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## ON REGULARIZABLE BIRATIONAL MAPS

JULIE DÉSERTI

ABSTRACT. BEDFORD asked if there exists a birational self map  $f$  of the complex projective plane such that for any automorphism  $A$  of the complex projective plane  $A \circ f$  is not conjugate to an automorphism. In [3] BLANC gave such a  $f$  of degree 6 and asked if there exists an example of smaller degree. In this article we give an example of degree 5.

### 1. INTRODUCTION

Denote by  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^k)$  the group of all birational self maps of  $\mathbb{P}_{\mathbb{C}}^k$ , also called the  $k$ -dimensional CREMONA group. Let  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^k)$  be the algebraic variety of all birational self maps of  $\mathbb{P}_{\mathbb{C}}^k$  of degree  $d$ . When  $k = 2$  and  $d \geq 2$  these varieties have many distinct components, of various dimensions ([6, 2]). The group  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^k) = \text{PGL}(k+1, \mathbb{C})$  acts by left translations, by right translations, and by conjugacy on  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^k)$ . Since this group is connected, these actions preserve each connected component.

A birational map  $f: \mathbb{P}_{\mathbb{C}}^k \dashrightarrow \mathbb{P}_{\mathbb{C}}^k$  is *regularizable* if there exist a smooth projective variety  $V$  and a birational map  $g: V \dashrightarrow \mathbb{P}_{\mathbb{C}}^k$  such that  $g^{-1} \circ f \circ g$  is an automorphism of  $V$ . To any element  $f$  of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^k)$  we associate the set  $\text{Reg}(f)$  defined by

$$\text{Reg}(f) := \{A \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^k) \mid A \circ f \text{ is regularizable}\}.$$

DOLGACHEV asked whether there exists a birational self map of  $\mathbb{P}_{\mathbb{C}}^k$  of degree  $> 1$  such that  $\text{Reg}(f) = \text{Aut}(\mathbb{P}_{\mathbb{C}}^k)$ . In [5] we give a negative answer to this question. More precisely we prove

**Theorem 1.1** ([5]). *Let  $f$  be a birational self map of  $\mathbb{P}_{\mathbb{C}}^k$  of degree  $d \geq 2$ .*

*The set of automorphisms  $A$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that  $\deg((A \circ f)^n) \neq (\deg(A \circ f))^n$  for some  $n > 0$  is a countable union of proper ZARISKI closed subsets of  $\text{PGL}(k+1, \mathbb{C})$ .*

*In particular there exists an automorphism  $A$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that  $A \circ f$  is not regularizable.*

BEDFORD asked: does there exist a birational map  $f$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that  $\text{Reg}(f) = \emptyset$ ? We will focus on the case  $k = 2$ . According to [1, 7] if  $\deg f = 2$ , then  $\text{Reg}(f) \neq \emptyset$ . What about birational

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maps of degree 3 ? BLANC proves that the set

$$\{f \in \text{Bir}_3(\mathbb{P}_{\mathbb{C}}^2) \mid \text{Reg}(f) \neq \emptyset, \lim_{n \rightarrow +\infty} (\deg(f^n))^{1/n} > 1\}$$

is dense in  $\text{Bir}_3(\mathbb{P}_{\mathbb{C}}^2)$  and that its complement has codimension 1 (see [3]). BLANC also gives a positive answer to BEDFORD question in dimension 2: if  $\chi: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  is the birational map given by

$$\chi: (x : y : z) \dashrightarrow (xz^5 + (yz^2 + x^3)^2 : yz^5 + x^3z^3 : z^6)$$

then  $\text{Reg}(\chi) = \emptyset$ .

**Remark 1.2.** Note that  $\chi = (x + y^2, y) \circ (x, y + x^3)$  in the affine chart  $z = 1$ . Indeed BLANC example can be generalized as follows: the birational map given in the affine chart  $z = 1$  by

$$\chi_{n,p} = (x + y^n, y) \circ (x, y + x^p) = (x + (y + x^p)^n, y + x^p)$$

satisfies  $\text{Reg}(\chi_{n,p}) = \emptyset$  (see §3).

Then BLANC asked: does there exist  $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  such that  $\deg < 6$  and  $\text{Reg}(f) = \emptyset$  ? The following statement gives a positive answer to this question:

**Theorem A.** If  $\varphi: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  is the birational map given by

$$\varphi: (x : y : z) \dashrightarrow (y^3(z^2 - xy) : z^5 - y^5 - xyz^3 : y^2z(z^2 - xy)),$$

then  $\text{Reg}(\varphi) = \emptyset$ .

**Acknowledgements.** I would like to thank Serge CANTAT for many interesting discussions.

## 2. PROOF OF THEOREM A

Let  $S$  be a smooth projective surface. Let  $\phi: S \dashrightarrow S$  be a birational map. This map admits a resolution

$$\begin{array}{ccc} & Z & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ S & \dashrightarrow & S \\ & \phi & \end{array}$$

where  $\pi_1: Z \rightarrow S$  and  $\pi_2: Z \rightarrow S$  are finite sequences of blow-ups. The resolution is *minimal* if and only if no  $(-1)$ -curve of  $Z$  is contracted by both  $\pi_1$  and  $\pi_2$ . The *base-points*  $\text{Base}(\phi)$  of  $\phi$  are the points blown-up by  $\pi_1$ , which can be points of  $S$  or infinitely near points. The proper base-points of  $\phi$  are called *indeterminacy points* of  $\phi$  and form a set denoted  $\text{Ind}(\phi)$ . Finally we denote by  $\text{Exc}(\phi)$  the set of curves contracted by  $\phi$ .

Denote by  $\mathfrak{b}(\phi)$  the number of base-points of  $\phi$ ; note that  $\mathfrak{b}(\phi)$  is equal to the difference of the ranks of  $\text{Pic}(Z)$  and  $\text{Pic}(S)$  and thus equal to  $\mathfrak{b}(\phi^{-1})$ . Let us introduce the *dynamical number of the base-points* of  $\phi$ . Since  $\mathfrak{b}(\phi \circ \psi) \leq \mathfrak{b}(\phi) + \mathfrak{b}(\psi)$  for any birational self map  $\psi$  of  $S$ ,  $\mu(\phi)$  is a

non-negative real number. As  $\mathfrak{b}(\phi) = \mathfrak{b}(\phi^{-1})$  one gets  $\mu(\phi^k) = |k\mu(\phi)|$  for any  $k \in \mathbb{Z}$ . Furthermore if  $Z$  is a smooth projective surface and  $\psi: S \dashrightarrow Z$  a birational map, then for all  $n \in \mathbb{Z}$

$$-2\mathfrak{b}(\psi) + \mathfrak{b}(\phi^n) \leq \mathfrak{b}(\psi \circ \phi^n \circ \psi^{-1}) \leq 2\mathfrak{b}(\psi) + \mathfrak{b}(\phi^n);$$

hence  $\mu(\phi) = \mu(\psi \circ \phi \circ \psi^{-1})$ . One can thus state the following result:

**Lemma 2.1** ([4]). *The dynamical number of base-points is an invariant of conjugation. In particular if  $\phi$  is a regularizable birational self map of a smooth projective surface, then  $\mu(\phi) = 0$ .*

A base-point  $p$  of  $\phi$  is a *persistent base-point* if there exists an integer  $N$  such that for any  $k \geq N$

$$\begin{cases} p \in \text{Base}(\phi^k) \\ p \notin \text{Base}(\phi^{-k}) \end{cases}$$

Let  $p$  be a point of  $S$  or a point infinitely near  $S$  such that  $p \notin \text{Base}(\phi)$ . Consider a minimal resolution of  $\phi$

$$\begin{array}{ccc} & Z & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ S & \overset{\phi}{\dashrightarrow} & S \end{array}$$

Because  $p$  is not a base-point of  $\phi$  it corresponds via  $\pi_1$  to a point of  $Z$  or infinitely near; using  $\pi_2$  we view this point on  $S$  again maybe infinitely near and denote it  $\phi^\bullet(p)$ . For instance if  $S = \mathbb{P}_{\mathbb{C}}^2$ ,  $p = (1 : 0 : 0)$  and  $f$  is the birational self map of  $\mathbb{P}_{\mathbb{C}}^2$  given by

$$(z_0 : z_1 : z_2) \dashrightarrow (z_1 z_2 + z_0^2 : z_0 z_2 : z_2^2)$$

the point  $f^\bullet(p)$  is not equal to  $p = f(p)$  but is infinitely near to it. Note that if  $\psi$  is a birational self map of  $S$  and  $p$  is a point of  $S$  such that  $p \notin \text{Base}(\phi)$ ,  $\phi(p) \notin \text{Base}(\psi)$ , then  $(\psi \circ \phi)^\bullet(p) = \psi^\bullet(\phi^\bullet(p))$ . One can put an equivalence relation on the set of points of  $S$  or infinitely near  $S$ : the point  $p$  is *equivalent* to the point  $q$  if there exists an integer  $k$  such that  $(\phi^k)^\bullet(p) = q$ ; in particular  $p \notin \text{Base}(\phi^k)$  and  $q \notin \text{Base}(\phi^{-k})$ . Remark that the equivalence class is the generalization of set of orbits for birational maps.

Let us give the relationship between the dynamical number of base-points and the equivalence classes of persistent base-points:

**Proposition 2.2** ([4]). *Let  $S$  be a smooth projective surface. Let  $\phi$  be a birational self map of  $S$ .*

*Then  $\mu(\phi)$  coincides with the number of equivalence classes of persistent base-points of  $\phi$ . In particular  $\mu(\phi)$  is an integer.*

This interpretation of the dynamical number of base-points allows to prove the following result that gives a characterization of regularizable birational maps:

**Theorem 2.3** ([4]). *Let  $\phi$  be a birational self map of a smooth projective surface. Then  $\phi$  is regularizable if and only if  $\mu(\phi) = 0$ .*

The birational map

$$\varphi: (x : y : z) \dashrightarrow (y^3(z^2 - xy) : z^5 - y^5 - xyz^3 : y^2z(z^2 - xy))$$

blows down the conic  $\mathcal{C}$  given by  $z^2 - xy = 0$  onto the point  $p = (0 : 1 : 0)$  and the line  $L_y$  defined by  $y = 0$  onto  $p$ . Furthermore  $\varphi$  has only one point of indeterminacy which is  $q = (1 : 0 : 0) = L_y \cap \mathcal{C}$ . The inverse of  $\varphi$  is the map

$$\varphi^{-1}: (x : y : z) \dashrightarrow (x^2yz^2 - z^5 + x^5 : x^2(x^2y - z^3) : xz(x^2y - z^3))$$

which blows down  $\mathcal{C}'$  given by  $x^2y - z^3 = 0$  onto  $q$  and the line  $L_x$  defined by  $x = 0$  onto  $q$ . Moreover  $\text{Ind}(\varphi^{-1}) = \mathcal{C}' \cap L_x = \{p\}$ .

If  $A$  is an automorphism of  $\mathbb{P}_{\mathbb{C}}^2$  let us set  $\varphi_A = A \circ \varphi$ . We will prove the two following statements:

**Lemma 2.4.** *The positive orbit of any point  $p_i^{(1)} \in \text{Base}(\varphi_A^{-1})$  is infinite.*

**Lemma 2.5.** *The negative orbit of any point  $q_i \in \text{Base}(\varphi_A)$  is infinite.*

Lemmas 2.4 and 2.5 imply that  $\mu(\varphi_A) = 0$ ; Theorem A thus follows from Theorem 2.3. We will now prove Lemmas 2.4 and 2.5.

The set of base points of  $\varphi$  is

$$\text{Base}(\varphi) = \{q, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\}$$

and the set of base points of  $\varphi^{-1}$  is

$$\text{Base}(\varphi^{-1}) = \{p, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}.$$

We have

$$\text{Base}(\varphi_A) = \text{Base}(\varphi), \quad \text{Exc}(\varphi_A) = \text{Exc}(\varphi).$$

However

$$\text{Base}(\varphi_A^{-1}) = \{p^{(1)}, p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, p_4^{(1)}, p_5^{(1)}, p_6^{(1)}, p_7^{(1)}, p_8^{(1)}\}$$

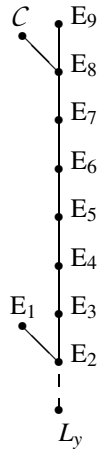
where  $p^{(1)} = A(p)$  and  $p_j^{(1)} = A(p_j)$ . Moreover

$$\text{Exc}(\varphi_A^{-1}) = \{A(L_x), A(\mathcal{C}')\}.$$

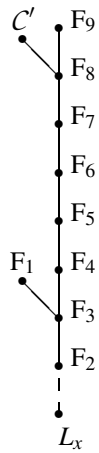
The map  $\varphi_A$  (resp.  $\varphi_A^{-1}$ ) has only one proper base point, and all its base points are in tower, that is  $q_i$  (resp.  $p_i$ ) is infinitely near to  $q_{i-1}$  (resp.  $p_{i-1}$ ) for  $i = 2, \dots, 8$ . We denote by  $\pi_1 : S \rightarrow \mathbb{P}_{\mathbb{C}}^2$  (resp.  $\pi_2 : S \rightarrow \mathbb{P}_{\mathbb{C}}^2$ ) the blow-up of the 8 base points of  $\varphi_A$  (resp.  $\varphi_A^{-1}$ ). We have

$$\begin{array}{ccc} & S & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}_{\mathbb{C}}^2 & \dashrightarrow_{\varphi_A} & \mathbb{P}_{\mathbb{C}}^2 \end{array}$$

We still denote by  $L_y$  and  $C$  (resp.  $L_x$  and  $C'$ ) the strict transform of  $L_y$  and  $C$  (resp.  $L_x$  and  $C'$ ). Let  $E_i \subset V_1$  (resp.  $F_i \subset V_2$ ) be the strict transform of the curve obtained by blowing up  $q_i$  (resp.  $p_i$ ). The configuration of the curves  $E_1, E_2, \dots, E_8, C$  and  $L_y$  on  $S$  is

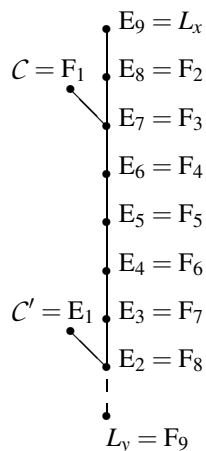


where two curves are connected by an edge if their intersection is positive. We will denote by  $\mathcal{T}'$  this tree. The configuration of the curves  $F_1, F_2, \dots, F_8, C'$  and  $L_x$  on  $S$  is:



Let us denote by  $\mathcal{T}$  this tree.

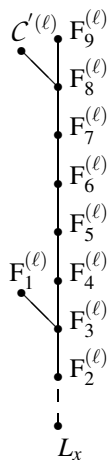
Because of the order of the curves contracted by  $\pi_2$  we get equalities between  $C, E_1, E_2, \dots, E_9$  and  $C', F_1, F_2, \dots, F_9$  according to the following figure:



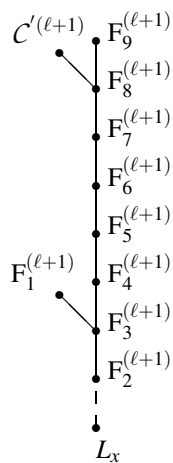
Furthermore  $\varphi$  sends

$L_y$ to $F_9$	$E_1$ to $C'$	$E_2$ to $F_8$	$E_3$ to $F_7$
$E_4$ to $F_6$	$E_5$ to $F_5$	$E_6$ to $F_4$	$E_7$ to $F_3$
$E_8$ to $F_2$	$E_9$ to $L_x$	$C$ to $F_1$	

Let us study the positive orbits of the base points of  $\text{Base}(\varphi_A^{-1})$ . Set  $p^{(k)} = \varphi_A^k(p^{(1)})$  and  $p_i^{(k)} = \varphi_A^k(p_i^{(1)})$ . As soon as  $p^{(\ell)}$  belongs to  $\mathbb{P}_{\mathbb{C}}^2 \setminus \{L_y, C\}$ ,  $\varphi_A$  sends the tree



onto the tree

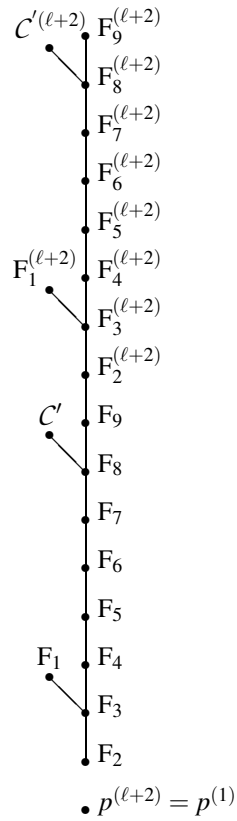


We will denote by  $\mathcal{T}^{(\ell)}$  (resp.  $\mathcal{T}'^{(\ell)}$ ) the tree above  $p^{(\ell)}$  (resp.  $q^{(\ell)}$ ) and we will say that  $\varphi_A$  sends  $(p^{(\ell)}, \mathcal{T}^{(\ell)})$  onto  $(p^{(\ell+1)}, \mathcal{T}^{(\ell+1)})$ .

The orbit of  $p^{(1)}$  is finite if there exists an integer  $\ell$  such that

- either  $p^{(\ell+1)}$  lies on  $L_y \setminus \{q\}$
- or  $p^{(\ell+1)}$  belongs to  $C \setminus \{q\}$ ;
- or  $p^{(\ell+1)} = q$ .

*Proof of Lemma 2.4.* (1) Let us first assume that  $p^{(\ell+1)}$  lies on  $L_y \setminus \{q\}$  or on  $C \setminus \{q\}$ . Then  $\varphi_A$  sends  $(p^{(\ell)}, \mathcal{T}^{(\ell)})$  onto  $(p^{(\ell+1)}, \mathcal{T}^{(\ell+1)})$  and  $(p^{(\ell+1)}, \mathcal{T}^{(\ell+1)})$  onto

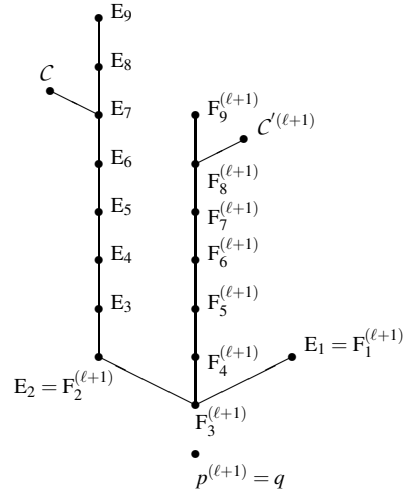


(2) Suppose finally that  $p^{(\ell+1)} = q$ . Since  $\varphi_A$  is a local diffeomorphism at  $p^{(\ell)}$  the map  $\varphi_A$  sends  $(p^{(\ell)}, \mathcal{T}^{(\ell)})$  onto  $(q = p^{(\ell+1)}, \mathcal{T}')$ .

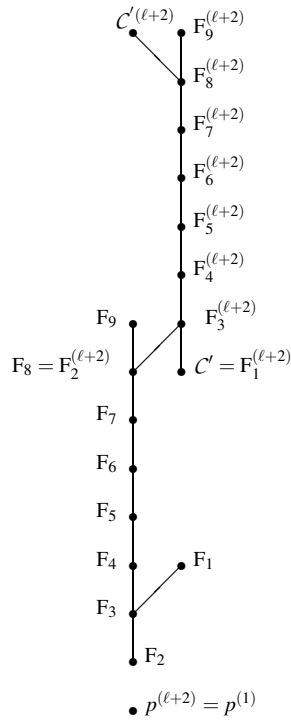
The curve  $F_1^{(\ell)}$  has to be sent onto  $E_1$  since  $F_1^{(\ell)}$  is the exceptional divisor obtained from the first blow up of  $p^{(\ell)}$ . Then

- either  $F_2^{(\ell)}$  is sent onto  $E_2$ ,
- or not.

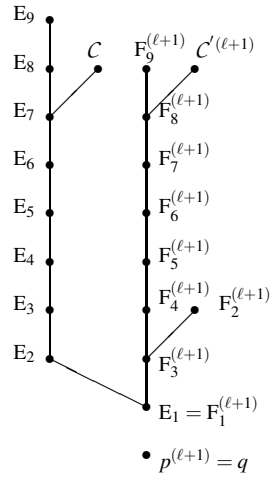
If  $F_2^{(\ell)}$  is sent onto  $E_2$ , then  $\varphi_A$  sends the tree



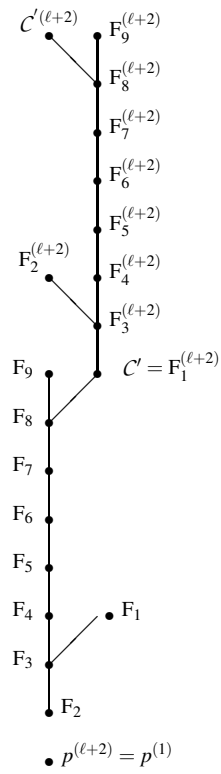
onto the tree



If  $F_2^{(\ell)}$  is not sent onto  $E_2$ , then  $\phi_A$  sends the tree



onto the tree



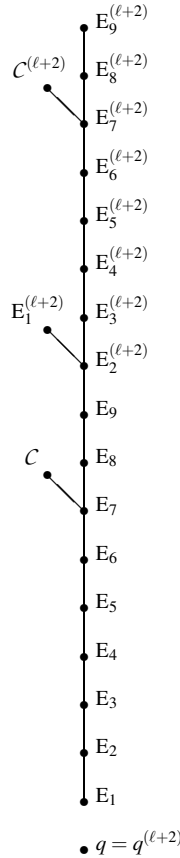
□

Similarly the study of the positive orbits of the base points of  $\text{Base}(\varphi_A^{-1})$  allows to prove Lemma 2.5. Let us denote by  $\{q^{(i)}\}$  the orbit of  $q$  under the action of  $\varphi_A^{-1}$ . As soon as  $q^{(\ell)}$  belongs to  $\mathbb{P}_{\mathbb{C}}^2 \setminus \{A(L_x), A(C)\}$  the map  $\varphi_A^{-1}$  sends  $(q^{(\ell)}, \mathcal{T}'^{(\ell)})$  onto  $(q^{(\ell+1)}, \mathcal{T}'^{(\ell+1)})$ .

The orbit of  $q$  is finite if one of the following holds

- $q^{(\ell+1)}$  lies on  $A(L_x) \setminus \{p^{(1)}\}$ ;
- $q^{(\ell+1)}$  belongs to  $A(C) \setminus \{p^{(1)}\}$ ;
- $q^{(\ell+1)} = p^{(1)}$ .

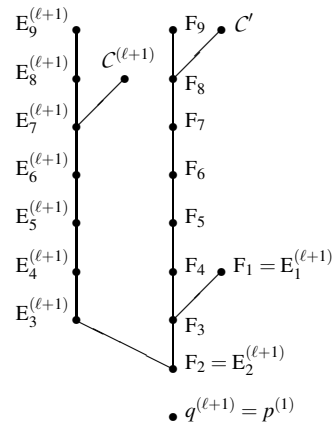
*Proof of Lemma 2.5.* (1) Let us first assume that  $q^{(\ell+1)}$  belongs to  $A(L_x) \setminus \{p^{(1)}\}$  or to  $A(C) \setminus \{p^{(1)}\}$ . Then  $\varphi_A^{-1}$  sends  $(q^{(\ell)}, \mathcal{T}'^{(\ell)})$  onto  $(q^{(\ell+1)}, \mathcal{T}'^{(\ell+1)})$  and  $(q^{(\ell+1)}, \mathcal{T}'^{(\ell+1)})$  onto the tree



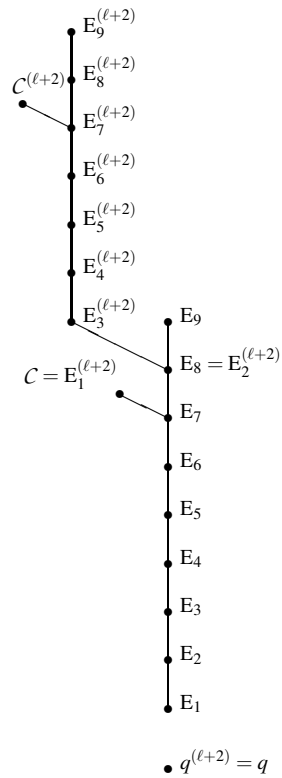
(2) Suppose finally that  $q^{(\ell+1)} = p^{(1)}$ . The curve  $E_1^{(\ell)}$  has to be sent onto  $F_1$  since  $E_1^{(\ell)}$  is the exceptional divisor obtained from the first blow up of  $q^{(\ell)}$ . Then

- either  $E_2^{(\ell)}$  is sent onto  $F_2$ ,
- or not.

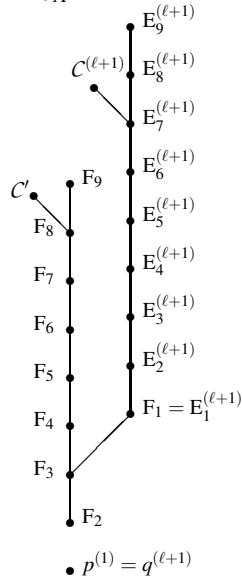
If  $E_2^{(\ell)}$  is sent onto  $F_2$ , then  $\varphi_A^{-1}$  sends



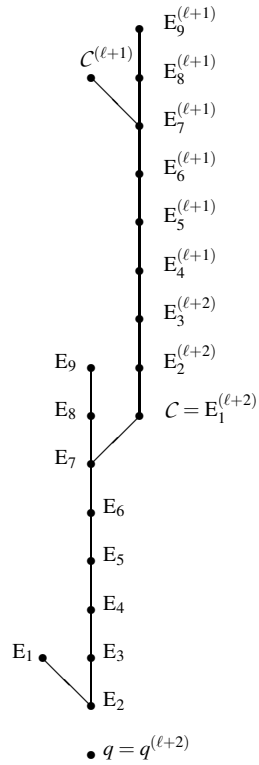
onto



If  $E_2^{(\ell)}$  is not sent onto  $F_2$ , then  $\phi_A^{-1}$  sends



onto the tree



□

### 3. BLANC EXAMPLE IN HIGHER DEGREE

Let us deal with Remark 1.2.

In [3] BLANC consider the birational map  $\chi_{23} = \varphi_2 \circ \psi_3$  with

$$\varphi_2: (x : y : z) \dashrightarrow (xz + y^2 : yz : z^2), \quad \psi_3: (x : y : z) \dashrightarrow (xz^2 : yz^2 + x^3 : z^3).$$

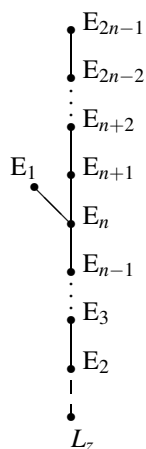
BLANC proves that for any  $A \in \text{PGL}(3, \mathbb{C})$

- the positive orbit of any point of  $\text{Base}((A \circ \chi_{23})^{-1})$  is infinite,
- the negative orbit of any point of  $\text{Base}(A \circ \chi_{23})$  is infinite.

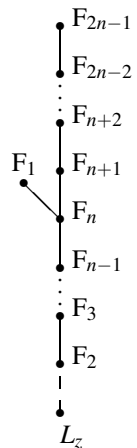
This implies that  $A \circ \chi_{23}$  is not regularizable, and so  $\text{Reg}(\chi_{23}) = \emptyset$ . It can be generalize in higher degree. Let us set

$$\varphi_n: (x : y : z) \dashrightarrow (xz^{n-1} + y^n : yz^{n-1} : z^n), \quad \psi_p: (x : y : z) \dashrightarrow (xz^{p-1} : yz^{p-1} + x^p : z^p).$$

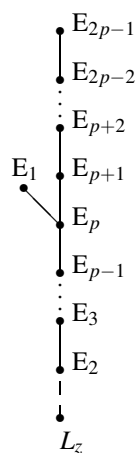
The tree of rational curves obtained by solving the indeterminacy of  $\varphi_n$  is



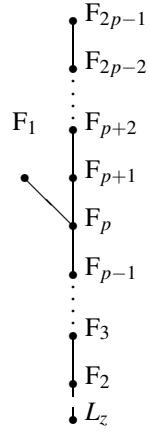
The tree of rational curves obtained by solving the indeterminacy of  $\varphi_n^{-1}$  is



The tree of rational curves obtained by solving the indeterminacy of  $\psi_p$  is

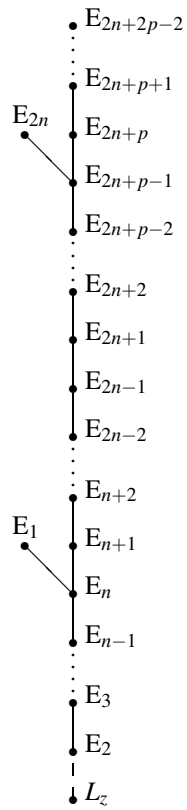


The tree of rational curves obtained by solving the indeterminacy of  $\psi_p^{-1}$  is

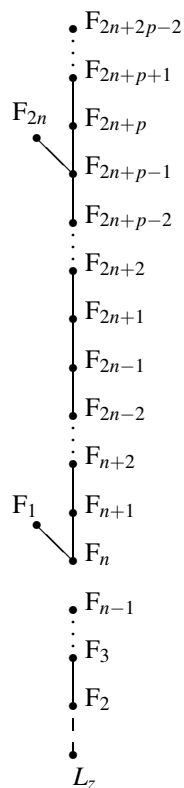


Let us now consider  $\chi_{n,p} = \varphi_n \circ \psi_p$ .

The tree of rational curves obtained by solving the indeterminacy of  $\chi_{n,p}$  is



The tree of rational curves obtained by solving the indeterminacy of  $\chi_{n,p}^{-1}$  is



Furthermore  $\chi_{n,p}$  sends

$L_z$ to $F_{2p+2n-2}$	$E_1$ to $F_p$	$E_{2n}$ to $F_1$	$E_{2p+2n-2}$ to $L_z$
$E_2$ to $F_{2p+2n-3}$	$E_3$ to $F_{2p+2n-4}$	$\dots$	$E_{n-1}$ to $F_{2p+n}$
$E_n$ to $F_{2p+n-1}$	$E_{n+1}$ to $F_{2p+n-2}$	$\dots$	$E_{2n+p-2}$ to $F_{p+1}$
$E_{p+2n-1}$ to $F_p$	$E_{p+2n}$ to $F_{p-1}$	$\dots$	$E_{2p+2n-1}$ to $F_2$

As a result using [3] we can state:

**Theorem 3.1.** *If*

$$\varphi_n : (x : y : z) \mapsto (xz^{n-1} + y^n : yz^{n-1} : z^n) \quad \psi_p : (x : y : z) \mapsto (xz^{p-1} : yz^{p-1} + x^p : z^p)$$

and  $\chi_{n,p} = \varphi_n \circ \psi_p$ , then  $\text{Reg}(\chi_{n,p}) = \emptyset$ .

#### REFERENCES

- [1] E. Bedford and K. Kim. Dynamics of rational surface automorphisms: linear fractional recurrences. *J. Geom. Anal.*, 19(3):553–583, 2009.
- [2] C. Bisi, A. Calabri, and M. Mella. On plane Cremona transformations of fixed degree. *J. Geom. Anal.*, 25(2):1108–1131, 2015.
- [3] J. Blanc. Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces. *Indiana Univ. Math. J.*, 62(4):1143–1164, 2013.
- [4] J. Blanc and J. Déserti. Degree growth of birational maps of the plane. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 14(2):507–533, 2015.
- [5] S. Cantat, J. Déserti, and J. Xie. In preparation.
- [6] D. Cerveau and J. Déserti. *Transformations birationnelles de petit degré*, volume 19 of *Cours Spécialisés*. Société Mathématique de France, Paris, 2013.
- [7] J. Diller. Cremona transformations, surface automorphisms, and plane cubics. *Michigan Math. J.*, 60(2):409–440, 2011. With an appendix by Igor Dolgachev.

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