HAL open science

DYNAMICS OF A FAMILY OF POLYNOMIAL AUTOMORPHISMS OF C3 , A PHASE TRANSITION

Julie Déserti, Martin Leguil

To cite this version:

Julie Déserti, Martin Leguil. DYNAMICS OF A FAMILY OF POLYNOMIAL AUTOMORPHISMS OF C3 , A PHASE TRANSITION. Journal of Geometric Analysis, 2018. hal-03000553

HAL Id: hal-03000553

https://hal.science/hal-03000553v1

Submitted on 12 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1606.09633v2 [math.DS] 1 Jul 2016

DYNAMICS OF A FAMILY OF POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^3$, A PHASE TRANSITION

JULIE DÉSERTI AND MARTIN LEGUIL

ABSTRACT. The polynomial automorphisms of the affine plane have been studied a lot: if f is such an automorphism, then either f preserves a rational fibration, has an uncountable centralizer and its first dynamical degree equals 1, or f preserves no rational curves, has a countable centralizer and its first dynamical degree is >1. In higher dimensions there is no such description. In this article we study a family $(\Psi_\alpha)_\alpha$ of polynomial automorphisms of $\mathbb{C}^3$. We show that the first dynamical degree of Ψ_α is >1, that Ψ_α preserves a unique rational fibration and has an uncountable centralizer. We then describe the dynamics of the family $(\Psi_\alpha)_\alpha$, in particular the speed of points escaping to infinity. We also observe different behaviors according to the value of the parameter α.

CONTENTS

1. Introduction 1
2. Invariant fibrations and degree growths 3
3. Centralizers 6
4. Dynamics on the invariant hypersurface $z_2=0$ 8
5. Points with bounded forward orbit, description of the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ 8
6. Birational conjugacy, dynamical properties of automorphisms of Hénon type 11
7. Definition of a Green function for Ψ_α 13
8. Analysis of the dynamics of the automorphism Ψ_α 14
References 25

1. INTRODUCTION

Hénon gave families of quadratic automorphisms of the plane which provide examples of dynamical systems with very complicated dynamics ([14, 15, 5, 8]). In [11] Friedland and Milnor proved that if f belongs to the group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$, then f is $\mathrm{Aut}(\mathbb{C}^2)$-conjugate either to an elementary automorphism (elementary in the sense that they preserve a rational fibration), or to an automorphism of Hénon type, *i.e.* to

$$g_1g_2\dots g_k, \qquad g_j\colon (z_0,z_1)\mapsto (z_1,P_j(z_1)-\delta_j z_0),\ \delta_j\in\mathbb{C}^*,\ P_j\in\mathbb{C}[z_1],\ \deg P_j\geq 2.$$

The topological entropy allows to measure chaotic behaviors. In dimension 1 the topological entropy of a rational fraction coincides with the logarithm of its degree; but the algebraic degree of a polynomial automorphism of $\mathbb{C}^2$ is not invariant under conjugacy so [20, 10] introduce the first dynamical degree. The topological entropy is equal to the logarithm of the first dynamical degree ([24, 4]). If f is of Hénon type, then the first dynamical degree of f is equal to its algebraic degree ≥ 2; on the other hand if f is conjugate to an elementary automorphism, then its first dynamical degree is 1 (*see* [11]). Another criterion to measure chaos is the size of the centralizer of an element. The group $\mathrm{Aut}(\mathbb{C}^2)$ has a structure of amalgamated product

([17]) hence according to [21] this group acts non trivially on a tree. Using this action Lamy proved that a polynomial automorphism is of Hénon type if and only if its centralizer is countable ([18]).

The group $\mathrm{Aut}(\mathbb{C}^3)$ and the dynamics of its elements are much less-known. In this article we study the properties of the family of polynomial automorphisms of $\mathbb{C}^3$ given by

$$\Psi_\alpha\colon (z_0,z_1,z_2)\mapsto (z_0+z_1+z_0^q z_2^d, z_0, \alpha z_2)$$

where α denotes a nonzero complex number with modulus ≤ 1, q an integer ≥ 2, and d an integer ≥ 1.

The automorphism Ψ_α can be seen as a skew product over the map $z_2\mapsto \alpha z_2$, and whose dynamics in the fibers is given by automorphisms of Hénon type. More precisely, if $z_2\in\mathbb{C}$, let us denote $\psi_{z_2}\colon (z_0,z_1)\mapsto (z_0+z_1+z_0^q z_2^d, z_0)$; then $\Psi_\alpha(z_0,z_1,z_2)=(\psi_{z_2}(z_0,z_1),\alpha z_2)$, and for every $n\geq 1$, we have $\Psi_\alpha^n(z_0,z_1,z_2)=((\psi_{z_2})_n(z_0,z_1),\alpha^n z_2)$, where

$$(\psi_{z_2})_n=\psi_{\alpha^{n-1}z_2}\circ\cdots\circ\psi_{\alpha z_2}\circ\psi_{z_2}. \tag{1.1}$$

If $\alpha\neq 0$, we also define the map $\phi_\alpha:=\alpha^l\psi_1$ where $l:=d/(q-1)$. We will see later on that Ψ_α is semi-conjugate to ϕ_α. The family of automorphisms $\{\Psi_\alpha\}_\alpha$ satisfies the following properties:

Proposition A. *Take $0<|\alpha|\leq 1$. Then*

- *the first dynamical degree of the automorphism Ψ_α (resp. Ψ_α^{-1}) is $q\geq 2$;*
- *the centralizer of Ψ_α is uncountable;*
- *if $0<|\alpha|\leq 1$, then Ψ_α preserves a unique rational fibration, $\{z_2=cst\}$.*

We then focus on the dynamics of Ψ_α, $0<|\alpha|\leq 1$. Let us introduce a definition. We denote by $\varphi:=\frac{1+\sqrt{5}}{2}$ the golden ratio. We say that the forward orbit of p *goes to infinity with Fibonacci speed* if the sequence $(\Psi_\alpha^n(p)\varphi^{-n})_{n\geq 0}$ converges and $\lim\limits_{n\to+\infty}\Psi_\alpha^n(p)\varphi^{-n}=p'\neq 0_{\mathbb{C}^3}$. In particular this implies

$$\|\Psi_\alpha^n(p)\|\sim\|p'\|\varphi^n.$$

The hypersurface $\{z_2=0\}$ is fixed by Ψ_α, and the induced map on it is a linear Anosov diffeomorphism. We see that for any $p\in\{z_2=0\}$, either its forward orbit goes to $0_{\mathbb{C}^3}$ exponentially fast, or it escapes to infinity with Fibonacci speed. Concerning points escaping to infinity with maximal speed, we prove:

Theorem B. *Fix $0<|\alpha|\leq 1$. For any point $p\in\mathbb{C}^3$, the limit $\lim\limits_{n\to+\infty}\frac{\log^+||\Psi_\alpha^n(p)||}{q^n}$ exists. The function*

$$G^+_{\Psi_\alpha}(p)=\lim_{n\to+\infty}\frac{\log^+||\Psi_\alpha^n(p)||}{q^n}$$

is plurisubharmonic, Hölder continuous, and satisfies $G^+_{\Psi_\alpha}\circ\Psi_\alpha=q\cdot G^+_{\Psi_\alpha}$. Set $\widetilde{l}:=2\max\left(\frac{d}{q-1},1\right)$; then

$$1\leq\limsup_{\|p\|\to+\infty}\frac{G^+_{\Psi_\alpha}(p)}{\log\|p\|}\leq\widetilde{l}.$$

Moreover, the set $\mathcal{E}:=\left\{p\in\mathbb{C}^3\mid G^+_{\Psi_\alpha}(p)>0\right\}$ of points escaping to infinity with maximal speed is open, connected, and has infinite Lebesgue measure on any complex line where $G^+_{\Psi_\alpha}$ is not identically zero. In particular, the set $\left\{p\in\mathbb{C}^3\mid \lim\limits_{n\to+\infty}\|\Psi_\alpha^n(p)\|=+\infty\right\}$ is of infinite measure. We also exhibit an explicit open set $\Omega\subset\mathcal{E}$.

Theorem C. *Assume $0<|\alpha|<1$. Then Ψ_α has a unique periodic point at finite distance, $0_{\mathbb{C}^3}=(0,0,0)$, which is a saddle point of index 2. The fixed hypersurface $\{z_2=0\}$ attracts any other point. Moreover, the*

set $K^+_{\Psi_\alpha}$ of points with bounded forward orbit is exactly the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, and the latter can be characterized analytically. The set $J^+_{\Psi_\alpha} := \partial K^+_{\Psi_\alpha}$ thus corresponds to $\overline{W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})}$.

We observe a phase transition in the dynamics of the family $\{\Psi_\alpha\}_{0<|\alpha|\leq 1}$ for the value $|\alpha| = \varphi^{(1-q)/d}$:

Theorem D. *Assume $0 < |\alpha| < \varphi^{(1-q)/d}$. The set $\mathcal{V} := \left\{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) = 0\right\}$ is a closed neighborhood of the hyperplane $\{z_2 = 0\}$. It consists in the disjoint union $\Omega'' \sqcup W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, where Ω'' has non-empty interior and the forward orbit of any point $p \in \Omega''$ goes to infinity with Fibonacci speed.*

Note that we also define an analytic function g whose domain of definition is equal to $\mathcal{V}$, and which parametrizes the stable manifold in the sense that $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ coincides with the zero set $\mathcal{Z}$ of g.

Theorem E. *Assume now $\varphi^{(1-q)/d} < |\alpha| < 1$. For any $p \in \mathbb{C}^3$, exactly one of the following cases occurs:*

- *either $p \in W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ and its forward orbit converges to $0_{\mathbb{C}^3}$ exponentially fast;*
- *or $p \in \{z_2 = 0\} \smallsetminus W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ and it goes to infinity with Fibonacci speed;*
- *or the speed explodes: $G^+_{\Psi_\alpha}(p) > 0$.*

In particular, contrary to the previous situation where the set Ω'' has non-empty interior, we see here that Fibonacci speed does not occur outside the hypersurface $\{z_2 = 0\}$.

Remark 1.1. We stress the fact that for any $0 < |\alpha| < 1$, the forward orbit of a point under Ψ_α is bounded if and only if it goes to $0_{\mathbb{C}^3}$. Moreover, we see that in the case the orbit is unbounded, it has to escape to infinity. This rigidity phenomenon is related to the properties of the automorphism of Hénon type ϕ_α to which Ψ_α is semi-conjugate, and which possesses an attractor at infinity such that the positive iterates of any point whose forward orbit is not bounded escape to it.

Theorem F. *Assume $|\alpha| = 1$. We define K_{Ψ_α} to be the set of points $p \in \mathbb{C}^3$ whose orbit $(\Psi^n_\alpha(p))_{n\in\mathbb{Z}}$ is bounded. Similarly to what we did above, we define the Green function $G^-_{\Psi_\alpha}$. Then for any point $p \in \mathbb{C}^3$, exactly one of the following assertions is satisfied:*

- *either the orbit of p is bounded, i.e. $p \in K_{\Psi_\alpha}$;*
- *or $p \in \{z_2 = 0\} \smallsetminus \{0_{\mathbb{C}^3}\}$ and either its forward orbit or its backward orbit escapes to infinity with Fibonacci speed;*
- *or $G^+_{\Psi_\alpha}(p) > 0$ or $G^-_{\Psi_\alpha}(p) > 0$; in this case, either its forward orbit or its backward orbit escapes to infinity with maximal speed.*

We define the associate Green currents $T^\pm_{\Psi_\alpha} := \mathrm{dd^c} G^\pm_{\Psi_\alpha}$, and we set $\mu_{\Psi_\alpha} := T^+_{\Psi_\alpha} \wedge T^-_{\Psi_\alpha} \wedge dz_2 \wedge d\overline{z_2}$. The measure μ_{Ψ_α} is invariant by Ψ_α and supported on the Julia set $J_{\Psi_\alpha} := \partial K_{\Psi_\alpha}$. For any $p_2 \neq 0$, the set $\mathcal{C}_{p_2} := \mathbb{C}^2 \times \left\{p_2 e^{ix} \mid x \in \mathbb{R}\right\}$ is invariant under Ψ_α. Define $\mathcal{J}_{p_2} := J_{\Psi_\alpha} \cap \mathcal{C}_{p_2}$; it is also invariant and we show that when α is not a root of unity, $(\Psi_\alpha|_{\mathcal{J}_{p_2}}, \mu_{\Psi_\alpha})$ is ergodic.

Acknowledgement. The first author would like to thank Artur Avila for helpful and fruitful discussions.

2. Invariant fibrations and degree growths

2.1. **Invariant fibrations.** Let us come back to dimension 2 for a while. As recalled in §1 if f is a polynomial automorphism of $\mathbb{C}^2$, then up to conjugacy either f is an elementary automorphism or f is of Hénon type. In the first case f preserves a rational fibration, whereas in the second one f does not preserve any rational curve ([6]). This gives a geometric criterion to distinguish maps of Hénon type and elementary

maps. What about dimension $n \geq 3$? Contrary to the 2-dimensional case we will see, as soon as $n = 3$, that "no invariant rational curve" does not mean "first dynamical degree > 1".

Assume that $0 < |\alpha| \leq 1$. Let Φ_α be the automorphism of $\mathbb{C}^3$ given by

$$\Phi_\alpha = (\alpha^l(z_0 + z_1 + z_0^q), \alpha^l z_0, \alpha z_2)$$

where $l := \frac{d}{q-1}$. It is possible to show that Ψ_α is conjugate to Φ_α through the *birational map* of $\mathbb{P}^3_{\mathbb{C}}$ given in the affine chart $z_3 = 1$ by

$$\theta = (z_0 z_2^l, z_1 z_2^l, z_2),$$

that is $\theta \circ \Psi_\alpha = \Phi_\alpha \circ \theta$. The advantage is that the action of Φ_α in the fibers is independent of the base point. Moreover, it has a lot of good properties; in particular, we will see that it is algebraically stable (§2.2). Nevertheless θ is birational so we might loose some information (§6).

Proposition 2.1. *For any $0 < |\alpha| \leq 1$, the polynomial automorphism Φ_α preserves a unique rational fibration, the fibration given by $\{z_2 = cst\}$.*

Corollary 2.2. *For any $0 < |\alpha| \leq 1$, the polynomial automorphism Ψ_α preserves a unique rational fibration, the fibration given by $\{z_2 = cst\}$.*

Proof of Proposition 2.1. Note that $\Phi_\alpha = (\phi_\alpha(z_0, z_1), \alpha z_2)$ where $\phi_\alpha \in \mathrm{Aut}(\mathbb{C}^2)$ is the automorphism of Hénon type given by $\phi_\alpha \colon (z_0, z_1) \mapsto \alpha^l(z_1 + z_0 + z_0^q, z_0)$. Since ϕ_α does not preserve rational curves ([6]) the only invariant rational fibration is $\{z_2 = \mathrm{cst}\}$. □

2.2. **Degrees and degree growths.** The results in this part hold for any $\alpha \neq 0$. As we saw in §1 the first dynamical degree is an important invariant; in this section we will thus compute $\lambda(\Psi_\alpha^{\pm 1})$ and $\lambda(\Phi_\alpha^{\pm 1})$. Let us first mention a big difference between dimension 2 and higher dimensions: if f belongs to $\mathrm{Aut}(\mathbb{C}^2)$ then $\deg f = \deg f^{-1}$. This equality does not necessarily hold in higher dimension; nevertheless if f belongs to $\mathrm{Aut}(\mathbb{C}^n)$, then $\deg f \leq (\deg f^{-1})^{n-1}$ and $\deg f^{-1} \leq (\deg f)^{n-1}$ (*see* [22]).

Lemma 2.3. *We have for any $n \geq 0$ both*

$$\deg(\Psi_\alpha^n) = q^n + d \times \frac{q^n - 1}{q - 1}$$

and

$$\deg(\Psi_\alpha^{-n}) = \deg(\Psi_\alpha^n).$$

Proof. Let us denote by $(P_\alpha^{(n)})_{n \geq -1}$ the sequence of polynomials where

$$\begin{cases} P_\alpha^{(-1)}(z_0, z_1, z_2) := z_1 \\ P_\alpha^{(0)}(z_0, z_1, z_2) := z_0 \\ \forall n \geq 0 \quad P_\alpha^{(n+1)} := P_\alpha^{(n)} + P_\alpha^{(n-1)} + (P_\alpha^{(n)})^q (\alpha^n z_2)^d. \end{cases}$$

In particular, for every $n \geq 0$, $\Psi_\alpha^n(z_0, z_1, z_2) = \left(P_\alpha^{(n)}(z_0, z_1, z_2), P_\alpha^{(n-1)}(z_0, z_1, z_2), \alpha^n z_2\right)$. Since the degree of the third component does not change, and the second component is just the first one at time $n-1$, the growth of the degree is supported by the first component, that is $\deg(\Psi_\alpha^n) = \deg(P_\alpha^{(n)})$. Let us then show the result by induction on n. The result is true for $n = 0$. If it holds for $n \geq 0$, then we have

$$\deg(P_\alpha^{(n+1)}) = \deg\left((P_\alpha^{(n)})^q (\alpha^n z_2)^d\right) = q\left(q^n + d \times \frac{q^n - 1}{q - 1}\right) + d = q^{n+1} + d \times \frac{q^{n+1} - 1}{q - 1}.$$

□

Since the degree is not invariant under conjugacy, [20, 10] introduce the first dynamical degree. If f is a polynomial automorphism of $\mathbb{C}^3$, the first dynamical degree of f is defined by

$$\lambda(f) = \lim_{n\to+\infty} (\deg f^n)^{1/n}.$$

It satisfies the following inequalities: $1 \leq \lambda(f) \leq \deg f$.

Corollary 2.4. *Since $q^n \leq \deg(\Psi_\alpha^n) \leq (d+1)q^n$, it follows that*

$$\lambda(\Psi_\alpha) = \lambda(\Psi_\alpha^{-1}) = q.$$

To any polynomial automorphism $f = (f_0, f_1, f_2)$ of $\mathbb{C}^3$ of degree d one can associate a birational self map of $\mathbb{P}^3_{\mathbb{C}}$ as follows

$$(z_0 : z_1 : z_2 : z_3) \dashrightarrow \left(z_3^d f_0\left(\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}\right) : z_3^d f_1\left(\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}\right) : z_3^d f_2\left(\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}\right) : z_3^d \right);$$

we still denote it by f.

If $g = (g_0 : g_1 : g_2 : g_3)$ is a birational self map of $\mathbb{P}^3_{\mathbb{C}}$, the *indeterminacy set* $\mathrm{Ind}(g)$ of g is the set where g is not defined, that is

$$\mathrm{Ind}(g) = \left\{ m \in \mathbb{P}^3_{\mathbb{C}} \,|\, g_0(m) = g_1(m) = g_2(m) = g_3(m) = 0 \right\}.$$

Remark that if we look at a birational map of $\mathbb{P}^3_{\mathbb{C}}$ that comes from a polynomial automorphism of $\mathbb{C}^3$ then its indeterminacy set is contained in $\{z_3 = 0\}$.

A polynomial automorphism of $\mathbb{C}^3$ is *algebraically stable* if for every $n \in \mathbb{N}$,

$$f^n\big(\{z_3 = 0\} \smallsetminus \mathrm{Ind}(f^n)\big) \not\subset \mathrm{Ind}(f).$$

Let us recall the following result.

Proposition 2.5 ([9])**.** *The map f is algebraically stable if and only if $\deg(f^n) = (\deg(f))^n$ for every $n \geq 1$.*

Lemma 2.3 and Proposition 2.5 imply that Ψ_α is not algebraically stable, as well as Ψ_α^{-1}. It can also be seen directly from the definition. Indeed the map

$$\Psi_\alpha = \big((z_0 + z_1)z_3^{q+d-1} + z_0^q z_2^d : z_0 z_3^{q+d-1} : \alpha z_2 z_3^{q+d-1} : z_3^{q+d}\big)$$

sends $z_3 = 0$ onto $(1:0:0:0)$ and $\mathrm{Ind}(\Psi_\alpha) = \{z_0 = 0,\ z_3 = 0\} \cup \{z_2 = 0,\ z_3 = 0\}$. Similarly, we see that

$$\mathrm{Ind}(\Psi_\alpha^{-1}) = \{z_1 = 0,\ z_3 = 0\} \cup \{z_2 = 0,\ z_3 = 0\},$$

and Ψ_α^{-1} sends $z_3 = 0$ onto $(0:1:0:0) \in \mathrm{Ind}(\Psi_\alpha^{-1})$. On the other hand $\Phi_\alpha(\{z_0 \neq 0,\ z_3 = 0\}) = (1:0:0:0)$ does not belong to $\mathrm{Ind}(\Phi_\alpha)$ and $(1:0:0:0)$ is a fixed point of Φ_α hence $\Phi_\alpha^n(\{z_0 \neq 0, z_3 = 0\}) = (1:0:0:0)$ for any $n \geq 1$. In particular, Φ_α is algebraically stable. We have for every $n \geq 0$, $\deg(\Phi_\alpha^n) = q^n$. Notice that for $n \geq 3$, there exist examples of maps $f \in \mathrm{Aut}(\mathbb{C}^3)$ which are algebraically stable but whose inverse f^{-1} is not algebraically stable (let us mention the following example due to Guedj: $f = (z_0^2 + \lambda z_1 + a z_2, \lambda^{-1} z_0^2 + z_1, z_0)$ with a and λ in $\mathbb{C}^*$). Yet this is not the case for Φ_α. Indeed,

$$\Phi_\alpha^{-1}(z_0 : z_1 : z_2 : z_3) = \left(\frac{z_1 z_3^{q-1}}{\alpha^l} : -\frac{z_1 z_3^{q-1}}{\alpha^l} + \frac{z_0 z_3^{q-1}}{\alpha^l} - \frac{z_1^q}{\alpha^{lq}} : \frac{z_2 z_3^{q-1}}{\alpha} : z_3^q \right)$$

so $\Phi_\alpha^{-1}(\{z_1 \neq 0,\ z_3 = 0\}) = (0:1:0:0)$ does not belong to $\mathrm{Ind}(\Phi_\alpha^{-1}) = \{z_1 = 0, z_3 = 0\}$ and is fixed by Φ_α^{-1}. Hence Φ_α^{-1} is also algebraically stable. As a result one can state:

Proposition 2.6. *For any integer $n \geq 1$ the following equalities hold*

$$\deg(\Phi_\alpha^n) = \deg(\Phi_\alpha^{-n}) = q^n, \qquad \lambda(\Phi_\alpha) = \lambda(\Phi_\alpha^{-1}) = q.$$

3. Centralizers

If G is a group and f an element of G, we denote by $\mathrm{Cent}(f, \mathrm{G})$ the centralizer of f in G, that is

$$\mathrm{Cent}(f, \mathrm{G}) := \{g \in \mathrm{G} \,|\, fg = gf\}.$$

The description of centralizers of discrete dynamical systems is an important problem in real and complex dynamics: Julia ([16]) and Ritt ([19]) showed that the centralizer of a rational function f of $\mathbb{P}^1$ is in general the set of iterates of f (we then say that the centralizer of f is trivial) except for some very special f. Later Smale asked if the centralizer of a generic diffeomorphism of a compact manifold is trivial ([23]). Since then a lot of mathematicians have looked at this question in different contexts; for instance as recalled in §1 Lamy has proved that the centralizer of a polynomial automorphism of $\mathbb{C}^2$ of Hénon type is in general trivial ([18]).

Fix α with $0 < |\alpha| \leq 1$. We would like to describe $\mathrm{Cent}(\Phi_\alpha, \mathrm{Aut}(\mathbb{C}^3))$. Of course it contains $\{\Phi_\alpha^n \,|\, n \in \mathbb{Z}\}$ but also the following one-parameter family

$$\{(\eta z_0, \eta z_1, \nu z_2) \,|\, \nu \in \mathbb{C}^*, \eta \text{ a } (q-1)\text{-th root of unity}\}.$$

We show that the centralizer is essentially reduced to the iterates of Φ_α and such maps. Since the automorphism $\phi_\alpha = \alpha^l(z_1 + z_0 + z_0^q, z_0)$ is of Hénon type, it follows from a result of Lamy [18] that $\mathrm{Cent}(\phi_\alpha, \mathrm{Aut}(\mathbb{C}^2)) \simeq \mathbb{Z} \rtimes \mathbb{Z}_n$ for some $n \in \mathbb{N}$.

Let $f \in \mathrm{Cent}(\Phi_\alpha, \mathrm{Aut}(\mathbb{C}^3))$; we write $f = (f_0, f_1, f_2)$.

Lemma 3.1. *We have $\frac{\partial f_2}{\partial z_0} = 0$, $\frac{\partial f_2}{\partial z_1} = 0$. Therefore, the last component f_2 only depends on z_2, and in fact it is a homothety:*

$$f_2(z_0, z_1, z_2) = f_2(z_2) = \mu z_2, \qquad \mu \in \mathbb{C}^*.$$

Proof. If we focus on the third coordinate in relation $\Phi_\alpha \circ f = f \circ \Phi_\alpha$, we get $\alpha f_2 = f_2 \circ \Phi_\alpha$, that is, for every $(z_0, z_1, z_2) \in \mathbb{C}^3$,

$$\alpha f_2(z_0, z_1, z_2) = f_2(\alpha^l z_0 + \alpha^l z_1 + \alpha^l z_0^q, \alpha^l z_0, \alpha z_2).$$

Taking the derivatives in the different coordinates, we obtain:

$$\left\{ \begin{array}{lcl} \alpha \frac{\partial f_2}{\partial z_0} & = & \alpha^l (1 + q z_0^{q-1}) \frac{\partial f_2}{\partial z_0} \circ \Phi_\alpha + \alpha^l \frac{\partial f_2}{\partial z_1} \circ \Phi_\alpha, \\ \alpha \frac{\partial f_2}{\partial z_1} & = & \alpha^l \frac{\partial f_2}{\partial z_0} \circ \Phi_\alpha, \\ \alpha \frac{\partial f_2}{\partial z_2} & = & \alpha \frac{\partial f_2}{\partial z_2} \circ \Phi_\alpha. \end{array} \right. \tag{3.1}$$

Let us consider the first coordinate, and assume that $\frac{\partial f_2}{\partial z_0} \neq 0$; we will get a contradiction by looking at highest-order terms in z_0. Since $f_2 \in \mathbb{C}[z_1, z_2][z_0]$, we can write $f_2(z_0, z_1, z_2) = \sum_{k \leq k_0} R_k(z_1, z_2) z_0^k$, where the R_k are polynomials and k_0 is the degree in z_0 of f_2. From our hypothesis, $k_0 \geq 1$. We also look at the expansion of $R_{k_0} \neq 0$:

$$R_{k_0}(z_1, z_2) = \sum_{m \leq m_0} Q_m(z_2) z_1^m, \quad Q_{m_0} \neq 0.$$

For the three terms, we look at the term of highest order in z_0:

$$\left\{\begin{array}{rcl} \alpha\frac{\partial f_2}{\partial z_0}(z_0,z_1,z_2) & = & \alpha k_0 R_{k_0}(z_1,z_2)z_0^{k_0-1}+\dots \\ \alpha^l(1+qz_0^{q-1})\frac{\partial f_2}{\partial z_0}\circ\Phi_\alpha(z_0,z_1,z_2) & = & qk_0\alpha^{l(k_0+m_0)}Q_{m_0}(\alpha z_2)z_0^{qk_0+m_0-1}+\dots \\ \alpha^l\frac{\partial f_2}{\partial z_1}\circ\Phi_\alpha(z_0,z_1,z_2) & = & m_0\alpha^{l(k_0+m_0)}Q_{m_0}(\alpha z_2)z_0^{qk_0+m_0-1}+\dots \end{array}\right.$$

Since we assume $k_0\geq 1$, and $q>1$, we have $qk_0+m_0-1>k_0-1$ so the coefficient of the term in $z_0^{qk_0+m_0-1}$ must vanish. But this coefficient is $(qk_0+m_0)\alpha^{l(k_0+m_0)}Q_{m_0}(\alpha z_2)\neq 0$, a contradiction. Hence $\frac{\partial f_2}{\partial z_0}=0$, and it follows from the second equation of (3.1) that $\frac{\partial f_2}{\partial z_1}=0$ as well. Therefore, $f_2=f_2(z_2)$.

Now, since $f\in\mathrm{Aut}(\mathbb{C}^3)$, we know that f_2 is of degree at most 1. The map f commutes with Φ_α, so it must preserve its fixed point $0_{\mathbb{C}^3}$, and we conclude that $f_2\colon z_2\mapsto\mu z_2$ for some $\mu\in\mathbb{C}^*$. □

Recall that $\phi_\alpha\colon(z_0,z_1)\mapsto\alpha^l(z_1+z_0+z_0^q,z_0)$. Let us denote $\widetilde{f}:=(f_0,f_1)$. By projecting the commutation relation on the first two coordinates, we get

$$\phi_\alpha\circ\widetilde{f}=\widetilde{f}\circ\Phi_\alpha. \tag{3.2}$$

Lemma 3.2. *The map $\widetilde{f}$ only depends on the first two coordinates.*

Proof. We rewrite (3.2) as the following system:

$$\left\{\begin{array}{rcl} \alpha^l f_0+\alpha^l f_1+\alpha^l f_0^q & = & f_0\circ\Phi_\alpha \\ \alpha^l f_0 & = & f_1\circ\Phi_\alpha. \end{array}\right. \tag{3.3}$$

We then get:

$$\alpha^l f_0\circ\Phi_\alpha+\alpha^{2l}f_0+\alpha^l f_0^q\circ\Phi_\alpha=f_0\circ\Phi_\alpha^2.$$

Let d_0 be the degree of $f_0\in\mathbb{C}[z_0,z_1][z_2]$. Since Φ_α does not change the degree in z_2, we obtain

$$\deg(\alpha^l f_0\circ\Phi_\alpha)=\deg(\alpha^{2l}f_0)=\deg(f_0\circ\Phi_\alpha^2)=d_0,\quad \deg(\alpha^l f_0^q\circ\Phi_\alpha)=d_0^q,$$

but $q>1$, which implies that $d_0=0$: f_0 does not depend on z_2. Using the second equation of (3.3), we see that f_1 does not depend on z_2 either. □

Therefore, Equation (3.2) can be rewritten:

$$\phi_\alpha\circ\widetilde{f}=\widetilde{f}\circ\phi_\alpha.$$

But ϕ_α is a Hénon automorphism, so according to [18] one has:

Corollary 3.3. *The map $\widetilde{f}$ belongs to the countable set* $\mathrm{Cent}(\phi_\alpha,\mathrm{Aut}(\mathbb{C}^2))\simeq\mathbb{Z}\rtimes\mathbb{Z}_n$, $n\in\mathbb{N}$.

We have seen that for any $f=(f_0,f_1,f_2)\in\mathrm{Cent}(\Phi_\alpha,\mathrm{Aut}(\mathbb{C}^3))$, (f_0,f_1) depends only on (z_0,z_1) and belongs to $\mathrm{Cent}(\phi_\alpha,\mathrm{Aut}(\mathbb{C}^2))$, and that f_2 depends only on z_2 and is a homothety. We conclude:

Proposition 3.4. *The centralizer of Φ_α in $\mathrm{Aut}(\mathbb{C}^3)$ is uncountable. More precisely*

$$\mathrm{Cent}(\Phi_\alpha,\mathrm{Aut}(\mathbb{C}^3))=\mathrm{Cent}(\phi_\alpha,\mathrm{Aut}(\mathbb{C}^2))\times\left\{z_2\mapsto\mu z_2\mid\mu\in\mathbb{C}^*\right\}\simeq(\mathbb{Z}\rtimes\mathbb{Z}_n)\times\mathbb{C}^*,\ n\in\mathbb{N}.$$

Corollary 3.5. *The centralizer of Ψ_α in $\mathrm{Aut}(\mathbb{C}^3)$ is uncountable.*

4. Dynamics on the invariant hypersurface $z_2 = 0$

The following holds for any $\alpha \neq 0$. Let us recall that the Fibonacci sequence is the sequence $(F_n)_n$ defined by: $F_0 = 0$, $F_1 = 1$ and for all $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}.$$

The hypersurface $\{z_2 = 0\}$ is invariant, and when $|\alpha| < 1$, it attracts every point $p \in \mathbb{C}^3$. On restriction to this hypersurface, the growth is given by the Fibonacci numbers $(F_n)_n$:

$$\Psi_\alpha^n|_{z_2=0} = (F_{n+1}z_0 + F_n z_1, F_n z_0 + F_{n-1}z_1), \quad n \geq 1. \tag{4.1}$$

Since

$$\Psi_\alpha^{-1}(z_0, z_1, z_2) = \left(z_1, -z_1 + z_0 - z_1^q \frac{z_2^d}{\alpha^d}, \frac{z_2}{\alpha}\right),$$

similarly, we have

$$\Psi_\alpha^{-n}|_{z_2=0} \colon (z_0, z_1) \mapsto (-1)^n (F_{n-1}z_0 - F_n z_1, -F_n z_0 + F_{n+1}z_1), \quad n \geq 1. \tag{4.2}$$

Moreover, it is easy to see that any periodic point of Ψ_α belongs to the hypersurface $\{z_2 = 0\}$. In fact, Ψ_α has a unique fixed point at finite distance, $0_{\mathbb{C}^3} = (0,0,0)$, and has no periodic point of period larger than 1. Let $\varphi := \frac{1+\sqrt{5}}{2}$ be the golden ratio and $\varphi' := -1/\varphi$. Since

$$F_n = \frac{\varphi^n - (\varphi')^n}{\sqrt{5}} = \frac{\varphi^n}{\sqrt{5}} + o(1),$$

we deduce from (4.1) that any point $(\varphi' z, z, 0)$ with $z \in \mathbb{C}$ converges to $0_{\mathbb{C}^3}$ when we iterate Ψ_α, while any other point of the form $(\beta z, z, 0)$ with $z \neq 0$ and $\beta \neq \varphi'$ goes to infinity. Likewise, we see from (4.2) that any point $(\varphi z, z, 0)$ with $z \in \mathbb{C}$ converges to $0_{\mathbb{C}^3}$ when we iterate Ψ_α^{-1}, while any other point of the form $(\beta z, z, 0)$ with $z \neq 0$ and $\beta \neq \varphi$ goes to infinity. Furthermore, in both cases, the speed of the convergence is exponential since it is in $O(|\varphi|^{-n})$ with $|\varphi| > 1$. In other terms, the linear map $\Psi_\alpha|_{z_2=0} \colon (z_0, z_1) \mapsto (z_0 + z_1, z_0)$ is hyperbolic, with a unique fixed point $0_{\mathbb{C}^2} = (0,0)$ of saddle type, and whose stable, respectively unstable manifolds correspond to the following lines:

$$W^s_{\Psi_\alpha|_{z_2=0}}(0_{\mathbb{C}^2}) = \Delta_{\varphi'} := \left\{ (\varphi' z, z) \,\middle|\, z \in \mathbb{C} \right\}, \quad W^u_{\Psi_\alpha|_{z_2=0}}(0_{\mathbb{C}^2}) = \Delta_\varphi := \left\{ (\varphi z, z) \,\middle|\, z \in \mathbb{C} \right\}.$$

Moreover, φ and φ' are just the eigenvalues of $\Psi_\alpha|_{z_2=0}$, and Δ_φ, $\Delta_{\varphi'}$ the corresponding eigenspaces.

5. Points with bounded forward orbit, description of the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$

When $0 < |\alpha| < 1$, we remark that $0_{\mathbb{C}^3}$ is a hyperbolic fixed point of saddle type. The tangent space at $0_{\mathbb{C}^3}$ can be written as $T_{0_{\mathbb{C}^3}}(\mathbb{C}^3) = E^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) \oplus E^u_{\Psi_\alpha}(0_{\mathbb{C}^3})$, where the stable, respectively unstable spaces are given by

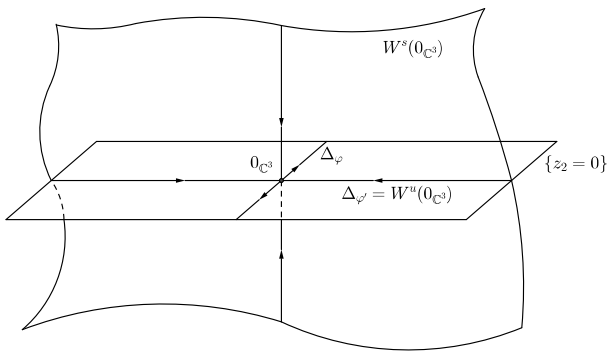
$$E^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) = \Delta_{\varphi'} \times \{0\} \oplus \{0_{\mathbb{C}^2}\} \times \mathbb{C}, \qquad E^u_{\Psi_\alpha}(0_{\mathbb{C}^3}) = \Delta_\varphi \times \{0\}.$$

These spaces integrate to stable and unstable manifolds

$$W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) := \left\{ p \in \mathbb{C}^3 \mid \lim_{n\to+\infty} \Psi_\alpha^n(p) = 0_{\mathbb{C}^3} \right\}, \quad W^u_{\Psi_\alpha}(0_{\mathbb{C}^3}) := \left\{ p \in \mathbb{C}^3 \mid \lim_{n\to+\infty} \Psi_\alpha^{-n}(p) = 0_{\mathbb{C}^3} \right\}$$

which are invariant by the dynamics; furthermore, $W^u_{\Psi_\alpha}(0_{\mathbb{C}^3}) = \Delta_\varphi \times \{0\}$, while $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ is biholomorphic to $\mathbb{C}^2$ (see [22]). Note that $\left(\Delta_{\varphi'} \times \{0\}\right) \cup \left(\{0_{\mathbb{C}^2}\} \times \mathbb{C}\right) \subset W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, but it is easy to see [1] that $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) \neq \Delta_{\varphi'} \times \mathbb{C}$.

1. Indeed if $p = (p_0, p_1, p_2)$ satisfies $p_2 \neq 0$ and $p_1 = -\varphi p_0$, we see that $P_\alpha^{(0)}(p) + \varphi P_\alpha^{(1)}(p) = p_0(1 + \varphi - \varphi^2) + \varphi p_0^q p_2^d = \varphi p_0^q p_2^d \neq 0$ hence $\Delta_{\varphi'} \times \mathbb{C}$ is not left invariant by Ψ_α.



In the next statement, we introduce a series that encodes the growth of forward iterates of a point.

Lemma 5.1. *Let $p = (p_0, p_1, p_2) \in \mathbb{C}^3$. For every $n \geq 0$ and any $\alpha \in \mathbb{C}$, we have*

$$P_\alpha^{(n+1)}(p) + \varphi^{-1} P_\alpha^{(n)}(p) = \varphi^n \left(\varphi p_0 + p_1 + p_2^d \sum_{j=0}^{n} \left(P_\alpha^{(j)}(p) \right)^q \varphi^{-j} \alpha^{jd} \right). \tag{5.1}$$

Proof. For $n \geq 0$ we have the following set of equalities:

$$\begin{array}{lrcccccl}
 & P_\alpha^{(n+1)}(p) & = & P_\alpha^{(n)}(p) & + & P_\alpha^{(n-1)}(p) & + & (P_\alpha^{(n)}(p))^q (\alpha^n p_2)^d \\
(\times\, \varphi) & P_\alpha^{(n)}(p) & = & P_\alpha^{(n-1)}(p) & + & P_\alpha^{(n-2)}(p) & + & (P_\alpha^{(n-1)}(p))^q (\alpha^{n-1} p_2)^d \\
(\times\, \varphi^2) & P_\alpha^{(n-1)}(p) & = & P_\alpha^{(n-2)}(p) & + & P_\alpha^{(n-3)}(p) & + & (P_\alpha^{(n-2)}(p))^q (\alpha^{n-2} p_2)^d \\
 & \vdots & = & \vdots & + & \vdots & + & \vdots \\
(\times\, \varphi^{n-1}) & P_\alpha^{(2)}(p) & = & P_\alpha^{(1)}(p) & + & p_0 & + & (P_\alpha^{(1)}(p))^q (\alpha p_2)^d \\
(\times\, \varphi^n) & P_\alpha^{(1)}(p) & = & p_0 & + & p_1 & + & (P_\alpha^{(0)}(p))^q (p_2)^d
\end{array}$$

Summing up, and because $\varphi^2 - \varphi - 1 = 0$, we obtain

$$P_\alpha^{(n+1)}(p) + \varphi^{-1} P_\alpha^{(n)}(p) = \varphi^n \left(\varphi p_0 + p_1 + p_2^d \sum_{j=0}^{n} \left(P_\alpha^{(j)}(p) \right)^q \varphi^{-j} \alpha^{jd} \right).$$

□

For every $n \geq -1$, we define the polynomial $g_n \in \mathbb{C}[z] = \mathbb{C}[z_0, z_1, z_2]$ by

$$g_n(z) := \varphi z_0 + z_1 + z_2^d \sum_{j=0}^{n} \left(P_\alpha^{(j)}(z) \right)^q \varphi^{-j} \alpha^{jd} = \left(P_\alpha^{(n+1)}(z) + \varphi^{-1} P_\alpha^{(n)}(z) \right) \varphi^{-n}.$$

We also introduce the power series

$$g(z) := \varphi z_0 + z_1 + z_2^d \sum_{j=0}^{+\infty} \left(P_\alpha^{(j)}(z) \right)^q \varphi^{-j} \alpha^{jd} = \varphi z_0 + z_1 + \sum_{j=-1}^{+\infty} \varphi^{-(j+1)} \left(P_\alpha^{(j+2)}(z) - P_\alpha^{(j+1)}(z) - P_\alpha^{(j)}(z) \right).$$

Let us denote by $\mathcal{D}$ its domain of definition, that is the set of $p \in \mathbb{C}^3$ such that the series $\sum_j \left(P_\alpha^{(j)}(p)\right)^q \varphi^{-j}\alpha^{jd}$ converges, and let

$$\mathcal{Z} := \left\{p \in \mathcal{D} \mid g(p) = 0\right\}$$

be the set of its zeroes. It is easy to check that both $\mathcal{D}$ and $\mathcal{Z}$ are invariant by the dynamics, that is $\Psi_\alpha(\mathcal{D}) \subset \mathcal{D}$ and $\Psi_\alpha(\mathcal{Z}) \subset \mathcal{Z}$. Moreover, if $p \in \mathcal{D}$, we denote by $r_n(p) := \sum_{j \geq n+1} \left(P_\alpha^{(j)}(p)\right)^q \varphi^{-j}\alpha^{jd}$ the tail of the corresponding series.

Corollary 5.2. *Suppose $0 < |\alpha| \leq 1$. Let $K^+_{\Psi_\alpha}$ denote the set of points $p = (p_0, p_1, p_2)$ whose forward orbit $\{\Psi^n_\alpha(p),\ n \geq 0\}$ is bounded. This is equivalent to the fact that the sequence $\left(|P_\alpha^{(n)}(p)|\right)_{n\geq 0}$ is bounded. If $p \in K^+_{\Psi_\alpha}$, then for every $n \geq 0$, we have*

$$|g_n(p)| = O(\varphi^{-n}). \tag{5.2}$$

In particular we deduce that

$$K^+_{\Psi_\alpha} \subset \mathcal{Z}, \quad \text{and} \quad |r_n(p)| = O(\varphi^{-n}). \tag{5.3}$$

Proof. It follows immediately from Lemma 5.1. Indeed under our assumptions we have:

$$|g_n(p)| \leq (|P_\alpha^{(n+1)}(p)| + \varphi^{-1}|P_\alpha^{(n)}(p)|)\varphi^{-n} = O(\varphi^{-n}).$$

This implies $p \in \mathcal{Z}$. Then we also have $g_n(p) = g(p) - p_2^d r_n(p) = -p_2^d r_n(p)$ and $|r_n(p)| = O(\varphi^{-n})$. □

Remark 5.3. We can see (5.3) as a codimension one condition that points with bounded forward orbit have to satisfy. Also, we see from (5.2) that locally, such points are close to the analytic manifold $\mathcal{Z}_n := \left\{p \in \mathbb{C}^3 \mid g_n(p) = 0\right\}$ for $n \geq 0$ big. If $p = (p_0, p_1, 0)$, we recover from (5.3) that p has bounded forward orbit if and only if it belongs to the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) \cap \{z_2 = 0\} = \Delta_{\varphi'} \times \{0\}$.

When $0 < |\alpha| < 1$, we have the following analytic characterization of the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$.

Proposition 5.4. *Assume $0 < |\alpha| < 1$. The point $p = (p_0, p_1, p_2) \in \mathbb{C}^3$ belongs to the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ if and only if the following properties hold:*

- *$p \in \mathcal{Z}$;*
- *the series $\sum_j |r_j(p)|\varphi^j$ is convergent.*

Equivalently, $p \in W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ if and only if $\sum_j |g_j(p)|\varphi^j$ converges.

Proof. If p belongs to the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, then its forward orbit is bounded and Corollary 5.2 tells us that $p \in \mathcal{Z}$. Moreover we have

$$|r_n(p)| = O\left(\sum_{j\geq n+1} \varphi^{-j}\alpha^{jd}\right) = O(\varphi^{-n}|\alpha|^{nd}),$$

hence $|r_n(p)|\varphi^n = O(|\alpha|^{nd})$ and the series $\sum_j |r_j(p)|\varphi^j$ converges.

For the other implication, we get from Lemma 5.1 that for every $j \geq 0$,

$$P_\alpha^{(j+1)}(p) + \varphi^{-1}P_\alpha^{(j)}(p) = -p_2^d\varphi^j r_j(p). \tag{5.4}$$

Now let $n \geq 0$. Write equations (5.4) for $j = 0, \ldots, n$ and combine them to obtain

$$P_\alpha^{(n+1)}(p) = \frac{(-1)^{n+1}}{\varphi^{n+1}}p_0 + p_2^d \sum_{j=0}^{n} (-1)^{n+j+1} r_j(p)\varphi^j\varphi^{j-n}.$$

The first term of the right hand side goes to 0 with n; we split the sum as follows:

$$\sum_{j=0}^{n}(-1)^{n+j+1}r_j(p)\varphi^j\varphi^{j-n} = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{n+j+1}r_j(p)\varphi^j\varphi^{j-n} + \sum_{j=\lfloor n/2\rfloor+1}^{n}(-1)^{n+j+1}r_j(p)\varphi^j\varphi^{j-n}.$$

We get

$$\left|\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{n+j+1}r_j(p)\varphi^j\varphi^{j-n}\right| \leq \varphi^{\lfloor n/2\rfloor-n}\sum_{j=0}^{+\infty}|r_j(p)|\varphi^j$$

hence it goes to 0 with respect to n. For the remaining term, we estimate

$$\left|\sum_{j=\lfloor n/2\rfloor+1}^{n}(-1)^{n+j+1}r_j(p)\varphi^j\varphi^{j-n}\right| \leq \sum_{j=\lfloor n/2\rfloor+1}^{+\infty}|r_j(p)|\varphi^j,$$

which goes to 0 as well. We conclude that $\lim_{n\to+\infty}P_\alpha^{(n)}(p)=0$, hence $p\in W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$.

The other equivalence follows from the fact that for $p\in\mathcal{Z}$, we have $g_n(p)=-p_2^d r_n(p)$. □

Corollary 5.5. *Assume $0<|\alpha|<1$. Then the forward orbit of a point $p=(p_0,p_1,p_2)$ is bounded if and only if p belongs to the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$; in other terms, $K^+_{\Psi_\alpha}=W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$.*

Proof. Let $p\in K^+_{\Psi_\alpha}$. We have already seen in Corollary 5.2 that $p\in\mathcal{Z}$. Moreover, if we denote $r_n(p):=\sum_{j\geq n+1}(P_\alpha^{(j)}(p))^q\varphi^{-j}\alpha^{jd}$, then $\sum_j|r_j(p)|\varphi^j$ is convergent since $|r_j(p)|=O(\varphi^{-j}|\alpha|^{jd})$. Then, Proposition 5.4 tells us that $p\in W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$. The other implication is straightforward. □

Lemma 5.6. *Assume now $|\alpha|=1$. We have seen that $p\in K^+_{\Psi_\alpha}$ implies that $p\in\mathcal{Z}$ and $|r_n(p)|=O(\varphi^{-n})$. Conversely, if $p\in\mathcal{Z}$ and $\sum_j|r_j(p)|\varphi^j$ is convergent, then $p\in K^+_{\Psi_\alpha}$.*

Proof. Assume that $p\in\mathcal{Z}$ and that $\sum_j|r_j(p)|\varphi^j$ converges. As previously, we have for every $n\geq 0$:

$$P_\alpha^{(n)}(p)=\frac{(-1)^n}{\varphi^n}p_0+p_2^d\sum_{j=0}^{n-1}(-1)^{n+j}r_j(p)\varphi^j\varphi^{j-(n-1)}.$$

From our assumption we deduce that $|P_\alpha^{(n)}(p)|\leq|p_0|+|p_2|^d\sum_{j=0}^{+\infty}|r_j(p)|\varphi^j$, hence $|P_\alpha^{(n)}(p)|=O(1)$. □

6. Birational conjugacy, dynamical properties of automorphisms of Hénon type

Assume $0<|\alpha|\leq 1$. We recall here some facts concerning the dynamics of the Hénon automorphism ϕ_α, and so on the dynamics of $\Phi_\alpha=(\phi_\alpha,\alpha z_2)$. Denote by $F^+_{\phi_\alpha}$ the largest open set on which $(\phi_\alpha^n)_n$ is locally equicontinuous, by $K^+_{\phi_\alpha}$ the set of points $p\in\mathbb{C}^2$ such that $(\phi_\alpha^n(p))_{n\geq 0}$ is bounded and by $J^+_{\phi_\alpha}$ its topological boundary. The automorphism ϕ_α is *regular*, that is, $\mathrm{Ind}(\phi_\alpha)\cap\mathrm{Ind}(\phi_\alpha^{-1})=\emptyset$. In particular, it is algebraically stable, hence the following limit exists and defines a Green function:

$$G^+_{\phi_\alpha}(z_0,z_1):=\lim_{n\to+\infty}\frac{\log^+||\phi_\alpha^n(z_0,z_1)||}{q^n}.$$

It satisfies the invariance property $G^+_{\phi_\alpha}\circ\phi_\alpha = q\cdot G^+_{\phi_\alpha}$. We define the associate current $T^+_{\phi_\alpha} = \mathrm{dd}^c G^+_{\phi_\alpha}$, where $\mathrm{d}^c = \frac{\mathbf{i}(\bar\partial-\partial)}{2\pi}$. Of course there are similar objects $F^-_{\phi_\alpha}$, $K^-_{\phi_\alpha}$, $J^-_{\phi_\alpha}$, $G^-_{\phi_\alpha}$ and $T^-_{\phi_\alpha}$ associated to the inverse map ϕ_α^{-1}; we also set $K_{\phi_\alpha} := K^+_{\phi_\alpha}\cap K^-_{\phi_\alpha}$. One inherits a probability measure $\mu_{\phi_\alpha} = T^+_{\phi_\alpha}\wedge T^-_{\phi_\alpha}$ which is invariant by ϕ_α. Set $\phi := \phi_1$. If $|\alpha|>1$, then $\phi_\alpha = \alpha^l\phi = ((\alpha^{-1})^l\phi^{-1})^{-1}$, where $\alpha^{-1}<1$ and ϕ^{-1} is of the same form as ϕ, so that we can restrict ourselves to the case where $|\alpha|\leq 1$. Besides, according to [3, 9, 4, 1, 2] the following properties hold: for $0<\alpha\leq 1$,

- the function $G^+_{\phi_\alpha}$ is Hölder continuous;
- we have the following characterization of points with bounded orbit:
$$K^+_{\phi_\alpha} = \left\{p\in\mathbb{C}^2 \,|\, G^+_{\phi_\alpha}(p)=0\right\}; \tag{6.1}$$
this tells us that points either have bounded foward orbit, or escape to infinity with maximal speed;
- let p be a saddle point of ϕ_α, then $J^+_{\phi_\alpha}$ is the closure of the stable manifold $W^s_{\phi_\alpha}(p)$;
- the support of $T^+_{\phi_\alpha}$ coincides with the boundary of $K^+_{\phi_\alpha}$, that is $J^+_{\phi_\alpha}$;
- the current $T^+_{\phi_\alpha}$ is extremal among positive closed currents in $\mathbb{C}^2$ and is – up to a multiplicative constant – the unique positive closed current supported on $K^+_{\phi_\alpha}$;
- the measure μ_{ϕ_α} has support in the compact set ∂K_{ϕ_α}, is mixing, maximises entropy and is well approximated by Dirac masses at saddle points.

One introduces analogous objects for the automorphism Φ_α. In particular, Φ_α is algebraically stable so we can also define the Green function
$$G^+_{\Phi_\alpha} := \lim_{n\to+\infty}\frac{\log^+||\Phi^n_\alpha||}{q^n}.$$
In fact, for any $(z_0,z_1,z_2)\in\mathbb{C}^3$, $G^+_{\Phi_\alpha}(z_0,z_1,z_2) = G^+_{\phi_\alpha}(z_0,z_1)$ because if $z_2\neq 0$, $\lim\limits_{n\to+\infty}\frac{\log|\alpha^n z_2|}{q^n}=0$. We introduce the holomorphic map
$$h\colon \mathbb{C}^3\to\mathbb{C}^2, \qquad h\colon (z_0,z_1,z_2)\mapsto (z_0z_2^l, z_1z_2^l).$$
It follows from the previous remark that $G^+_{\Phi_\alpha}\circ\theta = G^+_{\phi_\alpha}\circ h$ where $\theta = (z_0z_2^l, z_1z_2^l, z_2)$ conjugates Φ_α to Ψ_α (*i.e.* $\theta\Psi_\alpha = \Phi_\alpha\theta$). Moreover, for $0<|\alpha|<1$, one gets that
$$K^+_{\Phi_\alpha} = K^+_{\phi_\alpha}\times\mathbb{C}, \qquad K^-_{\Phi_\alpha} = K^-_{\phi_\alpha}\times\{0\}, \qquad K_{\Phi_\alpha} = K_{\phi_\alpha}\times\{0\},$$
while for $|\alpha|=1$,
$$K^+_{\Phi_\alpha} = K^+_{\phi_\alpha}\times\mathbb{C}, \qquad K^-_{\Phi_\alpha} = K^-_{\phi_\alpha}\times\{\mathbb{C}\}, \qquad K_{\Phi_\alpha} = K_{\phi_\alpha}\times\{\mathbb{C}\}.$$

When f is an algebraically stable polynomial automorphism of $\mathbb{C}^3$ one can inductively define the analytic sets $X_j(f)$ by
$$\begin{cases} X_1(f) = \overline{f\big(\{z_3=0\}\smallsetminus \mathrm{Ind}(f)\big)} \\ X_{j+1}(f) = \overline{f\big(X_j(f)\smallsetminus \mathrm{Ind}(f)\big)} \quad \forall\, j\geq 1\end{cases}$$
The sequence $(X_j(f))_j$ is decreasing, $X_j(f)$ is non-empty since f is algebraically stable, so it is stationary. Denote by $X(f)$ the corresponding limit set. An algebraically stable polynomial automorphism f of $\mathbb{C}^3$ is *weakly regular* if $X(f)\cap\mathrm{Ind}(f)=\emptyset$. For instance elements of $\mathcal{H}$ are weakly regular. A weakly regular automorphism is algebraically stable. Moreover $X(f)$ is an attracting set for f; in other words there exists an open neighborhood $\mathcal{V}$ of $X(f)$ such that $f(\mathcal{V})\Subset\mathcal{V}$ and $\bigcap\limits_{j=1}^{+\infty} f^j(\mathcal{V}) = X(f)$.

Both Φ_α and Φ_α^{-1} are weakly regular; furthermore $X(\Phi_\alpha) = (1:0:0:0)$ and $X(\Phi_\alpha^{-1}) = (0:1:0:0)$. Therefore $(1:0:0:0)$ (resp. $(0:1:0:0)$) is an attracting point for Φ_α (resp. Φ_α^{-1}). From [12, Corollary 1.8] the basin of attraction of $(1:0:0:0)$ is biholomorphic to $\mathbb{C}^3$.

As recalled above, the automorphisms $\phi_\alpha^\pm$ are regular, hence weakly regular, and similarly to $\Phi_\alpha^\pm$, they possess attractors $X(\phi_\alpha^+) = (1:0:0)$ and $X(\phi_\alpha^-) = (0:1:0)$ whose basins are biholomorphic to $\mathbb{C}^2$.

We do not inherit these properties for Ψ_α. Indeed remark that K_{Φ_α}, $X(\Phi_\alpha)$ and $X(\Phi_\alpha^{-1})$ are contained in $\{z_2 = 0\}$; but $\{z_2 = 0\}$ is contracted by θ^{-1} (recall that $\theta\Psi_\alpha = \Phi_\alpha\theta$) onto $\{z_2 = z_3 = 0\}$ and $\{z_2 = z_3 = 0\} = \mathrm{Ind}(\Psi_\alpha)$.

7. Definition of a Green function for Ψ_α

In this part, we assume $0 < |\alpha| \leq 1$. As for ϕ_α and Φ_α, and despite the fact that Ψ_α is not algebraically stable, we will see that it is possible to define a Green function for the automorphism Ψ_α which has almost as good properties. In particular, we will see that this function carries a lot of information about the dynamics of the automorphism Ψ_α.

Let $p = (p_0, p_1, p_2) \in \mathbb{C}^3$, and define $C = C(p_2) := 3\max(1, |p_2|^d) > 0$. We remark that for $n \geq 0$, we have $|\alpha^n p_2|^d \leq C$, hence $\max(\|\Psi_\alpha^{n+1}(p)\|, 1) \leq C\max(\|\Psi_\alpha^n(p)\|, 1)^q$. We deduce that for every $n \geq 0$,

$$\left| \frac{\log^+ \|\Psi_\alpha^{n+1}(p)\|}{q^{n+1}} - \frac{\log^+ \|\Psi_\alpha^n(p)\|}{q^n} \right| \leq \frac{\log(C)}{q^{n+1}},$$

hence

$$\lim_{n\to+\infty} \frac{\log^+ \|\Psi_\alpha^n(p)\|}{q^n} = \lim_{n\to+\infty} \frac{\log^+ \max(|P_\alpha^{(n)}(p)|, |P_\alpha^{(n-1)}(p)|)}{q^n} =: G_{\Psi_\alpha}^+(p)$$

exists. By construction, the function $G_{\Psi_\alpha}^+$ satisfies $G_{\Psi_\alpha}^+ \circ \Psi_\alpha = q \cdot G_{\Psi_\alpha}^+$. We note that on restriction to the hypersurface $\{z_2 = 0\}$,

$$G_{\Psi_\alpha}^+\big|_{\{z_2=0\}} = 0.$$

Indeed, if $p = (p_0, p_1, 0)$, then $\log^+ |P_\alpha^{(n)}(p)| = O(n)$ since we have seen that the forward iterates of p grow at most with Fibonacci speed.

For every $n \geq 0$, we have $\theta \circ \Psi_\alpha^n = \Phi_\alpha^n \circ \theta$. In particular, the following limits exist and satisfy:

$$\lim_{n\to+\infty} \frac{\log^+ ||\theta \circ \Psi_\alpha^n||}{q^n} = \lim_{n\to+\infty} \frac{\log^+ ||\Phi_\alpha^n \circ \theta||}{q^n} = G_{\Phi_\alpha}^+ \circ \theta. \tag{7.1}$$

Define the open set $\mathcal{U} := \left\{(z_0, z_1, z_2) \in \mathbb{C}^3 \,|\, z_2 \neq 0\right\}$ and let $p = (p_0, p_1, p_2) \in \mathcal{U}$. For every $n \geq 0$, recall that

$$\theta \circ \Psi_\alpha^n(p) = \theta\Big(P_\alpha^{(n)}(p), P_\alpha^{(n-1)}(p), \alpha^n p_2\Big) = \Big(P_\alpha^{(n)}(p)(\alpha^n p_2)^l, P_\alpha^{(n-1)}(p)(\alpha^n p_2)^l, \alpha^n p_2\Big).$$

For $j \in \{n-1, n\}$, we have

$$\log^+ |P_\alpha^{(j)}(p)| - l\log^+ |\alpha^n p_2| \leq \log^+ |P_\alpha^{(j)}(p)(\alpha^n p_2)^l| \leq \log^+ |P_\alpha^{(j)}(p)| + l\log^+ |\alpha^n p_2|,$$

so that

$$\frac{\log^+ ||\theta \circ \Psi_\alpha^n(p)||}{q^n} = \frac{\log^+ ||\Psi_\alpha^n(p)||}{q^n} + o(1).$$

We deduce from (7.1) that

$$G_{\Psi_\alpha}^+(p) = \lim_{n\to+\infty} \frac{\log^+ ||\Psi_\alpha^n(p)||}{q^n} = \lim_{n\to+\infty} \frac{\log^+ ||\theta \circ \Psi_\alpha^n(p)||}{q^n} = G_{\Phi_\alpha}^+ \circ \theta(p) = G_{\phi_\alpha}^+ \circ h(p). \tag{7.2}$$

Now if $p = (p_0, p_1, 0)$, then we have seen that $G_{\Psi_\alpha}^+(p) = 0$. Note that $h(p) = 0_{\mathbb{C}^2}$ and $\theta(p) = 0_{\mathbb{C}^3}$; therefore, $G_{\Phi_\alpha}^+ \circ \theta(p) = G_{\phi_\alpha}^+ \circ h(p) = 0$. We conclude that (7.2) holds for any point $p \in \mathbb{C}^3$.

The function $G^+_{\Psi_\alpha}$ is not $\equiv -\infty$, it is upper semicontinuous and satisfies the sub-mean value property (since $G^+_{\Phi_\alpha}$ does and $\theta\colon \mathbb{C}^3 \to \mathbb{C}^3$ is holomorphic); in other terms, $G^+_{\Psi_\alpha}$ is plurisubharmonic. Moreover, we know that $G^+_{\phi_\alpha}$ is Hölder continuous, and h is holomorphic, hence $G^+_{\Psi_\alpha}$ is Hölder continuous as well. We have shown:

Proposition 7.1. *For any point $p \in \mathbb{C}^3$, the limit*

$$\lim_{n\to+\infty} \frac{\log^+ ||\Psi^n_\alpha(p)||}{q^n} =: G^+_{\Psi_\alpha}(p)$$

exists; the function $G^+_{\Psi_\alpha} = G^+_{\Phi_\alpha} \circ \theta = G^+_{\phi_\alpha} \circ h$ is plurisubharmonic, Hölder continuous, and satisfies $G^+_{\Psi_\alpha} \circ \Psi_\alpha = q \cdot G^+_{\Psi_\alpha}$. We can then define the positive current $T^+_{\Psi_\alpha} := \mathrm{dd^c} G^+_{\Psi_\alpha}$. The maps $\theta|_{\mathcal{U}}$ and $h|_{\mathcal{U}}$ are submersions, and $T^+_{\Psi_\alpha}\big|_{\mathcal{U}} = (\theta|_{\mathcal{U}})^*(T^+_{\Phi_\alpha}\big|_{\mathcal{U}}) = (h|_{\mathcal{U}})^*(T^+_{\phi_\alpha}\big|_{\mathcal{U}})$. We also have $\Psi^*_\alpha(T^+_{\Psi_\alpha}) = q \cdot T^+_{\Psi_\alpha}$.

Remark 7.2. We observe that contrary to the case of ϕ_α, the set $K^+_{\Psi_\alpha}$ of points whose forward orbit is bounded is stricly contained in $\left\{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) = 0\right\}$; indeed, we have seen that the latter always contains $\{z_2 = 0\} \not\subset K^+_{\Psi_\alpha}$.

8. Analysis of the dynamics of the automorphism Ψ_α

In this section, we further analyze the dynamics of the automorphism Ψ_α, distinguishing between the value of $0 < |\alpha| \leq 1$. In particular, we want to describe what happens outside the invariant hypersurface $\{z_2 = 0\}$, where we have seen that the dynamics corresponds to the one of a linear Anosov diffeomorphism.

We see that a transition occurs for $|\alpha| = \varphi^{(1-q)/d}$. Indeed, when $|\alpha| < \varphi^{(1-q)/d}$, we observe different behaviors in the escape speed outside $\{z_2 = 0\}$ according to the choice of the starting point $p = (p_0, p_1, p_2)$: Fibonacci, or bigger than η^{q^n} for some $\eta > 1$. On the contrary, for $|\alpha| > \varphi^{(1-q)/d}$, we see that it is impossible to escape to infinity with Fibonacci speed, while the second case persists.

Let us say a few words about the critical value $\varphi^{(1-q)/d}$ where the transition happens. We define the cocycle $A\colon \mathbb{C}^3 \to \mathrm{GL}_2(\mathbb{C})$ by:

$$A(z_0, z_1, z_2) := \begin{pmatrix} 1 + z_0^{q-1} z_2^d & 1 \\ 1 & 0 \end{pmatrix},$$

and if $M \in \mathfrak{M}_2(\mathbb{C})$ and $v = (v_0, v_1) \in \mathbb{C}^2$, we set $M \cdot v := vM^T$. Recall that for every $z_2 \in \mathbb{C}$, we consider $\psi_{z_2} = (z_0 + z_1 + z_0^q z_2^d, z_0)$. We remark that for every $p = (p_0, p_1, p_2) \in \mathbb{C}^3$, $\psi_{p_2}(p_0, p_1) = A(p) \cdot (p_0, p_1)$. As usual we denote $A_0(p) := \mathrm{Id}$ and for $n \geq 1$,

$$A_n(p) := A(\Psi^{n-1}_\alpha(p)) \cdot A(\Psi^{n-2}_\alpha(p)) \ \dots \ A(\Psi_\alpha(p)) \cdot A(p).$$

In particular, for every $n \geq 0$, $\Psi^n_\alpha(p) = (A_n(p) \cdot (p_0, p_1), \alpha^n p_2)$; equivalently, $(P^{(n)}_\alpha(p), P^{(n-1)}_\alpha(p)) = A_n(p) \cdot (p_0, p_1)$. Note that

$$A(\Psi^n_\alpha(p)) = \begin{pmatrix} 1 + (P^{(n)}_\alpha(p))^{q-1} \alpha^{nd} p_2^d & 1 \\ 1 & 0 \end{pmatrix}.$$

- If $\lim\limits_{n\to+\infty} \left(P^{(n)}_\alpha(p)\right)^{q-1} \alpha^{nd} = 0$, then $\lim\limits_{n\to+\infty} A(\Psi^n_\alpha(p)) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, whose largest eigenvalue is φ. Then the growth will be exactly Fibonacci unless the initial point belongs to $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$. But then

$$\left(\varphi^{q-1} |\alpha|^d\right)^n = O\left(|P^{(n)}_\alpha(p)|^{q-1} |\alpha|^{nd}\right) = o(1),$$

  and necessarily $|\alpha| < \varphi^{(1-q)/d}$.

– If $|\alpha| > \varphi^{(1-q)/d}$, Fibonacci growth is impossible; indeed if $|P_\alpha^{(n)}(p)| \geq C\varphi^n$ with $C > 0$, then

$$|P_\alpha^{(n)}(p)|^{q-1}|\alpha|^{nd} \geq C^{q-1}\big(\varphi^{q-1}|\alpha|^d\big)^n$$

but here $\eta := \varphi^{q-1}|\alpha|^d > 1$ so that $|P_\alpha^{(n+1)}(p)| \gtrsim C^{q-1}\eta^n|P_\alpha^{(n)}(p)|$ and the growth is much more in fact.

We will also see that this transition reflects an analogous change in the dynamics of the Hénon automorphism ϕ_α: for $|\alpha| < \varphi^{(1-q)/d}$, the point $0_{\mathbb{C}^2}$ is a sink of ϕ_α, while for $|\alpha| > \varphi^{(1-q)/d}$, the point $0_{\mathbb{C}^2}$ becomes a saddle fixed point.

The following general lemma will be useful in the analysis that follows.

Lemma 8.1. *Assume that $0 < |\alpha| \leq 1$, and that $p = (p_0, p_1, p_2)$ satisfies:*

$$|P_\alpha^{(n)}(p)| = O([(1-\varepsilon)\varphi]^n).$$

Then with our previous notations, $p \in \mathcal{Z}$.

Proof. This follows again from Lemma 5.1. Indeed for every $n \geq 0$,

$$|g_n(p)| \leq \big(|P_\alpha^{(n+1)}(p)| + \varphi^{-1}|P_\alpha^{(n)}(p)|\big)\varphi^{-n} = O\big((1-\varepsilon)^n\big)$$

hence $g(p) = \lim_{n\to+\infty} g_n(p) = 0$. □

8.1. **Points escaping to infinity with maximal speed.** The results of this subsection hold for any $0 < |\alpha| \leq 1$. We start by exhibiting an explicit non-empty open set of points escaping to infinity very fast; then we state some facts concerning the set of points going to infinity with maximal speed, and show how they can be derived from the properties of the Green function $G^+_{\Psi_\alpha}$.

Set $\gamma := \frac{\ln|\alpha|}{\ln(\varphi)}$. We choose $M \geq 0$ sufficiently large so that $M(q-1) + d\gamma > 0$ (this is possible since by hypothesis, $q - 1 > 0$).

Proposition 8.2. *We define the open set*

$$\Omega := \big\{p = (p_0, p_1, p_2) \in \mathbb{C}^3 \mid |p_0| > |p_1| > 0 \text{ and } |p_1|^{q-1}|p_2|^d > 2 + \varphi^M\big\}.$$

Then for any point $p \in \Omega$, we have $G^+_{\Psi_\alpha}(p) > 0$; moreover the sequence $(|P_\alpha^{(n)}(p)|)_n$ is increasing.

The proof splits in two lemmas that we are going to detail now.

Lemma 8.3. *For any point p in Ω the escape speed is superpolynomial: for any $n \geq -1$,*

$$|P_\alpha^{(n)}(p)| \geq |p_1|\varphi^{Mn}. \tag{8.1}$$

Moreover the sequence $(|P_\alpha^{(n)}(p)|)_n$ is increasing.

Proof. The proof is by induction on $n \geq -1$. Let $p \in \Omega$; we will show that $(|P_\alpha^{(n)}(p)|)_n$ is increasing and that (8.1) holds. It follows from our assumptions that

– $|P_\alpha^{(-1)}(p)| = |p_1| \geq |p_1|\varphi^{-M}$.
– $|P_\alpha^{(0)}(p)| = |p_0| \geq |p_1|$.

– Take $n \geq 0$ and assume that $|P_\alpha^{(n)}(p)| \geq |p_1|\varphi^{Mn}$ and $|P_\alpha^{(n)}(p)| \geq |P_\alpha^{(n-1)}(p)|$. We estimate:

$$\begin{aligned}|P_\alpha^{(n+1)}(p)| &\geq |P_\alpha^{(n)}(p)|(|P_\alpha^{(n)}(p)|^{q-1}|\alpha^n p_2|^d - 2)\\ &\geq |P_\alpha^{(n)}(p)|(|p_1|^{q-1}|p_2|^d\varphi^{(M(q-1)+d\gamma)n} - 2)\\ &\geq |p_1|\varphi^{Mn}(|p_1|^{q-1}|p_2|^d - 2)\\ &\geq |p_1|\varphi^{M(n+1)}\end{aligned}$$

because $M(q-1)+d\gamma > 0$ and $|p_1|^{q-1}|p_2|^d - 2 > \varphi^M$. Since $|p_1|^{q-1}|p_2|^d - 2 \geq 1$, the previous inequalities also show that $|P_\alpha^{(n+1)}(p)| \geq |P_\alpha^{(n)}(p)|$, which concludes the induction.

□

Lemma 8.4. *Recall that $\gamma := \frac{\ln|\alpha|}{\ln(\varphi)}$ and that $M \geq 0$ is chosen such that $M(q-1)+d\gamma > 0$. Take $p \in \mathbb{C}^3$ such that the sequence $(|P_\alpha^{(n)}(p)|)_{n\geq 0}$ is increasing, and assume that there exists $n_0 \geq 1$ such that for every $n \geq n_0$, the following inequality holds*[2]*:*

$$|P_\alpha^{(n)}(p)| \geq \varphi^{Mn}.$$

Then the escape speed is much bigger in fact: there exist $n_1 \geq n_0$ and $\eta > 1$ such that for every $n \geq n_1$,

$$|P_\alpha^{(n)}(p)| \geq \eta^{q^n}.$$

In terms of the Green function introduced above, we then get $G^+_{\Psi_\alpha}(p) > 0$.

Proof. Since $(|P_\alpha^{(n)}(p)|)_{n\geq 0}$ is increasing, we have for every $n \geq 0$,

$$|P_\alpha^{(n)}(p)|^q(|\alpha|^n|p_2|)^d = |P_\alpha^{(n+1)}(p) - P_\alpha^{(n)}(p) - P_\alpha^{(n-1)}(p)| \leq 3|P_\alpha^{(n+1)}(p)|. \tag{8.2}$$

Set $x_n := \ln|P_\alpha^{(n)}(p)|$. From our hypotheses, we know that for every $n \geq n_0$, $x_n \geq Mn\ln\varphi$. Since $M(q-1)+d\gamma > 0$, we can take $\varepsilon > 0$ small such that we still have $M(q-1-\varepsilon)+d\gamma > 0$. Let $n_0' \geq n_0$ be chosen such that for $n \geq n_0'$, $n(M(q-1-\varepsilon)+d\gamma)\ln\varphi + d\ln|p_2| - \ln 3 \geq 0$. Thanks to (8.2), we get: for every $n \geq n_0'$,

$$\begin{aligned}x_{n+1} &\geq qx_n + nd\gamma\ln\varphi + d\ln|p_2| - \ln 3\\ &\geq (1+\varepsilon)x_n + (n(M(q-1-\varepsilon)+d\gamma)\ln\varphi + d\ln|p_2| - \ln 3)\\ &\geq (1+\varepsilon)x_n.\end{aligned}$$

We then obtain: for every $n \geq n_0'$,

$$x_n \geq (1+\varepsilon)^{n-n_0'}x_{n_0'} \geq (1+\varepsilon)^{n-n_0'}Mn_0'\ln\varphi.$$

As a result there exists $n_1 \geq n_0'$ such that for $n \geq n_1$,

$$\frac{x_n}{n^2} \geq \frac{(q/2)^{n-n_0'}Mn_0'\ln\varphi}{n^2} \geq -(nd\gamma\ln\varphi + d\ln|p_2| - \ln 3).$$

We can then refine the previous inequalities: for $n \geq n_1$,

$$x_{n+1} \geq qx_n + nd\gamma\ln\varphi + d\ln|p_2| - \ln 3 \geq q\left(1 - \frac{1}{n^2}\right)x_n.$$

Let $C := \prod_{n\geq n_1}\left(1-\frac{1}{n^2}\right)x_{n_1}q^{-n_1} > 0$; for every $n \geq n_1$, we have $x_n \geq Cq^n$, hence $|P_\alpha^{(n)}(p)| \geq \eta^{q^n}$, where $\eta := e^C > 1$. □

2. In particular, this is satisfied for points $p \in \Omega$ as we have seen in Lemma 8.3.

Remark 8.5. We have seen that the automorphism ϕ_α possesses an attractor at infinity $X(\phi_\alpha) = (1:0:0)$ whose basin is biholomorphic to $\mathbb{C}^2$. Then there exists $C = C(\alpha) > 0$ such that the forward orbit of any point $\widetilde{p} = (p_0, p_1) \in \mathbb{C}^2$ such that $|p_0| \gg |p_1|$ and $\|\widetilde{p}\| \geq C$ is attracted by $X(\phi_\alpha)$; in particular, $\widetilde{p} \notin K^+_{\phi_\alpha}$, and thus, $G^+_{\phi_\alpha}(\widetilde{p}) > 0$. If $p = (p_0, p_1, p_2) \in \mathbb{C}^3$ satisfies $|p_0| \gg |p_1|$ and $\|(p_0, p_1)\| \cdot |p_2|^l \geq C$, we see that $\|h(p)\| \geq C$, hence $G^+_{\phi_\alpha}(h(p)) > 0$, and $G^+_{\Psi_\alpha}(p) > 0$ as well. The definition of the set Ω in Proposition 8.2 is coherent with this observation.

Proposition 8.6. *Set $\widetilde{l} := 2\max(l, 1)$. We have*

$$1 \leq \limsup_{\|p\| \to +\infty} \frac{G^+_{\Psi_\alpha}(p)}{\log \|p\|} \leq \widetilde{l}.$$

The set $\mathcal{E} := \left\{ p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) > 0 \right\}$ of points escaping to infinity with maximal speed is open, connected and of infinite measure on any complex line where $G^+_{\Psi_\alpha}$ is not identically zero. In particular, the set

$$\left\{ p \in \mathbb{C}^3 \mid \lim_{n \to +\infty} \|\Psi^n_\alpha(p)\| = +\infty \right\}$$

of points whose forward orbit goes to infinity is of infinite measure.

Proof. The openness of $\mathcal{E}$ follows directly from the continuity of $G^+_{\Psi_\alpha}$, shown in Proposition 7.1.

The proof of the fact that $\mathcal{E}$ has infinite measure follows arguments given by Guedj-Sibony [12]. Since ϕ_α is algebraically stable, we know from Proposition 1.3 in [12] that $\limsup_{\|\widetilde{p}\| \to +\infty} \frac{G^+_{\phi_\alpha}(\widetilde{p})}{\log \|\widetilde{p}\|} = 1$. Therefore

$$\limsup_{\|p\| \to +\infty} \frac{G^+_{\Psi_\alpha}(p)}{\log \|p\|} = \limsup_{\|p = (p_0, p_1, p_2)\| \to +\infty} \frac{G^+_{\phi_\alpha} \circ h(p)}{\log \|h(p)\|} \times \frac{l \log |p_2| + \log \|(p_0, p_1)\|}{\log \|p\|} \leq \widetilde{l} \limsup_{\|\widetilde{p}\| \to +\infty} \frac{G^+_{\phi_\alpha}(\widetilde{p})}{\log \|\widetilde{p}\|} = \widetilde{l}.$$

For the other inequality, we remark that

$$\limsup_{\|p\| \to +\infty} \frac{G^+_{\Psi_\alpha}(p)}{\log \|p\|} \geq \limsup_{\|p = (p_0, p_1, 1)\| \to +\infty} \frac{G^+_{\phi_\alpha} \circ h(p)}{\log \|h(p)\|} \times \frac{\log \|(p_0, p_1)\|}{\log \|(p_0, p_1, 1)\|} = \limsup_{\|\widetilde{p}\| \to +\infty} \frac{G^+_{\phi_\alpha}(\widetilde{p})}{\log \|\widetilde{p}\|} = 1.$$

Assume that $p \in \mathbb{C}^3$ satisfies $G^+_{\Psi_\alpha}(p) > 0$, and for some $v \neq 0_{\mathbb{C}^3}$, consider the line $L := \left\{ p + tv \mid t \in \mathbb{C} \right\}$. Denote by $m(r)$ the Lebesgue measure of the set $\left\{ e^{ix},\ x \in \mathbb{R} \mid G^+_{\Psi_\alpha}(p + re^{ix}v) > 0 \right\}$. From what precedes, we know that there exists $C > 0$ such that for every $r \geq 0$,

$$G^+_{\Psi_\alpha}(p + re^{ix}v) \leq \widetilde{l} \log^+(r) + C.$$

By the sub-mean value property,

$$0 < G^+_{\Psi_\alpha}(p) \leq \frac{1}{2\pi} \int_0^{2\pi} G^+_{\Psi_\alpha}(p + re^{ix}v) dx \leq \frac{1}{2\pi} (\widetilde{l} \log^+(r) + C) m(r).$$

Therefore, $m(r) \geq \frac{2\pi G^+_{\Psi_\alpha}(p)}{\widetilde{l} \log^+(r) + C}$, and integrating over r, we get that the set of points p in L such that $G^+_{\Psi_\alpha}(p) > 0$ has infinite measure. The proof of connectivity is also based on the slow growth of $G^+_{\Psi_\alpha}$ and follows from similar arguments (see [13]). $\square$

8.2. **General remarks when $0<|\alpha|<1$.** In this case, $0_{\mathbb{C}^3}$ is a hyperbolic fixed point of Ψ_α of saddle type, and Corollary 5.5 tells us that the set $K^+_{\Psi_\alpha}$ of points with bounded forward orbit is exactly the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$. The positive Julia set $J^+_{\Psi_\alpha}$ thus corresponds to $\partial K^+_{\Psi_\alpha}=\overline{W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})}$. Let $p=(p_0,p_1,p_2)\in\mathbb{C}^3$. For $n\geq 0$,

$$\theta\circ\Psi^n_\alpha(p)=\big(P^{(n)}_\alpha(p)(\alpha^n p_2)^l,P^{(n-1)}_\alpha(p)(\alpha^n p_2)^l,\alpha^n p_2\big)=\big(\phi^n_\alpha\circ h(p),\alpha^n p_2\big). \tag{8.3}$$

From (8.3), we get that $h(W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}))=W^s_{\phi_\alpha}(0_{\mathbb{C}^2})=K^+_{\phi_\alpha}$.[3] We deduce that $J^+_{\phi_\alpha}=\partial W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$. Besides, we know that $K^+_{\phi_\alpha}=\big\{\widetilde{p}\in\mathbb{C}^2\mid G^+_{\phi_\alpha}(\widetilde{p})=0\big\}$, and this set is closed by continuity of $G^+_{\phi_\alpha}$. In particular, $W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$ is closed. For any $\widetilde{p}\in\mathbb{C}^2$, there are two possible behaviors: either $p\in W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$ and then its forward iterates converge to $0_{\mathbb{C}^2}$ exponentially fast, or they go to infinity with maximal speed.

8.3. **Analysis of the dynamics in the case where $0<|\alpha|<\varphi^{(1-q)/d}$.** We show that under this assumption, we can construct a set of points with non-empty interior for which the escape speed is much smaller, in fact Fibonacci.

From our hypothesis on α, we can take $\varepsilon>0$ small enough so that $\eta:=((1+\varepsilon)\varphi)^q|\alpha|^d<\varphi$.

Proposition 8.7. *Assume $0<|\alpha|<\varphi^{(1-q)/d}$. We consider the following open neighborhood of the hypersurface $\{z_2=0\}$:*

$$\Omega':=\big\{p=(p_0,p_1,p_2)\in\mathbb{C}^3\,|\,(|p_0|+|p_1|)^{q-1}|p_2|^d<\varphi\varepsilon\big\}.$$

If $p\in\Omega'$, there are two possible behaviors:

- *either p belongs to the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ and then its forward iterates converge to $0_{\mathbb{C}^3}$ exponentially fast;*
- *or p goes to infinity with Fibonacci speed: $\big(P^{(n)}_\alpha(p)\varphi^{-n}\big)_{n\geq 0}$ converges and we have*

$$\lim_{n\to+\infty}P^{(n)}_\alpha(p)\varphi^{-n}\in\mathbb{C}^*.$$

Remark 8.8. The last result tells us that if we start close enough to $\{z_2=0\}\subset\Omega'$, the dynamics is similar to the one we observe on restriction to this invariant hypersurface: either the starting point belongs to $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ and in this case its forward orbit converges to $0_{\mathbb{C}^3}$ with exponential speed, or the iterates escape to infinity with speed exactly Fibonacci. We remark that when $|\alpha|$ is small, we can choose $\varepsilon>0$ reasonably large, so that the set Ω' becomes larger and larger. This is coherent with the fact that the smaller $|\alpha|$ is, the faster we converge to the hypersurface $\{z_2=0\}$.

We start by showing that the speed cannot be more than Fibonacci.

Lemma 8.9. *Any point $p\in\Omega'$ grows at most with Fibonacci speed, that is, there exists $C=C(p_0,p_1)>0$ such that for any $n\geq 0$,*

$$|P^{(n)}_\alpha(p)|\leq C\varphi^n.$$

Proof. Let $\widetilde{C}=\widetilde{C}(p_0,p_1):=|p_0|+|p_1|$ and $\varepsilon>0$ be chosen as explained above. We first show that for every $n\geq 0$,

$$|P^{(n)}_\alpha(p)|\leq\widetilde{C}((1+\varepsilon)\varphi)^n.$$

The result is clearly true for $n=0$, and for $n=1$ we have

$$|P^{(1)}_\alpha(p)|\leq|p_0|+|p_1|+|p_0|^q|p_2|^d\leq\widetilde{C}(1+\widetilde{C}^{q-1}|p_2|^d)\leq\widetilde{C}(1+\varepsilon)\varphi.$$

3. Indeed, if $p\in W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, then $h(p)\in W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$; conversely, if $(p_0,p_1)\in K^+_{\phi_\alpha}$, then $(p_0,p_1)=h(p_0,p_1,1)$ and $(p_0,p_1,1)\in K^+_{\Psi_\alpha}=W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$.

Suppose that it holds for $n-1$ and n, that is

$$|P_\alpha^{(n-1)}(p)| \leq \widetilde{C}((1+\varepsilon)\varphi)^{n-1}, \qquad |P_\alpha^{(n)}(p)| \leq \widetilde{C}((1+\varepsilon)\varphi)^{n}.$$

We then have

$$\begin{aligned}
|P_\alpha^{(n+1)}(p)| &\leq |P_\alpha^{(n)}(p)| + |P_\alpha^{(n-1)}(p)| + |(P_\alpha^{(n)}(p))^q(\alpha^n p_2)^d| \\
&\leq \widetilde{C}\big((1+\varepsilon)\varphi\big)^n + \widetilde{C}\big((1+\varepsilon)\varphi\big)^{n-1} + \widetilde{C}^q|p_2|^d\big(((1+\varepsilon)\varphi)^q|\alpha|^d\big)^n \\
&\leq \widetilde{C}(1+\varepsilon)^n\varphi^{n+1} + \widetilde{C}\varphi\varepsilon\eta^n \\
&\leq \widetilde{C}(1+\varepsilon)^n\varphi^{n+1} + \widetilde{C}\varepsilon(1+\varepsilon)^n\varphi^{n+1} \\
&= \widetilde{C}\big((1+\varepsilon)\varphi\big)^{n+1},
\end{aligned}$$

which concludes the induction.

Using this fact, we obtain a good control on the non-linear term: for any $n \geq 0$,

$$|(P_\alpha^{(n)}(p))^q(\alpha^n p_2)^d| \leq \widetilde{C}^q|p_2|^d\big(((1+\varepsilon)\varphi)^q|\alpha|^d\big)^n = C_0\eta^n,$$

where $C_0 = C_0(p_0,p_1) := \widetilde{C}\varphi\varepsilon$. For every $n \geq 0$, we have

$$\begin{aligned}
|P_\alpha^{(n+1)}(p)| &\leq |P_\alpha^{(n)}(p)| + |P_\alpha^{(n-1)}(p)| + |(P_\alpha^{(n)}(p))^q(\alpha^n p_2)^d| \\
&\leq |P_\alpha^{(n)}(p)| + |P_\alpha^{(n-1)}(p)| + C_0\eta^n.
\end{aligned}$$

Thanks to the same trick as in the proof of Lemma 5.1, we obtain:

$$|P_\alpha^{(n+1)}(p)| + (\varphi-1)|P_\alpha^{(n)}(p)| \leq \varphi^n\left(\varphi|p_0| + |p_1| + C_0\sum_{j=0}^{n}\left(\frac{\eta}{\varphi}\right)^j\right).$$

Since $\eta < \varphi$, we can set $C = C(p_0,p_1) := \varphi^{-1}\left(\varphi|p_0| + |p_1| + C_0\sum_{j=0}^{+\infty}\left(\frac{\eta}{\varphi}\right)^j\right)$. We then get: for every $n \geq 0$,

$$|P_\alpha^{(n)}(p)| \leq C\varphi^n.$$

□

The proof of Proposition 8.7 is the combination of Lemma 8.9 and of the next result.

Lemma 8.10. *Assume $0 < |\alpha| < \varphi^{(1-q)/d}$ and take $p \in \mathbb{C}^3 \smallsetminus W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ with speed less than Fibonacci, i.e., there exists $C > 0$ such that for every $n \geq 0$, $|P_\alpha^{(n)}(p)| \leq C\varphi^n$. Then p goes to infinity with speed exactly Fibonacci: $\big(P_\alpha^{(n)}(p)\varphi^{-n}\big)_{n\geq 0}$ converges and we have*

$$\lim_{n\to+\infty} P_\alpha^{(n)}(p)\varphi^{-n} \in \mathbb{C}^*.$$

Proof. Take $p \in \mathbb{C}^3 \smallsetminus W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ such that for every $n \geq 0$, $|P_\alpha^{(n)}(p)| \leq C\varphi^n$, $C > 0$. We first show that p goes to infinity with speed at least Fibonacci too, i.e., there exists $C' = C'(p_0,p_1) > 0$ such that for every $n \geq 0$, $|P_\alpha^{(n)}(p)| \geq C'\varphi^n$. Recall that if $(z_0,z_1,z_2) \in \mathbb{C}^3$, we denote

$$A(z_0,z_1,z_2) := \begin{pmatrix} 1+z_0^{q-1}z_2^d & 1 \\ 1 & 0 \end{pmatrix},$$

and that for $n \geq 0$, $\left(P_\alpha^{(n)}(p), P_\alpha^{(n-1)}(p)\right) = A_n(p) \cdot (p_0, p_1)$, where

$$A_n(p) := A(\Psi_\alpha^{n-1}(p)) \cdot A(\Psi_\alpha^{n-2}(p)) \ \dots \ A(\Psi_\alpha(p)) \cdot A(p).$$

Note that for every $j \geq 0$,

$$A(\Psi_\alpha^j(p)) = \begin{pmatrix} 1 + (P_\alpha^{(j)}(p))^{q-1} \alpha^{jd} p_2^d & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.4}$$

Since $|P_\alpha^{(j)}(p)| \leq C\varphi^j$, we see that

$$|(P_\alpha^{(j)}(p))^{q-1} \alpha^{jd} p_2^d| \leq C^{q-1} |p_2|^d (\varphi^{q-1} |\alpha|^d)^j \leq \nu \eta^j, \tag{8.5}$$

where $\eta := \varphi^{q-1}|\alpha|^d < 1$ and $\nu := C^{q-1}|p_2|^d \geq 0$. Set $M_0 := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. If $\varepsilon > 0$, let us consider $B_\varepsilon(M_0) := \left\{ M \in \mathfrak{M}_2(\mathbb{C}) \mid \|M - M_0\|_\infty < \varepsilon \right\}$. We see from (8.4) and (8.5) that for every $j \geq 0$, $A(\Psi_\alpha^j(p)) \in B_{\nu\eta^j}(M_0)$. For $\varepsilon > 0$ small, every matrix $M \in B_\varepsilon(M_0)$ is hyperbolic with eigenvalues close to φ and φ'; moreover we can choose a family of cones $(\mathscr{C}_\varepsilon)_{\varepsilon > 0}$ around Δ_φ satisfying the following: there exist $C_0, C_1 > 0$ such that for each $M \in B_\varepsilon(M_0)$, every vector $v \in \mathscr{C}_\varepsilon$ will be expanded by a factor close to φ:

$$(1 - C_0\varepsilon)\varphi\|v\| \leq \|M \cdot v\| \leq (1 + C_1\varepsilon)\varphi\|v\|.$$

Since $p \notin W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, the iterates of (p_0, p_1) are expanded and accumulate on the unstable space $\Delta_\varphi = \left\{ (\varphi z, z) \mid z \in \mathbb{C} \right\}$ of M_0. In fact the angle $\angle(\Delta_\varphi, A_j(p) \cdot (p_0, p_1))$ decreases exponentially fast, and we can assume that for some $n_0 \geq 0$ and for every $j \geq n_0$, $A(\Psi_\alpha^j(p)) \in B_{\nu\eta^j}(M_0)$ maps $\mathscr{C}_{\nu\eta^j}$ to $\mathscr{C}_{\nu\eta^{j+1}}$. We deduce that for every $n \geq n_0$,

$$\prod_{j=n_0}^{n-1} (1 - C_0 \nu \eta^j) \|A_{n_0}(p) \cdot (p_0, p_1)\| \leq \frac{\|A_n(p) \cdot (p_0, p_1)\|}{\varphi^{n-n_0}} \leq \prod_{j=n_0}^{n-1} (1 + C_1 \nu \eta^j) \|A_{n_0}(p) \cdot (p_0, p_1)\|.$$

Let $C' := \varphi^{n_0} \prod_{j=n_0}^{+\infty} (1 - C_0 \nu \eta^j) \|A_{n_0}(p) \cdot (p_0, p_1)\| > 0$. We have thus obtained: for every $n \geq 0$,

$$\|(P_\alpha^{(n)}(p), P_\alpha^{(n-1)}(p))\| \geq C' \varphi^n.$$

Now, we note that p belongs to $\mathcal{D}$, the domain of definition of the series g introduced earlier. Indeed, for any $j \geq 0$, $|P_\alpha^{(j)}(p)|^q \varphi^{-j} |\alpha|^{jd} \leq C^q (\varphi^{q-1}|\alpha|^d)^j$ and $\varphi^{q-1}|\alpha|^d < 1$. We have shown that the sequence $(P_\alpha^{(n)}(p), P_\alpha^{(n-1)}(p))_n$ accumulates on the unstable direction $\left\{ (\varphi z, z) \mid z \in \mathbb{C} \right\}$ of M_0; therefore, we get

$$\lim_{n \to +\infty} \frac{P_\alpha^{(n)}(p)}{P_\alpha^{(n-1)}(p)} = \varphi. \tag{8.6}$$

Recall that for $n \geq 0$, $g_n(p) := (P_\alpha^{(n+1)}(p) + \varphi^{-1} P_\alpha^{(n)}(p)) \varphi^{-n}$. We have seen that $p \in \mathcal{D}$, and then, $(g_n(p))_n$ converges. From (8.6) we deduce that for every $n \geq 0$,

$$g_n(p) = \left(\frac{P_\alpha^{(n+1)}(p)}{P_\alpha^{(n)}(p)} + \varphi^{-1} \right) P_\alpha^{(n)}(p) \varphi^{-n} \sim \left(\varphi + \varphi^{-1} \right) P_\alpha^{(n)}(p) \varphi^{-n}.$$

This implies that $(P_\alpha^{(n)}(p)\varphi^{-n})_{n \geq 0}$ converges. But we also know from what precedes that $\lim_{n \to +\infty} |P_\alpha^{(n)}(p)\varphi^{-n}| > 0$, so that $\lim_{n \to +\infty} P_\alpha^{(n)}(p)\varphi^{-n} \in \mathbb{C}^*$, which concludes. $\square$

Recall that we take $\varepsilon > 0$ small enough so that $((1+\varepsilon)\varphi)^q|\alpha|^d < \varphi$, and that $\Omega' := \{(p_0,p_1,p_2) \in \mathbb{C}^3 \,|\, (|p_0|+|p_1|)^{q-1}|p_2|^d < \varphi\varepsilon\}$. We deduce from Proposition 8.7 that

$$G^+_{\Psi_\alpha}\big|_{\Omega'} = 0. \tag{8.7}$$

Indeed, the forward iterates of any point in Ω' grow at most with Fibonacci speed as we have seen. In particular, the set $\{p \in \mathbb{C}^3 \mid G_{\Psi^+_\alpha}(p) = 0\}$ has non-empty interior. Set $\delta := (\varphi\varepsilon)^{1/(q-1)}$ and define the open ball

$$B_\delta := \{\widetilde{p} = (p_0,p_1) \in \mathbb{C}^2 \mid \|\widetilde{p}\|_1 = |p_0|+|p_1| < \delta\}.$$

Recall that $l = \frac{d}{q-1}$ and that $h\colon (z_0,z_1,z_2) \mapsto (z_0z_2^l, z_1z_2^l)$. Remark that $h(\Omega') \subset B_\delta$. Indeed, if $p = (p_0,p_1,p_2) \in \Omega'$, then

$$\|h(p)\|_1 = |p_0p_2^l| + |p_1p_2^l| = \Big((|p_0|+|p_1|)^{q-1}|p_2|^d\Big)^{1/(q-1)} < (\varphi\varepsilon)^{1/(q-1)} = \delta.$$

Conversely, if $(p_0,p_1) \in B_\delta$, then $(p_0,p_1) = h(p_0,p_1,1)$ with $(p_0,p_1,1) \in \Omega'$, so that $h(\Omega') = B_\delta$ in fact. Since $G^+_{\Psi_\alpha} = G^+_{\phi_\alpha} \circ h$, we deduce from (8.7) that

$$G^+_{\phi_\alpha}\big|_{B_\delta} = 0.$$

But as we have seen, $K^+_{\phi_\alpha} = \{(p_0,p_1) \in \mathbb{C}^2 \mid G^+_{\phi_a}(p_0,p_1) = 0\}$. We conclude that for $|\alpha| < \varphi^{(1-q)/d}$, any point $(p_0,p_1) \in B_\delta$ has bounded forward orbit under ϕ_α. Actually we can say more. Recall that $\phi_\alpha = \alpha^l(z_0+z_1+z_0^q, z_0)$. We see that $0_{\mathbb{C}^2}$ is a sink of ϕ_α; indeed, the largest eigenvalue of the Jacobian is $\alpha^l\varphi$, which is stricly smaller than 1 from the assumption we made on α. For any $p \in \mathbb{C}^3$ and $n \geq 0$,

$$\theta \circ \Psi^n_\alpha(p) = \big(P^{(n)}_\alpha(p)(\alpha^n p_2)^l, P^{(n-1)}_\alpha(p)(\alpha^n p_2)^l, \alpha^n p_2\big) = \big(\phi^n_\alpha \circ h(p), \alpha^n p_2\big). \tag{8.8}$$

If $p \in \Omega'$, we know from Proposition 8.7 that there exists $C > 0$ such that for any $n \geq 0$, $|P^{(n)}_\alpha(p)| \leq C\varphi^n$. But then, $|P^{(n)}_\alpha(p)(\alpha^n p_2)^l| \leq C|p_2|^l(\varphi^{q-1}|\alpha|^d)^{n/(q-1)}$, and $\varphi^{q-1}|\alpha|^d < 1$, so we deduce from (8.8) and the equality $B_\delta = h(\Omega')$ that any point in B_δ goes to $0_{\mathbb{C}^2}$ by forward iteration of ϕ_α; equivalently, the basin of attraction $W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$ of the sink $0_{\mathbb{C}^2}$ contains the ball B_δ. Recall also that it is a general fact that for a sink p of ϕ_α, $W^s_{\phi_\alpha}(p)$ is biholomorphic to $\mathbb{C}^2$.

In the following, we will see how the previous results enable us to give a description of the dynamics of Ψ_α in the case where $0 < |\alpha| < \varphi^{(1-q)/d}$. Let $p \in \mathbb{C}^3$. We have shown previously that $K^+_{\Psi_\alpha} = W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, so assume that the forward orbit of p under Ψ_α is not bounded. There are two possibilities:

- either $G^+_{\Psi_\alpha}(p) > 0$ and the iterates of p go to infinity with maximal speed; in this case, we also have $G^+_{\phi_\alpha}(h(p)) = G^+_{\Psi_\alpha}(p) > 0$;
- or $G^+_{\Psi_\alpha}(p) = 0$ and $G^+_{\phi_\alpha}(h(p)) = G^+_{\Psi_\alpha}(p) = 0$ too. But then we know from the general properties of ϕ_α that $h(p) \in K^+_{\phi_\alpha}$. From §8.2, we also have $K^+_{\phi_\alpha} = W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$. It follows from (8.8) that $\lim\limits_{n\to+\infty} |P^{(n)}_\alpha(p)(\alpha^n p_2)^l|$ exists and vanishes. Since $l = d/(q-1)$, we deduce that $\lim\limits_{n\to+\infty} |P^{(n)}_\alpha(p)|^{q-1}|\alpha^n p_2|^d = 0$, hence $\lim\limits_{n\to+\infty} A(\Psi^n_\alpha(p)) = M_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ where A is the cocycle introduced earlier. Reasoning as before, and since by assumption $p \notin W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, we conclude accordingly that p escapes to infinity with Fibonacci speed.

We have thus shown:

Proposition 8.11. *When $0 < |\alpha| < \varphi^{(1-q)/d}$, the point $0_{\mathbb{C}^3}$ is a saddle fixed point of Ψ_α of index 2, and $J^+_{\Psi_\alpha} := \partial K^+_{\Psi_\alpha} = \overline{W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})}$. Moreover, $\{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) = 0\} = h^{-1}(W^s_{\phi_\alpha}(0_{\mathbb{C}^2})) = \Omega'' \sqcup W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, where*

Ω'' has non-empty interior (it contains the set $\Omega' \smallsetminus W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$), and the forward orbit of points that belong to it goes to infinity with Fibonacci speed. Moreover, $W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$ is biholomorphic to $\mathbb{C}^2$, and

$$K^+_{\phi_\alpha} = W^s_{\phi_\alpha}(0_{\mathbb{C}^2}) = \left\{\widetilde{p} \in \mathbb{C}^2 \mid G^+_{\phi_\alpha}(\widetilde{p}) = 0\right\}.$$

We summarize this as follows:

$$\begin{array}{rcl} & h & \\ \{z_2 = 0\} & \to & \{0_{\mathbb{C}^2}\}; \\ K^+_{\Psi_\alpha} = W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) & \to & K^+_{\phi_\alpha} = W^s_{\phi_\alpha}(0_{\mathbb{C}^2}); \\ \Omega'' & \to & W^s_{\phi_\alpha}(0_{\mathbb{C}^2}); \\ \left\{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) > 0\right\} & \to & \left\{\widetilde{p} \in \mathbb{C}^2 \mid G^+_{\phi_\alpha}(\widetilde{p}) > 0\right\}. \end{array} \tag{8.9}$$

Thanks to the last statement, we now give an alternative description of the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ in terms of the set $\mathcal{Z}$ of zeros of the series g introduced previously.

Proposition 8.12. *Assume $0 < |\alpha| < \varphi^{(1-q)/d}$. Set $\mathcal{V} := \left\{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) = 0\right\} = \Omega'' \sqcup W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$. Then $\mathcal{V}$ coincides with the domain of definition $\mathcal{D}$ of the series g introduced earlier. Moreover, we have the following parametrization of the stable manifold:*

$$W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) = \mathcal{Z} = \left\{p \in \mathcal{V} \mid g(p) = 0\right\} = \bigcup_{n \geq 0} \Psi_\alpha^{-n}(\Omega' \cap \mathcal{Z}).$$

Proof. Let $p \in \mathcal{V}$. We know from Proposition 8.11 that there exists $C > 0$ such that for any $j \geq 0$, $|P^{(j)}_\alpha(p)| \leq C\varphi^j$. Then p belongs to the domain of definition of g since in this case, $|P^{(j)}_\alpha(p)|^q \varphi^{-j} |\alpha|^{jd} \leq C^q (\varphi^{q-1}|\alpha|^d)^j$ and $\varphi^{q-1}|\alpha|^d < 1$. It is also clear that if $G^+_{\Psi_\alpha}(p) > 0$, then $p \notin \mathcal{D}$.

Recall that for any $n \geq 0$,

$$g_n(z) = \varphi z_0 + z_1 + z_2^d \sum_{j=0}^{n} \left(P^{(j)}_\alpha(z)\right)^q \varphi^{-j} \alpha^{jd} = \left(P^{(n+1)}_\alpha(z) + \varphi^{-1} P^{(n)}_\alpha(z)\right) \varphi^{-n}. \tag{8.10}$$

If $p \in \Omega'' = \mathcal{V} \smallsetminus W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$, then as in the proof of Lemma 8.10, we see that the sequence $(P^{(n)}_\alpha(p)\varphi^{-n})_{n \geq 0}$ converges; moreover, we get from (8.10):

$$\lim_{n \to +\infty} g_n(p) = \lim_{n \to +\infty} \left(\frac{P^{(n+1)}_\alpha(p)}{P^{(n)}_\alpha(p)} + \varphi^{-1} \right) P^{(n)}_\alpha(p)\varphi^{-n} = \left(\varphi + \varphi^{-1}\right) \lim_{n \to +\infty} P^{(n)}_\alpha(p)\varphi^{-n}.$$

But we also know that $\lim_{n \to +\infty} |P^{(n)}_\alpha(p)\varphi^{-n}| > 0$, so that $g(p) = \lim_{n \to +\infty} g_n(p) \neq 0$ and $p \notin \mathcal{Z}$. This shows that if $p \in \mathcal{Z}$ then $p \in W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$; the other implication is always true.

The last point follows from the fact that for $\Omega' \subset \mathcal{V}$, we have $\Omega' \cap W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) = \Omega' \cap \mathcal{Z}$. Moreover, Ω' contains a neighborhood of $0_{\mathbb{C}^3}$ so the orbit of any point $p \in W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$ will eventually reach Ω'. To conclude, we note that by invariance of the stable manifold, we have $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) = \cup_{n \geq 0} \Psi_\alpha^{-n}(\Omega' \cap W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}))$. □

8.4. **Analysis of the dynamics in the case where $\varphi^{(1-q)/d} < |\alpha| < 1$.** Thanks to previous results, we show the following intermediate result concerning the dynamics of Ψ_α in this case.

Proposition 8.13. *Assume $\varphi^{(1-q)/d} < |\alpha| < 1$. For $p \in \mathcal{U} = \{z_2 = 0\}^c$, we show the following trichotomy:*

- *either p belongs to the stable manifold $W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$; in this case, its forward iterates converge to $0_{\mathbb{C}^3}$ with exponential speed;*

– *or there exist* $\varepsilon > 0$ *and* $n_0 \geq 0$ *such that for* $n \geq n_0$,
$$|P_\alpha^{(n)}(p)| \leq ((1-\varepsilon)\varphi)^n;$$
in this case $p \in \mathcal{Z}$. *Furthermore,*
$$\limsup_{n\to+\infty} |P_\alpha^{(n)}(p)|^{q-1}|\alpha|^{nd} > 0;$$

– *or the orbit of* p *escapes to infinity very fast:* $G^+_{\Psi_\alpha}(p) > 0$. *Moreover the sequence* $(|P_\alpha^{(n)}(p)|)_n$ *is increasing after a certain time.*

Proof. Take $0 < \varepsilon < 1 - \varphi^{-1}|\alpha|^{d/(1-q)}$. Note that $\mu := |\alpha|^d((1-\varepsilon)\varphi)^{q-1} > 1$. Let $p = (p_0, p_1, p_2) \notin W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})$; according to Corollary 5.5 its forward orbit is unbounded. Suppose that there exists $n_0 \geq 0$ such that for every $n \geq n_0$, $|P_\alpha^{(n)}(p)| \leq ((1-\varepsilon)\varphi)^n$. From Lemma 8.1, we know that $p \in \mathcal{Z}$. Moreover, if $\limsup_{n\to+\infty} |P_\alpha^{(n)}(p)|^{q-1}|\alpha|^{nd} = 0$, then with our previous notations, $\lim_{n\to+\infty} A(\Psi_\alpha^n(p)) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and the growth is at least Fibonacci, which is excluded. We are then in the second case of Proposition 8.13.

Let us handle the remaining case. In particular, we can take $n_0 \geq 0$ as big as we want such that $|P_\alpha^{(n_0)}(p)| > ((1-\varepsilon)\varphi)^{n_0}$. Note that since by assumption $|\alpha| > \varphi^{(1-q)/d}$, and we have $(q-1) + d\gamma > 0$ (with the notations of Lemma 8.4). In particular, $M = 1$ satisfies the hypotheses of this lemma. Take $n_0 \geq 0$ sufficiently large such that $|P_\alpha^{(n_0)}(p)| > ((1-\varepsilon)\varphi)^{n_0}$ and $\mu^{n_0}|p_2|^d \geq 2 + \varphi$. We can always assume that $|P_\alpha^{(n_0)}(p)| \geq |P_\alpha^{(n_0-1)}(p)|$.[4] Since $\left|(P_\alpha^{(n_0)}(p))^{q-1}(\alpha^{n_0})^d\right| \geq \mu^{n_0}$, we deduce
$$\left|\frac{P_\alpha^{(n_0+1)}(p)}{P_\alpha^{(n_0)}(p)}\right| = \left|(P_\alpha^{(n_0)}(p))^{q-1}(\alpha^{n_0})^d|p_2|^d + 1 + \frac{P_\alpha^{(n_0-1)}(p)}{P_\alpha^{(n_0)}(p)}\right| \geq \mu^{n_0}|p_2|^d - 2 \geq \varphi.$$
This shows that after time at most n_0, the sequence $(|P_\alpha^{(n)}(p)|)_n$ is increasing, moreover, there exists $C_0 > 0$ such that for any $n \geq 0$,
$$|P_\alpha^{(n)}(p)| \geq C_0\varphi^n.$$
The assumptions of Lemma 8.4 are satisfied, and we thus get the desired estimate on the speed. □

Remark 8.14. Denote by $\mathcal{S}$ the set of points corresponding to the second case described in Proposition 8.13. A priori, points in $\mathcal{S}$ might exhibit a rather complicated dynamics: their forward orbit is not bounded, still, it could happen that it does not escape to infinity. We will see that in fact this behavior does not occur: $\mathcal{S} = \emptyset$. This is related to the properties of the Hénon map ϕ_α to which Ψ_α is semi-conjugate: ϕ_α possesses an attractor at infinity which attracts any point whose forward orbit is not bounded.

Let us see how the previous result enables us to conclude the analysis of the dynamics of Ψ_α when $\varphi^{(1-q)/d} < |\alpha| < 1$. Note that in this case, $0_{\mathbb{C}^2}$ becomes a saddle point for ϕ_α. We have seen in §8.2 that $W^s_{\phi_\alpha}(0_{\mathbb{C}^2}) = K^+_{\phi_\alpha}$ is closed. Therefore, we recover the fact recalled above in the particular case of the saddle fixed point $0_{\mathbb{C}^2}$, and which asserts that $J^+_{\phi_\alpha} = \overline{W^s_{\phi_\alpha}(0_{\mathbb{C}^2})} = W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$. Let $p = (p_0, p_1, p_2) \in \mathbb{C}^3$. For $n \geq 0$,
$$\theta \circ \Psi_\alpha^n(p) = \left(P_\alpha^{(n)}(p)(\alpha^n p_2)^l, P_\alpha^{(n-1)}(p)(\alpha^n p_2)^l, \alpha^n p_2\right) = \left(\phi_\alpha^n \circ h(p), \alpha^n p_2\right). \tag{8.11}$$
Recall that $\mathcal{U} := \{z_2 = 0\}^c$ and that $\mathcal{S} \subset \mathcal{U}$ denotes the set of points whose behavior is described in the second item of Proposition 8.13. From the estimate on the speed we obtained, we know that
$$\mathcal{S} \subset \left\{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) = 0\right\}.$$

4. Else there exists $n_1 < n_0$ such that $|P_\alpha^{(n_1)}(p)| > ((1-\varepsilon)\varphi)^{n_0}$ and $|P_\alpha^{(n_1)}(p)| \geq |P_\alpha^{(n_1-1)}(p)|$ and we consider n_1 instead of n_0.

Since $G^+_{\Psi_\alpha} = G_{\phi_\alpha} \circ h$, we deduce that $h(\mathcal{S}) \subset \{\widetilde{p} \in \mathbb{C}^2 \mid G^+_{\phi_\alpha}(\widetilde{p}) = 0\} = W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$. Assume that $\mathcal{S}$ is non-empty and take $p \in \mathcal{S}$. From (8.11), and because by definition $\mathcal{S} \subset \mathcal{U}$, we see that $\lim_{n\to+\infty} |P^{(n)}_\alpha(p)| \cdot |\alpha|^{nl}$ exists and vanishes. Since $l = d/(q-1)$, this is in contradiction with the estimate $\limsup_{n\to+\infty} |P^{(n)}_\alpha(p)|^{q-1} \cdot |\alpha|^{nd} > 0$ given in Proposition 8.13. Let us rephrase what we have obtained:

Proposition 8.15. *When $\varphi^{(1-q)/d} < |\alpha| < 1$, the automorphism Ψ_α shares a certain number of properties with the Hénon automorphism ϕ_α. The point $0_{\mathbb{C}^3}$ is a fixed point of Ψ_α of saddle type, and $J^+_{\Psi_\alpha} := \partial K^+_{\Psi_\alpha} = \overline{W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})}$. Moreover, it follows from the previous discussion that*

$$\begin{array}{rcl} & h & \\ \{z_2 = 0\} & \to & \{0_{\mathbb{C}^2}\}; \\ K^+_{\Psi_\alpha} = W^s_{\Psi_\alpha}(0_{\mathbb{C}^3}) & \to & K^+_{\phi_\alpha} = W^s_{\phi_\alpha}(0_{\mathbb{C}^2}); \\ \{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) > 0\} & \to & \{\widetilde{p} \in \mathbb{C}^2 \mid G^+_{\phi_\alpha}(\widetilde{p}) > 0\}. \end{array} \tag{8.12}$$

In this situation, we see that the set Ω'' introduced in the case where $0 < |\alpha| < \varphi^{(1-q)/d}$ shrinks to the hyperplane $\{z_2 = 0\}$ which is contracted by h; in particular, it has empty interior.

8.5. **A few words on the case where $|\alpha| = 1$.** Note that in this case, the point $0_{\mathbb{C}^3}$ is still fixed by Ψ_α but it is no longer hyperbolic. We show the following trichotomy:

Proposition 8.16. *Let $p \in \mathcal{U} = \{z_2 = 0\}^c$. We have three possibilities:*

- *either $p \in K^+_{\Psi_\alpha}$, that is, its forward orbit is bounded;*
- *or $p \in \mathcal{Z} \smallsetminus K^+_{\Psi_\alpha}$; in particular, $|P^{(n)}_\alpha(p)| = o(\varphi^{n/q})$;*
- *or $G^+_{\Psi_\alpha}(p) > 0$; moreover the sequence $(|P^{(n)}_\alpha(p)|)_n$ is increasing after a certain time.*

Proof. Assume that $p \notin K^+_{\Psi_\alpha}$ and $p \in \mathcal{D}$. This implies $|P^{(n)}_\alpha(p)| = o(\varphi^{n/q})$. From Lemma 8.1, and since $q > 1$, we deduce that $p \in \mathcal{Z}$ and we are in the second case.

Let us then assume that $p \notin \mathcal{D}$ and fix $\varepsilon > 0$ such that $(1-\varepsilon)\varphi > 1$. From Lemma 8.1, we see that for every $n_0 \geq 0$, it is possible to find $n \geq n_0$ such that $|P^{(n)}_\alpha(p)| \geq ((1-\varepsilon)\varphi)^n$. Arguing as in the proof of Proposition 8.13, we see that the assumptions of Lemma 8.4 are satisfied after a certain time, and we conclude that we are in the third case described above. □

Remark 8.17. Note that for any $p \in \mathbb{C}^3$, either its forward orbit escapes to infinity with maximal speed (this corresponds to the third case), or $p \in \mathcal{Z}$. We denote by $\mathcal{S}'$ the set of points corresponding to the second case described in Proposition 8.16. We will show later that in fact $\mathcal{S}' = \emptyset$.

When $\alpha^n = 1$ for some $n \geq 1$, we see that the dynamics of Ψ_α is essentially given by the one of the Hénon automorphism $\phi = \phi_1$, so we assume in the following that α is not a root of unity.

Reasoning as before, (8.11) tells us that $h(K^+_{\Psi_\alpha}) = K^+_{\phi_\alpha} = \{\widetilde{p} \in \mathbb{C}^2 \mid G^+_{\phi_\alpha}(\widetilde{p}) = 0\}$, but now, $h(W^s_{\Psi_\alpha}(0_{\mathbb{C}^3})) \subsetneq W^s_{\phi_\alpha}(0_{\mathbb{C}^2})$. Again, $K^+_{\Psi_\alpha} \neq \{p \in \mathbb{C}^3 \mid G^+_{\Psi_\alpha}(p) = 0\}$ since there are points in $\{z_2 = 0\}$ escaping to infinity with Fibonacci speed. The point $0_{\mathbb{C}^2}$ is still a saddle point of ϕ_α, hence $J^+_{\phi_\alpha} := \partial K^+_{\phi_\alpha} = \overline{W^s_{\phi_\alpha}(0_{\mathbb{C}^2})}$. The map Ψ_α^{-1} is of the same form as Ψ_α, and similarly, we have $h(K^-_{\Psi_\alpha}) = K^-_{\phi_\alpha} = \{\widetilde{p} \in \mathbb{C}^2 \mid G^-_{\phi_\alpha}(\widetilde{p}) = 0\}$ as well as $J^-_{\phi_\alpha} := \partial K^-_{\phi_\alpha} = \overline{W^u_{\phi_\alpha}(0_{\mathbb{C}^2})}$.

We define $K_{\Psi_\alpha} := K^+_{\Psi_\alpha} \cap K^-_{\Psi_\alpha}$; note that $K_{\Psi_\alpha} \cap \{z_2 = 0\} = (\Delta_\varphi \times \{0\}) \cap (\Delta_{\varphi'} \times \{0\}) = \{0_{\mathbb{C}^3}\}$, and that $h(K_{\Psi_\alpha}) = K_{\phi_\alpha}$. We also see that $\theta|_{\mathcal{U}}$ maps bijectively $K_{\Psi_\alpha} \cap \mathcal{U}$ onto $K_{\Phi_\alpha} \cap \mathcal{U} = K_{\phi_\alpha} \times \mathbb{C}^*$. In particular,

$K_{\Psi_\alpha} = \{0_{\mathbb{C}^3}\} \cup \theta^{-1}(K_{\phi_\alpha} \times \mathbb{C}^*)$. Since $\theta_{|\mathcal{U}}$ is a biholomorphism, we deduce that:

$$J_{\Psi_\alpha} := \partial K_{\Psi_\alpha} = \partial(\{0_{\mathbb{C}^3}\} \cup \theta^{-1}(K_{\phi_\alpha} \times \mathbb{C}^*)) = \{0_{\mathbb{C}^3}\} \cup \overline{\theta^{-1}(J_{\phi_\alpha} \times \mathbb{C}^*)}.$$

Now, Proposition 8.16 implies that for any $p \in \mathcal{U}$, either $|P_\alpha^{(n)}(p)| = O(\varphi^{j/q})$ (this corresponds to the first and the second cases described in this proposition), or $G^+_{\Psi_\alpha}(p) > 0$. With the notations of Proposition 8.16, assume that $\mathcal{S}' \neq \emptyset$ and let $p \in \mathcal{S}' \subset (K^+_{\Psi_\alpha})^c$. In particular, $G^+_{\Psi_\alpha}(p) = 0$. But $G^+_{\Psi_\alpha}(p) = G^+_{\phi_\alpha}(h(p))$, so $h(p) \in \left\{\widetilde{p} \in \mathbb{C}^2 \mid G^+_{\phi_\alpha}(\widetilde{p}) = 0\right\} = K^+_{\phi_\alpha}$. Then Equation (8.11) implies that $p \in K^+_{\Psi_\alpha}$, a contradiction: we conclude that $\mathcal{S}' = \emptyset$.

We define a Green function $G^-_{\Psi_\alpha}$ in the same way as we did before, as well as a current $T^-_{\Psi_\alpha} := \mathrm{dd^c}(G^-_{\Psi_\alpha})$. We note that $T^\pm_{\Psi_\alpha}|_{\mathcal{U}} = (\theta_{|\mathcal{U}})^*(T^\pm_{\Phi_\alpha}|_{\mathcal{U}}) = (h_{|\mathcal{U}})^*(T^\pm_{\phi_\alpha}|_{\mathcal{U}})$, and by construction, the currents $T^\pm_{\Psi_\alpha}$ satisfy $\Psi_\alpha^*(T^\pm_{\Psi_\alpha}) = q^{\pm 1} \cdot T^\pm_{\Psi_\alpha}$. The measure

$$\mu_{\Psi_\alpha} := T^+_{\Psi_\alpha} \wedge T^-_{\Psi_\alpha} \wedge dz_2 \wedge d\overline{z_2}$$

is invariant by Ψ_α. Moreover, if we denote $\mu_{\Phi_\alpha} := \mu_{\phi_\alpha} \wedge dz_2 \wedge d\overline{z_2}$, then

$$\mu_{\Psi_\alpha|\mathcal{U}} = (h_{|\mathcal{U}})^*(\mu_{\phi_\alpha|\mathcal{U}}) \wedge dz_2 \wedge d\overline{z_2} = (\theta_{|\mathcal{U}})^*(\mu_{\Phi_\alpha|\mathcal{U}}). \tag{8.13}$$

Since μ_{ϕ_α} has support in the compact set $J_{\phi_\alpha} := \partial K_{\phi_\alpha}$, we deduce from (8.13) that μ_{Ψ_α} is supported on $J_{\Psi_\alpha} = \{0_{\mathbb{C}^3}\} \cup \overline{\theta^{-1}(J_{\phi_\alpha} \times \mathbb{C}^*)}$.

For every $p_2 \neq 0$, the set $\mathcal{C}_{p_2} := \mathbb{C}^2 \times \left\{p_2 e^{ix} \mid x \in \mathbb{R}\right\}$ is invariant both by Ψ_α and Φ_α. We know that $(\phi_\alpha, \mu_{\phi_\alpha})$ is mixing (in particular, weakly mixing), and for any $p_2 \neq 0$, the restriction of $z_2 \mapsto \alpha z_2$ to $\mathcal{C}_{p_2}$ is ergodic for $dz_2 \wedge d\overline{z_2}$, hence $(\Phi_\alpha|_{\mathcal{C}_{p_2}}, \mu_{\Phi_\alpha})$ is ergodic (see [7] for instance). We define $\mathcal{J}_{p_2} := J_{\Psi_\alpha} \cap \mathcal{C}_{p_2}$; this set is invariant, and we know that $\mu_{\Psi_\alpha}|_{\mathcal{J}_{p_2}}$ is supported on it. By (8.13), we conclude that $(\Psi_\alpha|_{\mathcal{J}_{p_2}}, \mu_{\Psi_\alpha})$ is ergodic too. Yet there is no hope to get mixing properties for Ψ_α since by projection on the third coordinate, $z_2 \mapsto \alpha z_2$ is a quasiperiodic factor of the dynamics. We have thus obtained:

Proposition 8.18. *For any point $p \in \mathbb{C}^3$, we are in exactly one of the following cases:*

- *either the orbit of p is bounded, i.e. $p \in K_{\Psi_\alpha}$;*
- *or $p \in \{z_2 = 0\} \setminus \{0_{\mathbb{C}^3}\}$;*
- *or $G^+_{\Psi_\alpha}(p) > 0$ or $G^-_{\Psi_\alpha}(p) > 0$.*

The measure μ_{Ψ_α} is invariant by Ψ_α and supported on the Julia set $J_{\Psi_\alpha} = \{0_{\mathbb{C}^3}\} \cup \overline{\theta^{-1}(J_{\phi_\alpha} \times \mathbb{C}^*)}$. Moreover, when $p_2 \neq 0$ and α is not a root of unity, $(\Psi_\alpha|_{\mathcal{J}_{p_2}}, \mu_{\Psi_\alpha})$ is ergodic.

References

[1] E. Bedford, M. Lyubich, and J. Smillie. Distribution of periodic points of polynomial diffeomorphisms of $\mathbf{C}^2$. *Invent. Math.*, 114(2):277–288, 1993.

[2] E. Bedford, M. Lyubich, and J. Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$. IV. The measure of maximal entropy and laminar currents. *Invent. Math.*, 112(1):77–125, 1993.

[3] E. Bedford and J. Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$. II. Stable manifolds and recurrence. *J. Amer. Math. Soc.*, 4(4):657–679, 1991.

[4] E. Bedford and J. Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$. III. Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.*, 294(3):395–420, 1992.

[5] M. Benedicks and L. Carleson. The dynamics of the Hénon map. *Ann. of Math. (2)*, 133(1):73–169, 1991.

[6] M. Brunella. Minimal models of foliated algebraic surfaces. *Bull. Soc. Math. France*, 127(2):289–305, 1999.

[7] L. A. Bunimovich, S. G. Dani, R. L. Dobrushin, I. P. Jakobson, M. V. Kornfeld, N. B. Maslova, Ya. B. Pesin, J. Smillie, Yu. M. Sukhov, and A. M. Vershik. *Dynamical systems, ergodic theory and applications*, volume 100. Springer Science & Business Media, 2000.

[8] R. Devaney and Z. Nitecki. Shift automorphisms in the Hénon mapping. *Comm. Math. Phys.*, 67(2):137–146, 1979.

[9] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimensions. In *Complex potential theory (Montreal, PQ, 1993)*, volume 439 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 131–186. Kluwer Acad. Publ., Dordrecht, 1994. Notes partially written by Estela A. Gavosto.

[10] S. Friedland. Entropy of algebraic maps. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993)*, number Special Issue, pages 215–228, 1995.

[11] S. Friedland and J. Milnor. Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Systems*, 9(1):67–99, 1989.

[12] V. Guedj and N. Sibony. Dynamics of polynomial automorphisms of $\mathbf{C}^k$. *Ark. Mat.*, 40(2):207–243, 2002.

[13] M. Heins. On a notion of convexity connected with a method of Carleman. *Journal d'Analyse Mathématique*, 7(1):53–77, 1959.

[14] M. Hénon. Numerical study of quadratic area-preserving mappings. *Quart. Appl. Math.*, 27:291–312, 1969.

[15] M. Hénon. A two-dimensional mapping with a strange attractor. *Comm. Math. Phys.*, 50(1):69–77, 1976.

[16] G. Julia. Mémoire sur la permutabilité des fractions rationnelles. *Ann. Sci. École Norm. Sup. (3)*, 39:131–215, 1922.

[17] H. W. E. Jung. Über ganze birationale Transformationen der Ebene. *J. Reine Angew. Math.*, 184:161–174, 1942.

[18] S. Lamy. L'alternative de Tits pour $\mathrm{Aut}[\mathbb{C}^2]$. *J. Algebra*, 239(2):413–437, 2001.

[19] J. F. Ritt. Permutable rational functions. *Trans. Amer. Math. Soc.*, 25(3):399–448, 1923.

[20] A. Russakovskii and B. Shiffman. Value distribution for sequences of rational mappings and complex dynamics. *Indiana Univ. Math. J.*, 46(3):897–932, 1997.

[21] J.-P. Serre. *Arbres, amalgames*, SL_2. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[22] N. Sibony. Dynamique des applications rationnelles de $\mathbf{P}^k$. In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris, 1999.

[23] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

[24] J. Smillie. The entropy of polynomial diffeomorphisms of $\mathbf{C}^2$. *Ergodic Theory Dynam. Systems*, 10(4):823–827, 1990.

UNIVERSITÉ PARIS DIDEROT, SORBONNE PARIS CITÉ, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, UMR 7586, CNRS, SORBONNE UNIVERSITÉS, UPMC UNIV PARIS 06, F-75013 PARIS, FRANCE.

E-mail address: `deserti@math.univ-paris-diderot.fr`

E-mail address: `martin.leguil@imj-prg.fr`