Y_1 = (0, 1, 0, 0)\ .$$

Suppose first that the hyperplane H_{Y_2} shares the same point at infinity p with H_{X_1} and H_{Y_1}. We can assume $p = [0 : 0 : 1 : 1] \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$. Together with the orthogonality of Y_2 with X_1, this implies (up to changing the sign if necessary)

$$Y_2 = (0, 1, a, -a)$$

for some parameter $a \neq 0$. Applying the orthogonality of X_2 with Y_1 and Y_2, we now find (always up to a sign)

$$X_2 = (1, 0, b, -b)$$

for some b, which implies that H_{X_2} also shares the point $p = [0 : 0 : 1 : 1]$. Thus we still have a cusp group and we are in the case of item (2).

Suppose instead that H_{Y_2} does not share the same point at infinity with H_{X_1} and H_{Y_1}. In other words, we have two geodesics in H_{X_1} (which is a copy of $\mathbb{H}^2$): $\ell_1 = H_{Y_1} \cap H_{X_1}$ and $\ell_2 = H_{Y_2} \cap H_{X_1}$. There are two possibilities: either ℓ_1 and ℓ_2 intersect in H_{X_1}, or they are ultraparallel. See Figure 4 and the related Figure 3.

If ℓ_1 and ℓ_2 intersect in H_{X_1}, we can assume that $\ell_1 \cap \ell_2 = \{[0 : 0 : 0 : 1]\}$. Equivalently,

$$Y_2 = (0, \cos\theta, \sin\theta, 0)\ ,$$

where θ is the angle between the two geodesics in H_{X_1}. In this case, the two timelike planes H_{Y_1} and H_{Y_2} have timelike intersection by Lemma 2.6 (the intersection is indeed the timelike geodesic $[\cos(s) : 0 : 0 : \sin(s)]$). Imposing the orthogonality of H_{X_2} with H_{Y_1} and H_{Y_2}, we

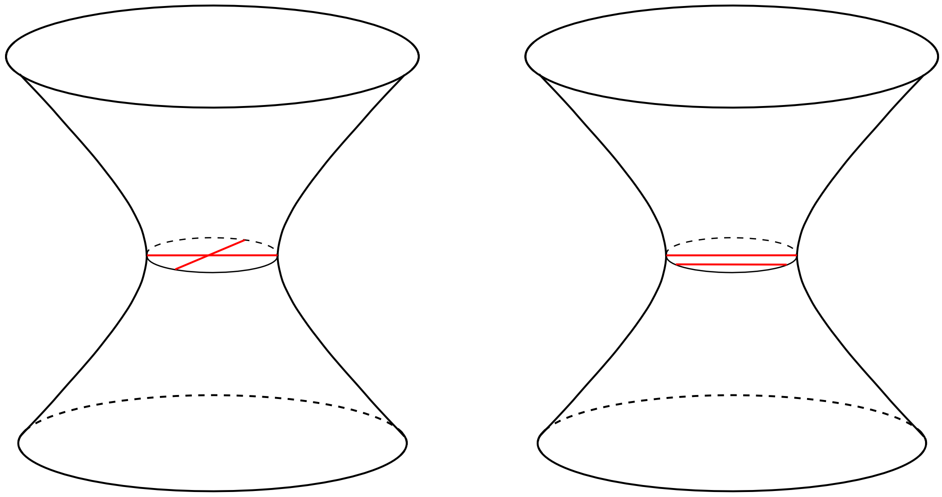

FIGURE 4. The two configurations for the geodesics ℓ_1 and ℓ_2 in the "horizontal" spacelike plane H_{X_1}, as in the proof of Proposition 3.7. The two timelike planes H_{Y_i} and containing ℓ_i and orthogonal to H_{X_1} are as in Figure 3, left and right figure respectively.

find (up to a sign)

$$X_2 = (\cos\varphi, 0, 0, \sin\varphi) \ ,$$

which means that H_{X_1} and H_{X_2} are disjoint in $\overline{\mathbb{A}\mathrm{d}\mathbb{S}}^3$ by Lemma 2.5. (The parameter φ is indeed the timelike distance between H_{X_1} and H_{X_2}, which is achieved on the timelike geodesic we have just introduced.) So, in this case item (3) of the statement holds.

If ℓ_1 and ℓ_2 are ultraparallel, we can assume

$$Y_2 = (0, \cosh\theta, 0, \sinh\theta) \ ,$$

where θ is now the distance between the two aforementioned geodesics in H_{X_1}. In this case, H_{Y_1} and H_{Y_2} have spacelike intersection (which is the geodesic $[\cosh(s) : 0 : \sinh(s) : 0]$, see Lemma 2.6). Imposing again the orthogonality of H_{X_2} with H_{Y_1} and H_{Y_2}, and changing sign if necessary, we find

$$X_2 = (\cosh\varphi, 0, \sinh\varphi, 0) \ ,$$

namely, H_{X_1} and H_{X_2} intersect in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ by Lemma 2.5 (the parameter φ now being their angle of intersection). Thus, item (4) of the statement holds. This concludes the proof. □

Remark 3.8. In the proof of Proposition 3.7 we have used only that ρ can be continuously deformed to ρ'. Hence the conclusions of Proposition 3.7 and Proposition 3.4 actually hold on the entire connected component of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{\mathrm{cube}}, G)$ containing ρ.

3.2. **Rigidity in dimension four.** Let us now move to dimension four.

Let Γ_{cube} be the group generated by the reflections in the faces of a Euclidean cube. The group Γ_{cube} has 6 generators, one for each face, and 12 commutation relations, one for each edge of the cube, involving the two faces adjacent to that edge. Of course, there is also a square-type relation for each generator. There is no relation between the generators corresponding to opposite faces.

Definition 3.9 (Cusp group in dimension 4)**.** The image of a representation of Γ_{cube} into $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or $\mathrm{Isom}(\mathbb{H}^4)$ is called a *cusp group* if the 6 generators are sent to reflections in 6 distinct hyperplanes which share the same point at infinity.

In the AdS case, among these 6 hyperplanes, two opposite hyperplanes are necessarily spacelike, while the other 4 are timelike.

The following proposition is the fundamental property that can be roughly rephrased as: "cusp groups stay cusp groups". Its hyperbolic counterpart is proved in [KS10, Lemma 5.3].

Proposition 3.10. *Let* $\rho\colon \Gamma_{\mathrm{cube}} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ *be a representation whose image is a cusp group. Then all nearby representations are cusp groups.*

Similarly to dimension three, we will obtain Proposition 3.10 as a special case of a more general statement including the collapsed case. Let us first give the definition of collapsed cusp group, where two non-commuting generators can be sent to the same reflection (along a spacelike hyperplane in the AdS case):

Definition 3.11 (Collapsed cusp group in dimension 4)**.** The image of a representation of Γ_{cube} into $\mathrm{Isom}(\mathbb{H}^4)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ is called a *collapsed cusp group* if the 6 generators are sent to reflections along 5 distinct hyperplanes which share the same point at infinity. In the AdS case, we require that the unique reflection associated to two generators is along a spacelike hyperplane.

Let us now formulate and prove the more general version of Proposition 3.10.

Proposition 3.12. *Let* $\rho\colon \Gamma_{\mathrm{cube}} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ *be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations* ρ'*, exactly one of the following possibilities holds:*

(1) If $\boldsymbol{s}_1$ *and* $\boldsymbol{s}_2$ *are generators such that* $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$ *is a reflection in a spacelike hyperplane, then* $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$*.*

(2) The image of ρ' *is a cusp group.*

Proof. Similarly to the three-dimensional case treated in the previous section, any representation ρ' nearby ρ lies in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{\mathrm{cube}}, G)$, hence it sends the 6 standard generators of Γ_{cube} to reflections. Moreover, the hyperplanes of ρ' have the same type (spacelike or timelike) as for ρ.

Let us pick a lift $f'\colon S \to \mathbb{R}^5$ of ρ', for S the standard generating set of Γ_{cube}. Denote by $\boldsymbol{s}_1, \boldsymbol{s}_2$ the two generators corresponding to opposite faces of the cube which are sent to reflections in spacelike hyperplanes, and $X_i = f'(\boldsymbol{s}_i)$ (so that $q_{-1}(X_i) = -1$). Similarly we define Y_i and Z_i for $i = 1, 2$, on which q_{-1} takes value 1. Hence each of this 6 vectors is orthogonal to 4 of the others: more precisely, A_i is orthogonal to all the others except A_j, for $A \in \{X, Y, Z\}$ and $i, j = 1, 2$.

Now, let us assume that the hyperplanes H_{X_1} and H_{X_2} do not coincide, that is $X_1 \neq \pm X_2$. We shall show that the image of ρ' is still a cusp group.

Let us start by considering the intersection with H_{X_1}, which is a copy of $\mathbb{H}^3$. Here we see the (two-dimensional) planes $H_{Y_1} \cap H_{X_1}$, $H_{Y_2} \cap H_{X_1}$, $H_{Z_1} \cap H_{X_1}$ and $H_{Z_2} \cap H_{X_1}$, whose associated reflections give a representation $\Gamma_{\mathrm{rect}} \to \mathrm{Isom}(\mathbb{H}^3)$ which is nearby a (rectangular) cusp group. As in the proof of Proposition 3.5, it is easy to see that if this representation of Γ_{rect} is a cusp group in H_{X_1}, then necessarily also H_{X_2} shares the same point at infinity with H_{Y_1}, H_{Y_2}, H_{Z_1}, H_{Z_2}. Therefore the image of Γ_{cube} is still a cusp group, since we are assuming that $H_{X_2} \neq H_{X_1}$.

Hence let us assume that the representation of Γ_{rect} is not a cusp group, and we will derive a contradiction. By Proposition 3.3 (up to relabelling) we may assume that $H_{Y_1} \cap H_{X_1}$ and $H_{Y_2} \cap H_{X_1}$ intersect in H_{X_1}, while $H_{Z_1} \cap H_{X_1}$ and $H_{Z_2} \cap H_{X_1}$ are disjoint in H_{X_1}. This implies that $H_{Y_1} \cap H_{Y_2}$ is a timelike plane (i.e. a copy of $\mathbb{A}\mathrm{d}\mathbb{S}^2$), while $H_{Z_1} \cap H_{Z_2}$ is spacelike (i.e. a copy of $\mathbb{H}^2$). To see this, one can in fact assume that, up to the signs,

$$X_1 = (1,0,0,0,0) \qquad Y_1 = (0,1,0,0,0) \qquad Y_2 = (0,\cos\theta,\sin\theta,0,0)\ ,$$

and apply Lemma 2.6 — and similarly for Z_1 and Z_2.

Now, let us consider the intersection with H_{Y_1}, which is a copy of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. We have thus a representation of Γ_{rect} acting on this copy of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ as a cusp group or collapsed cusp group, with generators which are reflections in $H_{Z_1} \cap H_{Y_1}$, $H_{Z_2} \cap H_{Y_1}$, $H_{X_1} \cap H_{Y_1}$ and $H_{X_2} \cap H_{Y_1}$. Since $H_{Z_1} \cap H_{Z_2}$ is spacelike, then $H_{Z_1} \cap H_{Z_2} \cap H_{Y_1}$ is also spacelike, and therefore we are in the situation of Proposition 3.7 item (4), recalling that $H_{X_1} \neq H_{X_2}$ by our assumption. This implies that $H_{X_1} \cap H_{Y_1}$ and $H_{X_2} \cap H_{Y_1}$ intersect in $H_{Y_1} \subset \mathbb{A}\mathrm{d}\mathbb{S}^4$.

On the other hand, considering the intersection with H_{Z_1}, which is again a copy of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, since $H_{Y_1} \cap H_{Y_2}$ is timelike, we find that $H_{Y_1} \cap H_{Y_2} \cap H_{Z_1}$ is a timelike geodesic. By Proposition 3.7 item (3), $H_{X_1} \cap H_{Z_1}$ and $H_{X_2} \cap H_{Z_1}$ do not intersect in H_{Z_1}, which in turn implies (since H_{X_1} and H_{X_2} are both orthogonal to H_{Z_1}) that H_{X_1} and H_{X_2} are disjoint in $\mathbb{A}\mathrm{d}\mathbb{S}^4$. This contradicts the conclusion of the previous paragraph. □

Remark 3.13. In case (1) of Proposition 3.12, i.e. when $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$, the following possibility is not excluded: for some deformation ρ' of ρ, the remaining 4 generators $\boldsymbol{s}_3, \dots, \boldsymbol{s}_6$ (which are sent by ρ to a rectangular cusp group in a copy of $\mathbb{H}^3$) are *not* sent by ρ' to a cusp group.

The analogous property for $\mathbb{H}^4$, which is a generalisation of [KS10, Lemma 5.3] can be proved along the same lines. We state it here:

Proposition 3.14. *Let $\rho\colon \Gamma_{\mathrm{cube}} \to \mathrm{Isom}(\mathbb{H}^4)$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ' exactly one of the following possibilities holds:*

(1) If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$.

(2) The image of ρ' is a cusp group.

4. The hyperbolic and AdS character variety of Γ_{22}

In this section, we study the $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ character varieties of the group Γ_{22} near the conjugacy classes of the holonomy representations ρ_t found in [KS10, RS]. The prove here Theorem 1.1 in the hyperbolic and AdS case.

4.1. The group Γ_{22}.

As in [KS10], we define

$$\Gamma_{22} < \mathrm{Isom}(\mathbb{H}^4)$$

to be the group generated by the hyperbolic reflections along the hyperplanes determined by the 22 vectors in Table 1. These hyperplanes bound a right-angled polytope in $\mathbb{H}^4$ of infinite volume, which is obtained by "removing two opposite walls" from the ideal right-angled 24-cell.

All the dihedral angles between two intersecting hyperplanes are right. Therefore Γ_{22} is a right-angled Coxeter group. We will consider Γ_{22} as an abstract group, that is the right-angled Coxeter groups on 22 generators

$$\boldsymbol{0}^+, \dots, \boldsymbol{7}^+, \boldsymbol{0}^-, \dots, \boldsymbol{7}^-, \boldsymbol{A}, \dots, \boldsymbol{F}$$

satisfying the following relations:

- $\boldsymbol{s}^2 = 1$ for each generator $\boldsymbol{s}$,

$$\begin{aligned}
\mathbf{0}^+ &= \left(\sqrt{2},+1,+1,+1,+1\right), & \mathbf{0}^- &= \left(\sqrt{2},+1,+1,+1,-1\right),\\
\mathbf{1}^+ &= \left(\sqrt{2},+1,-1,+1,-1\right), & \mathbf{1}^- &= \left(\sqrt{2},+1,-1,+1,+1\right),\\
\mathbf{2}^+ &= \left(\sqrt{2},+1,-1,-1,+1\right), & \mathbf{2}^- &= \left(\sqrt{2},+1,-1,-1,-1\right),\\
\mathbf{3}^+ &= \left(\sqrt{2},+1,+1,-1,-1\right), & \mathbf{3}^- &= \left(\sqrt{2},+1,+1,-1,+1\right),\\
\mathbf{4}^+ &= \left(\sqrt{2},-1,+1,-1,+1\right), & \mathbf{4}^- &= \left(\sqrt{2},-1,+1,-1,-1\right),\\
\mathbf{5}^+ &= \left(\sqrt{2},-1,+1,+1,-1\right), & \mathbf{5}^- &= \left(\sqrt{2},-1,+1,+1,+1\right),\\
\mathbf{6}^+ &= \left(\sqrt{2},-1,-1,+1,+1\right), & \mathbf{6}^- &= \left(\sqrt{2},-1,-1,+1,-1\right),\\
\mathbf{7}^+ &= \left(\sqrt{2},-1,-1,-1,-1\right), & \mathbf{7}^- &= \left(\sqrt{2},-1,-1,-1,+1\right),\\
\boldsymbol{A} &= \left(1,+\sqrt{2},0,0,0\right), & \boldsymbol{B} &= \left(1,0,+\sqrt{2},0,0\right),\\
\boldsymbol{C} &= \left(1,0,0,+\sqrt{2},0\right), & \boldsymbol{D} &= \left(1,0,0,-\sqrt{2},0\right),\\
\boldsymbol{E} &= \left(1,0,-\sqrt{2},0,0\right), & \boldsymbol{F} &= \left(1,-\sqrt{2},0,0,0\right).
\end{aligned}$$

TABLE 1. The 22 unit vectors defining the bounding hyperplanes of a right-angled polytope in $\mathbb{H}^4$. The reflections in these hyperplanes generate the Coxeter group Γ_{22}. Adding the vectors $(1,0,0,0,\pm\sqrt{2})$ to this list one obtains the ideal right-angled 24-cell.

- $\boldsymbol{s}_1\boldsymbol{s}_2 = \boldsymbol{s}_2\boldsymbol{s}_1$ for each pair $\boldsymbol{s}_1, \boldsymbol{s}_2$ of generators such that the corresponding vectors in Table 1 are orthogonal with respect to the bilinear form b_1.

The generators are partitioned into 3 *types*: *positive* $\mathbf{0}^+,\dots,\mathbf{7}^+$, *negative* $\mathbf{0}^-,\dots,\mathbf{7}^-$, and *letters* $\boldsymbol{A},\dots,\boldsymbol{F}$. The type is inherited from the standard 3-colouring of the facets of the 24-cell (see [KS10] for more details).

The reader can check from Table 1 that there are no commutation condition between two generators of the same type, that every $\boldsymbol{i}^+$ commutes with 4 vectors of type $\boldsymbol{j}^-$ (including $\boldsymbol{i}^-$), and every $\boldsymbol{X} \in \{\boldsymbol{A},\dots,\boldsymbol{F}\}$ commutes with $\boldsymbol{i}^-$ and $\boldsymbol{i}^+$ for 4 choices of $\boldsymbol{i} \in \{\mathbf{0},\dots,\mathbf{7}\}$. Hence there are $8\cdot 4 + 6\cdot 8 = 80$ commutation relations. Altogether, there are $102 = 22 + 80$ relations.

We would like to stress once more that throughout the following (with a few exceptions which will be remarked opportunely) we will use the symbols $\boldsymbol{i}^+ \in \{\mathbf{0}^+,\dots,\mathbf{7}^+\}$, $\boldsymbol{i}^- \in \{\mathbf{0}^-,\dots,\mathbf{7}^-\}$ and $\boldsymbol{X} \in \{\boldsymbol{A},\dots,\boldsymbol{F}\}$ to denote the 22 abstract generators of Γ_{22} (rather than vectors in $\mathbb{R}^5$).

4.2. **A curve of geometric representations.** Let us now introduce the representations of our interest, which appear in the statement of Theorem 1.1. Unlike the introduction, we will omit the superscript G hereafter, and the ambient geometry we consider will be clear from the context.

Definition 4.1 (The two paths ρ_t)**.** For $t \in (-1,1)$, we define ρ_t to be the representation of Γ_{22} in $\mathrm{Isom}(\mathbb{H}^4)$ (resp. $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$) sending each generator $\boldsymbol{s}$ of Γ_{22} to the hyperbolic (resp. AdS) reflection $r_{f_t(\boldsymbol{s})}$ associated to the corresponding vector $f_t(\boldsymbol{s})$ of Table 2 (resp. Table 3).

Some comments are necessary to explain Definition 4.1 and the tables involved:

$$
\begin{aligned}
f_t(\mathbf{0}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, +t, +t, +t, +1\right), & f_t(\mathbf{0}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, +1, +1, +1, -t\right),\\
f_t(\mathbf{1}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, +t, -t, +t, -1\right), & f_t(\mathbf{1}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, +1, -1, +1, +t\right),\\
f_t(\mathbf{2}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, +t, -t, -t, +1\right), & f_t(\mathbf{2}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, +1, -1, -1, -t\right),\\
f_t(\mathbf{3}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, +t, +t, -t, -1\right), & f_t(\mathbf{3}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, +1, +1, -1, +t\right),\\
f_t(\mathbf{4}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, -t, +t, -t, +1\right), & f_t(\mathbf{4}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, -1, +1, -1, -t\right),\\
f_t(\mathbf{5}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, -t, +t, +t, -1\right), & f_t(\mathbf{5}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, -1, +1, +1, +t\right),\\
f_t(\mathbf{6}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, -t, -t, +t, +1\right), & f_t(\mathbf{6}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, -1, -1, +1, -t\right),\\
f_t(\mathbf{7}^+) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}\,t, -t, -t, -t, -1\right), & f_t(\mathbf{7}^-) &= \frac{1}{\sqrt{1+t^2}}\left(\sqrt{2}, -1, -1, -1, +t\right),\\
f_t(\boldsymbol{A}) &= \left(1, +\sqrt{2}, 0, 0, 0\right), & f_t(\boldsymbol{B}) &= \left(1, 0, +\sqrt{2}, 0, 0\right),\\
f_t(\boldsymbol{C}) &= \left(1, 0, 0, +\sqrt{2}, 0\right), & f_t(\boldsymbol{D}) &= \left(1, 0, 0, -\sqrt{2}, 0\right),\\
f_t(\boldsymbol{E}) &= \left(1, 0, -\sqrt{2}, 0, 0\right), & f_t(\boldsymbol{F}) &= \left(1, -\sqrt{2}, 0, 0, 0\right).
\end{aligned}
$$

TABLE 2. The list of vectors X, satisfying $q_1(X) = 1$, in Definition 4.1. The representation ρ_t maps each generator $\boldsymbol{s}$ to the hyperbolic reflection in the orthogonal complement of $f_t(\boldsymbol{s})$.

$$
\begin{aligned}
f_t(\mathbf{0}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, +t, +t, +t, +1\right), & f_t(\mathbf{0}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, +1, +1, +1, +t\right),\\
f_t(\mathbf{1}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, +t, -t, +t, -1\right), & f_t(\mathbf{1}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, +1, -1, +1, -t\right),\\
f_t(\mathbf{2}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, +t, -t, -t, +1\right), & f_t(\mathbf{2}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, +1, -1, -1, +t\right),\\
f_t(\mathbf{3}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, +t, +t, -t, -1\right), & f_t(\mathbf{3}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, +1, +1, -1, -t\right),\\
f_t(\mathbf{4}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, -t, +t, -t, +1\right), & f_t(\mathbf{4}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, -1, +1, -1, +t\right),\\
f_t(\mathbf{5}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, -t, +t, +t, -1\right), & f_t(\mathbf{5}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, -1, +1, +1, -t\right),\\
f_t(\mathbf{6}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, -t, -t, +t, +1\right), & f_t(\mathbf{6}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, -1, -1, +1, +t\right),\\
f_t(\mathbf{7}^+) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}\,t, -t, -t, -t, -1\right), & f_t(\mathbf{7}^-) &= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{2}, -1, -1, -1, -t\right),\\
f_t(\boldsymbol{A}) &= \left(1, +\sqrt{2}, 0, 0, 0\right), & f_t(\boldsymbol{B}) &= \left(1, 0, +\sqrt{2}, 0, 0\right),\\
f_t(\boldsymbol{C}) &= \left(1, 0, 0, +\sqrt{2}, 0\right), & f_t(\boldsymbol{D}) &= \left(1, 0, 0, -\sqrt{2}, 0\right),\\
f_t(\boldsymbol{E}) &= \left(1, 0, -\sqrt{2}, 0, 0\right), & f_t(\boldsymbol{F}) &= \left(1, -\sqrt{2}, 0, 0, 0\right).
\end{aligned}
$$

TABLE 3. The list of vectors for the definition of ρ_t, in the AdS case. The quadratic form q_{-1} takes value -1 on the vectors $f_t(\boldsymbol{i}^+)$, and $+1$ on the vectors $f_t(\boldsymbol{i}^-)$ and $f_t(\boldsymbol{X})$.

(1) It can be checked that all the orthogonality relations (with respect to the bilinear form b_1) between vectors in Table 1 are maintained for the vectors in Table 2 with respect to b_1, and in Table 3 with respect to b_{-1}. This shows that Definition 4.1 is well-posed, meaning that ρ_t are representations of Γ_{22} by Lemma 2.3.
(2) By construction the representations ρ_t are in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$, for $G = \mathrm{Isom}(\mathbb{H}^n)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$ (see Definition 2.8). Tables 2 and 3 exhibit continuous lifts $f_t \colon S \to \mathbb{R}^{110}$ as in Lemma 2.10, taking values in a subset of $\mathbb{R}^{110}$ defined by the vanishing of 102 quadratic conditions, for S the standard generating set of Γ_{22}.
(3) The vectors of Table 2 coincide with those of Table 1 for $t = 1$. Hence, in the hyperbolic case, the path of representations ρ_t is a deformation of the reflection group of the aforementioned right-angled polytope with 22 facets. For $t \in (0,1)$, this coincides with the path of representations exhibited in [KS10]. For $t \in (-1,0)$, the representation ρ_t is obtained by conjugating ρ_{-t} by the reflection r in the "horizontal" hyperplane $x_4 = 0$

$$r\colon (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0, x_1, x_2, x_3, -x_4)\ . \tag{5}$$

(This is seen immediately using Remark 2.11.)
(4) On the Anti-de Sitter side, the path ρ_t has been exhibited in [RS] for $t \in (-1,0)$. Again, the path is extended here for positive times by conjugation by r.
(5) Both these paths occur as the holonomy representations of a deformation of hyperbolic and Anti-de Sitter cone-orbifold structures. The purpose of our previous work [RS] was to describe the geometric transition from hyperbolic ($t > 0$) to Anti-de Sitter ($t < 0$) structures. Since here we are interested in the $\mathrm{Isom}(\mathbb{H}^4)$- and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-chatacter varieties on their own, we found more useful to treat the two paths ρ_t separately, and extend each of them by orientation-reversing conjugation also for negative (resp. positive) times.

4.3. **The collapsed representation and the cuboctahedron.** For $t = 0$ the hyperbolic and Anti-de Sitter representations ρ_0 take value in the stabiliser of the hyperplane $\{x_4 = 0\}$, which is a a totally geodesic copy of $\mathbb{H}^3$ in both $\mathbb{H}^4$ and $\mathbb{A}\mathrm{d}\mathbb{S}^4$. Unlike the case $t \neq 0$, these representations are not holonomies of hyperbolic/AdS orbifold structures, but correspond to what we call the *collapse* of the respective geometric structures.

If we consider $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ as subgroups of $\mathrm{PGL}(5,\mathbb{R})$, then the representations ρ_0 agree for the hyperbolic and AdS case. The stabiliser

$$G_0 = \mathrm{Stab}_{\mathrm{Isom}(\mathbb{H}^4)}(\{x_4 = 0\}) = \mathrm{Stab}_{\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)}(\{x_4 = 0\})$$

is indeed the common subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ consisting of projective classes in $\mathrm{PGL}(5,\mathbb{R})$ of matrices in the block form:

$$\left[\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & \pm 1 \end{array}\right].$$

The stabiliser G_0 is isomorphic to $\mathrm{Isom}(\mathbb{H}^3) \times \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}/2\mathbb{Z}$-factor is generated by the reflection r of Equation (5), which acts by switching the two sides of $\{x_4 = 0\}$. Under this isomorphism, the representation ρ_0 reads as:

$$
\begin{aligned}
v_{\mathbf{0}} &= \left(\sqrt{2},+1,+1,+1\right), & & \\
v_{\mathbf{1}} &= \left(\sqrt{2},+1,-1,+1\right), & v_{\boldsymbol{A}} &= \left(1,+\sqrt{2},0,0\right), \\
v_{\mathbf{2}} &= \left(\sqrt{2},+1,-1,-1\right), & v_{\boldsymbol{B}} &= \left(1,0,+\sqrt{2},0\right), \\
v_{\mathbf{3}} &= \left(\sqrt{2},+1,+1,-1\right), & v_{\boldsymbol{C}} &= \left(1,0,0,+\sqrt{2}\right), \\
v_{\mathbf{4}} &= \left(\sqrt{2},-1,+1,-1\right), & v_{\boldsymbol{D}} &= \left(1,0,0,-\sqrt{2}\right), \\
v_{\mathbf{5}} &= \left(\sqrt{2},-1,+1,+1\right), & v_{\boldsymbol{E}} &= \left(1,0,-\sqrt{2},0\right), \\
v_{\mathbf{6}} &= \left(\sqrt{2},-1,-1,+1\right), & v_{\boldsymbol{F}} &= \left(1,-\sqrt{2},0,0\right). \\
v_{\mathbf{7}} &= \left(\sqrt{2},-1,-1,-1\right), & &
\end{aligned}
$$

TABLE 4. The vectors $v_{\boldsymbol{i}}, v_{\boldsymbol{X}} \in \mathbb{R}^{1,3}$ defining the bounding planes of an ideal right-angled cuboctahedron in $\mathbb{H}^3$. These vectors are involved also in the Definition 6.3, introducting the cocycles τ_λ in the vector space $Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$.

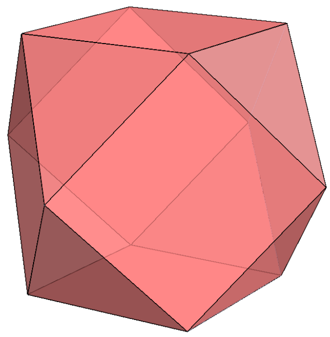

FIGURE 5. A cuboctahedron is the convex envelop of the midpoints of the edges of a regular cube (or octahedron). It is realised in $\mathbb{H}^3$ as an ideal right-angled polytope.

$$
\begin{aligned}
\rho_0(\boldsymbol{i}^+) &= r && \text{for each } \boldsymbol{i} \in \{\mathbf{0},\dots,\mathbf{7}\} \\
\rho_0(\boldsymbol{i}^-) &= r_{v_{\boldsymbol{i}}} && \text{for each } \boldsymbol{i} \in \{\mathbf{0},\dots,\mathbf{7}\} \\
\rho_0(\boldsymbol{X}) &= r_{v_{\boldsymbol{X}}} && \text{for each } \boldsymbol{X} \in \{\boldsymbol{A},\dots,\boldsymbol{F}\}
\end{aligned}
\tag{6}
$$

where the vectors $v_{\boldsymbol{i}}, v_{\boldsymbol{X}} \in \mathbb{R}^{1,3}$, collected in Table 4, define the bounding planes $H_{v_{\boldsymbol{i}}}$ and $H_{v_{\boldsymbol{X}}}$ of an ideal right-angled cuboctahedron in $\mathbb{H}^3$. The triangular faces of this cuboctahedron are of type $\boldsymbol{i}$, while the quadrilateral faces are of type $\boldsymbol{X}$ (see Figure 5).

4.4. **The conjugacy action.** We are ready to start the study of the hyperbolic and AdS character varieties of Γ_{22} near the representation ρ_t introduced in Definition 4.1. We are mostly interested in the topology of the character variety, rather than its structure of (semi)algebraic affine set, so we shall avoid the language of GIT.

We can define the character variety as a "Hausdorff quotient" of the representation variety by conjugation:

Definition 4.2 (Character variety)**.** Let G be $\mathrm{Isom}(\mathbb{H}^n)$, G_{HP^n} or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, and G^+ denote its subgroup of orientation-preserving transformations. Given a finitely generated group Γ, let also $\mathrm{Hom}(\Gamma, G)^*$ be the subset of $\mathrm{Hom}(\Gamma, G)$ consisting of points with closed orbits for the conjugacy action of G^+. (Note that the action preserves $\mathrm{Hom}(\Gamma, G)^*$.) We call *character variety* of Γ in G, denoted by

$$X(\Gamma, G) = \mathrm{Hom}(\Gamma, G) /\!\!/ G^+ \ ,$$

the quotient of $\mathrm{Hom}(\Gamma, G)^*$ by the action of G^+ by conjugation.

In the special case $\Gamma = \Gamma_{22}$, we now describe some properties of the action of G by conjugation on $\mathrm{Hom}(\Gamma_{22}, G)$. For $t \neq 0$, nearby ρ_t the action of G is "good", namely is free and proper, as we will see in the following two lemmas.

Lemma 4.3. *For $t \neq 0$, the stabiliser of ρ_t in G is trivial. The stabiliser of ρ_0 in G is the order-two subgroup generated by the reflection r in the hyperplane $\mathbb{H}^3 = \{x_4 = 0\}$.*

Proof. We give the proof for the hyperbolic and AdS case at the same time, since they are completely analogous. By Remark 2.11, any element in the stabiliser of ρ_t is induced by a matrix $A \in \mathrm{O}(q_{\pm 1})$ which maps every vector $f_t(\boldsymbol{s})$ in Table 2 or Table 3 either to itself or to its opposite. Since the 6 vectors $f_t(\boldsymbol{A}), \ldots, f_t(\boldsymbol{F})$ do not depend on t and generate the orthogonal complement of the hyperplane $\{x_4 = 0\}$, the matrix A must preserve the hyperplane $\{x_4 = 0\}$.

Moreover, let $\mathcal{P}_t$ be the polytope bounded by the 22 hyperplanes orthogonal to the vectors of Tables 2 or 3. It was proved in [MR18, Proposition 3.19] and [RS, Proposition 7.21] that the intersection of $\mathcal{P}_t$ with the hyperplane defined by the equation $x_4 = 0$ (a totally geodesic copy of $\mathbb{H}^3$) is constant and is an ideal right-angled cuboctahedron (see Section 4.3). Since the action of A on the projectivisation of $\{x_4 = 0\}$ necessarily preserves each face of the cuboctahedron, it follows that A must act on the linear hyperplane $\{x_4 = 0\}$ as $\pm\mathrm{id}$.

This shows that the only non-trivial candidates for A are $\pm r$, where r is the reflection of Equation (5). For $t = 0$, the reflection r preserves all the hyperplanes orthogonal to the vectors of Tables 2 or 3, hence the associated element in G generates the stabiliser of ρ_0. When $t \neq 0$, the reflection r does not preserve any of the hyperplanes of the form $H_{f_t(\boldsymbol{i}^+)}$ and $H_{f_t(\boldsymbol{i}^-)}$, hence the stabiliser of ρ_t is trivial in this case. □

The next lemma will be useful to show that the action of G^+ by conjugation is proper, in a suitable region of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$.

Lemma 4.4. *Suppose that η_n is a sequence in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$ converging to some ρ_t, and h_n is a sequence in G such that $h_n \cdot \eta_n$ converges. Then h_n has a subsequence that converges in G.*

Proof. Suppose that $\eta_n \to \rho_t$ and h_n is a sequence in G such that $h_n \cdot \eta_n \to \eta_\infty$. Since $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$ is clopen in the representation variety, the limit point η_∞ is in $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$. Passing to the finite cover $g^{-1}(0)$ of Lemma 2.10, and up to taking subsequences, we can then assume to have a sequence f_n in $g^{-1}(0)$ (projecting to η_n) such that $f_n \to f_\infty$ and $h_n \cdot f_n \to \widehat{f}_\infty$. Here we are thinking of $f_n, f_\infty, \widehat{f}_\infty$ as functions from the standard generators of Γ_{22} to $\mathbb{R}^5$, and (by a small abuse of notation) h_n is a sequence in $\mathrm{O}(q_{\pm 1})$ acting by the obvious action on $\mathbb{R}^5$ (see Remark 2.11).

We have to show that h_n converges in $\mathrm{O}(q_{\pm 1})$ up to subsequences. Recall that f_∞ is a lift in $g^{-1}(0)$ of ρ_t, and therefore (up to changes of sign) the vectors $f_\infty(\boldsymbol{s})$ are given by Table 2 or Table 3 for some value of t. Take five generators $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_5$ of Γ_{22} such that $f_\infty(\boldsymbol{s}_1), \ldots, f_\infty(\boldsymbol{s}_5)$ are linearly independent, for instance $\mathbf{0}^-, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$. Since linear independence is an open condition, $\{f_n(\boldsymbol{s}_1), \ldots, f_n(\boldsymbol{s}_5)\}$ forms a basis of $\mathbb{R}^5$ for large n.

The linear isometry $h_n \in \mathrm{O}(q_{\pm 1})$, considered as a 5-by-5 matrix, is therefore determined by the condition that h_n sends the basis $\{f_n(\boldsymbol{s}_1), \ldots, f_n(\boldsymbol{s}_5)\}$ to $\{h_n \cdot f_n(\boldsymbol{s}_1), \ldots, h_n \cdot f_n(\boldsymbol{s}_5)\}$. More concretely, we can write h_n (as a matrix) as $(h_{n,1})^{-1} \circ h_{n,2}$, where $h_{n,1}$ is the matrix sending the standard basis to the basis $\{f_n(\boldsymbol{s}_1), \ldots, f_n(\boldsymbol{s}_5)\}$, and $h_{n,2}$ is the matrix sending the standard basis to the basis $\{h_n \cdot f_n(\boldsymbol{s}_1), \ldots, h_n \cdot f_n(\boldsymbol{s}_5)\}$. Since f_n and $h_n \cdot f_n$ are converging sequences, we have that $h_{n,1} \to h_{\infty,1}$ and $h_{n,2} \to h_{\infty,2}$, and moreover $h_{\infty,1}$ is invertible since $f_\infty(\boldsymbol{s}_1), \ldots, f_\infty(\boldsymbol{s}_5)$ is a basis.

Therefore h_n converges to a 5-by-5 matrix $h_\infty = (h_{\infty,1})^{-1} \circ h_{\infty,2}$, which is still in $\mathrm{O}(q_{\pm 1})$ since $\mathrm{O}(q_{\pm 1})$ is closed in the space of 5-by-5 matrices. This concludes the proof. □

Remark 4.5. In the portion of the character variety of our interest, no non-Hausdorff pathological situation arises. More precisely, the GIT quotient $\mathrm{Hom}(\Gamma_{22}, G) /\!\!/ G^+$ coincides with the ordinary topological quotient in a neighbourhood of each $[\rho_t]$. (This holds similarly for $\mathrm{Hom}(\Gamma_{22}, G) /\!\!/ G$.) For, it follows from Lemma 4.4 that:

- The G^+–action is proper on $G^+ \cdot \{\rho_t \mid t \in (-1, 1)\}$.
- For each t, the G^+–orbit of ρ_t is closed. (This follows by applying Lemma 4.4 to the constant sequence $\eta_n \equiv \rho_t$).

Actually the latter is true in a neighborhood of $\{\rho_t \mid t \in (-1, 1)\}$, since in the proof of Lemma 4.4 we only used that, for five generators $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_5$ of Γ_{22}, the corresponding vectors in $\mathbb{R}^5$ are linearly independent, and this is still true in an open neighborhood.

In fact, our argument shows a little more, namely that if ρ is in such a neighbourhood, then $[\rho]$ is separated from any other point in the topological quotient $\mathrm{Hom}(\Gamma_{22}, G)/G^+$. This is because, if $[\rho]$ were not separated from $[\rho']$, we would have a sequence $\rho_n \to \rho$ and a sequence h_n such that $h_n \rho_n h_n^{-1}$ converges to ρ'. But Lemma 4.4 shows that $h_n \to h_\infty$ up to subsequences, hence by continuity h_∞ conjugates ρ and ρ', namely $[\rho] = [\rho']$.

4.5. A smoothness result.

In the hyperbolic case, the smoothness of the $\mathrm{Isom}(\mathbb{H}^4)$-character variety near the points $[\rho_t]$ with $t \neq 0$ has been proved in [KS10, Theorem 12.3]:

Proposition 4.6. *For $t \in (0, 1)$, the space $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{H}^4))$ is a smooth 11-dimensional manifold near ρ_t.*

(Recalling that for negative times ρ_t is a conjugate of ρ_{-t}, the result holds for $t \in (-1, 0)$ as well.)

Our main purpose is to extend and generalise the analysis for $t = 0$, and do similarly for the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-character variety.

Let us first briefly sketch the lines of the proof Proposition 4.6 given in [KS10]. By Lemma 2.10 (recall $g \colon \mathbb{R}^{(n+1)|S|} \to \mathbb{R}^{|S|+|R|}$ from the proof), it suffices to show that $g^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^{110}$ near any preimage of ρ_{t_0}, for all $t_0 \in (0, 1)$.

Let
$$f_t \colon \{\text{standard generators of } \Gamma_{22}\} \to \mathbb{R}^5 \tag{7}$$
be as in Table 2, so giving an embedding of $(0, 1)$ into $g^{-1}(0) \subset \mathbb{R}^{110}$ going through a preimage of ρ_{t_0}. The proof in [KS10] essentially consists in showing that the kernel of $g \colon \mathbb{R}^{110} \to \mathbb{R}^{102}$ is 11-dimensional for $t \in (0, 1)$. Since there is a 10-dimensional smooth orbit given by the action of $\mathrm{Isom}^+(\mathbb{H}^4)$, the proof boils down to showing that the tangent space to the orbit has a 1-dimensional complement, which is indeed given by the tangent space to the 1-dimensional submanifold $\{f_t \mid t \in (0, 1)\}$.

Since the action of $\mathrm{Isom}^+(\mathbb{H}^4)$ is smooth, it then follows that the $\mathrm{Isom}^+(\mathbb{H}^4)$-orbit of the curve $\{\rho_t \mid t \in (0,1)\}$ is a smooth 11-dimensional manifold, on which the $\mathrm{Isom}^+(\mathbb{H}^4)$-action by conjugation is free and proper by Lemma 4.3 and Lemma 4.4. Hence it follows from Proposition 4.6 that $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{H}^4))$ is a 1-dimensional smooth manifold near $[\rho_t]$, for $t \in (0,1)$.

In the next sections, we will prove the analogous of Proposition 4.6 for the AdS case. However, we are interested also in the study of the character variety near "the collapse", that is the point $[\rho_0]$. Hence we will prove a more detailed statement.

Let G be as usual $\mathrm{Isom}(\mathbb{H}^4)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$.

Definition 4.7 (The space Hom_0)**.** We define $\mathrm{Hom}_0(\Gamma_{22}, G)$ as the subset of $\mathrm{Hom}_{\mathrm{refl}}(\Gamma_{22}, G)$ of representations ρ such that the following holds. Let $\boldsymbol{s}_1, \boldsymbol{s}_2$ be any pair of generators of Γ_{22} such that the hyperplanes fixed by $\rho_t(\boldsymbol{s}_1)$ and $\rho_t(\boldsymbol{s}_2)$ are either tangent at infinity or equal for some $t \neq 0$. Then, so are the hyperplanes fixed by $\rho(\boldsymbol{s}_1)$ and $\rho(\boldsymbol{s}_2)$.

Recall from Lemmas 2.4 and 2.5 that two hyperplanes are tangent at infinity or equal if and only if, using the bilinear form b_1 for $\mathbb{H}^4$ and b_{-1} for $\mathbb{A}\mathrm{d}\mathbb{S}^4$, the product of their orthogonal unit vectors is 1 in absolute value. It is thus easy to check from Tables 2 and 3 that this condition is preserved by the deformation f_t for all t both in the hyperbolic and AdS case, and thus the definition is well-posed (i.e. it does not depend on the choice of $t \neq 0$).

In the setting of Lemma 2.10, $\mathrm{Hom}_0(\Gamma_{22}, G)$ corresponds to a subset of $g^{-1}(0) \subset \mathbb{R}^{110}$ defined by the vanishing of 36 more quadratic conditions. Indeed, for each of the 12 ideal vertices of the polytope $\mathcal{P}_t$ bounded by the hyperplanes of Tables 2 and 3, we have 3 tangency conditions (see [RS, Proposition 7.13]). Hence $\mathrm{Hom}_0(\Gamma_{22}, G)$ is locally homeomorphic to the zero locus of a function $g_0 \colon \mathbb{R}^{110} \to \mathbb{R}^{138}$ extending g. More precisely:

Lemma 4.8. *The space* $\mathrm{Hom}_0(\Gamma_{22}, G)$ *is finitely covered by a disjoint union of subsets of* $\mathbb{R}^{110}$ *defined by the vanishing of* 138 *quadratic conditions.*

Remark 4.9. For simplicity of exposition, from now on we will work in the AdS setting, i.e. in the case $G = \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$. All what follows can be easily adapted to the hyperbolic case. We will therefore omit the proofs and only highlight the points where differences with respect to the AdS case occur.

The essential property we will prove is that near each of the representations ρ_t the variety $\mathrm{Hom}_0(\Gamma_{22}, G)$ is smooth. Hence the goal of the next two sections is to prove the following:

Proposition 4.10. *For* $t \in (-1,1)$*, the space* $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ *is a smooth 11-dimensional manifold near* ρ_t*.*

The proof of Proposition 4.10 will be given at the end of Section 4.7. From the results on cusp rigidity established in Section 3.2, we obtain the smoothness of $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ for $t \neq 0$ as a direct corollary:

Corollary 4.11. *For* $t \in (-1,1) \setminus \{0\}$*, the space* $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ *is a smooth 11-dimensional manifold near* ρ_t*.*

Proof. It is not difficult to check that, when $t \neq 0$, for every pair of generators $\boldsymbol{s}_1, \boldsymbol{s}_2$ of Γ_{22} such that the associated hyperplanes $H_{f_t(\boldsymbol{s}_1)}$ and $H_{f_t(\boldsymbol{s}_2)}$ are tangent at infinity, there are 4 other generators $\boldsymbol{s}_3, \ldots, \boldsymbol{s}_6$ such that the reflections $r_{\boldsymbol{s}_1}, \ldots, r_{\boldsymbol{s}_6}$ generate a cusp group in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$. By Lemma 3.10, the tangencies at infinity are preserved since cusp groups stay cusp groups under small deformations. Hence a neighbourhood of ρ_t in $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$

is actually contained in $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$. The proof now follows from Proposition 4.10. □

The next sections will be devoted to the proof of Proposition 4.10. We will adapt some of the ideas of [KS10, Sections 5, 11, 12] used in the proof of Proposition 4.6 in the hyperbolic case. An analogous argument shows that the statement of Proposition 4.10 holds also for the $\mathbb{H}^4$-character variety, which for $t = 0$ is new with respect to the results of [KS10].

4.6. **Infinitesimal deformations of the "letter" generators.** Recall Lemma 4.8. Throughout this and the following sections, we denote by

$$g_0 \colon \mathbb{R}^{110} \to \mathbb{R}^{138}$$

the quadratic function defining the clopen subset of $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ that contains the lifts of the representations ρ_t. A continuous lift of the path $t \mapsto \rho_t$ is defined by f_t in Table 3. To prove Proposition 4.10 in the AdS case, it then suffices to show that for all $t \in (-1, 1)$ the set $g_0^{-1}(0) \subset \mathbb{R}^{110}$ is a smooth 11-dimensional manifold near each f_t.

Notation. Let us fix $t \in (-1, 1)$. For simplicity, by an abuse of notation, in this and next section we denote $f_t(\boldsymbol{s}) \in \mathbb{R}^5$ by $\boldsymbol{s}$. In other words, in what follows $\boldsymbol{s} \in \mathbb{R}^5$ denotes a vector (of q_{-1}-norm 1 or -1 depending whether the corresponding hyperplane in $\mathbb{A}\mathrm{d}\mathbb{S}^4$ is timelike or spacelike, respectively) from Table 3, and is therefore implicitly considered as a function of t. Its derivative in t will be denoted by $\dot{\boldsymbol{s}}$. The symbol $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ will denote the corresponding element of $g_0^{-1}(0) \subset \mathbb{R}^{110}$, as a function from the standard generators of Γ_{22} to $\mathbb{R}^5$, while $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ will denote an element in the kernel of the differential of g_0 at $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$, and will be called an *infinitesimal deformation* of $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$.

Observe that the vectors $\boldsymbol{A}, \dots, \boldsymbol{F}$ of Table 3 are constant in t, hence the derivative of the path in $g_0^{-1}(0)$ provided by Table 3 satisfies $\dot{\boldsymbol{X}} = 0$ for all $\boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}$.

By Remark 2.11, the natural $\mathrm{O}(q_{-1})$–action on $g_0^{-1}(0)$ is given by $\boldsymbol{s} \mapsto A \cdot \boldsymbol{s}$ for $A \in \mathrm{O}(q_{-1})$. Therefore the tangent space to the orbit of an element $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ of $g_0^{-1}(0)$ consists precisely of the elements of the kernel of dg_0 of the form

$$\boldsymbol{s} \mapsto \dot{\boldsymbol{s}} = \mathfrak{a} \cdot \boldsymbol{s} \ , \tag{8}$$

where $\boldsymbol{s}$ varies in $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ and $\mathfrak{a} = \frac{d}{dt}\big|_{t=0} A_t \in \mathfrak{so}(q_{-1})$, for any smooth path $t \mapsto A_t$ in $\mathrm{O}(q_{-1})$ with $A_0 = \mathrm{id}$.

The first step in the proof of Proposition 4.10 is to show that, up to this infinitesimal action, we can assume that *any* infinitesimal deformation vanishes at least on 4 elements of $\{\boldsymbol{A}, \dots, \boldsymbol{F}\}$.

Lemma 4.12. *Fix $t \in (-1, 1)$, and let $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ be an infinitesimal deformation of $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$. Up to the action of $\mathfrak{a} \in \mathfrak{so}(q_{-1})$ as in (8), we can assume that*

$$\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = \dot{\boldsymbol{D}} = 0 \ , \tag{9}$$

and that

$$\dot{\boldsymbol{E}} = (0, 0, 0, 0, \epsilon) \qquad \text{and} \qquad \dot{\boldsymbol{F}} = (0, 0, 0, 0, \phi) \tag{10}$$

for some $\epsilon, \phi \in \mathbb{R}$.

The analogous lemma in the hyperbolic case, for $t \neq 0$, has been proved in [KS10, Proposition 11.1], and in fact the arguments here follow roughly the same lines as their proof. However, the first part of their proof uses a nice geometric argument which would be complicated to adapt to AdS geometry. For this reason, we rather use a linear algebra argument here.

Notation. To simplify the notation, from here to the end of Section 4.7, we denote by $\langle\cdot,\cdot\rangle$ the bilinear form b_{-1}. If one wants to repeat the proof for $G = \mathrm{Isom}(\mathbb{H}^4)$, then $\langle\cdot,\cdot\rangle$ should denote b_1. The reader should pay attention that in Section 6 the bracket $\langle\cdot,\cdot\rangle$ will instead be used to denote the Minkowski bilinear form on $\mathbb{R}^n$.

Proof. The proof will follow from three claims.

First we claim that we can assume $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = 0$. Equivalently, given any infinitesimal deformation $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$, we want to show that there exists $\mathfrak{a} \in \mathfrak{so}(q_{-1})$ such that

$$\mathfrak{a}\cdot\boldsymbol{A} = \dot{\boldsymbol{A}}\ , \qquad \mathfrak{a}\cdot\boldsymbol{B} = \dot{\boldsymbol{B}}\ , \qquad \mathfrak{a}\cdot\boldsymbol{C} = \dot{\boldsymbol{C}}\ . \tag{11}$$

Indeed, if (11) is true, we can then subtract to $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ the element in the tangent space to the orbit of the form (8) (i.e. given by $\dot{\boldsymbol{s}} = \mathfrak{a}\cdot\boldsymbol{s}$) and obtain a new infinitesimal deformation for which $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = 0$.

To show the first claim, consider the basis $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, e_4\}$ of $\mathbb{R}^5$, where $e_4 = (0,0,0,0,1)$. Recall that matrices $\mathfrak{a}$ in the Lie algebra $\mathfrak{so}(q_{-1})$ are characterised by the condition that

$$\langle\mathfrak{a}\cdot u, w\rangle + \langle u, \mathfrak{a}\cdot w\rangle = 0 \tag{12}$$

for every u, w, and that it suffices in fact to check the condition for all pair of elements u, w of our fixed basis. Moreover, to define the matrix $\mathfrak{a}$ in $\mathfrak{so}(q_{-1})$, it suffices to define it on 4 vectors of the basis of $\mathbb{R}^5$, such that (12) holds when u, w are chosen among these 4 vectors. The definition of $\mathfrak{a}$ on the last vector of the basis is then uniquely determined by (12).

Let us now apply these preliminary remarks. By differentiating the conditions

$$\langle\boldsymbol{A}, \boldsymbol{A}\rangle = \langle\boldsymbol{B}, \boldsymbol{B}\rangle = \langle\boldsymbol{C}, \boldsymbol{C}\rangle = 1$$

we obtain

$$\langle\boldsymbol{A}, \dot{\boldsymbol{A}}\rangle = \langle\boldsymbol{B}, \dot{\boldsymbol{B}}\rangle = \langle\boldsymbol{B}, \dot{\boldsymbol{B}}\rangle = 0\ . \tag{13}$$

By differentiating the tangency conditions

$$\langle\boldsymbol{A}, \boldsymbol{B}\rangle = \langle\boldsymbol{A}, \boldsymbol{C}\rangle = \langle\boldsymbol{B}, \boldsymbol{C}\rangle = -1$$

we get the conditions

$$\langle\boldsymbol{A}, \dot{\boldsymbol{B}}\rangle + \langle\dot{\boldsymbol{A}}, \boldsymbol{B}\rangle = 0, \quad \langle\boldsymbol{A}, \dot{\boldsymbol{C}}\rangle + \langle\dot{\boldsymbol{A}}, \boldsymbol{C}\rangle = 0, \quad \langle\boldsymbol{B}, \dot{\boldsymbol{C}}\rangle + \langle\dot{\boldsymbol{B}}, \boldsymbol{C}\rangle = 0\ . \tag{14}$$

Equations (13) and (14) show that a linear transformation $\mathfrak{a}$ sending $\boldsymbol{A}$ to $\dot{\boldsymbol{A}}$, $\boldsymbol{B}$ to $\dot{\boldsymbol{B}}$ and $\boldsymbol{C}$ to $\dot{\boldsymbol{C}}$ satisfies the conditions of (12) for all pairs of u, w chosen in $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$. Imposing (12) one can also define $\mathfrak{a}$ on the two remaining elements $\boldsymbol{D}$ and e_4 of the fixed basis so as to satisfy (12) for all u, w (with one parameter of freedom). This shows that we can find $\mathfrak{a}$ satisfying Equation (11), and our first claim is proved.

Second, we claim that we can further assume that

$$\langle\dot{\boldsymbol{D}}, e_4\rangle = 0\ .$$

To see this second claim, by repeating the same reasoning as in the beginning of this proof, it suffices to find another $\mathfrak{a}' \in \mathfrak{so}(q_{-1})$ so that

$$\mathfrak{a}'\cdot\boldsymbol{A} = \mathfrak{a}'\cdot\boldsymbol{B} = \mathfrak{a}'\cdot\boldsymbol{C} = 0 \qquad \text{and} \qquad \mathfrak{a}'\cdot\boldsymbol{D} = \langle\dot{\boldsymbol{D}}, e_4\rangle e_4\ . \tag{15}$$

Indeed, if (15) holds then the conditions (12) are satisfied for u, w chosen in $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$, and we have already remarked that $\mathfrak{a}'\cdot e_4$ will then be uniquely determined by (12) in such a way that $\mathfrak{a}' \in \mathfrak{so}(q_{-1})$. This shows our second claim.

Finally we claim that, under the above assumptions, necessarily $\dot{\boldsymbol{D}} = 0$, $\dot{\boldsymbol{E}} = (0,0,0,0,\epsilon)$ and $\dot{\boldsymbol{F}} = (0,0,0,0,\phi)$. This part of the proof follows closely [KS10, Proposition 11.1].

As observed in the proof of Corollary 4.11, since $\langle \boldsymbol{A}, \boldsymbol{D} \rangle = -1$, the vectors $\boldsymbol{A}$ and $\boldsymbol{D}$ play the role of two non-commuting generators (reflections along two timelike hyperplanes that are tangent at infinity) of a cusp group generated by the images of $\boldsymbol{A}$, $\boldsymbol{D}$, $\mathbf{3}^+$, $\mathbf{3}^-$, $\mathbf{2}^+$, and $\mathbf{2}^-$. By the assumption that tangencies at infinity are preserved (recall that we are in Hom_0), any deformation of $\boldsymbol{A}$ and $\boldsymbol{D}$ satisfies $\langle \boldsymbol{A}, \boldsymbol{D} \rangle = -1$. So, by differentiating and using $\dot{\boldsymbol{A}} = 0$, we obtain $\langle \boldsymbol{A}, \dot{\boldsymbol{D}} \rangle = 0$. Analogously, $\langle \boldsymbol{B}, \dot{\boldsymbol{D}} \rangle = 0$.

Together with $\langle \boldsymbol{D}, \dot{\boldsymbol{D}} \rangle = 0$ (which follows from $\langle \boldsymbol{D}, \boldsymbol{D} \rangle = 1$) and the assumption $\langle \dot{\boldsymbol{D}}, v \rangle = 0$, we have necessarily

$$\dot{\boldsymbol{D}} = (\sqrt{2}\delta, \delta, \delta, -\delta, 0)$$

for some δ. Similarly for $\dot{\boldsymbol{E}}$, using that $\langle \boldsymbol{A}, \dot{\boldsymbol{E}} \rangle = \langle \boldsymbol{C}, \dot{\boldsymbol{E}} \rangle = \langle \boldsymbol{E}, \dot{\boldsymbol{E}} \rangle = 0$, we find

$$\dot{\boldsymbol{E}} = (\sqrt{2}\epsilon', \epsilon', -\epsilon', \epsilon', \epsilon) \ .$$

For $\boldsymbol{F}$, from $\langle \boldsymbol{B}, \dot{\boldsymbol{F}} \rangle = \langle \boldsymbol{C}, \dot{\boldsymbol{F}} \rangle = \langle \boldsymbol{F}, \dot{\boldsymbol{F}} \rangle = 0$ we find

$$\dot{\boldsymbol{F}} = (\sqrt{2}\phi', -\phi', \phi', \phi', \phi) \ .$$

Now using that $\boldsymbol{D}$ and $\boldsymbol{E}$ remain tangent at infinity, and similarly for the pairs $\{\boldsymbol{D}, \boldsymbol{F}\}$ and $\{\boldsymbol{E}, \boldsymbol{F}\}$, we have the relations

$$\langle \boldsymbol{D}, \dot{\boldsymbol{E}} \rangle + \langle \dot{\boldsymbol{D}}, \boldsymbol{E} \rangle = 0 \ , \quad \langle \boldsymbol{D}, \dot{\boldsymbol{F}} \rangle + \langle \dot{\boldsymbol{D}}, \boldsymbol{F} \rangle = 0 \ , \quad \langle \boldsymbol{E}, \dot{\boldsymbol{F}} \rangle + \langle \dot{\boldsymbol{E}}, \boldsymbol{F} \rangle = 0 \ ,$$

which read as:

$$2\sqrt{2}\,\delta + 2\sqrt{2}\,\epsilon' = 0 \ , \qquad 2\sqrt{2}\,\delta + 2\sqrt{2}\,\phi' = 0 \ , \qquad 2\sqrt{2}\,\epsilon' + 2\sqrt{2}\,\phi' = 0 \ .$$

Hence $\delta = \epsilon' = \phi' = 0$, and this shows the claim. The proof of Lemma 4.12 is complete. □

4.7. **Infinitesimal deformations of the "positive" and "negative" generators.** We conclude in this section the proof of Proposition 4.10.

A direct computation from Table 3 shows that the tangent vector to our explicit path f_t in $g_0^{-1}(0)$ is given by:

$$\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^- \qquad \dot{\boldsymbol{i}}^- = \lambda \boldsymbol{i}^+ \qquad \dot{\boldsymbol{X}} = 0 \tag{16}$$

where

$$\lambda = \frac{1}{(1-t^2)^{3/2}} \ ,$$

for all $\boldsymbol{i} \in \{\mathbf{0}, \dots, \mathbf{7}\}$ and $\boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}$. (In the hyperbolic case, from Table 2, one would instead obtain $\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^-$, $\dot{\boldsymbol{i}}^- = -\lambda \boldsymbol{i}^+$ and $\dot{\boldsymbol{X}} = 0$ for $\lambda = (1+t^2)^{-3/2}$.)

We shall now show that, under the assumption in the statement of Lemma 4.12, *every* infinitesimal deformation $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ of $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ satisfies (16) for some λ. Again, the proof follows roughly the lines of [KS10, Section 12], with the necessary adaptations to the AdS setting, and some simplifications.

Lemma 4.13. *Fix $t \in (-1,1)$, and let $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ be an infinitesimal deformation of the normalised vectors $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ satisfying (9) and (10). Then*

$$\begin{aligned} \dot{\mathbf{0}}^+ &= \lambda \mathbf{0}^- \qquad & \dot{\mathbf{0}}^- &= \lambda \mathbf{0}^+ \\ \dot{\mathbf{3}}^+ &= \lambda \mathbf{3}^- \qquad & \dot{\mathbf{3}}^- &= \lambda \mathbf{3}^+ \end{aligned} \tag{17}$$

for some $\lambda \in \mathbb{R}$ (depending on t).

Proof. Using the assumptions $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = 0$, the derivatives of the relations $\langle \mathbf{0}^+, \boldsymbol{A} \rangle = \langle \mathbf{0}^+, \boldsymbol{B} \rangle = \langle \mathbf{0}^+, \boldsymbol{C} \rangle = 0$ yield

$$\langle \dot{\mathbf{0}}^+, \boldsymbol{A} \rangle = \langle \dot{\mathbf{0}}^+, \boldsymbol{B} \rangle = \langle \dot{\mathbf{0}}^+, \boldsymbol{C} \rangle = 0 \ . \tag{18}$$

Together with

$$\langle \dot{\mathbf{0}}^+, \mathbf{0}^+ \rangle = 0 \;, \tag{19}$$

we obtain $\dot{\mathbf{0}}^+ = \lambda_0^+ \mathbf{0}^-$ for some λ_0^+.

Indeed, the vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\mathbf{0}^+$ are linearly independent, and $\mathbf{0}^-$ satisfies all the four linear conditions (18) and (19), hence $\mathbf{0}^-$ spans the space of solutions. Similarly for $\mathbf{0}^-$, we obtain $\dot{\mathbf{0}}^- = \lambda_0^- \mathbf{0}^+$, and repeating the same argument for $\mathbf{3}^+$ and $\mathbf{3}^-$ (replacing the role of $\boldsymbol{C}$ by $\boldsymbol{D}$) we find $\dot{\mathbf{3}}^+ = \lambda_3^+ \mathbf{3}^-$ and $\dot{\mathbf{3}}^- = \lambda_3^- \mathbf{3}^+$.

Now, differentiating the relation $\langle \mathbf{0}^+, \mathbf{0}^- \rangle = 0$, we get

$$0 = \langle \dot{\mathbf{0}}^+, \mathbf{0}^- \rangle + \langle \mathbf{0}^+, \dot{\mathbf{0}}^- \rangle = \lambda_0^+ \langle \mathbf{0}^-, \mathbf{0}^- \rangle + \lambda_0^- \langle \mathbf{0}^+, \mathbf{0}^+ \rangle = \lambda_0^+ - \lambda_0^-$$

which implies $\lambda_0^+ = \lambda_0^-$. Similarly we have $\lambda_3^+ = \lambda_3^-$. Finally by differentiating $\langle \mathbf{3}^+, \mathbf{0}^- \rangle = 0$ we find

$$0 = \langle \dot{\mathbf{3}}^+, \mathbf{0}^- \rangle + \langle \mathbf{3}^+, \dot{\mathbf{0}}^- \rangle = \lambda_3^+ \langle \mathbf{3}^-, \mathbf{0}^- \rangle + \lambda_0^- \langle \mathbf{3}^+, \mathbf{0}^+ \rangle = \lambda_0^- - \lambda_3^+$$

whence $\lambda_0^- = \lambda_3^+$. This concludes the proof. □

We remark that in the hyperbolic case the same computation shows that $\dot{\mathbf{0}}^+ = \lambda \mathbf{0}^-$, $\dot{\mathbf{0}}^- = -\lambda \mathbf{0}^+$, $\dot{\mathbf{3}}^+ = \lambda \mathbf{3}^-$ and $\dot{\mathbf{3}}^- = -\lambda \mathbf{3}^+$ for some $\lambda \in \mathbb{R}$, as the only differences with respect to the AdS argument is that $\langle \mathbf{0}^+, \mathbf{0}^+ \rangle = \langle \mathbf{3}^+, \mathbf{3}^+ \rangle = 1$ and $\langle \mathbf{3}^+, \mathbf{0}^+ \rangle = -1$ from Table 2.

So, using the assumption $\dot{\boldsymbol{A}} = \dot{\boldsymbol{B}} = \dot{\boldsymbol{C}} = \dot{\boldsymbol{D}} = 0$, we have proved that (16) holds for $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \mathbf{3}^+\}$ and $\boldsymbol{i}^- \in \{\mathbf{0}^-, \mathbf{3}^-\}$. If we knew that $\dot{\boldsymbol{E}} = \dot{\boldsymbol{F}} = 0$, we could repeat a similar argument to show that (16) holds also for the remaining $\boldsymbol{i}^\pm$'s. It thus essentially remains to show that $\dot{\boldsymbol{E}} = \dot{\boldsymbol{F}} = 0$.

Lemma 4.14. *Fix $t \in (-1, 1)$, and let $(\{\dot{\boldsymbol{i}}^+\}, \{\dot{\boldsymbol{j}}^-\}, \{\dot{\boldsymbol{X}}\})$ be an infinitesimal deformation of the normalised vectors $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\})$ satisfying (9) and (10). Then $\dot{\boldsymbol{E}} = \dot{\boldsymbol{F}} = 0$ and there exists $\lambda \in \mathbb{R}$ such that, for every $\boldsymbol{i} \in \{\mathbf{0}, \dots, \mathbf{7}\}$,*

$$\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^- \quad \textit{and} \quad \dot{\boldsymbol{i}}^- = \lambda \boldsymbol{i}^+ \;.$$

Proof. Let $\lambda \in \mathbb{R}$ be as in the conclusion of Lemma 4.13. Let us first focus on the variations of $\mathbf{1}$ and $\mathbf{2}$, similarly to the proof of Lemma 4.13. Taking the derivatives of the relations $\langle \mathbf{1}^+, \boldsymbol{A} \rangle = \langle \mathbf{1}^+, \boldsymbol{C} \rangle = 0$ and using $\dot{\boldsymbol{A}} = \dot{\boldsymbol{C}} = 0$, we have

$$\langle \dot{\mathbf{1}}^+, \boldsymbol{A} \rangle = \langle \dot{\mathbf{1}}^+, \boldsymbol{C} \rangle = 0$$

whereas from $\langle \mathbf{1}^+, \mathbf{1}^+ \rangle = -1$ we derive

$$\langle \dot{\mathbf{1}}^+, \mathbf{1}^+ \rangle = 0 \;.$$

Here we do not know $\dot{\boldsymbol{E}} = 0$ yet, hence we cannot argue that $\langle \dot{\mathbf{1}}^+, \boldsymbol{E} \rangle = 0$, which would imply that $\dot{\mathbf{1}}^+$ is a multiple of $\mathbf{1}^-$. However, observing that $\boldsymbol{A}, \boldsymbol{C}$ and $\mathbf{1}^+$ are linearly independent, and that the linear system for the $\dot{\mathbf{1}}^+$ given by the above three conditions is satisfied by the vectors $\mathbf{0}^-$ and $\mathbf{1}^-$ (which are linearly independent), by a dimension argument $\dot{\mathbf{1}}^+$ is necessarily a linear combination of $\mathbf{0}^-$ and $\mathbf{1}^-$. Analogously one gets that $\dot{\mathbf{1}}^-$ is necessarily a linear combination of $\mathbf{0}^+$ and $\mathbf{1}^+$, and replacing $\boldsymbol{C}$ by $\boldsymbol{D}$, and $\mathbf{0}$ by $\mathbf{3}$, we find similar relations for $\dot{\mathbf{2}}^-$ and $\dot{\mathbf{2}}^+$. Let us summarise them here:

$$\begin{aligned}\dot{\mathbf{1}}^- &= \lambda_1^- \mathbf{1}^+ + \mu_1^- \mathbf{0}^+ \ , \\ \dot{\mathbf{1}}^+ &= \lambda_1^+ \mathbf{1}^- + \mu_1^+ \mathbf{0}^- \ , \\ \dot{\mathbf{2}}^- &= \lambda_2^- \mathbf{2}^+ + \mu_2^- \mathbf{3}^+ \ , \\ \dot{\mathbf{2}}^+ &= \lambda_2^+ \mathbf{2}^- + \mu_2^+ \mathbf{3}^- \ .\end{aligned}$$

We claim here that, as expected from (16), $\lambda_1^- = \lambda_1^+ = \lambda_2^- = \lambda_2^+ = \lambda$ and $\mu_1^- = \mu_1^+ = \mu_2^- = \mu_2^+ = 0$, and moreover $\dot{\boldsymbol{E}} = 0$. In fact, it will suffice to show $\mu_1^+ = 0$. Indeed, recalling the assumption $\dot{\boldsymbol{E}} = (0,0,0,0,\epsilon)$, the derivative of the relation $\langle \boldsymbol{E}, \mathbf{1}^+ \rangle = 0$ gives

$$0 = \langle \dot{\boldsymbol{E}}, \mathbf{1}^+ \rangle + \langle \boldsymbol{E}, \dot{\mathbf{1}}^+ \rangle = \frac{1}{\sqrt{1-t^2}} (\epsilon - 2\sqrt{2}\mu_1^+) \ , \tag{20}$$

hence we will obtain $\epsilon = 0$, namely $\dot{\boldsymbol{E}} = 0$. Once we have $\dot{\boldsymbol{E}} = 0$, we can proceed exactly as in Lemma 4.13 to deduce that $\mu_1^- = \mu_1^+ = \mu_2^- = \mu_2^+ = 0$ and then $\lambda_1^- = \lambda_1^+ = \lambda_2^- = \lambda_2^+ = \lambda$ (which also follows from Equation (21) below).

We shall need one more intermediate step. By differentiating the relation $\langle \mathbf{0}^-, \mathbf{1}^+ \rangle = 0$, we find

$$0 = \langle \dot{\mathbf{0}}^-, \mathbf{1}^+ \rangle + \langle \mathbf{0}^-, \dot{\mathbf{1}}^+ \rangle = \lambda \langle \mathbf{0}^+, \mathbf{1}^+ \rangle + \lambda_1^+ \langle \mathbf{0}^-, \mathbf{1}^- \rangle + \mu_1^+ \langle \mathbf{0}^-, \mathbf{0}^- \rangle = \lambda - \lambda_1^+ + \mu_1^+ \ .$$

Using similarly the relations $\langle \mathbf{0}^+, \mathbf{1}^- \rangle = \langle \mathbf{2}^-, \mathbf{3}^+ \rangle = \langle \mathbf{2}^+, \mathbf{3}^- \rangle = 0$ we find three analogous identities. We summarise these four important identities here:

$$\lambda = \lambda_1^- - \mu_1^- = \lambda_1^+ - \mu_1^+ = \lambda_2^- - \mu_2^- = \lambda_2^+ - \mu_2^+ \ . \tag{21}$$

We can now focus on proving that $\mu_1^+ = 0$. Differentiating $\langle \mathbf{1}^+, \mathbf{2}^- \rangle = 0$ we see that

$$0 = \lambda_1^+ \langle \mathbf{1}^-, \mathbf{2}^- \rangle + \mu_1^+ \langle \mathbf{0}^-, \mathbf{2}^- \rangle + \lambda_2^- \langle \mathbf{1}^+, \mathbf{2}^+ \rangle + \mu_2^- \langle \mathbf{1}^+, \mathbf{3}^+ \rangle \ .$$

Using $\langle \mathbf{1}^-, \mathbf{2}^- \rangle = -1$, $\langle \mathbf{1}^+, \mathbf{2}^+ \rangle = 1$ and an explicit computation for the other two terms, we obtain:

$$\lambda_2^- - \lambda_1^+ = \frac{3+t^2}{1-t^2}\mu_1^+ + \frac{1+3t^2}{1-t^2}\mu_2^- \ .$$

On the other hand, from Equation (21) we have $\lambda_2^- - \lambda_1^+ = \mu_2^- - \mu_1^+$, whence

$$\mu_1^+ + t^2 \mu_2^- = 0 \ . \tag{22}$$

If $t = 0$, we are done. Otherwise, we will combine (22) with the derivative of the relation $\langle \boldsymbol{E}, \mathbf{2}^- \rangle = 0$, namely

$$\frac{t}{\sqrt{1-t^2}} (-\epsilon - 2\sqrt{2}\mu_2^-) = 0 \ ,$$

which together with Equation (20) gives $\mu_1^+ + \mu_2^- = 0$. Together with (22), this shows that $\mu_1^+ = 0$.

Having proved that $\dot{\boldsymbol{E}} = 0$, the proof that $\dot{\boldsymbol{F}} = 0$ follows exactly the same lines, with $\mathbf{4}$ and $\mathbf{5}$ playing the role of $\mathbf{1}$ and $\mathbf{2}$. Arguing as in Lemma 4.13 one then shows that $\dot{\boldsymbol{i}}^+ = \lambda \boldsymbol{i}^-$ and $\dot{\boldsymbol{i}}^- = \lambda \boldsymbol{i}^+$ for all $\boldsymbol{i} \in \{\mathbf{1}, \ldots, \mathbf{7}\}$. ∎

This provides the conclusion of the proof of Proposition 4.10.

Proof of Proposition 4.10. Let us fix $t \in (-1,1)$. We now show that the kernel of the differential of $g_0 \colon \mathbb{R}^{110} \to \mathbb{R}^{138}$ is 11-dimensional at $(\{\boldsymbol{i}^+\}, \{\boldsymbol{j}^-\}, \{\boldsymbol{X}\}) \in g_0^{-1}(0)$.

Lemmas 4.12, 4.13 and 4.14 showed that every element in the kernel of dg_0 is of the form (16) up to adding an element of the form (8), that is an element tangent to the orbit of the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-action. It is also easy to see that such element in the tangent space of the orbit is unique, for if two elements $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have this property, it follows that $\mathfrak{a} := \mathfrak{a}_1 - \mathfrak{a}_2$ satisfies $\mathfrak{a} \cdot \boldsymbol{X} = 0$ for $\boldsymbol{X} = \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and the characterising conditions (12) (already used in Lemma 4.12) show that $\mathfrak{a} = 0$. The very same argument shows that the map defined in (8) from the Lie algebra $\mathfrak{isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ into the kernel of the differential of g_0 (whose image is the tangent space to the orbit of the $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$-action) is injective.

In other words, the 10-dimensional tangent space of the orbit has a 1-dimensional complement, consisting precisely of the elements of the form (16), hence the kernel of the differential of g_0 has dimension 11. By the constant rank theorem, $g_0^{-1}(0)$ is a manifold of dimension 11 near the elements in the orbit of ρ_t. □

4.8. **Proof of Theorem 1.1 in the AdS case.** We can finally conclude the proof of Theorem 1.1 in the AdS setting. We state it again here (removing the superscript G) for convenience:

Theorem 1.1 (AdS case)**.** *The point* $[\rho_0] \in X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ *has a neighbourhood* $\mathcal{U} = \mathcal{V} \cup \mathcal{H}$ *homeomorphic to the set* $\mathcal{S} = \{(x_1^2 + \ldots + x_{12}^2) \cdot x_{13} = 0\} \subset \mathbb{R}^{13}$*, where:*

- $[\rho_0]$ *corresponds to the origin,*
- *the curve* $\mathcal{V} = \{[\rho_t]\}_{t\in(-1,1)}$ *corresponds to the* x_{13}*-axis, and*
- $\mathcal{H}$ *corresponds to* $\{x_{13} = 0\}$*, identified to a neighbourhood of the complete hyperbolic orbifold structure of the ideal right-angled cuboctahedron in its deformation space.*

The group $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)/\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4) \cong \mathbb{Z}/2\mathbb{Z}$ *acts on* $\mathcal{S}$ *by changing sign to* x_{13}*.*

See Figure 1 for an illustration of this description of the character variety.

Remark 4.15. We showed in Remark 4.5 that the orbits of the points in a neighborhood of the curve $\{\rho_t\}$ are closed, hence for the purpose of Theorem 1.1, the GIT quotient in the standard definition of character variety coincides (at the topological level) with the ordinary topological quotient $\mathrm{Hom}(\Gamma_{22}, G)/G^+$. In fact, in Remark 4.5 we explained directly that the points in such neighborhood are separated from all the other points of $\mathrm{Hom}(\Gamma_{22}, G)/G^+$.

We decided to give a proof only in the AdS case, since the fact that the points $[\rho_t]$ for $t > 0$ form a smooth curve (Proposition 4.6) has already been proved in [KS10], while its AdS counterpart is completely new. The proof for the hyperbolic case is analogous (recall Remark 4.9). Moreover, the description of the collapse (namely, at the representation ρ_0) is also new in both (hyperbolic and AdS) cases.

Proof of Theorem 1.1 — AdS case. We split the proof into several steps.

***Step 1*:** As a first step, let us define $\widetilde{\mathcal{V}} \subset \mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ as the $\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$–orbit of the curve $\{\rho_t\}_{t\in(-1,1)}$. Let us also observe that $\widetilde{\mathcal{V}}$ is contained in the subset $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ introduced in Definition 4.7.

Since by Lemma 4.4 the $\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$–action by conjugation is free on $\{\rho_t\}_{t\in(-1,1)}$, the map $(g, t) \mapsto g \cdot \rho_t$ defines a continuous injection

$$\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4) \times (-1, 1) \to \mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)) \ ,$$

where by Proposition 4.10 the latter is a smooth 11-dimensional manifold. By the invariance of domain, this injection is a homeomorphism onto its image, which is $\widetilde{\mathcal{V}}$. By Lemma 4.3 and Lemma 4.4, the $\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4)$–action by conjugation is free and proper on $\widetilde{\mathcal{V}}$ thus the projection in the quotient $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ is

$$\mathcal{V} := \{[\rho_t] \mid t \in (-1, 1)\} \ ,$$

which is homeomorphic to a line.

***Step 2*:** The second component $\mathcal{H}$ is defined as follows.

Recall from Section 4.3 that we have a fixed spacelike hyperplane $\mathbb{H}^3 \subset \mathbb{A}\mathrm{d}\mathbb{S}^4$ defined by the equation $x_4 = 0$, which is the fixed point set of the reflection r. The stabiliser of this hyperplane in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ is isomorphic to $\mathrm{Isom}(\mathbb{H}^3) \times \langle r \rangle$, where $\mathrm{Isom}(\mathbb{H}^3)$ acts by isometries on $\{x_4 = 0\}$ and does not switch the two sides. We will thus consider $\mathrm{Isom}(\mathbb{H}^3)$ as a subgroup of $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$.

Consider the reflection group Γ_{co} of the ideal right-angled cuboctahedron. We define the map

$$\Psi\colon \mathrm{Hom}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)) \to \mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$$

that associates to $\eta\colon \Gamma_{\mathrm{co}} \to \mathrm{Isom}(\mathbb{H}^3)$ the representation $\Psi_\eta\colon \Gamma_{22} \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ sending each of the generators $\boldsymbol{0}^+, \dots, \boldsymbol{7}^+$ of Γ_{22} to the reflection r, and each of the generators $\boldsymbol{0}^-, \dots, \boldsymbol{7}^-, \boldsymbol{A}, \dots, \boldsymbol{F}$ to the corresponding element of $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ through η. It is then straightforward to check that:

(1) The map Ψ is well-defined and equivariant for the conjugacy action of $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$.

(2) The following induced map is injective

$$\widehat{\Psi}\colon X(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)) \to X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))\ .$$

Indeed, (1) holds because, using that r commutes with the elements of $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, the images of the generators in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ through Ψ_η satisfy the relations of Γ_{22}, so that Ψ_η is indeed a representation of Γ_{22} in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$. The equivariance of Ψ is clear using again that r commutes with $\mathrm{Isom}(\mathbb{H}^3)$.

Moreover, (2) holds because if two representations Ψ_{η_1} and Ψ_{η_2} in the image of Ψ are conjugate by some $g \in \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, then $\Psi_{\eta_1}(\boldsymbol{i}^+) = \Psi_{\eta_2}(\boldsymbol{i}^+) = r$, hence g must fix the hyperplane $\mathbb{H}^3 \subset \mathbb{A}\mathrm{d}\mathbb{S}^4$, and therefore $g \in \mathrm{Stab}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3) \times \langle r \rangle$. Moreover, up to composing with r, which commutes with both Ψ_{η_i}, we can also assume that g belongs to the subgroup $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, hence η_1 and η_2 are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$.

The representation ρ_0 is clearly in the image of Ψ, as $\rho_0 = \Psi_{\eta_0}$ where η_0 is the holonomy representation of the complete hyperbolic orbifold structure of the cuboctahedron, as expressed by Equation (6). The variety $X(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ is a 12-dimensional manifold in a neighborhood (say $\mathcal{H}_0$) of $[\eta_0]$, since it corresponds to a neighbourhood of the complete hyperbolic orbifold structure of the right-angled cuboctahedron in its deformation space. To show this, the same proof of [KS10, Proposition 5.2] applies, as a well-known "reflective" orbifold version of Thurston's hyperbolic Dehn filling [Thu79] (note that the ideal cuboctahedron has 12 cusps).

Therefore a neighborhood $\mathcal{H}_0$ of $[\rho_0]$ in $X(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ is homeomorphic to $\mathbb{R}^{12}$, and we can also assume that $\widehat{\Psi}|_{\mathcal{H}_0}$ is a homeomorphism onto its image. Then let us define $\mathcal{H} := \widehat{\Psi}(\mathcal{H}_0)$.

***Step 3*:** We claim that the intersection of $\mathcal{H}$ and $\mathcal{V}$ consists only of the point $[\rho_0]$.

Indeed, suppose $[\rho] \in \mathcal{H} \cap \mathcal{V}$, for ρ in $\mathrm{Hom}(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$. On the one hand $\rho = \Psi(\eta)$, where $\eta \in \mathrm{Hom}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ is a deformation of the orbifold fundamental group of the cuboctahedron. On the other hand ρ lies in $\widetilde{\mathcal{V}} \subset \mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$. In particular, η maps each peripheral subgroup of Γ_{co} to a cusp group.

By the Mostow–Prasad rigidity, η is conjugate to the holonomy representation η_0 of the complete right-angled ideal cuboctahedron. Since both ρ and ρ_0 send each of the generators

$\mathbf{0}^+, \dots, \mathbf{7}^+$ to r, which commutes with $\mathrm{Isom}(\mathbb{H}^3) < \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, the representations ρ and ρ_0 are also conjugate in $\mathrm{Isom}(\mathbb{H}^3)$, and therefore $[\rho] = [\rho_0]$.

***Step 4*:** Let us now show that the point $[\rho_0] \in X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ has a neighbourhood $\mathcal{U}$ which is contained in the union of the two components $\mathcal{V}$ and $\mathcal{H}$.

To see this, let ρ be a representation nearby ρ_0. We claim that if two generators which are sent by ρ_0 to the same reflection r (hence necessarily of the form $\boldsymbol{i}^+$ and $\boldsymbol{j}^+$) are sent to reflections in coinciding hyperplanes also by ρ, then all generators $\mathbf{0}^+, \dots, \mathbf{7}^+$ are sent by ρ to the same reflection. That is, if $\rho(\boldsymbol{i}^+) = \rho(\boldsymbol{j}^+)$ for some $\boldsymbol{i}, \boldsymbol{j}$, then $\rho(\boldsymbol{i}^+) = \rho(\boldsymbol{j}^+)$ for all $\boldsymbol{i}, \boldsymbol{j}$. This will show our thesis by the rigidity property of Proposition 3.12: if $[\rho]$ is not on the "horizontal" component $\mathcal{H}$, then no two letter generators are sent to the same reflection, and thus all the collapsed cusp groups of ρ_0 are cusp groups for ρ. That is, ρ is in $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ and thus in the "vertical" component $\widetilde{\mathcal{V}}$, since $\widetilde{\mathcal{V}}$ is open in $\mathrm{Hom}_0(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$.

To prove the claim, suppose that two generators $\boldsymbol{i}^+$ and $\boldsymbol{j}^+$ are such that $\rho(\boldsymbol{i}^+) = \rho(\boldsymbol{j}^+)$. By the symmetries of the polytope $\mathcal{P}_t$ (see [RS, Lemma 7.6]) and Proposition 3.12, we can assume the two generators are $\mathbf{0}^+$ and $\mathbf{1}^+$. Up to conjugation in $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, we can also assume $\rho(\mathbf{0}^+) = \rho(\mathbf{1}^+) = r$. To simplify the notation, let f be a preimage of ρ in $g^{-1}(0)$, which associates to each generator of Γ_{22} a vector in $\mathbb{R}^5$ of square norm 1 or -1 with respect to q_{-1}.

Up to changing the sign if necessary, $f(\mathbf{0}^+) = f(\mathbf{1}^+) = e_4 = (0, 0, 0, 0, 1)$. From the relations in Γ_{22}, the vector $f(\mathbf{2}^+)$ is necessarily orthogonal to $f(\mathbf{1}^-)$, $f(\mathbf{2}^-)$, $f(\mathbf{3}^-)$ and $f(\boldsymbol{A})$. But by the assumption $f(\mathbf{0}^+) = f(\mathbf{1}^+) = e_4$ and the relations involving $\mathbf{0}^+$, the vector e_4 is orthogonal to $f(\mathbf{1}^-)$, $f(\mathbf{3}^-)$ and $f(\boldsymbol{A})$, while from the relations involving $\mathbf{1}^+$, the vector e_4 is orthogonal to $f(\mathbf{2}^-)$.

For a small deformation of ρ_0, the vectors $f(\mathbf{1}^-)$, $f(\mathbf{2}^-)$, $f(\mathbf{3}^-)$ and $f(\boldsymbol{A})$ are linearly independent, because they are for ρ_0 (see Table 3). Hence the conditions of being orthogonal to these 4 vectors define a linear system of 4 independent equations, which are satisfied by e_4. Hence $f(\mathbf{2}^+)$, which is a solution of the system, coincides with e_4 up to rescaling. Since $q(f(\mathbf{2}^+)) = -1$, we can assume that $f(\mathbf{2}^+) = e_4$. Namely, $\rho(\mathbf{2}^+) = r$. By arguing similarly for $\mathbf{3}^+$ and then for all the other generators, one easily finds sufficiently many relations to show that $\rho(\boldsymbol{i}^+) = r$ for each generator $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$, and therefore ρ is in $\mathcal{H}$. This proves the claim.

***Step 5*:** Summarising the previous steps, we have shown that the class $[\rho_0]$ has a neighborhood $\mathcal{U}$ which only consists of points of $\mathcal{H}$ and $\mathcal{V}$. Since we already know that $\mathcal{H}$ and $\mathcal{V}$ are smooth submanifolds outside of ρ_0, it is harmless to enlarge $\mathcal{U}$ so that it contains entirely $\mathcal{H}$ and $\mathcal{V}$.

We have therefore obtained a neighborhood $\mathcal{U}$ of $[\rho_0]$ in $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ homeomorphic to

$$(\{0\} \times \mathbb{R}) \ \cup \ (\mathbb{R}^{12} \times \{0\}) \ \subset \ \mathbb{R}^{13} \ ,$$

where the two components are precisely $\mathcal{H}$ and $\mathcal{V}$.

***Step 6*:** It remains to prove the last sentence about the action of the group

$$\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)/\mathrm{Isom}^+(\mathbb{A}\mathrm{d}\mathbb{S}^4) \cong \mathbb{Z}/2\mathbb{Z}$$

generated by the coset of the reflection r.

This is now simple: on the one hand, as observed after Definition 4.1, conjugation by r acts on $\mathcal{V}$, which is homeomorphic to $(-1,1)$, by $[\rho_t] \mapsto [\rho_{-t}]$. On the other hand, by construction of $\mathcal{H}$, conjugation by r fixes pointwise the elements in $\mathcal{H}$, which are of the form Ψ_η for some $\eta\colon \Gamma_{co} \to \mathrm{Isom}(\mathbb{H}^3)$. This concludes the proof. □

We conclude the section with a couple of observations on the nature of the fixed points for the action of G on $\mathrm{Hom}(\Gamma_{22}, G)$.

Lemma 4.3 shows that the stabiliser of each point ρ_t in $\mathrm{Hom}(\Gamma_{22}, G)$, for the conjugacy action of G, is trivial, except ρ_0 which has stabiliser $\langle r\rangle$. In fact, a small adaptation of the proof shows that, in a neighborhood of ρ_0, the stabiliser of all points in the the horizontal component $\mathcal{H}$ is as well the group $\mathbb{Z}/2\mathbb{Z}$ generated by r. This is because we can find a neighborhood of ρ is in the image of Ψ such that, for a lift f of ρ, the vectors $f(\boldsymbol{A}), f(\boldsymbol{B}), f(\boldsymbol{C}), f(\boldsymbol{D}) \in \mathbb{R}^5$ are linearly independent. Indeed the vectors $f_0(\boldsymbol{A}), f_0(\boldsymbol{B}), f_0(\boldsymbol{C}), f_0(\boldsymbol{D})$ are linearly independent, and being independent is an open condition. By the structure of the group Γ_{22}, the vectors $f(\boldsymbol{A}), f(\boldsymbol{B}), f(\boldsymbol{C}), f(\boldsymbol{D})$ are necessarily orthogonal to $(0,0,0,0,1)$, since ρ maps each generator $\boldsymbol{i}^+$ to r. Hence one can repeat the proof of Lemma 4.3 and see that an element in the stabiliser of ρ must necessarily fix $\{x_4 = 0\}$ setwise, and moreover must act trivially on $\{x_4 = 0\}$. Hence the only possible candidates are the identity and r, both of which fix ρ by definition of Ψ.

In conclusion, let us consider the full quotient $\mathrm{Hom}(\Gamma_{22}, G)/\!\!/G$, which is a $\mathbb{Z}/2\mathbb{Z}$–quotient of $X(\Gamma_{22}, G)$, where $\mathbb{Z}/2\mathbb{Z} \cong G/G^+$. A local picture of this full quotient is given in Figure 1 (right), as a consequence of the fact that the generator of $\mathbb{Z}/2\mathbb{Z}$ acts by changing sign to the x_{13}-coordinate, hence as a "reflection" with respect to the horizontal component $\mathcal{H}$. The "horizontal" component (which is the projection of $\mathcal{H}$ to the full quotient $\mathrm{Hom}(\Gamma_{22}, G)/\!\!/G$) entirely consists of points with associated group $\mathbb{Z}/2\mathbb{Z}$. They are "double" points in a suitable sense, which reminds "mirror" points in the language of orbifolds.

5. Cusp groups in HP geometry

In this section we introduce half-pipe geometry, discuss its relations with Minkowski geometry, and prove the half-pipe version of the flexibility and rigidity statements for right-angled cusp groups.

5.1. **Half-pipe geometry.** Let us denote by q_0 the following degenerate bilinear form on $\mathbb{R}^{n+1}$:

$$q_0(x) = -x_0^2 + x_1^2 + \ldots + x_{n-1}^2 \ .$$

Then *half-pipe space* of dimension n is defined as

$$\mathsf{HP}^n = \mathrm{P}\{x \in \mathbb{R}^{n+1} \,|\, q_0(x) < 0\} \subset \mathbb{RP}^n \ ,$$

and the group of *half-pipe transformations* is

$$G_{\mathsf{HP}^n} = \{[A] \in \mathrm{PO}(q_0) \,|\, Ae_n = \pm e_n\}$$

(here $e_0, \ldots, e_n$ is the canonical basis of $\mathbb{R}^{n+1}$).

Explicitely, an element $[A] \in G_{\mathsf{HP}^n}$ has the form

$$A = \left[\begin{array}{ccc|c} & & & 0 \\ & \widehat{A} & & \vdots \\ & & & 0 \\ \hline \star & \ldots & \star & \pm 1 \end{array}\right]$$

for some n-by-n matrix $\widehat{A}$ which preserves the bilinear form of signature $(-,+,\dots,+)$ on $\mathbb{R}^n$, where the stars denote the entries of any vector in $\mathbb{R}^n$ and the square brackets denote the projective class of a matrix in $\mathrm{GL}_{n+1}(\mathbb{R})$.

The *boundary at infinity* of HP^n is as usual

$$\partial\mathsf{HP}^n = \mathrm{P}\{x \in \mathbb{R}^{n+1} \,|\, q_0(x) = 0\} ,$$

and can be visualised as the union of a cylinder constituted by those $[x] \in \partial\mathsf{HP}^n$ such that $(x_0,\dots,x_{n-1})$ does not vanish, and the point $[e_n]$ which is the one-point compactification of the previous cylinder. The point $[e_n] \in \partial\mathsf{HP}^n$ is a distinguished point, since it is preserved by the action of every element of G_{HP^n} on $\partial\mathsf{HP}^n$.

Also, we remark that there is a natural map from HP^n to $\mathrm{P}\{x \in \mathbb{R}^{n+1} \,|\, q_0(x) < 0, x_n = 0\}$, which is a copy of $\mathbb{H}^{n-1}$, given simply by $(x_0,\dots,x_n) \mapsto (x_0,\dots,x_{n-1},0)$. We shall call this map the *projection*

$$\pi\colon \mathsf{HP}^n \to \mathbb{H}^{n-1} .$$

Its fibers are called *degenerate lines*, since they are projective lines in $\mathbb{RP}^n$ going through $[e_n]$, and the restriction of the bilinear form b_0 associated to q_0 is degenerate. Degenerate lines are preserved by the action of G_{HP^n}.

Finally, the projection map π is equivariant with respect to the obvious epimorphism $G_{\mathsf{HP}^n} \to \mathrm{Isom}(\mathbb{H}^{n-1})$ and extends to a map from $\partial\mathsf{HP}^n \smallsetminus \{[e_n]\}$ to $\partial\mathbb{H}^{n-1}$.

5.2. **Duality with Minkowski space.** We will find comfortable to exploit the well-known duality between half-pipe and Minkowski geometry. We will not provide details of the proofs here, see [BF, FS19, RS] for a more complete treatment.

The fundamental observation is that HP^n is identified to the space of spacelike affine hyperplanes in Minkowski space $\mathbb{R}^{1,n-1} := (\mathbb{R}^n, q_1)$ where q_1 is the non-degenerate bilinear form on $\mathbb{R}^n$ introduced in Section 2.1. The correspondence is given by associating to a point $[x] \in \mathsf{HP}^n$ the affine hyperplane of $\mathbb{R}^{1,n}$ defined by the equation

$$b_1((x_0,\dots,x_{n-1}),(y_0,\dots,y_{n-1})) + x_n = 0 , \qquad (23)$$

for b_1 the bilinear form associated to q_1. Clearly the correspondence is well-defined in $\mathsf{HP}^n \subset \mathbb{RP}^n$, and the condition that such affine hyperplane is spacelike is equivalent to $[x] \in \mathsf{HP}^n$.

The isometry group $\mathrm{Isom}(\mathbb{R}^{1,n-1}) \cong \mathrm{O}(q_1) \ltimes \mathbb{R}^n$ acts naturally on the space of spacelike affine hyperplanes, and the correspondence is also well-behaved with respect to the group actions, as we summarise in the following lemma (see for instance [BF], [FS19] or [RS, Lemma 2.8]).

Lemma 5.1. *There is a "duality" homeomorphism*

$$\{\textit{spacelike affine hyperplanes in } \mathbb{R}^{1,n-1}\} \cong \mathsf{HP}^n$$

which is equivariant with respect to a group isomorphism

$$\phi\colon \mathrm{Isom}(\mathbb{R}^{1,n-1}) \to G_{\mathsf{HP}^n} .$$

In this work, we will adopt almost entirely this "dual" point of view for half-pipe geometry. In this setting, the boundary $\partial\mathsf{HP}^n$ has a natural identification:

$$\partial\mathsf{HP}^n \cong \{\text{lightlike affine hyperplanes in } \mathbb{R}^{1,n-1}\} \cup \{\infty\} , \qquad (24)$$

where the point $[e_n]$ in $\partial\mathsf{HP}^n$ corresponds to ∞ on the right-hand side, while $\partial\mathsf{HP}^n \smallsetminus \{[e_n]\}$ identifies to the space of lightlike affine hyperplanes using again (23). Geometrically, the decomposition in the right-hand side of (24) reflects the fact that, up to taking a subsequence, a sequence of spacelike affine hyperplanes in $\mathbb{R}^{1,n-1}$ may either converge to a lightlike hyperplane or escape from all compact subsets.

The projection π is interpreted in this dual setting as the map which associates to a spacelike affine hyperplane in $\mathbb{R}^{1,n-1}$ its unique parallel linear hyperplane. Equivalently, thinking of π with values in $\mathbb{H}^{n-1} \subset \mathbb{RP}^{n-1}$, it associates to a spacelike affine hyperplane its normal direction with respect to the Minkowski product b_1. Of course π extends to the complement of ∞ in $\partial\mathsf{HP}^n$, with values in $\partial\mathbb{H}^{n-1}$.

5.3. **Hyperplanes.** Let us now consider hyperplanes in half-pipe geometry.

Definition 5.2 (HP hyperplane)**.** A *half-pipe hyperplane* is the intersection of HP^n with a projective hyperplane in $\mathbb{RP}^n$. It is called *degenerate* if it contains a degenerate line of HP^n; *non-degenerate* otherwise.

From now on, we will always think of HP^n dually as the space of spacelike affine hyperplanes in $\mathbb{R}^{1,n-1}$, using Lemma 5.1. For more details on the proofs of the following statements, see [RS, Section 4.3].

Lemma 5.3. *Any non-degenerate hyperplane of* HP^n *is dual to the set of spacelike affine hyperplanes going through a given point* $p \in \mathbb{R}^{1,n-1}$*.*

We will refer to the point p as the *dual* point to the non-degenerate hyperplane, and conversely we will make reference to the hyperplane *dual* to a point of $\mathbb{R}^{1,n-1}$. With this duality approach, it is very easy to describe the relative position of non-degenerate hyperplanes:

Lemma 5.4. *Given two points* $p, q \in \mathbb{R}^{1,n-1}$*, their dual hyperplanes*

- *intersect in* HP^n *if and only if* $p - q$ *is spacelike,*
- *are disjoint in* HP^n *but their closures intersect in* $\partial\mathsf{HP}^n$ *if and only if* $p-q$ *is lightlike,*
- *have disjoint closures in* $\mathsf{HP}^n \cup \partial\mathsf{HP}^n$ *if and only if* $p - q$ *is timelike.*

In half-pipe geometry, the situation for degenerate and non-degenerate hyperplanes is qualitatively different, as we shall see also in Section 5.4 below. Let us first characterise degenerate hyperplanes in terms of Minkowski geometry:

Lemma 5.5. *Any degenerate hyperplane of* HP^n *is the preimage of a hyperplane in* $\mathbb{H}^{n-1}$ *by the projection map* $\pi\colon \mathsf{HP}^n \to \mathbb{H}^{n-1}$*. That is, it is dual to the set of spacelike affine hyperplanes having normal direction in a given hyperplane of* $\mathbb{H}^{n-1}$*.*

There are three possibilities for the relative position of two degenerate hyperplanes $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$ in HP^n:

- If S_1 and S_2 intersect in $\mathbb{H}^{n-1}$, then $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$ intersect in HP^n in the subset $\pi^{-1}(S_1 \cap S_2)$;
- If S_1 and S_2 intersect in $\partial\mathbb{H}^{n-1} \smallsetminus \mathbb{H}^{n-1}$, then the closures of $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$ intersect in a degenerate line of $\partial\mathsf{HP}^n$;
- Finally, if S_1 and S_2 have disjoint closures in $\mathbb{H}^{n-1} \cup \partial\mathbb{H}^{n-1}$, then the closures of $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$ only intersect in $\{\infty\} \in \partial\mathsf{HP}^n$.

5.4. **Reflections.** Like in pseudo-Riemannian geometry, a *reflection* in HP^n is a non-trivial involution in G_{HP^n} that fixes pointwise a hyperplane.

We shall again distinguish two cases:

Proposition 5.6. *There exists a unique reflection in* G_{HP^n} *fixing a given non-degenerate hyperplane in* HP^n*.*

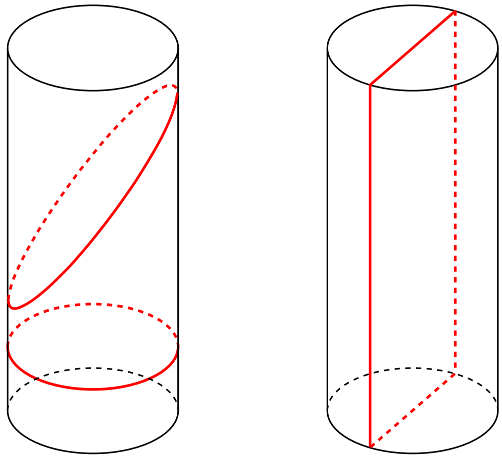

FIGURE 6. Hyperplanes in the affine (cylindric) model of HP^n: on the left, two spacelike hyperplanes, on the right, a degenerate hyperplane.

Proof. By Lemma 5.3, a reflection in G_{HP^n} is induced by an element of $\mathrm{Isom}(\mathbb{R}^{1,n-1})$ that fixes setwise all the spacelike hyperplanes going through a point $p \in \mathbb{R}^{1,n-1}$. The involution $\phi(-\mathrm{id}, 2p)$ therefore has such a property. It is the only reflection fixing the hyperplane dual to p. Indeed, for a transformation $\phi(A, v)$ with this property, the linear part A must fix all the timelike directions in $\mathbb{R}^{1,n-1}$, hence $A = \pm\mathrm{id}$, but the choice $A = \mathrm{id}$ implies necessarily $v = 0$ because $\phi(A, v)$ has order two, and therefore gives a trivial transformation. □

Let us now consider degenerate hyperplanes:

Proposition 5.7. *There exists a one-parameter family of reflections in G_{HP^n} fixing a given degenerate hyperplane in HP^n.*

Proof. From Lemma 5.5, a degenerate hyperplane in HP^n has the form $\pi^{-1}(H_X)$ where, using the notation of Section 2.2, X denotes a vector in $\mathbb{R}^{1,n-1}$ such that $q_1(X) = 1$ and H_X is the hyperplane in $\mathbb{H}^{n-1}$ induced by the orthogonal complement $X^{\perp_1}$. Any reflection in G_{HP^n} fixing $\pi^{-1}(H_X)$ pointwise must be of the form $\phi(A, v)$ where the linear part A fixes $X^{\perp_1}$ pointwise. Hence the only possible candidates for A are the identity and the Minkowski reflection in H_X, which we denote by r_X. Since (A, v) is assumed to be an involution, $A = \mathrm{id}$ only gives the trivial transformation (i.e. $v = 0$). On the other hand, imposing the involutive condition for the choice $A = r_X$ we obtain the reflections $\phi(r_X, v)$ for any $v \in \mathrm{Span}(X)$. These are indeed reflections in the half-pipe hyperplane $\pi^{-1}(H_X)$, since they fix setwise all spacelike hyperplanes of $\mathbb{R}^{1,n-1}$ with normal direction in H_X. □

Finally, it is necessary to analyse conditions which assure that two reflections commute. From Proposition 5.6, it is clear that two reflections $\phi(-\mathrm{id}, 2p)$ and $\phi(-\mathrm{id}, 2q)$ in non-degenerate hyperplanes do not commute unless $p = q$, i.e. unless the hyperplanes of reflection coincide.

By Proposition 5.7, reflections in degenerate hyperplanes are induced by Minkowski reflections in timelike hyperplanes. Hence two reflections $\phi(r_{X_1}, v_1)$ and $\phi(r_{X_2}, v_2)$ commute if and only if their linear parts commute.

The remaining case is considered in the following lemma, which is straightforward:

Lemma 5.8. *Let v, w, X be vectors in $\mathbb{R}^{1,n-1}$, with $q_1(X) = 1$ and $v \in \mathrm{Span}(X)$. The Minkowski isometries (r_X, v) and $(-\mathrm{id}, w)$ commute if and only if $w = v + u$ with $u \in X^{\perp_1}$.*

Proof. An easy computation shows that (r_X, v) and $(-\mathrm{id}, w)$ commute if and only if

$$(\mathrm{id} - r_X)(w) = 2v \ . \tag{25}$$

Writing $w = \lambda X + u$ for $\lambda \in \mathbb{R}$ and $u \in X^{\perp_1}$, we have $r_X(w) = -\lambda X + u$, hence the condition (25) is equivalent to $\lambda X = v$. □

5.5. **Cusp groups in half-pipe geometry.** Let us now discuss the properties of flexibility and rigidity of cusp representations for half-pipe geometry, similarly to what we did for hyperbolic and AdS geometry in Section 3. The statements will be completely analogous, but the proofs simpler than their AdS (and hyperbolic) counterparts above.

The definitions of cusp groups and collapsed cusp groups are parallel to the AdS case:

Definition 5.9 (Cusp groups for HP^3)**.** The image of a representation of Γ_{rect} into G_{HP^3} is called:

- a *cusp group* if the four generators are sent to reflections in four distinct planes which share the same point in $\partial\mathsf{HP}^n$;
- a *collapsed cusp group* if the four generators are sent to reflections along three distinct planes, two degenerate and one non-degenerate, which share the same point in $\partial\mathsf{HP}^n$.

It follows from the discussion of the previous section that a cusp group representation must necessarily map two generators corresponding to opposite sides of the rectangle to reflections in degenerate hyperplanes, and the other two generators to reflections in non-degenerate hyperplanes.

The following example describes the structure of a (possibly collapsed) cusp group in HP geometry. By the non-uniqueness of half-pipe reflections in a degenerate plane (Proposition 5.7), we need to describe not only the planes fixed by the reflections associated to each generators, but also the reflections themselves.

Example 5.10. Let the image of $\rho\colon \Gamma_{\mathrm{rect}} \to G_{\mathsf{HP}^3}$ be a cusp group or collapsed cusp group, let $\boldsymbol{s}_1, \boldsymbol{s}_2$ be the generators such that $\rho(\boldsymbol{s}_1), \rho(\boldsymbol{s}_2)$ are reflections in a non-degenerate plane, and $\boldsymbol{t}_1, \boldsymbol{t}_2$ those such that $\rho(\boldsymbol{t}_1), \rho(\boldsymbol{t}_2)$ are reflections in a degenerate plane. Up to conjugacy, we can assume that $\rho(\boldsymbol{s}_1) = \phi(-\mathrm{id}, 0)$, that is, $\rho(\boldsymbol{s}_1)$ is the unique reflection in the dual plane to the origin of $\mathbb{R}^{1,2}$.

Using Lemma 5.8, $\rho(\boldsymbol{t}_1)$ and $\rho(\boldsymbol{t}_2)$ are necessarily of the form $\phi(r_{X_i}, 0)$, for X_i a unit spacelike vector in $\mathbb{R}^{1,2}$. This means that the two degenerate planes fixed by $\rho(\boldsymbol{t}_i)$ are of the form $\pi^{-1}(H_{X_i})$, for $i = 1, 2$. Since the four planes are assumed to meet in a single point in $\partial\mathsf{HP}^3$, necessarily the geodesics H_{X_1} and H_{X_2} of $\mathbb{H}^2$ meet in $\partial\mathbb{H}^2$. This means that $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a lightlike line in $\mathbb{R}^{1,2}$.

Finally, by Lemma 5.8 $\rho(\boldsymbol{s}_2)$ must be of the form $\phi(-\mathrm{id}, w)$ for some $w \in X_1^{\perp_1} \cap X_2^{\perp_1}$. This means that the non-degenerate plane fixed by $\rho(\boldsymbol{s}_2)$ is the dual of the point $w/2 \in \mathbb{R}^{1,2}$. If $w = 0$, then we have a collapsed cusp group, otherwise a cusp group. See Figure 7 on the left.

Let us now prove the HP analogue of Propositions 3.3 and 3.5 (see Figure 7).

Proposition 5.11. *Let $\rho\colon \Gamma_{\mathrm{rect}} \to G_{\mathsf{HP}^3}$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

1. *If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$.*
2. *The image of ρ' is a cusp group.*

(3) A pair of opposite planes intersect in HP^3*, while the other pair of opposite planes have disjoint closures in* $\mathsf{HP}^3 \cup \partial\mathsf{HP}^3 \smallsetminus \{\infty\}$*.*

Proof. Let $\rho' : \Gamma_{\mathrm{rect}} \to G_{\mathsf{HP}^3}$ be a representation nearby ρ. As in Example 5.10, we can assume that the reflection associated to one of the generators $\boldsymbol{s}_1$ of Γ_{rect} is $\rho'(\boldsymbol{s}_1) = \phi(-\mathrm{id}, 0)$, so that its fixed plane is the dual plane to the origin of $\mathbb{R}^{1,2}$. Repeating the argument of Example 5.10, we have $\rho'(\boldsymbol{t}_i) = \phi(r_{X_i}, 0)$ for some unit spacelike vectors X_i, and $\rho'(\boldsymbol{s}_2) = \phi(-\mathrm{id}, w)$ for some $w \in X_1^{\perp_1} \cap X_2^{\perp_1}$.

If $w = 0$, we are in case (1). Let us therefore assume $w \neq 0$. If the geodesics H_{X_1} and H_{X_2} intersect in $\partial\mathbb{H}^2$, then $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a lightlike geodesic, hence the image of ρ' is a cusp group as in Example 5.10 and we are in case (2).

If H_{X_1} and H_{X_2} intersect in $\mathbb{H}^2$, then $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a timelike geodesic, hence w is timelike. By Lemma 5.4, the fixed planes of $\rho'(\boldsymbol{s}_1)$ and $\rho'(\boldsymbol{s}_2)$ are disjoint, while the degenerate hyperplanes fixed by $\rho'(\boldsymbol{t}_1)$ and $\rho'(\boldsymbol{t}_2)$, namely $\pi^{-1}(H_{X_1})$ and $\pi^{-1}(H_{X_2})$, intersect in HP^3 (along a degenerate line). Hence point (3) is fulfilled.

Finally, if H_{X_1} and H_{X_2} are ultraparallel geodesics, then the closures of $\pi^{-1}(H_{X_1})$ and $\pi^{-1}(H_{X_2})$ only intersect in $\{\infty\}$. In this case $X_1^{\perp_1} \cap X_2^{\perp_1}$ is a spacelike geodesic, hence by Lemma 5.4 the fixed planes of $\rho'(\boldsymbol{s}_1)$ and $\rho'(\boldsymbol{s}_2)$ intersect. Therefore point (3) is fulfilled again. □

Moving to dimension four, we define cusp groups in half-pipe geometry:

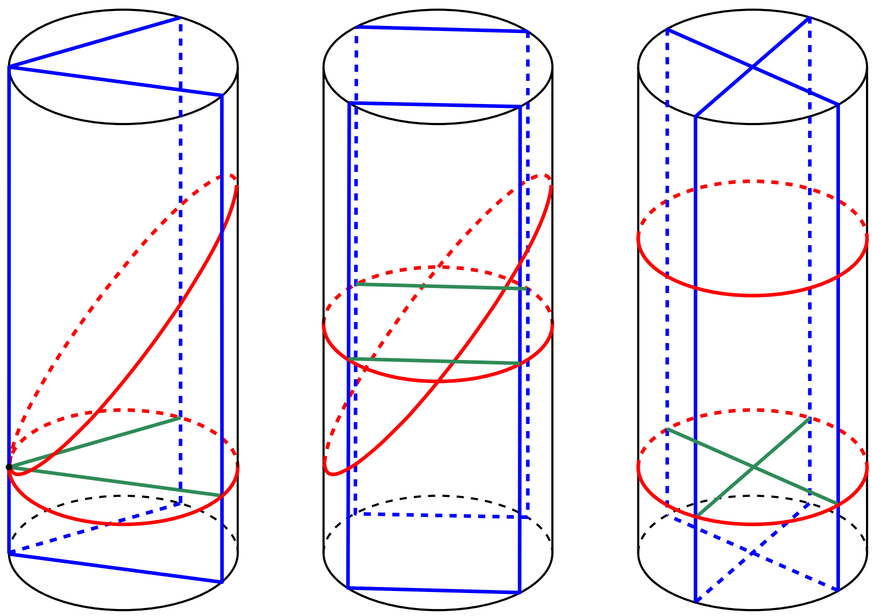

FIGURE 7. Three possibilities for a representation of Γ_{rect} in G_{HP^3}, as in the proof of Proposition 5.11. In red, two non-degenerate planes, in blue two degenerate planes, and in green their intersections, which are geodesics in a copy of $\mathbb{H}^2$. On the left, the green geodesics are tangent at infinity and we have a cusp group. In the middle, they are ultra-parallel, so the degenerate blue planes of HP^3 are disjoint, while the non-degenerate red planes intersect. On the right, the green geodesics intersect, so do the blue (degenerate) planes, while the red (non-degenerate) planes are disjoint.

Definition 5.12 (Cusp groups for HP^4)**.** The image of a representation of Γ_{cube} into G_{HP^4} is called:

- a *cusp group* if the 6 generators are sent to reflections in 6 distinct hyperplanes which share the same point at infinity;
- a *collapsed cusp group* if the 6 generators are sent to reflections along five distinct hyperplanes, four degenerate and one spacelike, which share the same point at infinity.

The half-pipe version of Proposition 3.12 and 3.14 is now proved along the same lines:

Proposition 5.13. *Let $\rho\colon \Gamma_{\text{cube}} \to G_{\mathsf{HP}^4}$ be a representation whose image is a cusp group or a collapsed cusp group. For all nearby representations ρ', exactly one of the following possibilities holds:*

(1) If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are generators such that $\rho(\boldsymbol{s}_1) = \rho(\boldsymbol{s}_2)$ is a reflection in a non-degenerate hyperplane, then $\rho'(\boldsymbol{s}_1) = \rho'(\boldsymbol{s}_2)$.

(2) The image of ρ' is a cusp group.

Proof. Let us denote by $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ the generators of Γ_{cube} (corresponding to opposite faces of the cube) that are sent by ρ to reflections in a non-degenerate hyperplane; by $\boldsymbol{t}_1, \boldsymbol{t}_2$ and $\boldsymbol{u}_1, \boldsymbol{u}_2$ the other two pairs of opposite generators, which are necessarily sent to reflections in degenerate hyperplanes. By continuity, the same holds for ρ'.

Up to conjugation we can assume that $\rho'(\boldsymbol{s}_1) = \phi(-\mathrm{id}, 0)$, and therefore by Lemma 5.8 $\rho'(\boldsymbol{t}_i) = \phi(r_{X_i}, 0)$ and $\rho'(\boldsymbol{u}_i) = \phi(r_{Y_i}, 0)$, for X_i, Y_i unit spacelike vectors. The restriction of ρ' to the subgroup generated by these four elements gives a representation of Γ_{rect} in a copy of $\mathrm{Isom}(\mathbb{H}^3)$, and is nearby a 3-dimensional cusp group.

Suppose first $\rho'|_{\Gamma_{\text{rect}}}$ is a cusp group in $\mathrm{Isom}(\mathbb{H}^3)$. This means that $X_1^{\perp_1} \cap X_2^{\perp_1} \cap Y_1^{\perp_1} \cap Y_2^{\perp_1}$ is a lightlike line in $\mathbb{R}^{1,3}$. Then $\rho'(\boldsymbol{s}_2)$ is of the form $\phi(-\mathrm{id}, w)$ and by Lemma 5.8 $w \in X_1^{\perp_1} \cap X_2^{\perp_1} \cap Y_1^{\perp_1} \cap Y_2^{\perp_1}$. Hence ρ gives a cusp group in G_{HP^4} and we are in point (2).

If $\rho'|_{\Gamma_{\text{rect}}}$ does not give a cusp group in $\mathbb{H}^3$, by Proposition 3.3 two planes intersect in $\mathbb{H}^3$, while the other two are disjoint in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. We will derive a contradiction. Up to relabelling, we can assume that the planes H_{X_1} and H_{X_2} intersect in $\mathbb{H}^3$, while the closures of H_{Y_1} and H_{Y_2} are disjoint. Hence in the degenerate subspace $\pi^{-1}(H_{X_1})$ (which is a copy of HP^3), the sets $\pi^{-1}(H_{Y_1}) \cap \pi^{-1}(H_{X_1})$ and $\pi^{-1}(H_{Y_2}) \cap \pi^{-1}(H_{X_1})$ are disjoint. Applying Proposition 5.11 to the restriction of ρ' to the subgroup generated by $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{u}_1, \boldsymbol{u}_2$, the fixed planes of $\rho'(\boldsymbol{s}_1)$ and $\rho'(\boldsymbol{s}_2)$ intersect in $\pi^{-1}(H_{X_1})$ (and thus in HP^4).

On the other hand, in $\pi^{-1}(H_{Y_1})$ (which is again a copy of HP^3), $\pi^{-1}(H_{X_1}) \cap \pi^{-1}(H_{Y_1})$ and $\pi^{-1}(H_{X_2}) \cap \pi^{-1}(H_{Y_1})$ intersect. In fact H_{X_1} and H_{X_2} intersect in $\mathbb{H}^3$, hence also in H_{Y_1} since H_{X_1} and H_{X_2} are orthogonal to H_{Y_1}. By Proposition 5.11 again, the fixed planes of $\rho'(\boldsymbol{s}_1)$ and $\rho'(\boldsymbol{s}_2)$ are disjoint in HP^4, which contradicts the conclusion of the previous paragraph. □

6. Group cohomology and the HP character variety

The goal of this section is to prove the half-pipe part of Theorem 1.1. An essential step is an explicit computation of the first cohomology group $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ in Proposition 6.5, a result for which we will give other applications in Section 7.

6.1. The first cohomology group.

We recall here a few notions of group cohomology.

Let Γ be a group, V a finite-dimensional real vector space, and $\varrho\colon \Gamma \to \mathrm{GL}(V)$ a representation. The *first cohomology group* of Γ associated to ϱ is the quotient

$$H^1_\varrho(\Gamma, V) = Z^1_\varrho(\Gamma, V)/B^1_\varrho(\Gamma, V) ,$$

where

- the space of *cocycles* is
$$Z^1_\varrho(\Gamma, V) = \{\tau \colon \Gamma \to V \mid \forall \gamma, \eta \in \Gamma \quad \tau(\gamma\eta) = \varrho(\gamma)\tau(\eta) + \tau(\gamma)\} ,$$
- the space of *coboundaries* is
$$B^1_\varrho(\Gamma, V) = \{\tau \colon \Gamma \to V \mid \exists v \in V \ \forall \gamma \in \Gamma \quad \tau(\gamma) = \varrho(\gamma)v - v\} .$$

The space $Z^1_\varrho(\Gamma, V)$ coincides with the space of *affine deformations* of ϱ, namely the functions $\tau \colon \Gamma \to V$ such that (ϱ, τ) gives a representation of Γ to $\mathrm{GL}(V) \ltimes V$. The difference $\tau - \tau'$ of two cocycles is a coboundary if and only if the corresponding representations (ϱ, τ) and (ϱ, τ') are conjugate in V. We have in summary:

Lemma 6.1. *The vector space $H^1_\varrho(\Gamma, V)$ parameterises the representations of Γ in $\mathrm{GL}(V) \ltimes V$ having linear part ϱ, up to conjugation.*

When Γ is a right-angled Coxeter group, the space $Z^1_\varrho(\Gamma, V)$ has the following description in terms of generators and relations.

Lemma 6.2. *Let Γ be a right-angled Coxeter group as in Defintion 2.7, and $\varrho \colon \Gamma \to \mathrm{GL}(V)$ a representation. Then $Z^1_\varrho(\Gamma, V)$ is isomorphic to the vector space of functions $\tau \colon S \to V$ such that:*

- $\tau(\boldsymbol{s}) \in \mathrm{Ker}(\mathrm{id} + \varrho(\boldsymbol{s}))$ *for all $\boldsymbol{s} \in S$, and*
- $(\mathrm{id} - \varrho(\boldsymbol{s}_i))\tau(\boldsymbol{s}_j) = (\mathrm{id} - \varrho(\boldsymbol{s}_j))\tau(\boldsymbol{s}_i)$ *for all $(\boldsymbol{s}_i, \boldsymbol{s}_j) \in R$.*

Proof. Clearly a cocycle in $Z^1_\varrho(\Gamma, V)$ is determined by its values on the generators. The conditions that have to be satisfied by τ for each relation are $0 = \tau(\boldsymbol{s}^2) = \varrho(\boldsymbol{s})\tau(\boldsymbol{s}) + \tau(\boldsymbol{s})$, from which we get the first point, and $\tau(\boldsymbol{s}_i\boldsymbol{s}_j) = \tau(\boldsymbol{s}_j\boldsymbol{s}_i)$ for every $(\boldsymbol{s}_i, \boldsymbol{s}_j) \in R$. Expanding $\tau(\boldsymbol{s}_i\boldsymbol{s}_j) = \varrho(\boldsymbol{s}_i)\tau(\boldsymbol{s}_j) + \tau(\boldsymbol{s}_i)$ we obtain the second point. □

6.2. **Half-pipe representations.** We now introduce the half-pipe representations of our interest, which have been computed in [RS, Remark 7.16] by applying a rescaling argument to the hyperbolic or AdS holonomy representations ρ_t.

Notation. Throughout the following, we will denote by $\langle \cdot, \cdot \rangle$ the Minkowski product of $\mathbb{R}^{1,3}$ (previously denoted by b_1) and by $v^\perp \subset \mathbb{R}^{1,3}$ the orthogonal complement of $v \in \mathbb{R}^{1,3}$ with respect to the Minkowski product.

Recall that by Lemma 5.1 the transformation group G_{HP^4} is isomorphic to $\mathrm{Isom}(\mathbb{R}^{1,3}) \cong \mathrm{O}(1,3) \ltimes \mathbb{R}^4$. We will exhibit the half-pipe holonomies as representations in $\mathrm{Isom}(\mathbb{R}^{1,3})$.

Definition 6.3 (The HP representation ρ_λ)**.** Given $\lambda \in \mathbb{R}$, we define a representation

$$\rho_\lambda = (\varrho_0, \tau_\lambda) \colon \Gamma_{22} \to \mathrm{O}(1,3) \ltimes \mathbb{R}^4$$

on the standard generators of Γ_{22} as follows. The linear part ϱ_0 is independent of λ and is defined by:

$$\begin{aligned} \varrho_0(\boldsymbol{i}^+) &= -\mathrm{id} && \text{for each } \boldsymbol{i} \in \{\boldsymbol{0}, \ldots, \boldsymbol{7}\} \\ \varrho_0(\boldsymbol{i}^-) &= r_{v_i} && \text{for each } \boldsymbol{i} \in \{\boldsymbol{0}, \ldots, \boldsymbol{7}\} \\ \varrho_0(\boldsymbol{X}) &= r_{v_{\boldsymbol{X}}} && \text{for each } \boldsymbol{X} \in \{\boldsymbol{A}, \ldots, \boldsymbol{F}\} \end{aligned} \tag{26}$$

while the translation part is:

$$\begin{aligned} \tau_\lambda(\boldsymbol{i}^+) = \tau_\lambda(\boldsymbol{i}^-) = (-1)^i \lambda v_{\boldsymbol{i}} \quad & \text{for each } \boldsymbol{i} \in \{\boldsymbol{0}, \dots, \boldsymbol{7}\} \\ \tau_\lambda(\boldsymbol{X}) = 0 \quad & \text{for each } \boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\} \end{aligned} \tag{27}$$

where the vectors $v_{\boldsymbol{s}}$ are defined in Table 4.

Recall that the vectors in Table 4 definine the bounding planes of an ideal right-angled cuboctahedron in $\mathbb{H}^3$. Moreover, r_v denotes the reflection in $\mathrm{O}(1,3)$ in the hyperplane $v^\perp$, namely, the linear transformation acting on $v^\perp$ as the identity and on the subspace generated by v as minus the identity.

Remark 6.4. When $\lambda = 0$, the representation ρ_0 is naturally identified to those introduced in Definition 4.1 for $t = 0$. Indeed, recall from Section 4.3 that in the hyperbolic and AdS case ρ_0 takes value in the stabiliser G_0 of the hyperplane $\{x_4 = 0\}$, and G_0 is a common subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$, both seen as a subgroups of $\mathrm{PGL}(5,\mathbb{R})$.

Now, the representation $\rho_0 = (\varrho_0, 0)$ introduced in Definition 6.3 also takes value in the stabiliser of $\{x_4 = 0\}$ in G_{HP^4} which coincides again with the subgroup G_0 of $\mathrm{PGL}(5,\mathbb{R})$. Under the isomorphism with $\mathrm{Isom}(\mathbb{R}^{1,3})$, the group G_0 is dually identified with the stabiliser of the origin in $\mathbb{R}^{1,3}$, namely the linear subgroup $\mathrm{O}(1,3) < \mathrm{Isom}(\mathbb{R}^{1,3})$. The explicit isomorphism $\mathrm{O}(1,3) \cong G_0$ is

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in G_0 \ . \tag{28}$$

(It is important to pay attention that G_0 is a subgroup of $\mathrm{PGL}(5,\mathbb{R})$, hence the expression on the right hand-side is a projective class of matrix: for instance, the reflection r of (5) is the image of $A = -\mathrm{id}$ in the isomorphism (28).)

Under the isomorphism (28), the representation $\rho_0 = (\varrho_0, 0)$ of Definition 6.3 (with zero translation part) coincides with the "collapsed" representation expressed in (6). This justifies that in the statement of Theorem 1.1 we refer to the *same* representation ρ_0 in all three geometries.

The goal of the following section is to compute the first cohomology group associated to the representation $\varrho_0 \colon \Gamma_{22} \to \mathrm{O}(1,3)$ of Definition 6.3. Applications of the result will then be given in Sections 6.4 and 7.1.

6.3. **The "geometric" cocycle is a generator.** Recall Definition 6.3. The goal of this section is to prove the following:

Proposition 6.5. *The vector space $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ has dimension one.*

To prove Proposition 6.5, we will show that every cohomology class in $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ is represented by a cocycle τ_λ of the form (27), for some $\lambda \in \mathbb{R}$.

We already know from [RS, Remark 7.16] that $\rho_\lambda = (\varrho_0, \tau_\lambda)$ of Definition 6.3 is a representation of Γ_{22}, hence τ_λ is a cocycle. This can however be checked directly from (26) and (27) using Lemma 6.2. Let us introduce some additional notation:

Definition 6.6 (The subspace U_0)**.** We denote by U_0 the 1-dimensional vector subspace of $Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ composed of cocycles of the form (27), for some $\lambda \in \mathbb{R}$.

Let us observe that τ_λ vanishes on all the letter generators and that $\tau_\lambda(\boldsymbol{i}^-)$ and $\tau_\lambda(\boldsymbol{i}^+)$ are all vectors of norm $|\lambda|$ for the Minkowski product on $\mathbb{R}^{1,3}$, since all the $v_{\boldsymbol{i}}$ have unit Minkowski norm.

The ultimate goal will be to show that any cocycle $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ has a unique decomposition $\tau = \tau_\lambda - \eta$, for some $\eta \in B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ and $\tau_\lambda \in U_0$. The proof will follow from a sequence of computational lemmas.

Lemma 6.7. *Let* $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$. *Then,*

$$\tau(\boldsymbol{i}^-) \in \mathrm{Span}(v_{\boldsymbol{i}}) \quad \text{for each } \boldsymbol{i}^- \in \{\boldsymbol{0}^-, \dots, \boldsymbol{7}^-\}, \text{ and}$$
$$\tau(\boldsymbol{X}) \in \mathrm{Span}(v_{\boldsymbol{X}}) \quad \text{for each } \boldsymbol{X} \in \{\boldsymbol{A}, \dots, \boldsymbol{F}\}.$$

Proof. By Lemma 6.2 we get $\tau(\boldsymbol{i}^-) \in \mathrm{Ker}(\mathrm{id} + \varrho_0(\boldsymbol{i}^-))$. This kernel equals the subspace generated by $v_{\boldsymbol{i}}$ since $\varrho_0(\boldsymbol{i}^-)$ is the Minkowski reflection fixing the hyperplane $v_{\boldsymbol{i}}^\perp$. The proof for the letter generators is the same. □

The following step is a first reduction of the problem.

Lemma 6.8. *Let* $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$. *Then there exists a unique* $\eta \in B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ *such that, if* $\hat\tau = \tau - \eta$, *then*

$$\hat\tau(\boldsymbol{A}) = \hat\tau(\boldsymbol{B}) = \hat\tau(\boldsymbol{C}) = \hat\tau(\boldsymbol{D}) = 0\ . \tag{29}$$

After the proof of Lemma 6.8, we will show that if $\hat\tau$ satisfies (29), then it is of the form (27) for some $\lambda \in \mathbb{R}$. Together with Lemma 6.8, this will imply that

$$Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) = U_0 \oplus B^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$$

and therefore that $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ is one-dimensional.

Proof of Lemma 6.8. Let τ be any cocycle. By Lemma 6.7, we have that $\tau(\boldsymbol{X}) \in \mathrm{Span}(v_{\boldsymbol{X}})$ for all $\boldsymbol{X} \in \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$. Define the linear map

$$L\colon \mathbb{R}^{1,3} \to \mathrm{Span}(v_{\boldsymbol{A}}) \oplus \mathrm{Span}(v_{\boldsymbol{B}}) \oplus \mathrm{Span}(v_{\boldsymbol{C}}) \oplus \mathrm{Span}(v_{\boldsymbol{D}})$$

by

$$L(w) = (\varrho_0(\boldsymbol{A})w - w, \varrho_0(\boldsymbol{B})w - w, \varrho_0(\boldsymbol{C})w - w, \varrho_0(\boldsymbol{D})w - w)\ .$$

The proof follows if we show that L is invertible.

Let us write the matrix associated to L in the basis $\{v_{\boldsymbol{A}}, v_{\boldsymbol{B}}, v_{\boldsymbol{C}}, v_{\boldsymbol{D}}\}$ on the source and on the target. Recalling that the $v_{\boldsymbol{X}}$ are all unit vectors for the Minkowski product $\langle \cdot, \cdot \rangle$ and that $\rho_0(\boldsymbol{X})$ is the reflection in $v_{\boldsymbol{X}}^\perp$, we have

$$\varrho_0(\boldsymbol{X})v_{\boldsymbol{Y}} - v_{\boldsymbol{Y}} = \varrho_0(\boldsymbol{X})\left(\langle v_{\boldsymbol{Y}}, v_{\boldsymbol{X}}\rangle v_{\boldsymbol{X}}\right) - \langle v_{\boldsymbol{Y}}, v_{\boldsymbol{X}}\rangle v_{\boldsymbol{X}} = -2\langle v_{\boldsymbol{Y}}, v_{\boldsymbol{X}}\rangle v_{\boldsymbol{X}}\ .$$

This shows that the associated matrix of L is

$$-2\begin{pmatrix} \langle v_{\boldsymbol{A}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{A}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{A}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{A}}, v_{\boldsymbol{D}}\rangle \\ \langle v_{\boldsymbol{B}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{B}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{B}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{B}}, v_{\boldsymbol{D}}\rangle \\ \langle v_{\boldsymbol{C}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{C}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{C}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{C}}, v_{\boldsymbol{D}}\rangle \\ \langle v_{\boldsymbol{D}}, v_{\boldsymbol{A}}\rangle & \langle v_{\boldsymbol{D}}, v_{\boldsymbol{B}}\rangle & \langle v_{\boldsymbol{D}}, v_{\boldsymbol{C}}\rangle & \langle v_{\boldsymbol{D}}, v_{\boldsymbol{D}}\rangle \end{pmatrix},$$

which is invertible by the non-degeneracy of the Minkowski product. □

Let us now compute the cocycle condition which arises from any orthogonality condition in Γ_{22}.

Lemma 6.9. *Let* $\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$.

- *For any relation in* Γ_{22} *of the form* $\boldsymbol{i}^+\boldsymbol{j}^- = \boldsymbol{j}^-\boldsymbol{i}^+$, *we have that* $\tau(\boldsymbol{i}^+) - \tau(\boldsymbol{j}^-) \in v_{\boldsymbol{j}}^\perp$.
- *For any relation in* Γ_{22} *of the form* $\boldsymbol{i}^+\boldsymbol{X} = \boldsymbol{X}\boldsymbol{i}^+$, *we have that* $\tau(\boldsymbol{i}^+) - \tau(\boldsymbol{X}) \in v_{\boldsymbol{X}}^\perp$.

Proof. Let us show the first point, the second being completely analogous. By Lemma 6.2

$$\left(\mathrm{id} - \varrho_0(\boldsymbol{j}^-)\right)\tau(\boldsymbol{i}^+) = \left(\mathrm{id} - \varrho_0(\boldsymbol{i}^+)\right)\tau(\boldsymbol{j}^-) = 2\tau(\boldsymbol{j}^-) = \left(\mathrm{id} - \varrho_0(\boldsymbol{j}^-)\right)\tau(\boldsymbol{j}^-)\ ,$$

where we have used that $\varrho_0(\boldsymbol{i}^+) = -\mathrm{id}$, that $\varrho_0(\boldsymbol{j}^-)$ is the reflection in the Minkowski hyperplane $v_{\boldsymbol{j}}^\perp$, and that $\tau(\boldsymbol{j}^-) \in \mathrm{Span}(v_{\boldsymbol{j}})$ by Lemma 6.7. Hence $\tau(\boldsymbol{i}^+) - \tau(\boldsymbol{j}^-)$ is in the kernel of $\mathrm{id} - \varrho_0(\boldsymbol{j}^-)$, namely in $v_{\boldsymbol{j}}^\perp$. □

Let us now go back to showing that a cocycle $\hat\tau$ as in Lemma 6.8 is of the form (27). Our next step is:

Lemma 6.10. *Suppose $\hat\tau \in Z^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ satisfies (29). Then $\hat\tau(\mathbf{0}^+) = \hat\tau(\mathbf{0}^-) = \lambda v_{\mathbf{0}}$ and $\hat\tau(\mathbf{3}^+) = \hat\tau(\mathbf{3}^-) = -\lambda v_{\mathbf{3}}$ for some $\lambda \in \mathbb{R}$.*

Proof. It follows from Lemma 6.7 that $\hat\tau(\mathbf{0}^-) = \mu_0 v_{\mathbf{0}}$, and similarly $\hat\tau(\mathbf{3}^-) = \mu_3 v_{\mathbf{3}}$. We remark that we have no similar condition on the $\boldsymbol{i}^+$ coming from the relation that $\boldsymbol{i}^+$ squares to the identity.

However, we claim that in our assumption also $\hat\tau(\mathbf{0}^+) \in \mathrm{Span}(v_{\mathbf{0}})$ and $\hat\tau(\mathbf{3}^+) \in \mathrm{Span}(v_{\mathbf{3}})$. Indeed, applying Lemma 6.9 to the relation $\mathbf{0}^+\boldsymbol{A} = \boldsymbol{A}\mathbf{0}^+$ and using that $\hat\tau(\boldsymbol{A}) = 0$ by hypothesis, we get $\hat\tau(\mathbf{0}^+) \in v_{\boldsymbol{A}}^\perp$. Similarly, from $\mathbf{0}^+\boldsymbol{B} = \boldsymbol{B}\mathbf{0}^+$ and $\mathbf{0}^+\boldsymbol{C} = \boldsymbol{C}\mathbf{0}^+$, we obtain that $\hat\tau(\mathbf{0}^+)$ is in $v_{\boldsymbol{B}}^\perp$ and $v_{\boldsymbol{C}}^\perp$. Now, $v_{\boldsymbol{A}}$, $v_{\boldsymbol{B}}$ and $v_{\boldsymbol{C}}$ are linearly independent, hence $v_{\boldsymbol{A}}^\perp \cap v_{\boldsymbol{B}}^\perp \cap v_{\boldsymbol{C}}^\perp$ is 1-dimensional and therefore coincides with $\mathrm{Span}(v_{\mathbf{0}})$, since $v_{\mathbf{0}}$ is orthogonal to all of them. By applying the same argument to $\hat\tau(\mathbf{3}^+)$ and the letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$ (since by hypothesis $\hat\tau$ vanishes on $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$), we obtain that $\hat\tau(\mathbf{3}^+) \in \mathrm{Span}(v_{\mathbf{3}})$.

Hence we have shown that $\hat\tau(\mathbf{0}^+) = \lambda_0 v_{\mathbf{0}}$ and $\hat\tau(\mathbf{3}^+) = \lambda_3 v_{\mathbf{3}}$. We have to show that $\lambda_0 = \mu_0 = -\lambda_3 = -\mu_3$. Let us apply Lemma 6.9 to the relation $\mathbf{0}^+\mathbf{0}^- = \mathbf{0}^-\mathbf{0}^+$. We obtain

$$\hat\tau(\mathbf{0}^+) - \hat\tau(\mathbf{0}^-) \in v_{\mathbf{0}}^\perp\ ,$$

that is,

$$0 = \langle \lambda_0 v_{\mathbf{0}} - \mu_0 v_{\mathbf{0}}, v_{\mathbf{0}}\rangle = \lambda_0 - \mu_0\ ,$$

hence $\lambda_0 = \mu_0$. Analogously $\lambda_3 = \mu_3$. If we now apply Lemma 6.9 to the relation $\mathbf{0}^+\mathbf{3}^- = \mathbf{3}^-\mathbf{0}^+$ we get

$$\hat\tau(\mathbf{0}^+) - \hat\tau(\mathbf{3}^-) \in v_{\mathbf{3}}^\perp\ ,$$

which in turn gives

$$0 = \langle \lambda_0 v_{\mathbf{0}} - \lambda_3 v_{\mathbf{3}}, v_{\mathbf{3}}\rangle = \lambda_0\langle v_{\mathbf{0}}, v_{\mathbf{3}}\rangle - \lambda_3\langle v_{\mathbf{3}}, v_{\mathbf{3}}\rangle = -\lambda_0 - \lambda_3.$$

We conclude by setting $\lambda := \lambda_0 = -\lambda_3$. □

Remark 6.11. The proof of Lemma 6.10 only worked for $\boldsymbol{i} = \mathbf{0}, \mathbf{3}$ because we used that $\hat\tau$ vanishes on $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$, and we needed to pick three linearly independent vectors among these four. Once we show that $\hat\tau$ also vanishes on $\boldsymbol{E}$ and $\boldsymbol{F}$ (Lemma 6.12 below), the same argument will apply exactly in the same way to show that

$$\hat\tau(\boldsymbol{i}^+) = \hat\tau(\boldsymbol{i}^-) = \lambda v_{\boldsymbol{i}}$$

for $\boldsymbol{i}$ odd, and

$$\hat\tau(\boldsymbol{i}^+) = \hat\tau(\boldsymbol{i}^-) = -\lambda v_{\boldsymbol{i}}$$

for $\boldsymbol{i}$ even. This will therefore conclude the proof that $\hat\tau$ is in the form (27).

Lemma 6.12. *If $\hat\tau \in Z^1_{\rho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ satisfies (29), then $\hat\tau(\boldsymbol{E}) = \hat\tau(\boldsymbol{F}) = 0$.*

Proof. From Lemma 6.7, we know that

$$\hat\tau(\boldsymbol{E}) = e v_{\boldsymbol{E}} \qquad \text{and} \qquad \hat\tau(\boldsymbol{F}) = f v_{\boldsymbol{F}}\ .$$

We wish to show that $e = f = 0$. Let us first prove that $e = 0$.

Observe that $v_{\boldsymbol{A}}$, $v_{\boldsymbol{C}}$ and $v_{\boldsymbol{E}}$ are linearly independent, and they are all orthogonal to $v_{\mathbf{1}}$. Hence $\{v_{\mathbf{1}}, v_{\boldsymbol{A}}, v_{\boldsymbol{C}}, v_{\boldsymbol{E}}\}$ is a (non-orthogonal!) basis of unit vectors and we can decompose:

$$\hat{\tau}(\mathbf{1}^+) = \lambda_1 v_{\mathbf{1}} + \alpha v_{\boldsymbol{A}} + \gamma v_{\boldsymbol{C}} + \epsilon v_{\boldsymbol{E}} \ .$$

(We ultimately will get, at the end of the proof, that $\lambda_1 = -\lambda$ and $\alpha = \gamma = \epsilon = 0$, but we do not know this yet.) As a preliminary remark, observe that $\hat{\tau}(\mathbf{1}^-) = \lambda_1 v_{\mathbf{1}}$, since from the relation $\mathbf{1}^+\mathbf{1}^- = \mathbf{1}^-\mathbf{1}^+$ we obtain

$$\hat{\tau}(\mathbf{1}^+) - \hat{\tau}(\mathbf{1}^-) \in v_{\mathbf{1}}^{\perp} \ ,$$

and comparing with the above decomposition, necessarily $\hat{\tau}(\mathbf{1}^-) = \lambda_1 v_{\mathbf{1}}$.

Since $\hat{\tau}(\boldsymbol{A}) = 0$, from the relation $\mathbf{1}^+\boldsymbol{A} = \boldsymbol{A}\mathbf{1}^+$ we obtain

$$\hat{\tau}(\mathbf{1}^+) \in v_{\boldsymbol{A}}^{\perp} \ ,$$

namely,

$$0 = \langle \hat{\tau}(\mathbf{1}^+), v_{\boldsymbol{A}} \rangle = \alpha - \gamma - \epsilon \ . \tag{30}$$

From the same computation for the relation $\mathbf{1}^+\boldsymbol{C} = \boldsymbol{C}\mathbf{1}^+$ we derive

$$0 = \langle \hat{\tau}(\mathbf{1}^+), v_{\boldsymbol{C}} \rangle = -\alpha + \gamma - \epsilon \ . \tag{31}$$

Finally, the relation $\mathbf{1}^+\boldsymbol{E} = \boldsymbol{E}\mathbf{1}^+$ implies $\hat{\tau}(\mathbf{1}^+) - \hat{\tau}(\boldsymbol{E}) \in v_{\boldsymbol{E}}^{\perp}$, whence

$$e = \langle \hat{\tau}(\boldsymbol{E}), v_{\boldsymbol{E}} \rangle = \langle \hat{\tau}(\mathbf{1}^+), v_{\boldsymbol{E}} \rangle = -\alpha - \gamma + \epsilon \ . \tag{32}$$

From (30), (31) and (32) together we find

$$\alpha = \gamma = -\frac{e}{2} \qquad \epsilon = 0 \ . \tag{33}$$

On the other hand, consider the relation $\mathbf{1}^+\mathbf{2}^- = \mathbf{2}^-\mathbf{1}^+$. It implies

$$\hat{\tau}(\mathbf{1}^+) - \hat{\tau}(\mathbf{2}^-) \in v_{\mathbf{2}}^{\perp} \ ,$$

where we already know that $\hat{\tau}(\mathbf{2}^-) = \lambda_2 v_{\mathbf{2}}$. A direct computation gives

$$0 = \langle \lambda_1 v_{\mathbf{1}} + \alpha v_{\boldsymbol{A}} + \gamma v_{\boldsymbol{C}} + \epsilon v_{\boldsymbol{E}} - \lambda_2 v_{\mathbf{2}}, v_{\mathbf{2}} \rangle = -\lambda_1 - 2\sqrt{2}\gamma - \lambda_2 \ .$$

If we show that $\lambda_1 = -\lambda_2$, we are done for $\hat{\tau}(\boldsymbol{E})$, since $\gamma = 0$ implies $e = 0$ from (33).

To see this last point, recall that $\hat{\tau}(\mathbf{0}^+) = \lambda v_{\mathbf{0}}$ and $\hat{\tau}(\mathbf{3}^+) = -\lambda v_{\mathbf{3}}$ as proved in Lemma 6.10. Now, from the orthogonality relation $\mathbf{0}^+\mathbf{1}^- = \mathbf{1}^-\mathbf{0}^+$ we find $\hat{\tau}(\mathbf{0}^+) - \hat{\tau}(\mathbf{1}^-) \in v_{\mathbf{1}}^{\perp}$. Using the preliminary remark at the beginning of the proof,

$$0 = \lambda \langle v_{\mathbf{0}}, v_{\mathbf{1}} \rangle - \lambda_1 \langle v_{\mathbf{1}}, v_{\mathbf{1}} \rangle = -\lambda - \lambda_1 \ .$$

Thus $\lambda_1 = -\lambda$. Repeating the same argument to the relation $\mathbf{3}^+\mathbf{2}^- = \mathbf{2}^-\mathbf{3}^+$ one finds $\lambda_2 = \lambda$, and therefore $\lambda_1 = -\lambda_2$.

The proof that $f = 0$ follows the same lines, applied to $\mathbf{4}^+$ in place of $\mathbf{1}^+$, with the letters $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{F}$, and in the final part to $\mathbf{5}^-$ in place of $\mathbf{2}^-$. □

Having shown that $\hat{\tau}(\boldsymbol{X}) = 0$ for every $\boldsymbol{X}$, it remains to show that $\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-)$ has the form of (27). For $\boldsymbol{i} = \mathbf{0}, \mathbf{3}$, this is the content of Lemma 6.10. Following the same proof, one shows first that

$$\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-)$$

for every $\boldsymbol{i}$ (it suffices to modify the proof by picking three letters $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ so that $v_{\boldsymbol{X}}$, $v_{\boldsymbol{Y}}$ and $v_{\boldsymbol{Z}}$ are orthogonal to $v_{\boldsymbol{i}}$). Then using the crossed relations $\boldsymbol{i}^+\boldsymbol{j}^- = \boldsymbol{j}^-\boldsymbol{i}^+$ — it is easy to see that there are indeed enough of such relations — one mimics the second part of

Lemma 6.10 and obtains that

$$\hat{\tau}(\boldsymbol{i}^+) = \hat{\tau}(\boldsymbol{i}^-) = (-1)^i \lambda v_{\boldsymbol{i}} \ .$$

This concludes the proof of Proposition 6.5, namely that $\dim H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) = 1$.

6.4. **Proof of Theorem 1.1 in the HP case.** We are ready to conclude the proof of the half-pipe version of Theorem 1.1.

Recalling that $G_{\mathsf{HP}^4} \cong \mathrm{O}(1,3) \ltimes \mathbb{R}^{1,3}$, one has a natural map:

$$\mathcal{L} \colon X(\Gamma, G_{\mathsf{HP}^4}) \to X(\Gamma, \mathrm{O}(1,3))$$

which associates to the conjugacy class of a representation $\rho \colon \Gamma \to G_{\mathsf{HP}^4}$ the conjugacy class of the linear part of ρ.

Recalling Lemma 6.1, one has the identification

$$\mathcal{L}^{-1}([\varrho]) \cong H^1_{\varrho}(\Gamma, \mathbb{R}^{1,3}) \ . \tag{34}$$

Observe that if $\varrho' = h \circ \varrho \circ h^{-1}$ for $h \in \mathrm{O}(1,3)$, then $H^1_{\varrho}(\Gamma, \mathbb{R}^{1,3})$ and $H^1_{\varrho'}(\Gamma, \mathbb{R}^{1,3})$ are isomorphic by means of the map $\tau \mapsto h \circ \tau$.

Theorem 1.1 (HP case)**.** *The point $[\rho_0] \in X(\Gamma_{22}, G_{\mathsf{HP}^4})$ has a neighbourhood $\mathcal{U} = \mathcal{V} \cup \mathcal{H}$ homeomorphic to the set $\mathcal{S} = \{(x_1^2 + \ldots + x_{12}^2) \cdot x_{13} = 0\} \subset \mathbb{R}^{13}$, where:*

- *$[\rho_0]$ corresponds to the origin,*
- *$\mathcal{V}$ corresponds to the x_{13}-axis, and is identified to $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$,*
- *$\mathcal{H}$ corresponds to $\{x_{13} = 0\}$, identified to a neighbourhood of the complete hyperbolic orbifold structure of the ideal right-angled cuboctahedron in its deformation space.*

The group $G_{\mathsf{HP}^4}/G^+_{\mathsf{HP}^4} \cong \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{S}$ by changing sign to x_{13}.

Proof. The proof follows a similar strategy to the AdS (and hyperbolic) case, so we will split again the proof in several steps which are parallel to those given in Section 4.8. Most steps are much simpler here.

***Step 1*:** Let us define the vertical component $\mathcal{V}$ in $X(\Gamma_{22}, G_{\mathsf{HP}^4})$ as $\mathcal{L}^{-1}([\varrho_0])$, namely, $\mathcal{V}$ consists of all the conjugacy classes of representations with linear part in $[\varrho_0]$. By (34), $\mathcal{V}$ is identified to $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$, hence is homeomorphic to a line by Proposition 6.5. By construction, $\mathcal{V}$ contains the holonomy of the half-pipe orbifold structures we built in [RS].

***Step 2*:** The second component $\mathcal{H}$ is defined similarly to the AdS case. We define the map

$$\Psi \colon \mathrm{Hom}(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)) \to \mathrm{Hom}(\Gamma_{22}, G_{\mathsf{HP}^4})$$

sending a representation $\eta \colon \Gamma_{\mathrm{co}} \to \mathrm{Isom}(\mathbb{H}^3)$ to the representation $\Psi_\eta \colon \Gamma_{22} \to \mathrm{O}(1,3) < \mathrm{O}(1,3) \ltimes \mathbb{R}^{1,3}$ (hence with trivial translation part, which we omit) which sends each of the generators $\mathbf{0}^-, \ldots, \mathbf{7}^-, \boldsymbol{A}, \ldots, \boldsymbol{F}$ to the corresponding element of $\mathrm{O}(1,3)$, and each $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \ldots, \mathbf{7}^+\}$ to $-\mathrm{id}$.

Again, it is straightforward to check that the induced map

$$\widehat{\Psi} \colon X(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3)) \to X(\Gamma_{22}, G_{\mathsf{HP}^4}) \ .$$

is well-defined and injective.

The representation ρ_0 is clearly in the image of Ψ, since $\rho_0 = \Psi_{\eta_0}$ where η_0 is the holonomy representation of the complete hyperbolic orbifold structure of the cuboctahedron. As in the AdS case, $[\eta_0]$ has a neighborhood $\mathcal{H}_0$ in $X(\Gamma_{\mathrm{co}}, \mathrm{Isom}(\mathbb{H}^3))$ homeomorphic to $\mathbb{R}^{12}$ and on which $\widehat{\Psi}$ is a homeomorphism onto its image, and we define $\mathcal{H}$ to be the image of $\mathcal{H}_0$.

***Step 3*:** Clearly, the intersection of $\mathcal{H}$ and $\mathcal{V}$ consists only of the point $[\rho_0]$, since any element in $\mathcal{H}$ has trivial translation part (up to conjugacy).

***Step 4*:** We now show that the point $[\rho_0] \in X(\Gamma_{22}, G_{\mathsf{HP}^4})$ has a neighbourhood $\mathcal{U}$ which is contained in the union of the two components $\mathcal{V}$ and $\mathcal{H}$.

Let ρ be a nearby representation, with linear part $\mathcal{L}\rho$ and translation part $\tau\colon \Gamma_{22} \to \mathbb{R}^{1,3}$. Observe that, since $-\mathrm{id}$ is an isolated point in the representations of $\mathbb{Z}/2\mathbb{Z}$ into $\mathrm{O}(1,3)$, for each generator $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$ we have $\mathcal{L}\rho(\boldsymbol{i}^+) = -\mathrm{id}$.

We claim that if two distinct generators which are sent by ρ_0 to $-\mathrm{id}$ (hence necessarily of the form $\boldsymbol{i}^+$ and $\boldsymbol{j}^+$) are sent by ρ to the same reflection, than all the generators $\mathbf{0}^+, \dots, \mathbf{7}^+$ are sent by ρ to the same reflection. In other words, if $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+)$ for some $\boldsymbol{i}^+ \neq \boldsymbol{j}^+$, then $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+)$ for all $\boldsymbol{i}^+, \boldsymbol{j}^+$.

Assuming the claim, the proof then follows by the following argument. We can assume (up to conjugation) that $\tau(\boldsymbol{i}^+) = 0$ for all $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$. By Proposition 5.13, if some of the collapsed cusp groups of ρ_0 is not deformed to a cusp group, then up to conjugation ρ has the property that $\rho(\boldsymbol{i}^+) = (-\mathrm{id}, 0)$ for all $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$, and therefore $[\rho] \in \mathcal{H}$. On the other hand, if all the collapsed cusp groups of ρ_0 are deformed in ρ to cusp groups, then the linear part of ρ is of the form $\mathcal{L}\rho = \Psi_\eta$ for a representation $\eta\colon \Gamma_{\mathrm{co}} \to \mathrm{Isom}(\mathbb{H}^3)$ which sends all peripheral groups to (three-dimensional) cusp groups in $\mathbb{H}^3$, and therefore η is conjugate to η_0 in $\mathrm{Isom}(\mathbb{H}^3)$ by the the Mostow–Prasad rigidity. Thus $[\mathcal{L}\rho] = [\mathcal{L}\rho_0]$, which means that $[\rho] \in \mathcal{V}$.

To prove the claim, suppose that $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+)$. We can assume that $\tau(\boldsymbol{i}^+) = \tau(\boldsymbol{j}^+) = 0$ by conjugation. Analogously to the same step in the AdS case, by symmetry (see [RS, Lemma 7.6]) and Proposition 3.12, we can assume the two generators are $\mathbf{0}^+$ and $\mathbf{1}^+$. Hence we have $\rho(\mathbf{0}^+) = \rho(\mathbf{1}^+) = (-\mathrm{id}, 0)$. We see from the relations involving $\mathbf{0}^+$ that $\rho(\mathbf{0}^+)$ commutes with $\rho(\mathbf{1}^-)$, $\rho(\mathbf{3}^-)$ and $\rho(\boldsymbol{A})$, which have all linear part a reflection in $\mathbb{H}^3$. By Lemma 5.8, $\rho(\mathbf{1}^-)$, $\rho(\mathbf{3}^-)$ and $\rho(\boldsymbol{A})$ have zero translation part. Additionally, we see from the relations involving $\mathbf{1}^+$ that $\rho(\mathbf{2}^-)$ has zero translation part. Now, from the relations involving $\mathbf{2}^+$, we get that $\rho(\mathbf{2}^+)$ commutes with $\rho(\mathbf{1}^-)$, $\rho(\mathbf{2}^-)$, $\rho(\mathbf{3}^-)$ and $\rho(\boldsymbol{A})$. Observe that the linear part of $\rho(\mathbf{2}^+)$ is necessarily $-\mathrm{id}$, in a neighbourhood of ρ_0. Hence by applying Lemma 5.8 again, the translation part of $\rho(\mathbf{2}^+)$ is in the intersection of the hyperplanes of $\mathbb{R}^{1,3}$ fixed by $\rho(\mathbf{1}^-)$, $\rho(\mathbf{2}^-)$, $\rho(\mathbf{3}^-)$ and $\rho(\boldsymbol{A})$. The hyperplanes fixed by $\rho_0(\mathbf{1}^-)$, $\rho_0(\mathbf{2}^-)$, $\rho_0(\mathbf{3}^-)$ and $\rho_0(\boldsymbol{A})$ are $v_{\mathbf{1}}^\perp$, $v_{\mathbf{2}}^\perp$, $v_{\mathbf{3}}^\perp$ and $v_{\boldsymbol{A}}^\perp$, where the vectors $v_{\mathbf{1}}$, $v_{\mathbf{2}}$, $v_{\mathbf{3}}$ and $v_{\boldsymbol{A}}$ are listed in Table 4 and are linearly independent. Hence they remain linearly independent for ρ a deformation of ρ_0 in a small neighbourhood. This means that the translation part of $\rho(\mathbf{2}^+)$ is zero, since the only solution of the linear system which imposes the orthogonality to these four linearly independent vectors is the trivial solution. This shows that $\rho(\mathbf{2}^+) = (-\mathrm{id}, 0)$, which therefore coincides with $\rho(\mathbf{0}^+) = \rho(\mathbf{1}^+)$.

Similarly to the AdS case, one argues similarly for $\mathbf{3}^+$ and then for all the other generators, to show that $\rho(\boldsymbol{i}^+) = (-\mathrm{id}, 0)$ for each generator $\boldsymbol{i}^+ \in \{\mathbf{0}^+, \dots, \mathbf{7}^+\}$, and this concludes the claim.

***Step 5*:** In summary, we showed that $[\rho_0]$ has a neighborhood $\mathcal{U}$ in $X(\Gamma_{22}, \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4))$ which only consists of points of $\mathcal{H}$ and $\mathcal{V}$. Additionally, one can repeat the same reasoning in the first part of the previous step, to show that for any other $[\rho_0']$ in $\mathcal{V}$ (hence having the same linear part as ρ_0 and non-vanishing translation part) a neighbourhood of $[\rho_0']$ is contained in $\mathcal{V}$, as a consequence of the half-pipe cusp rigidity of Proposition 5.13 (the non-collapsed case). Hence by taking the union of all these neighbourhoods, one finds a $\mathcal{U}$ containing $[\rho_0]$ such that $\mathcal{U} = \mathcal{V} \cup \mathcal{H}$.

***Step 6*:** For the last statement, it is evident that conjugation by $\mathbb{Z}/2\mathbb{Z} \cong G_{\mathsf{HP}^4}/G^+_{\mathsf{HP}^4}$ acts by switching sign to the x_{13}-coordinate, since conjugation by $(-\mathrm{id}, 0)$, whose class generates $\mathbb{Z}/2\mathbb{Z}$, acts on $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ by changing the sign. This concludes the proof. □

7. Infinitesimal aspects

In this last section we give two additional applications of the previous group-cohomological considerations. In Section 7.1 prove Theorem 1.2, giving a description of the Zariski tangent space of the three character varieties at the "collapsed" representation $[\rho_0]$ from the point of view of real algebraic geometry. In Section 7.2, we provide the right generalisation of Danciger's condition (3) to arbitrary dimension.

7.1. The Zariski tangent space.

We prove here Theorem 1.2, describing the Zariski tangent space of $X(\Gamma_{22}, G)$ at the singular point $[\rho_0]$, where $G = \mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or G_{HP^4}, and show Theorem 1.2.

We shall apply the definition of first cohomology group given in Section 6.1 to the representation

$$\mathrm{Ad}\,\rho_0 \colon \Gamma_{22} \to \mathrm{GL}(\mathfrak{g}) ,$$

which is the composition of our $\rho_0 \colon \Gamma_{22} \to G$ and the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

In general, for a finitely presented group Γ with a given presentation with s generators and r relations, the set $\mathrm{Hom}(\Gamma, G)$ is identified to a subset of G^s defined by the vanishing of r conditions given by the relations. If we encode these conditions by $F\colon G^s \to G^r$, so as to identify $\mathrm{Hom}(\Gamma, G)$ with $F^{-1}(0)$, then it is known from [Gol84] that $Z^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{g})$ is isomorphic to the kernel of dF at ρ. The isomorphism essentially associates to a germ of paths at ρ represented by $t \mapsto \rho_t$ the cocycle τ defined by

$$\tau(\gamma) = \left.\frac{d}{dt}\right|_{t=0} \rho_t(\gamma)\rho_0(\gamma)^{-1} ,$$

which is therefore interpreted as an infinitesimal deformation of ρ. Moreover, if we suppose that the action of G^+ on $\mathrm{Hom}(\Gamma, G)$ by conjugation is free at ρ, then the subspace of $\mathrm{Ker}(dF)$ corresponding to the tangent space to the orbit of G^+ identifies to $B^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{g})$ under this correspondence. Thus the quotient $H^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{g}) = Z^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{g})/B^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{g})$ is naturally identified with the Zariski tangent space of $X(\Gamma, G)$ at $[\rho]$.

Let us now go back to the representation $\rho_0 \colon \Gamma_{22} \to G_0$ (see Section 6.1). There is a well-known splitting

$$\mathfrak{g} \cong \mathfrak{isom}(\mathbb{H}^{n-1}) \oplus \mathbb{R}^n . \tag{35}$$

When $G = \mathrm{Isom}(\mathbb{H}^n)$ or $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^n)$, the splitting is given by writing an element $\mathfrak{a}$ of $\mathfrak{g}$ as

$$\mathfrak{a} = \left(\begin{array}{ccc|c} & & & \vdots \\ & \mathfrak{a}_0 & & \mp w \\ & & & \vdots \\ \hline \dots & w^T J & \dots & 0 \end{array}\right) , \tag{36}$$

where $J = \mathrm{diag}(-1, 1, \dots, 1)$, for $\mathfrak{a}_0 \in \mathfrak{so}(1, n-1)$ and $w \in \mathbb{R}^n$. When $G = G_{\mathsf{HP}^n}$, the splitting (35) is even simpler to obtain, by using the isomorphism $G_{\mathsf{HP}^n} \cong \mathrm{O}(1, n-1) \ltimes \mathbb{R}^{1,n-1}$.

The splitting (35) is equivariant with respect to the three natural actions of G_0: the adjoint action on $\mathfrak{g}$, the adjoint action on $\mathfrak{isom}(\mathbb{H}^3)$ by means of the isomorphism $G_0 \cong \mathrm{Isom}(\mathbb{H}^3) \times (\mathbb{Z}/2\mathbb{Z})$, and the action on $\mathbb{R}^{1,3}$ by means of the isomorphism $G_0 \cong \mathrm{O}(1,3)$ of (28). We thus have a natural decomposition

$$H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g}) = H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)) \oplus H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) \ . \tag{37}$$

Here ρ_0 is the composition of $\varrho_0 \colon \Gamma_{22} \to G_0$ with the inclusion $G_0 \to G$. Using this decomposition, we can prove:

Proposition 7.1. *Let $G = \mathrm{Isom}(\mathbb{H}^4)$, $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^4)$ or G_{HP^4}. The Zariski tangent space of $X(\Gamma_{22}, G)$ at $[\rho_0]$ is isomorphic to*

$$H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g}) \cong \mathbb{R}^{13} \ .$$

In the natural direct sum decomposition (37) of $H^1_{\mathrm{Ad}\,\rho_0}(\Gamma_{22}, \mathfrak{g})$, the factor $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ is 1-dimensional, and the factor $H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$ is 12-dimensional.

Proof. Since we have proved that $\dim H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3}) = 1$ in Proposition 6.5, it will suffice to show that $\dim H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)) = 12$. For this purpose, we claim that the group $H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$ is isomorphic to $H^1_{\mathrm{Ad}\,\iota}(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3))$, where Γ_{co} is the reflection group of the right-angled cuboctahedron and ι is its inclusion into $\mathrm{Isom}(\mathbb{H}^3)$. The latter has dimension 12, since the character variety of Γ_{co} in $\mathrm{Isom}(\mathbb{H}^3)$ is smooth and 12-dimensional near $[\iota]$. We have already mentioned (in Section 4.8, Step 2) that this last fact is true by "reflective hyperbolic Dehn filling".

To show the claim, define a map

$$\psi \colon Z^1_{\mathrm{Ad}\,\iota}(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3)) \to Z^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$$

defined by, identifying Γ_{co} with the subgroup of Γ_{22} generated by $\mathbf{0}^-, \ldots \mathbf{7}^-, \boldsymbol{A}, \ldots, \boldsymbol{F}$,

$$\psi(\tau)(\boldsymbol{X}) = \tau(\boldsymbol{X}) \qquad \psi(\tau)(\boldsymbol{i}^-) = \tau(\boldsymbol{i}^-) \qquad \psi(\tau)(\boldsymbol{i}^+) = 0 \ ,$$

for every $\tau \in Z^1_{\mathrm{Ad}\,\iota}(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3))$. Let us justify that this map is well-defined. Indeed $\psi(\tau)(\boldsymbol{i}^+\boldsymbol{X}) = \psi(\tau)(\boldsymbol{X}\boldsymbol{i}^+)$ and $\psi(\tau)(\boldsymbol{i}^+\boldsymbol{j}^-) = \psi(\tau)(\boldsymbol{j}^-\boldsymbol{i}^+)$. This is easily checked using Lemma 6.2 and the fact that $\varrho_0(\boldsymbol{i}^+) = -\mathrm{id}$, thus $\mathrm{Ad}\,\varrho_0(\boldsymbol{i}^+) = \mathrm{id}$. Hence ψ maps cocycles to cocycles. Moreover, it maps coboundaries to coboundaries, for if $\tau(\boldsymbol{s}) = \mathrm{Ad}\,\varrho_0(\boldsymbol{s})\mathfrak{a} - \mathfrak{a}$ for some $\mathfrak{a} \in \mathfrak{isom}(\mathbb{H}^3)$, then of course $\psi(\tau)(\boldsymbol{s}) = \mathrm{Ad}\,\varrho_0(\boldsymbol{s})\mathfrak{a} - \mathfrak{a}$ for $\boldsymbol{s} = \boldsymbol{i}^-$ or $\boldsymbol{X}$. As $\mathrm{Ad}\,\varrho_0(\boldsymbol{i}^+) = \mathrm{id}$, the identity holds trivially also for $\boldsymbol{s} = \boldsymbol{i}^+$.

Thus ψ induces a map

$$\psi \colon H^1_{\mathrm{Ad}\,\iota}(\Gamma_{\mathrm{co}}, \mathfrak{isom}(\mathbb{H}^3)) \to H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$$

which is obviously injective. It remains to show that it is surjective. To see this, given any cocycle $\sigma \in Z^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3))$, Lemma 6.2 implies that $\sigma(\boldsymbol{i}^+)$ is in the kernel of $\mathrm{id} + \mathrm{Ad}\,\varrho(\boldsymbol{i}^+)$, which in fact equals $2\mathrm{id}$. Hence $\sigma(\boldsymbol{i}^+) = 0$ and σ is in the image of ψ. This concludes the proof. □

To conclude the proof of Theorem 1.2, it remains to show the following:

Proposition 7.2. *A vector in the Zariski tangent space of $X(\Gamma_{22}, G)$ at $[\rho_0]$ is integrable if and only if it lies in one of the two factors of the direct sum decomposition (37).*

Proof. The vectors in the subspace $\{0\} \oplus H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ are integrable, as they are tangent to the component $\mathcal{V}$.

More precisely, in the hyperbolic and AdS case, any generator of $\{0\} \oplus H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ can be shown to coincide with the derivative $\frac{d}{dt}\big|_{t=0} \rho_t \circ \rho_0^{-1}$ up to reparameterisation. For the half-pipe case, it is obvious that the vectors in $\{0\} \oplus H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$ are tangent to $\mathcal{V}$ itself, since $\mathcal{V}$ is identified to the vector space $H^1_{\varrho_0}(\Gamma_{22}, \mathbb{R}^{1,3})$.

Similarly, the vectors in $H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma_{22}, \mathfrak{isom}(\mathbb{H}^3)) \oplus \{0\}$, which are tangent to the deformations of the ideal right-angled cuboctahedron, are integrable, as they are tangent to the horizontal component.

As a consequence of our Theorem 1.1, these are the only integrable vectors in the Zariski tangent space. □

The proof of Theorem 1.2 is complete. We thus have a satisfying picture of the singularity that appears in the character variety at the collapse, also in the sense of (real) algebraic geometry.

7.2. **Danciger's condition in arbitrary dimension.** In [Dan13], Danciger introduced half-pipe geometry, mostly focusing the attention on three-dimensional transition. In his work, the infinitesimal study of the character variety of a group Γ is mostly interpreted in terms of the identifications $\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \cong \mathrm{PSL}_2(\mathbb{R} \oplus \mathbb{R}\sigma)$, where $\mathbb{R} \oplus \mathbb{R}\sigma$ is the algebra generated by 1 and σ with the relation $\sigma^2 = 1$.

In that approach, the "translation" part of half-pipe holonomies corresponds essentially to tangent vectors to the $\mathrm{PSL}_2(\mathbb{R})$ character variety. More precisely, these are elements in $H^1_{\mathrm{Ad}\,\varrho}(\Gamma, \mathfrak{sl}_2(\mathbb{R})) \cong H^1_{\mathrm{Ad}\,\varrho}(\Gamma, \mathfrak{so}(1,2))$ multiplied by $i = \sqrt{-1}$ in the hyperbolic case, and by σ in the Anti-de Sitter case, and are thus considered as elements of $H^1_{\mathrm{Ad}\,\varrho}(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ or $H^1_{\mathrm{Ad}\,\varrho}(\Gamma, \mathfrak{sl}_2(\mathbb{R} \oplus \mathbb{R}\sigma))$.

This might look different from the description we gave in the previous section. Hence the purpose of this last section is to briefly interpret the three-dimensional picture in our context.

When $n = 3$ there is a natural isomorphism between $\mathbb{R}^{1,2}$ and $\mathfrak{so}(1,2)$, which is $\mathrm{SO}(1,2)$-equivariant, and thus the splitting of the Lie algebra

$$\mathfrak{so}(1,3) \cong \mathfrak{so}(1,2) \oplus \mathbb{R}^{1,2} \tag{38}$$

induces a splitting of the first cohomology group of the form

$$\begin{aligned} H^1_{\mathrm{Ad}\,\rho_0}(\Gamma, \mathfrak{so}(1,3)) &= H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma, \mathfrak{so}(1,2)) \oplus H^1_{\varrho_0}(\Gamma, \mathbb{R}^{1,2}) \\ &\cong H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma, \mathfrak{sl}_2(\mathbb{R})) \oplus H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma, \mathfrak{sl}_2(\mathbb{R})) \ , \end{aligned}$$

thus in two copies of the Zariski tangent space to the $\mathrm{PSL}_2(\mathbb{R})$ character variety.

Now, it is an entertaining exercise to check that, under the isomorphism between $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{SO}(1,3)$ given by considering the action of $\mathrm{PSL}_2(\mathbb{C})$ on the vector space of 2-by-2 Hermitian matrices, which gives by differentiation an isomorphism between $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{so}(1,3)$, the splitting (38) corresponds to the splitting

$$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \oplus i\,\mathfrak{sl}_2(\mathbb{R}) \ .$$

By replacing the role of i by σ, one can check the analogous correspondence for $\mathrm{PSL}_2(\mathbb{R} \oplus \mathbb{R}\sigma)$ and $\mathrm{SO}(2,2)$. This explains, from our point of view, why in [Dan13] the author considers tangent vectors in $i\,H^1_{\mathrm{Ad}\,\varrho_0}(\Gamma, \mathfrak{sl}_2(\mathbb{R}))$ (and analogously in the AdS case, replacing i with σ).

In fact, it can be shown that in any dimension n the elements in the factor $H^1_{\varrho_0}(\Gamma, \mathbb{R}^{1,n-1})$ of the decomposition (37) are those that may arise geometrically as the rescaled limits of hyperbolic or AdS holonomies ρ_t, such that the "collapsed" ρ_0 has image in the group G_0 which preserves a copy of $\mathbb{H}^{n-1}$. (As in the previous section, here ϱ_0 is a representation in $\mathrm{O}(1, n-1)$ which corresponds to ρ_0 under the isomorphism $G_0 \cong \mathrm{Isom}(\mathbb{H}^{n-1}) \times (\mathbb{Z}/2\mathbb{Z})$.)

This justifies the claim that the condition

$$H^1_{\varrho_0}(\Gamma, \mathbb{R}^{1,n-1}) \cong \mathbb{R}$$

is the right generalisation to arbitrary dimension n of the condition (3) of Danciger mentioned in the introduction. If one wants to attempt a regeneration theorem in the spirit of [Dan13, Theorem 1.2] to higher dimension, this condition could be a natural hypothesis.

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