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# THE EMBEDDINGS OF THE HEISENBERG GROUP INTO THE CREMONA GROUP

JULIE DÉSERTI

ABSTRACT. In this note we describe the embeddings of the Heisenberg group into the Cremona group.

## INTRODUCTION

The Heisenberg group is the non-abelian nilpotent group given by

$$\mathcal{H} = \langle f, g \mid [f, g] = h, [f, h] = [g, h] = \text{id} \rangle.$$

It has two generators,  $f$  and  $g$ , and  $h$  is the generator of the center of  $\mathcal{H}$ .

The Cremona group is the group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  of birational maps of the projective plane  $\mathbb{P}_{\mathbb{C}}^2$  into itself. Such maps can be written in the form

$$(x : y : z) \dashrightarrow (P_0(x, y, z) : P_1(x, y, z) : P_2(x, y, z))$$

where  $P_0, P_1, P_2 \in \mathbb{C}[x, y, z]$  are homogeneous polynomials of the same degree, and this degree is the degree of the map, if the polynomials have no common factor (of positive degree). Recall that if  $\phi$  is a birational self map of the complex projective plane, then one of the following holds ([Giz80, Can01, DF01, BD15]):

- ◇ the sequence  $(\deg(\phi^n))_{n \in \mathbb{N}}$  is bounded, and  $\phi$  is said to be *elliptic*;
- ◇ the sequence  $(\deg(\phi^n))_{n \in \mathbb{N}}$  grows linearly with  $n$ , and  $\phi$  is said to be a *Jonquières twist*;
- ◇ the sequence  $(\deg(\phi^n))_{n \in \mathbb{N}}$  grows quadratically with  $n$ , and  $\phi$  is said to be a *Halphen twist*;
- ◇  $(\deg(\phi^n))_{n \in \mathbb{N}}$  grows exponentially fast with  $n$ , and  $\phi$  is said to be *hyperbolic*.

**Proposition A.** *Let  $\rho$  be an embedding of  $\mathcal{H}$  into the Cremona group. Then  $\rho(\mathcal{H})$  does not contain hyperbolic birational maps.*

*More precisely  $\rho(f)$  and  $\rho(g)$  are either elliptic birational maps, or Jonquières twists.*

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We describe the embeddings of  $\mathcal{H}$  into the Cremona group. In [DÓ6] we already looked at such embeddings but with the following assumption: the images of  $f$  and  $g$  are elliptic birational self maps. There are other embeddings of  $\mathcal{H}$  into the Cremona group:

**Theorem B.** *Let  $\rho$  be an embedding of  $\mathcal{H}$  into the Cremona group. Then up to birational conjugacy*

◇ *either  $\rho(\mathcal{H})$  is a subgroup of  $\mathrm{PGL}(3, \mathbb{C})$  and*

$$\rho(f) = (x + \alpha y, y + \beta) \quad \rho(g) = (x + \gamma y, y + \delta)$$

*with  $\alpha, \beta, \gamma, \delta$  in  $\mathbb{C}$  such that  $\alpha\delta - \beta\gamma = 1$ ;*

◇ *or  $\rho(\mathcal{H})$  is a subgroup of the group of polynomial automorphisms of  $\mathbb{C}^2$  and  $(\rho(f), \rho(g))$  is one of the following pairs*

$$\begin{aligned} & ((ax + Q(y), y + c), (\alpha x + P(y), y + \gamma)), \\ & \left( \left( ax + Q(y), by + \frac{\gamma(b-1)}{\beta-1} \right), (\alpha x + P(y), \beta y + \gamma) \right) \end{aligned}$$

*with  $\beta \in \mathbb{C}^* \setminus \{1\}$ ,  $a, \alpha, b$  in  $\mathbb{C}^*$ ,  $c, \gamma$  in  $\mathbb{C}$  and  $P, Q$  in  $\mathbb{C}[y]$ ;*

◇ *or  $\rho(f)$  is a Jonquières twist and  $(\rho(f), \rho(g))$  is one of the following pairs*

$$\begin{aligned} & \left( (x, \delta x^{\pm 1} y), (\gamma x, ya(x)) \right), & \left( (x, \delta x^{\pm 2} y), (\gamma x, ya(x)) \right), \\ & \left( (-x, \delta x^{\pm 1} y), (\gamma x, yb(x)) \right), & \left( (\lambda x, yc(x)), (\delta x, yd(x)) \right) \end{aligned}$$

*with  $\delta, \gamma \in \mathbb{C}^*$ ,  $\lambda \in \mathbb{C}^* \setminus \{1, -1\}$  and  $a, b, c, d \in \mathbb{C}(x)^*$  such that*

$$\frac{b(x)}{b(-x)} \in \mathbb{C}^*, \quad \frac{c(\delta x)d(x)}{c(x)d(\lambda x)} \in \mathbb{C}^*.$$

**Remark C.** *Note that the two last families are not empty. For instance*

$$\left( (-x, \alpha x^{\pm 1} y), (\beta x, \gamma x^2 y) \right), \quad \left( (\lambda x, \alpha x^p y), (\gamma x, \beta x^q y) \right)$$

*with  $\alpha, \beta, \gamma \in \mathbb{C}^*$ ,  $\lambda \in \mathbb{C}^* \setminus \{1, -1\}$ ,  $p, q \in \mathbb{N}$  are such pairs.*

## 1. SOME RECALLS

**1.1. About birational maps of the complex projective plane.** Let  $\phi$  be a birational self map of the complex projective plane. Then one of the following holds ([Giz80, Can01, DF01, BD15]):

◇  $\phi$  is *elliptic* if and only if the sequence  $(\deg(\phi^n))_{n \in \mathbb{N}}$  is bounded. In this case there exist a birational map  $\psi: S \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  and an integer  $k \geq 1$  such that  $\psi^{-1} \circ \phi^k \circ \psi$  belongs to the connected component of the identity of the group

$\text{Aut}(S)$ . Either  $\phi$  is of finite order, or  $\phi$  is conjugate to an automorphism of  $\mathbb{P}_{\mathbb{C}}^2$ , which restricts to one of the following automorphisms on some open subset isomorphic to  $\mathbb{C}^2$ :

- $(x, y) \mapsto (\alpha x, \beta y)$  where the kernel of the group morphism

$$\mathbb{Z}^2 \rightarrow \mathbb{C}^2 \quad (i, j) \mapsto \alpha^i \beta^j$$

is generated by  $(k, 0)$  for some  $k \in \mathbb{Z}$ ;

- $(x, y) \mapsto (\alpha x, y + 1)$  where  $\alpha \in \mathbb{C}^*$ .
- ◇  $\phi$  is *parabolic* if and only if the sequence  $(\deg(\phi^n))_{n \in \mathbb{N}}$  grows linearly or quadratically with  $n$ . If  $\phi$  is parabolic, there exist a birational map  $\psi: S \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  and a fibration  $\pi: S \rightarrow B$  onto a curve  $B$  such that  $\psi^{-1} \circ \phi \circ \psi$  permutes the fibers of  $\pi$ . If  $(\deg(\phi^n))_{n \in \mathbb{N}}$  grows linearly, then the fibration  $\pi$  is rational and  $\phi$  is said to be a *Jonquières twist*. If  $(\deg(\phi^n))_{n \in \mathbb{N}}$  grows quadratically, then the fibration  $\pi$  is elliptic and  $\phi$  is said to be a *Halphen twist*.
- ◇  $\phi$  is *hyperbolic* if and only if  $(\deg(\phi^n))_{n \in \mathbb{N}}$  grows exponentially fast with  $n$ : there is a constant  $c(\phi)$  such that  $\deg(\phi^n) = c(\phi)\lambda(\phi)^n + O(1)$ .

**1.2. About distorted elements.** If  $G$  is a group generated by a finite subset  $F \subset G$  the  $F$ -length  $|g|_F$  of an element  $g$  of  $G$  is defined as the least non-negative integer  $\ell$  such that  $g$  admits an expression of the form  $g = f_1 f_2 \dots f_\ell$  where each  $f_i$  belongs to  $F \cup F^{-1}$ . We say that  $g$  is *distorted* if  $\lim_{k \rightarrow +\infty} \frac{|g^k|_F}{k} = 0$  (note that the limit  $\lim_{k \rightarrow +\infty} \frac{|g^k|_F}{k}$  always exists and is a real number since the sequence  $k \mapsto |g^k|_F$  is subadditive). This notion actually does not depend on the chosen  $F$ , but only on the pair  $(g, G)$ .

If  $G$  is any group, an element  $g \in G$  is *distorted* if it is distorted in some finitely generated subgroup of  $G$ .

The element  $h$  of

$$\mathcal{H} = \langle f, g \mid [f, g] = h, [f, h] = [g, h] = \text{id} \rangle$$

satisfies the following property:

$$\forall k \in \mathbb{Z} \quad h^{k^2} = [f^k, g^k] = f^k g^k f^{-k} g^{-k}$$

so  $\|h^{k^2}\| \leq 4k$  and  $\lim_{k \rightarrow +\infty} \frac{\|h^{k^2}\|}{k^2} = 0$ . Hence  $h$  is distorted.

An element  $\phi \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is said to be *algebraic* if it is contained in an algebraic subgroup of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . By [BF13, §2.6] the map  $\phi \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is algebraic if and only if the sequence  $(\deg(\phi^n))_{n \in \mathbb{N}}$  is bounded. In other words elliptic elements and algebraic elements coincide. By [BD15, Proposition 2.3] this is also equivalent to say that  $\phi$  is of finite order or conjugate to an element of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$ . A straightforward

computation shows that every element of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  is distorted in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  (see [BF19, Lemma 4.40]). As a consequence every algebraic element of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is distorted. The converse statement also holds:

**Theorem 1.1** ([BF19, CC19]). *Any distorted element of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is elliptic.*

**Corollary 1.2.** *Let  $\rho$  be an embedding of  $\mathcal{H}$  into  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . Then  $\rho(\mathfrak{h})$  is elliptic.*

We will use this corollary and the following description of the centralizer of hyperbolic birational maps to prove Proposition A:

**Proposition 1.3** ([BC16]). *Let  $\phi$  be a birational map of the complex projective plane. If  $\phi$  is hyperbolic, then the infinite cyclic group generated by  $\phi$  is a finite index subgroup of the centralizer of  $\phi$  in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .*

*Proof of the first part of Proposition A.* Assume that  $\rho(\mathcal{H})$  contains an hyperbolic element  $\phi$ . Since  $\rho(\mathfrak{h})$  is the generator of the center of  $\rho(\mathcal{H})$ ,  $\rho(\mathfrak{h})$  commutes with  $\phi$ . Proposition 1.3 implies that either  $\rho(\mathfrak{h})$  is hyperbolic, or  $\rho(\mathfrak{h})$  is of finite order. But  $\rho(\mathfrak{h})$  is not hyperbolic Corollary (1.2) and by definition  $\rho(\mathfrak{h})$  is of infinite order. As a result  $\rho(\mathcal{H})$  does not contain hyperbolic element.  $\square$

**1.3. About centralizers of elliptic birational maps.** Let us recall the description of the centralizers of the elliptic birational self maps of infinite order of the complex projective plane obtained in [BD15].

Consider  $\phi$  an elliptic element of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . Assume that  $\phi$  is of infinite order. As recalled in §1.1 the map  $\phi$  is conjugate to an automorphism of  $\mathbb{P}_{\mathbb{C}}^2$  which restricts to one of the following automorphisms on some open subset isomorphic to  $\mathbb{C}^2$ :

- (1)  $(\alpha x, \beta y)$  where  $\alpha, \beta$  belong to  $\mathbb{C}^*$ ;
- (2)  $(\alpha x, y + 1)$  where  $\alpha \in \mathbb{C}^*$ .

In case (1) the centralizer of  $\phi$  in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is

$$\{(\eta(x), ya(x^k)) \mid a \in \mathbb{C}(x), \eta \in \text{PGL}(2, \mathbb{C}), \eta(\alpha x) = \alpha \eta(x)\};$$

in particular the elements of the centralizer of  $\phi$  are elliptic birational maps or Jonquières twists. In case (2) the centralizer of  $\phi$  in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is

$$\{(\eta(x), y + a(x)) \mid \eta \in \text{PGL}(2, \mathbb{C}), \eta(\alpha x) = \alpha \eta(x), a \in \mathbb{C}(x), a(\alpha x) = a(x)\};$$

in particular the elements of the centralizer of  $\phi$  are elliptic birational maps.

Corollary 1.2 and the previous description imply:

**Lemma 1.4.** *Let  $\rho$  be an embedding of  $\mathcal{H}$  into  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .*

*Then  $\rho(\mathfrak{h})$  is elliptic and up to birational conjugacy*

◇ either  $\rho(h) = (\alpha x, \beta y)$ , where the kernel of the group morphism

$$\mathbb{Z}^2 \rightarrow \mathbb{C}^2 \quad (i, j) \mapsto \alpha^i \beta^j$$

is generated by  $(k, 0)$  for some  $k \in \mathbb{Z}$  and both  $\rho(f)$ ,  $\rho(g)$  belong to

$$\{(\eta(x), \gamma a(x^k)) \mid a \in \mathbb{C}(x), \eta \in \text{PGL}(2, \mathbb{C}), \eta(\alpha x) = \alpha \eta(x)\};$$

◇ or  $\rho(h) = (\alpha x, y + 1)$ , and both  $\rho(f)$ ,  $\rho(g)$  belong to

$$\{(\eta(x), \gamma + a(x)) \mid \eta \in \text{PGL}(2, \mathbb{C}), \eta(\alpha x) = \alpha \eta(x), a \in \mathbb{C}(x), a(\alpha x) = a(x)\}.$$

In particular  $\rho(f)$  and  $\rho(g)$  are elliptic birational maps or Jonquières twists.

It ends the proof of Proposition A.

## 2. PROOF OF THEOREM B

**2.1. Assume that all the generators of  $\rho(\mathcal{H})$  are elliptic.** The group  $\text{Aut}(\mathbb{C}^2)$  of polynomial automorphisms of  $\mathbb{C}^2$  is a subgroup of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . It is generated by the group

$$A = \{(a_0x + a_1y + a_2, b_0x + b_1y + b_2) \mid a_i, b_i \in \mathbb{C}, a_0b_1 - a_1b_0 \neq 0\}$$

and

$$E = \{(\alpha x + P(y), \beta y + \gamma) \mid \alpha, \beta \in \mathbb{C}^*, \gamma \in \mathbb{C}, P \in \mathbb{C}[y]\}.$$

Let us recall the following result obtained when we study the embeddings of  $\text{SL}(n, \mathbb{Z})$  into the Cremona group:

**Lemma 2.1** ([DÓ6]). *Let  $\rho$  be an embedding of  $\mathcal{H}$  into  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .*

*If  $\rho(f)$ ,  $\rho(g)$  and  $\rho(h)$  are elliptic, then up to birational conjugacy*

◇ *either  $\rho(\mathcal{H})$  is a subgroup of  $\text{PGL}(3, \mathbb{C})$ , and more precisely*

$$\rho(f) = (x + \alpha y, y + \beta) \quad \rho(g) = (x + \gamma y, y + \delta)$$

*with  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$  such that  $\alpha\delta - \beta\gamma = 1$ ;*

◇ *or  $\rho(\mathcal{H})$  is a subgroup of  $E$  and  $\rho(h^2) = (x + P(y), y)$  for some  $P \in \mathbb{C}[y]$ .*

This statement implies the following one:

**Proposition 2.2.** *Let  $\rho$  be an embedding from  $\mathcal{H}$  into  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . Assume that  $\rho(f)$ ,  $\rho(g)$  and  $\rho(h)$  are elliptic.*

*Then up to birational conjugacy*

◇ *either  $\rho(\mathcal{H})$  is a subgroup of  $\text{PGL}(3, \mathbb{C})$ , more precisely*

$$\rho(f) = (x + \alpha y, y + \beta) \quad \rho(g) = (x + \gamma y, y + \delta)$$

*with  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$  such that  $\alpha\delta - \beta\gamma = 1$ ;*

◇ or  $\rho(\mathcal{H})$  is a subgroup of  $E$  and  $(\rho(f), \rho(g))$  is one of the following pairs

$$\begin{aligned} & ((ax + Q(y), y + c), (\alpha x + P(y), y + \gamma)) \\ & \left( \left( ax + Q(y), by + \frac{\gamma(b-1)}{\beta-1} \right), (\alpha x + P(y), \beta y + \gamma) \right) \end{aligned}$$

with  $a, \alpha, b$  in  $\mathbb{C}^*$ ,  $c, \gamma$  in  $\mathbb{C}$ ,  $\beta \in \mathbb{C}^* \setminus \{1\}$  and  $P, Q$  in  $\mathbb{C}[y]$ .

*Proof.* The first assertion follows from Lemma 2.1. Let us focus on the second one.

If  $\rho(h)$  belongs to  $E$  and  $\rho(h^2) = (x + P(y), y)$ , then  $\rho(h) = (\varepsilon x + Q(y), \eta(y))$  with  $\varepsilon^2 = 1$ ,  $Q \in \mathbb{C}[y]$  and  $\eta(y) \in \{-y + \gamma, y\}$ . But  $\rho(f)$  and  $\rho(g)$  belong to  $E$  so  $[\rho(f), \rho(g)] = \rho(h)$  implies that  $\varepsilon = 1$  and  $\eta(y) = y$ , i.e.  $\rho(h) = (x + Q(y), y)$ . Set

$$\rho(f) = (ax + R(y), by + c), \quad \rho(g) = (\alpha x + P(y), \beta y + \gamma).$$

The second component of  $\rho(f)\rho(g)$  has to be equal to the second component of  $\rho(h)\rho(g)\rho(f)$ , that is

$$\beta by + b\gamma + c = \beta by + \beta c + \gamma;$$

in other words either  $\beta = b = 1$ , or  $c = \frac{\gamma(b-1)}{\beta-1}$ .  $\square$

**2.2. Assume that  $\rho(f)$  is a Jonquières twist with trivial action on the basis of the fibration.** Since  $\rho(h)$  is elliptic, then up to birational conjugacy either  $\rho(h) = (\alpha x, \beta y)$ , or  $\rho(h) = (\alpha x, y + 1)$  (see §1.1). But  $\rho(f)$  belongs to the centralizer of  $\rho(h)$  and is a Jonquières twist; therefore according to §1.3 one has:  $\rho(h) = (\alpha x, \beta y)$ ,  $\rho(f)$  can be written as  $(x, ya(x))$  and  $\rho(g)$  as  $(\mu(x), yb(x))$  with  $\mu \in \text{PGL}(2, \mathbb{C})$  and  $a, b \in \mathbb{C}(x)^*$ .

Let us remark that if  $\mu = \text{id}$ , then  $[\rho(f), \rho(g)] = \rho(h)$  implies  $\alpha = \beta = 1$  so  $\mu \neq \text{id}$ .

The relation  $[\rho(f), \rho(g)] = \rho(h)$  implies that  $\alpha = 1$ , and  $a(\mu(x)) = \beta a(x)$ . Let us first look at polynomials  $P$  such that  $P(\mu(x)) = \beta P(x)$ :

**Claim 2.3.** *If  $P$  is a non-zero polynomial such that  $P(\mu(x)) = \lambda^2 P(x)$ ,  $\lambda^2 \neq 1$ , then one of the following holds:*

- ◇  $P(x) = \delta \left( \frac{\gamma}{\lambda^2 - 1} + x \right)$ ,  $\mu(x) = \gamma + \lambda^2 x$  with  $a, \delta \in \mathbb{C}$ ;
- ◇  $P(x) = \delta \left( \frac{\gamma}{\lambda + 1} - x \right)^2$ ,  $\mu(x) = \gamma - \lambda x$  with  $\gamma \in \mathbb{C}$ , and  $\delta \in \mathbb{C}^*$ ;
- ◇  $P(x) = \delta \left( \frac{\gamma}{\lambda - 1} + x \right)^2$ ,  $\mu(x) = \gamma + \lambda x$  with  $\gamma \in \mathbb{C}$ , and  $\delta \in \mathbb{C}^*$ .

*Proof.* Let us consider the set  $Z_P = \{z \mid P(z) = 0\}$  of roots of  $P$ . It is a finite set invariant by  $\mu$ . As a result  $\mu^n|_{Z_P} = \text{id}$  for some integer  $n$ .

If  $\#Z_P \geq 3$ , then  $\mu^n|_{Z_P} = \text{id}$  implies  $\mu^n = \text{id}$ . Recall that

$$\rho(f) = (x, ya(x)), \quad \rho(g) = (\mu(x), yb(x)), \quad \rho(h) = (\alpha x, \beta y)$$

so

$$\rho(f)^n = (x, yA(x)), \quad \rho(g)^n = (\mu^n(x), yB(x)) = (x, yB(x)), \quad \rho(h)^{n^2} = (\alpha^{n^2}x, \beta^{n^2}y).$$

Then  $[\rho(f)^n, \rho(g)^n] = \rho(h)^{n^2}$  implies  $\alpha^{n^2} = \beta^{n^2} = 1$ , that is  $\rho(h)$  is of finite order: contradiction.

Hence  $\#Z_P \leq 2$  so  $\deg P \leq 2$ . A straightforward computation implies the statement.  $\square$

Let us come back to  $a(\mu(x)) = \beta a(x)$ . As  $\beta$  is of infinite order and  $a$  belongs to  $\mathbb{C}(x)^*$  we can rewrite this equality as follows:  $\frac{P(\mu(x))}{Q(\mu(x))} = \frac{\lambda_1^2 P(x)}{\lambda_2^2 Q(x)}$  where

- ◇  $\lambda_1, \lambda_2$  are two elements of  $\mathbb{C} \setminus \{\pm 1\}$  such that  $\beta = \frac{\lambda_1^2}{\lambda_2^2}$ ;
- ◇  $P, Q$  are two polynomials without common factor.

As a result up to birational conjugacy  $(\rho(f), \rho(g))$  is one of the following pairs

$$\begin{array}{cc} \left( (x, \delta xy), (\lambda x, yb(x)) \right) & \left( (x, \delta x^2 y), (\lambda x, yb(x)) \right) \\ \left( \left( x, \frac{y}{\delta x} \right), (\lambda x, yb(x)) \right) & \left( \left( x, \frac{y}{\delta x^2} \right), (\lambda x, yb(x)) \right) \end{array}$$

with  $\delta \in \mathbb{C}^*$ ,  $\lambda \in \mathbb{C}^*$  of infinite order and  $b \in \mathbb{C}(x)$ .

We can thus state

**Proposition 2.4.** *Let  $\rho$  be an embedding of  $\mathcal{H}$  into  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .*

*If  $\rho(f)$  is a Jonquières twist with trivial action on the basis of the fibration, then up to birational conjugacy  $(\rho(f), \rho(g))$  is one of the following pairs*

$$\begin{array}{cc} \left( (x, \delta xy), (\lambda x, yb(x)) \right) & \left( (x, \delta x^2 y), (\lambda x, yb(x)) \right) \\ \left( \left( x, \frac{y}{\delta x} \right), (\lambda x, yb(x)) \right) & \left( \left( x, \frac{y}{\delta x^2} \right), (\lambda x, yb(x)) \right) \end{array}$$

with  $\delta \in \mathbb{C}^*$ ,  $\lambda \in \mathbb{C}^*$  of infinite order and  $b \in \mathbb{C}(x)$ .

**2.3. Assume that  $\rho(f)$  is a Jonquières twist with non-trivial action on the basis of the fibration.** Since  $\rho(h)$  is elliptic and of infinite order, then up to birational conjugacy either  $\rho(h) = (\alpha x, \beta y)$ , or  $\rho(h) = (\alpha x, y + 1)$  (see §1.1). But  $\rho(f)$  belongs to the centralizer of  $\rho(h)$  and is a Jonquières twist; therefore according to §1.3 one has:  $\rho(h) = (\alpha x, \beta y)$ ,  $\rho(f)$  can be written as  $(\eta(x), ya(x))$  and  $\rho(g)$  as  $(\mu(x), yb(x))$  with  $\eta, \mu$  in  $\text{PGL}(2, \mathbb{C})$  and  $a, b$  in  $\mathbb{C}(x)$ .

Up to conjugacy by an element of  $\left\{ \left( \frac{ax+b}{cx+d}, y \right) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}(2, \mathbb{C}) \right\}$  one can assume that either  $\eta(x) = x + 1$ , or  $\eta(x) = \lambda x$  (remark that this conjugacy doesn't preserve the first component of  $\rho(h)$ ).

Note that a direct computation implies

$$\{v \in \mathrm{PGL}(2, \mathbb{C}) \mid v(\alpha x) = \alpha v(x)\} = \begin{cases} \mathrm{PGL}(2, \mathbb{C}) & \text{if } \alpha = 1 \\ \{\beta x^{\pm 1} \mid \beta \in \mathbb{C}^*\} & \text{if } \alpha = -1 \\ \{\beta x \mid \beta \in \mathbb{C}^*\} & \text{if } \alpha^2 \neq 1 \end{cases} \quad (2.1)$$

so when  $\eta$  is an homothety we will have to distinguish the cases  $\lambda = -1$  and  $\lambda \neq -1$ .

2.3.1. *Assume that  $\eta(x) = x + 1$ .* Since  $\rho(f)$  and  $\rho(h)$  commute,  $\rho(h)$  can be written as  $(x + \gamma, \beta y)$ .

If  $\gamma \neq 0$ , then  $[\rho(f), \rho(h)] = \mathrm{id}$  leads to  $a(x + \gamma) = a(x)$ , that is  $a(x) = a \in \mathbb{C}$ : contradiction with the fact that  $\rho(f)$  is a Jonquières twist.

If  $\gamma = 0$ , then  $\rho(h) = (x, \beta y)$  and  $[\rho(f), \rho(g)] = \rho(h)$  leads to  $\rho(g) = (x + \mu, yb(x))$  and  $b(x)a(x + \mu) = \beta a(x)b(x + 1)$ . Let us write  $a$  as  $\frac{P}{Q}$  and  $b$  as  $\frac{R}{S}$  with  $P, Q, R, S \in \mathbb{C}[y]$  then  $b(x)a(x + \mu) = \beta a(x)b(x + 1)$  can be rewritten

$$P(x + \mu)Q(x)R(x)S(x + 1) = \beta P(x)Q(x + \mu)R(x + 1)S(x) \quad (2.2)$$

Denote by  $p_i$  (resp.  $q_\ell$ , resp.  $r_j$ , resp.  $s_k$ ) the coefficient of the highest term of  $P$  (resp.  $Q$ , resp.  $R$ , resp.  $S$ ). The coefficient of the highest term of the left-hand side of (2.2) has to be equal to the coefficient of the highest term of the right-hand side of (2.2), that is  $p_i q_\ell r_j s_k = \beta p_i q_\ell r_j s_k$ . So  $\beta = 1$ , i.e.  $\rho(h) = (x, y)$ : contradiction.

2.3.2. *Suppose that  $\eta(x) = -x$ , i.e.  $\rho(f) = (-x, ya(x))$ .*

**Remark 2.5.** The map  $\rho(f)^2 = (x, ya(x)a(-x))$  is a Jonquières twist that preserves fiberwise the rational fibration  $x = \mathrm{cst}$ ; consequently Proposition 2.4 says that  $\rho(f)^2$  is one of the following:

$$(x, \delta xy), \quad (x, \delta x^2 y), \quad \left(x, \delta \frac{y}{x}\right), \quad \left(x, \delta \frac{y}{x^2}\right)$$

with  $\delta \in \mathbb{C}^*$ . Let us try to determine  $\rho(f)$ . If  $\rho(f)^2 = (x, \delta xy)$ , then we have to consider the equation  $a(x)a(-x) = \delta x$ . The right-hand side of this equation is invariant by  $x \mapsto -x$  whereas the left-hand side not, so there is no solution. The same holds if  $\rho(f)^2 = (x, \delta \frac{y}{x})$ . Consequently  $\rho(f)^2$  is one of the following:

$$(x, \delta x^2 y), \quad \left(x, \delta \frac{y}{x^2}\right)$$

with  $\delta \in \mathbb{C}^*$  and  $\rho(f)$  is thus one of the following:

$$(-x, \zeta xy), \quad \left(-x, \zeta \frac{y}{x}\right)$$

with  $\zeta \in \mathbb{C}^*$ .

Since  $f$  and  $h$  commute, then (2.1) implies that either  $\rho(h) = (\frac{\alpha}{x}, \beta y)$ , or  $\rho(h) = (\alpha x, \beta y)$ . Let us consider these two cases.

- ◇ Assume first that  $\rho(h) = \left(\frac{\alpha}{x}, \beta y\right)$ . Note that  $\left(\frac{\alpha}{x}, \beta y\right)$  does not commute neither to  $(-x, \zeta xy)$ , nor to  $(-x, \zeta \frac{y}{x})$ : contradiction with  $[\rho(f), \rho(h)] = \text{id}$ .
- ◇ Suppose now that  $\rho(h) = (\alpha x, \beta y)$ .
  - If  $\alpha^2 \neq 1$ , then  $[\rho(g), \rho(h)] = \text{id}$  and (2.1) imply that  $\rho(g) = (\gamma x, yb(x))$ . Then  $[\rho(f), \rho(g)] = \rho(h)$  leads to  $\alpha = 1$ : contradiction with  $\alpha^2 \neq 1$ .
  - If  $\alpha = -1$ , that is  $\rho(h) = (-x, \beta y)$ , then according to  $[\rho(g), \rho(h)] = \text{id}$  and (2.1) we get that either  $\rho(g) = (\gamma x, yb(x))$ , or  $\rho(g) = \left(\frac{\gamma}{x}, yb(x)\right)$ . In both cases the relation  $[\rho(f), \rho(g)] = \rho(h)$  leads to a contradiction.
  - If  $\alpha = 1$ , *i.e.*  $\rho(h) = (x, \beta y)$ , then  $[\rho(f), \rho(g)] = \rho(h)$  implies that either  $\rho(g) = (\gamma x, yb(x))$  or  $\rho(g) = \left(\frac{\gamma}{x}, yb(x)\right)$ .  
First let us assume that  $\rho(g) = (\gamma x, yb(x))$ . If  $\rho(f) = (-x, \zeta xy)$ , then  $[\rho(f), \rho(g)] = \rho(h)$  leads to  $\gamma b(x) = \beta b(-x)$ , that is  $\frac{b(x)}{b(-x)}$  belongs to  $\mathbb{C}^*$ .  
If  $\rho(f) = (-x, \zeta \frac{y}{x})$ , then  $[\rho(f), \rho(g)] = \rho(h)$  implies  $b(x) = \beta \gamma b(-x)$ , that is  $\frac{b(x)}{b(-x)}$  belongs to  $\mathbb{C}^*$ .  
Finally suppose that  $\rho(g) = \left(\frac{\gamma}{x}, yb(x)\right)$ . If  $\rho(f) = (-x, \zeta xy)$ , then  $[\rho(f), \rho(g)] = \rho(h)$  leads to  $\gamma b(x) = \beta x^2 b(-x)$ . Write  $b$  as  $\frac{P}{Q}$  with  $P, Q$  in  $\mathbb{C}[x]$ ; then  $\gamma b(x) = \beta x^2 b(-x)$  is equivalent to

$$\gamma P(x)Q(-x) = \beta x^2 P(-x)Q(x)$$

and the degree of the left-hand side is  $\deg P + \deg Q$  whereas the degree of the right-hand side is  $\deg P + \deg Q + 2$ : contradiction. If  $\rho(f) = (-x, \zeta \frac{y}{x})$ , then a straightforward computation implies similarly a contradiction.

**Proposition 2.6.** *Let  $\rho$  be an embedding of  $\mathcal{H}$  into  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .*

*If  $\rho(f)$  is a Jonquieres twist with a order 2 action on the basis of the fibration, then up to birational conjugacy  $(\rho(f), \rho(g))$  is one of the following pairs*

$$\left((-x, \alpha xy), (\beta x, ya(x))\right), \quad \left(\left(-x, \alpha \frac{y}{x}\right), (\beta x, ya(x))\right)$$

*with  $\alpha, \beta \in \mathbb{C}^*$  and  $a \in \mathbb{C}(x)^*$  such that  $\frac{a(x)}{a(-x)} \in \mathbb{C}^*$ .*

2.3.3. *Assume that  $\eta(x) = \lambda x$ ,  $\lambda^2 \neq 1$ . Recall that*

$$\rho(f) = (\lambda x, ya(x)), \quad \rho(g) = (\mu(x), yb(x)), \quad \rho(h) = (\nu(x), \beta y)$$

*with  $\lambda$  in  $\mathbb{C}^* \setminus \{1, -1\}$ ,  $\beta$  in  $\mathbb{C}^*$ ,  $\mu, \nu$  in  $\text{PGL}(2, \mathbb{C})$ , and  $a, b$  in  $\mathbb{C}(x)^*$ .*

First note that since  $\rho(f)$  and  $\rho(h)$  commute,  $\nu(\lambda x) = \lambda \nu(x)$ . According to (2.1), the homography  $\nu$  is an homothety (recall that  $\lambda^2 \neq 1$ ):  $\nu(x) = \gamma x$  with  $\gamma \in \mathbb{C}^*$ .

The relations  $[\rho(f), \rho(g)] = \rho(h)$ ,  $[\rho(f), \rho(h)] = [\rho(g), \rho(h)] = \text{id}$  imply the following ones

$$a(x) = a(\gamma x) \quad (2.3)$$

$$b(x) = b(\gamma x) \quad (2.4)$$

$$\mu(\gamma x) = \gamma \mu(x) \quad (2.5)$$

$$\lambda \mu(x) = \gamma \mu(\lambda x) \quad (2.6)$$

$$b(x)a(\mu(x)) = \beta a(x)b(\lambda x) \quad (2.7)$$

We will distinguish the cases  $\gamma = 1$ ,  $\gamma = -1$ ,  $\gamma^2 \neq 1$ .

- ◇ Assume that  $\gamma^2 \neq 1$ . Then (2.1) and (2.5) lead to  $\mu(x) = \mu x$  with  $\mu \in \mathbb{C}^*$ . Equation (2.6) can be rewritten  $\lambda \mu x = \gamma \mu \lambda x$ , that is  $\gamma = 1$ : contradiction with the assumption  $\gamma^2 \neq 1$ .
- ◇ Suppose that  $\gamma = 1$ . Then  $\lambda^2 \neq 1$ , (2.1) and (2.6) lead to  $\mu(x) = \mu x$  with  $\mu \in \mathbb{C}^*$ . In other words

$$\rho(f) = (\lambda x, ya(x)), \quad \rho(g) = (\mu x, yb(x))$$

with  $\lambda$  in  $\mathbb{C}^* \setminus \{1, -1\}$ ,  $\mu$  in  $\mathbb{C}^*$ ,  $a, b$  in  $\mathbb{C}(x)$  such that  $\frac{a(\mu x)b(x)}{a(x)b(\lambda x)}$  belongs to  $\mathbb{C}^*$ .

- ◇ Assume that  $\gamma = -1$ . Then (2.1) and (2.5) imply that  $\mu(x) = \mu x^{\pm 1}$  with  $\mu \in \mathbb{C}^*$ . If  $\mu(x) = \mu x$ , then (2.6) can be rewritten  $\lambda \mu x = -\lambda \mu x$ : contradiction. If  $\mu(x) = \frac{\mu}{x}$ , then (2.6) can be rewritten  $\frac{\lambda \mu}{x} = -\frac{\mu}{\lambda x}$ ; hence  $\lambda^2 = -1$ .
  - If  $\lambda = \mathbf{i}$ , then  $\rho(f) = (\mathbf{i}x, ya(x))$  and  $\rho(f)^4 = (x, ya(x)a(\mathbf{i}x)a(-x)a(-\mathbf{i}x))$  preserves fiberwise the fibration  $x = \text{cst}$ . According to Proposition 2.4  $\rho(f)^4$  can be written as  $(x, \delta xy)$ , or  $(x, \delta x^2 y)$ , or  $(x, \frac{y}{\delta x})$ , or  $(x, \frac{y}{\delta x^2})$ . If  $\rho(f)^4 = (x, \delta xy)$ , then  $\delta x = a(x)a(\mathbf{i}x)a(-x)a(-\mathbf{i}x)$ ; but the right-hand side of this equality is invariant by  $x \mapsto \mathbf{i}x$  whereas the left-hand side is not. As a consequence  $\rho(f)^4$  can not be written  $(x, \delta xy)$ . Similarly one sees that  $\rho(f)^4$  can not be written  $(x, \delta x^2 y)$ ,  $(x, \frac{y}{\delta x})$ , and  $(x, \frac{y}{\delta x^2})$ . Thus  $\lambda \neq \mathbf{i}$ .
  - Similarly one gets that the case  $\lambda = -\mathbf{i}$  does not happen.

**Proposition 2.7.** *Let  $\rho$  be an embedding of  $\mathcal{H}$  into  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .*

*If  $\rho(f)$  is a Jonquieres twist with an action on the basis of the fibration that is neither trivial, nor of order 2, then up to birational conjugacy  $(\rho(f), \rho(g))$  is one of the following pairs*

$$((\lambda x, ya(x)), (\mu x, yb(x)))$$

*with  $\lambda \in \mathbb{C}^* \setminus \{1, -1\}$ ,  $\mu \in \mathbb{C}^*$ ,  $a, b \in \mathbb{C}(x)^*$  such that  $\frac{a(\mu x)b(x)}{a(x)b(\lambda x)} \in \mathbb{C}^*$ .*

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