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QUARTO-QUARTIC BIRATIONAL MAPS OF $\mathbb{P}_3(\mathbb{C})$

JULIE DÉSERTI AND FRÉDÉRIC HAN

ABSTRACT. We construct a determinantal family of quarto-quartic transformations of a complex projective space of dimension 3 from trigonal curves of degree 8 and genus 5. Moreover we show that the variety of $(4,4)$-birational maps of $\mathbb{P}_3(\mathbb{C})$ has at least four irreducible components and describe three of them. *Keywords: Birational transformation, Determinantal, ACM, quarto-quartic.*

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1. INTRODUCTION

Let $\mathbb{P}_3$ and $\mathbb{P}'_3$ be two complex projective spaces of dimension 3. Denote by $\mathrm{Bir}_d(\mathbb{P}_3,\mathbb{P}'_3)$ the set of birational maps from $\mathbb{P}_3$ to $\mathbb{P}'_3$ defined by a linear system of degree d with base locus of codimension at least 2. A rational map $\phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$ is said to have bidegree (d_1,d_2) if it is an element of $\mathrm{Bir}_{d_1}(\mathbb{P}_3,\mathbb{P}'_3)$ such that its inverse is in $\mathrm{Bir}_{d_2}(\mathbb{P}'_3,\mathbb{P}_3)$. Let $\mathrm{Bir}_{d_1,d_2}(\mathbb{P}_3,\mathbb{P}'_3)$ be the quasi projective variety ([13]) of birational maps from $\mathbb{P}_3$ to $\mathbb{P}'_3$ of bidegree (d_1,d_2). It is obvious that a general surface in a linear system of degree d defining an element of $\mathrm{Bir}_d(\mathbb{P}_3,\mathbb{P}'_3)$ must be rational, but it should have a strong impact for $d>3$. Our first motivations to study $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ were to understand some typical differences between $d\leq 3$ and $d>3$. For $d\leq 3$ a large amount of examples was already known a long time ago ([11]) and recent works such as [15] or [4] were more involved with classification problems. But for $d>3$ only few examples are known. Basically, for $d>3$, the classical examples are built by lifting a birational transformation of a projective space of lower dimension ([5, §7.2.3]). In particular, for $d\geq 2$ monoidal linear systems give the de Jonquières family $\mathcal{J}_{d,d}$ (*see* [13], [14]). Another classical family (denoted by $\mathcal{R}_{d,d}$) can be constructed with surfaces of degree d with a line of multiplicity $d-1$. As expected from the components of $\mathrm{Bir}_{3,3}(\mathbb{P}_3,\mathbb{P}'_3)$, we show that the closure of $\mathcal{R}_{4,4}$ is an irreducible component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ (Proposition 3.13). A first surprise arises from normality properties. Indeed, in $\mathrm{Bir}_3(\mathbb{P}_3,\mathbb{P}'_3)$ linear systems without a normal surface were exceptional and similar to elements of $\mathcal{R}_{3,3}$, but it turns out that any element of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ with a linear system containing a normal surface must be in $\mathcal{J}_{4,4}$ (Lemma 3.5). So the closure of $\mathcal{J}_{4,4}$ is an irreducible component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ (Corollary 3.6) and most of the work occurs with non normal surfaces. On the other hand $\mathcal{J}_{3,3}$ is not an irreducible component of $\mathrm{Bir}_{3,3}(\mathbb{P}_3,\mathbb{P}'_3)$ because it is in the boundary of the classical cubo-cubic determinantal family ([4, §4.2.2]). The classical cubo-cubic transformations ([5, 7.2.2]) are built from Arithmetically Cohen Macaulay curves of degree 6 and genus 3. This family is so particular ([12]) that it was unexpected to find in $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ another ACM family. Hence most of the present work is about the construction of the ACM family $\mathcal{D}_{4,4}$ and its geometric properties. We first define elements of $\mathcal{D}_{4,4}$ via a direct construction of the resolution of their base ideal. This

resolution is non generic among the maps of the following type

$$\mathcal{O}_{\mathbb{P}_3}^2(-H) \oplus \mathcal{O}_{\mathbb{P}_3}(-2H) \longrightarrow \mathcal{O}_{\mathbb{P}_3}^4$$

and we provide an explicit construction in § 4.3. Then we get from Corollary 4.14 and Proposition 4.20 the following geometric description of $\mathcal{D}_{4,4}$.

Theorem A. *Let Γ be a trigonal curve of genus 5, and let $\overline{\Gamma}$ be its embedding in $\mathbb{P}_3$ by a general linear system $|\mathcal{O}_\Gamma(H)|$ of degree 8. Then*

- *the quartic surfaces containing $\overline{\Gamma}$ and singular along its unique 5-secant line give an element $\phi_{\overline{\Gamma}}$ of $\mathcal{D}_{4,4}$;*
- *the inverse of $\phi_{\overline{\Gamma}}$ is obtained by the same construction with the same trigonal curve Γ but embedded in $\mathbb{P}'_3$ with $|\omega_\Gamma^{\otimes 2}(-H)|$.*

We also geometrically describe the P-locus of $\phi_{\overline{\Gamma}}$ (*see* § 4.4). Finally we exhibit an other family $\mathcal{C}_{4,4}$ constructed with linear systems of quartic surfaces with a double conic. While studying components of $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ it turns out that the following invariant allows to distinguish all the components of $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ we find. Again, it is different in $\mathrm{Bir}_3(\mathbb{P}_3, \mathbb{P}'_3)$ where this invariant is not meaningful because in most examples it is 1 ([4, Prop. 2.6]).

Definition 1.1. [11, Chapter IX] Let ϕ be an element of $\mathrm{Bir}_d(\mathbb{P}_3, \mathbb{P}'_3)$, and let H (resp. H') be a general hyperplane of $\mathbb{P}_3$ (resp. $\mathbb{P}'_3$). The ***genus of*** ϕ is the geometric genus of the curve $H \cap \phi^{-1}(H')$.

Theorem B. *The quasi projective variety $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ has at least four irreducible components. Three of them are the closure of the families $\mathcal{D}_{4,4}$, $\mathcal{J}_{4,4}$ and $\mathcal{R}_{4,4}$. The family $\mathcal{C}_{4,4}$ is in another irreducible component. Furthermore we have*

$$\begin{array}{cccc} \dim(\mathcal{R}_{4,4}) = 37, & \dim(\mathcal{C}_{4,4}) = 37, & \dim(\mathcal{D}_{4,4}) = 46, & \dim(\mathcal{J}_{4,4}) = 54, \\ g(\phi_{\mathcal{R}_{4,4}}) = 0, & g(\phi_{\mathcal{C}_{4,4}}) = 1, & g(\phi_{\mathcal{D}_{4,4}}) = 2, & g(\phi_{\mathcal{J}_{4,4}}) = 3, \end{array}$$

where $g(\phi_X)$ denotes the genus of an element ϕ_X general in the family X.

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2. Definitions and first properties

Let $\phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$ be a rational map given, for some choice of coordinates, by

$$(z_0 : z_1 : z_2 : z_3) \dashrightarrow \big(\phi_0(z_0,z_1,z_2,z_3) : \phi_1(z_0,z_1,z_2,z_3) : \phi_2(z_0,z_1,z_2,z_3) : \phi_3(z_0,z_1,z_2,z_3)\big)$$

where the ϕ_i's are homogeneous polynomials of the same degree d without common factors. The ***indeterminacy set*** of ϕ is the set of the common zeros of the ϕ_i's. Denote by $\mathscr{I}_\phi$ the ideal generated by the ϕ_i's. Let F_ϕ be the scheme defined by $\mathscr{I}_\phi$; it is called ***base locus*** or ***base scheme*** of ϕ. If $\dim F_\phi = 1$ then define F^1_ϕ as the maximal subscheme of F_ϕ of dimension 1 without isolated point and without embedded point. The ***P-locus*** of ϕ is the union of irreducible hypersurfaces that are blown down to subvarieties of codimension at least 2 (*see* [5, § 7.1.4]).

Lemma 2.1. *Let $\phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$ be a $(4,d)$ birational map. Let H' be a general hyperplane of $\mathbb{P}'_3$. Let γ be the reduced scheme with support the singular locus of $\phi^{-1}(H')$ and consider the secant variety*

$$\mathrm{Sec}(\gamma) = \overline{\bigcup_{\substack{p,q\in\gamma\\ p\neq q}} \delta_{p,q}} \subset \mathbb{P}_3$$

where $\delta_{p,q}$ denotes the line through p and q. Then the codimension of $\mathrm{Sec}(\gamma)$ in $\mathbb{P}_3$ is strictly positive.

Proof. Let p, q be two distinct points of γ. The line $\delta_{p,q}$ passing through p and q intersects F^1_ϕ with length at least 4 since we have the inclusion of ideals $\mathscr{I}_{F^1_\phi} \subset \mathscr{I}^2_\gamma$; the line $\delta_{p,q}$ is thus blown down by $|\mathscr{I}_{F^1_\phi}(4H)|$.

But ϕ is dominant, so a general point of $\mathbb{P}_3$ is not in the P-locus of ϕ. Hence $\mathrm{codim}_{\mathbb{P}_3}(\mathrm{Sec}(\gamma)) > 0$. □

Corollary 2.2. *Let $\phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$ be a $(4,d)$ birational map. Let H' be a general hyperplane of $\mathbb{P}'_3$. Let γ be the reduced scheme defined by the singular locus of $\phi^{-1}(H')$. If $\dim\gamma = 1$, then the 1-dimensional part of γ is either a line, or a conic (possibly degenerated), or 3 lines through a fixed point.*

Proof. A general hyperplane section of the quartic surface $\phi^{-1}(H')$ is an irreducible reduced plane quartic curve. So it has at most 3 singular points hence $\deg(\gamma) \leq 3$ when $\dim(\gamma) = 1$. Moreover γ can't contain a plane curve of degree 3 because $\phi^{-1}(H')$ is an irreducible quartic surface. From Lemma 2.1, neither a space cubic curve, nor two disjoint lines, nor a line and a smooth conic can be in γ. Hence only a line, a conic or 3 lines through a fixed point are possible. □

Unfortunately we have not studied [1] families with 3 lines through a point because we first missed such examples.

Example 2.3. (Loria 1890, [11, Chapter XIV §8. $T_{4,4}$ Steiner quartics]) Let l_0, l_1, l_2 be lines in $\mathbb{P}_3$ of ideal (z_1,z_2), (z_2,z_0), (z_0,z_1) and O_1, O_2, O_3 be three general points in the plane $z_3 = 0$. For

1. We don't expect that this example is related to one of the 4 components described in Th. B; it seems to us that it should give one more component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$.

$1 \leq j \leq 3$, let q_j be the quadric cone containing the lines l_0, l_1, l_2 and the two points of O_1, O_2, O_3 distinct from O_j. Then the following quartics

$$(q_1q_2, q_0q_2, q_0q_1, z_0z_1z_2z_3)$$

gives a quarto-quartic birational transformation; the general element of this linear system is a Steiner quartic.

2.1. Families in $\mathrm{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3)$ and semi-continuity.

Definition 2.4. Let p, q be integers such that $\mathrm{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3)$ is not empty, and let S be a reduced subscheme of $\mathrm{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3)$. From the natural embedding of

$$\mathrm{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3) \subset \mathbb{P}\left(\mathrm{Hom}\left(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(1)), \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(p))\right)\right)$$

we obtain by pull back from S to $S \times \mathbb{P}_3$ the tautological sequence

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(1)) \otimes \mathcal{O}_{S\times\mathbb{P}_3}(-1,0) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(p)) \otimes \mathcal{O}_{S\times\mathbb{P}_3}.$$

It gives the following two maps and we can globalize over S the previous definitions of the base scheme and its singular locus.

— Let F_S be the subscheme of $S \times \mathbb{P}_3$ defined by the vanishing of the tautological map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(1)) \otimes \mathcal{O}_{S\times\mathbb{P}_3}(-1,0) \longrightarrow \mathcal{O}_{S\times\mathbb{P}_3}(0,p)$$

— Let F_S^σ be the subscheme of $S \times \mathbb{P}_3$ defined by the vanishing of the Jacobian map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(1)) \otimes (\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(1)))^\vee \otimes \mathcal{O}_{S\times\mathbb{P}_3}(-1,0) \longrightarrow \mathcal{O}_{S\times\mathbb{P}_3}(0,p-1)$$

— For an hyperplane H of $\mathbb{P}_3$ we define the restrictions to H as the following intersections in $S \times \mathbb{P}_3$

$$F_S \cdot H = F_S \cap (S \times H), \quad F_S^\sigma \cdot H = F_S^\sigma \cap (S \times H)$$

Corollary 2.5. *The morphisms $F_S \to S$ and $F_S^\sigma \to S$ are projective morphisms, moreover all the fibers of $F_S \to S$ have the same dimension.*

— *If $q < p^2$ we have for all ϕ in S,* $\dim F_\phi = 1$ *(*see *Notation 3.4),*

— *if $q = p^2$ we have for all ϕ in S,* $\dim F_\phi = 0$.

Let us recall the following application of the semi-continuity of the dimension of the fibers of a coherent sheaf.

Lemma 2.6. *Let $f\colon X \to Y$ be a finite morphism of Noetherian schemes over $\mathbb{C}$. The map*

$$Y \to \mathbb{Z},\ y \mapsto \deg(X_y)$$

is upper semi-continuous.

Proof. Let $\mathbb{C}(y)$ be the residual field of Y at y. Remark that the degree of X_y is the dimension of the complex vector space $f_*(\mathcal{O}_X) \otimes \mathbb{C}(y)$ because f is finite. By [9, Th. III-8.8] the sheaf $f_*(\mathcal{O}_X)$ is coherent on Y, so we conclude by semi-continuity of the dimension of the fibers of a coherent sheaf (*see* [9, Ex. III-12.7.2]). □

Corollary 2.7. *With the previous notation, the following functions are upper semi-continuous*

(1) $\alpha\colon S \to \mathbb{Z},\ \phi \mapsto \deg(F_\phi)$

(2) $\beta\colon S \to \mathbb{Z},\ \phi \mapsto \dim(F_\phi^\sigma)$

(3) $\eta\colon \beta^{-1}(\{1\}) \to \mathbb{Z},\ \phi \mapsto \deg(F_\phi^\sigma)$

Proof. The semi-continuity of β is from Chevalley's theorem (*see* [8] 13.1.5) because $F_S^\sigma \to S$ is proper. Let us prove the semi-continuity of α by showing that for any n in $\mathbb{Z}$ the set $\alpha^{-1}(]-\infty,n])$ is open. If it is not empty, then consider an arbitrary element ψ such that $\alpha(\psi) \leq n$. We can chose an hyperplane H of $\mathbb{P}_3$ such that $F_\psi \cap H$ is artinian of length $\alpha(\psi) = \deg(F_\psi)$. By Corollary 2.5 the projective morphism $\pi_S\colon F_S \cdot H \to S$ is generically finite. According to Chevalley's theorem the set

$$U = \{\phi \in S, \dim(\pi_S^{-1}(\{\phi\})) \leq 0\}$$

is an open set of S containing ψ. Hence over U we have a finite morphism $\pi_U\colon\ F_U \cdot H \to U$. Applying Lemma 2.6 to π_U we have an open set U' such that

$$\forall \phi \in U',\ \alpha(\phi) = \deg(F_\phi) \leq \text{length}(F_\phi \cap H) \leq \text{length}(F_\psi \cap H) = \alpha(\psi) \leq n.$$

So for any element ψ of $\alpha^{-1}(]-\infty,n])$ there is an open set U' of S such that

$$\psi \in U',\ U' \subset \alpha^{-1}(]-\infty,n]).$$

In conclusion $\alpha^{-1}(]-\infty,n])$ is open and α is upper semi-continuous on S. The proof is the same for η because it is also a projective morphism such that all its fibers are one dimensional. □

Let us also recall the following

Lemma 2.8. *Let $\pi\colon X \to Y$ be a projective morphism of noetherian schemes, and Ω_π be the sheaf of relative differentials, then for all $n \in \mathbb{N}$ the set*

$$\{x \in X,\ \text{rank}((\Omega_\pi)_x) \geq n\}$$

is closed in X, so its image by π is closed in Y.

Proof. From [9, Rem. II-8.9.1] the sheaf Ω_π is coherent on X. As a result the rank of its fiber is upper semi-continuous by [9, Ex. III-12.7.2] and $\{x \in X,\ \text{rank}((\Omega_\pi)_x) \geq n\}$ is closed in X. But π is projective hence the image of this closed set is closed in Y. □

3. Two classical families in $\text{Bir}_{d,d}(\mathbb{P}_3, \mathbb{P}_3')$

It is a classical construction to obtain birational transformations over projective spaces from a factorization through the blow up of a linear space and a birational transformation between projective spaces of lower dimension ([5, Def. 7.2.9]). The following one is a typical example of this construction.

3.1. The Monoidal de Jonquières family $\mathcal{J}_{d,d}$.

Many equivalent properties are detailed in ([13] or [14]) so let us choose the following definition.

Definition 3.1. Let d be a non zero integer, and let p be a point of $\mathbb{P}_3$. Choose coordinates such that the ideal of p in $\mathbb{P}_3$ is $\mathscr{I}_p = (z_0, z_1, z_2)$. Consider $S_d = z_3 P_{d-1} + P_d$ and $S_{d-1} = z_3 Q_{d-2} + Q_{d-1}$ with P_i, Q_i in $\mathbb{C}[z_0, z_1, z_2]$ homogeneous of degree i such that $\gcd(S_{d-1}, S_d) = 1$ and $P_{d-1}Q_{d-1} \neq P_d Q_{d-2}$. The rational map $\phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}_3'$ defined by an isomorphism $\mathbb{P}_3' \simeq |\mathscr{I}_\phi(dH)|^\vee$ with

$$\mathscr{I}_\phi = S_{d-1} \cdot \mathscr{I}_p + (S_d)$$

is birational of bidegree (d,d). The base scheme F_ϕ is defined by the monoidal complete intersection (S_{d-1},S_d) with an immerged point at p. Denote by $\mathcal{J}_{d,d}$ the corresponding family of $\mathrm{Bir}_{d,d}(\mathbb{P}_3,\mathbb{P}'_3)$.

Remark 3.2. With the notation of Definition 3.1 we can find coordinates on $\mathbb{P}'_3$ such that the map ϕ can be written

$$\left(z_0 : z_1 : z_2 : \frac{P_{d-1}z_3+P_d}{Q_{d-2}z_3+Q_{d-1}}\right)$$

it is then clear that the inverse of ϕ

$$\left(z_0 : z_1 : z_2 : \frac{Q_{d-1}z_3-P_d}{-Q_{d-2}z_3+P_{d-1}}\right)$$

is also a monoidal de Jonquières element of $\mathrm{Bir}_{d,d}(\mathbb{P}'_3,\mathbb{P}_3)$.

Corollary 3.3 ([13])**.** *The family $\mathcal{J}_{d,d}\subset \mathrm{Bir}_{d,d}(\mathbb{P}_3,\mathbb{P}'_3)$ is irreducible of dimension $2d^2+2d+14$.*

Proof. The choice of S_{d-1} up to factor is $\binom{d}{d-2}+\binom{d+1}{d-1}-1$ dimensional. The choice of S_d up to factor with $S_d\notin S_{d-1}\cdot\mathbb{C}[z_0,z_1,z_2]$ has dimension $\binom{d+2}{d}+\binom{d+1}{d-1}-3-1$. With the choices of an automorphism of $\mathbb{P}'_3$ and of a point $p\in\mathbb{P}_3$ we have $\dim \mathcal{J}_{d,d}=\binom{d}{d-2}+2\binom{d+1}{d-1}+\binom{d+2}{d}+13$. □

Let us recall that $\mathcal{J}_{3,3}$ is not an irreducible component of $\mathrm{Bir}_{3,3}(\mathbb{P}_3,\mathbb{P}'_3)$ (*see* [4, §4.2.2]), so the next Lemma is quite unexpected.

Notation 3.4. Let $\phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$ be a birational transformation of bidegree (p,q). For a general line l' in $\mathbb{P}'_3$ we will denote by

— $\phi^{-1}(l')$ the complete intersection in $\mathbb{P}_3$ of degree p^2 defined by l'.
— $C_1=\phi_*^{-1}(l')\subset\mathbb{P}_3$ the closure of the image by ϕ^{-1} of points of l' where ϕ^{-1} is regular. As a result C_1 is an irreducible curve of degree q and geometric genus 0.
— C_2 the residual scheme of C_1 in $\phi^{-1}(l')$. (Note that F^1_ϕ and C_2 are set theoretically identical but in general F^1_ϕ is just a subscheme of C_2.)

Furthermore for any scheme Z we denote by $\mathcal{I}_Z$ its ideal.

Lemma 3.5. *Let $\phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$ be a $(4,4)$-birational map. Let H' be a general hyperplane of $\mathbb{P}'_3$ and denote by S the surface $\phi^{-1}(H')$. If S is normal, then C_1 is a plane rational quartic and ϕ belongs to $\mathcal{J}_{4,4}$.*

Proof. The curves C_1 and C_2 are geometrically linked by quartic surfaces, so the dualizing sheaf of $C_1\cup C_2$ is $\omega_{C_1\cup C_2}=\mathcal{O}_{C_1\cup C_2}(4H)$ and we have the liaison exact sequence

$$0\longrightarrow \mathcal{I}_{C_1\cup C_2}(4H)\longrightarrow \mathcal{I}_{C_2}(4H)\longrightarrow \mathcal{H}om(\mathcal{O}_{C_1},\mathcal{O}_{C_1\cup C_2}(4H))\longrightarrow 0.$$

The surface S is normal and contains F^1_ϕ. Hence F^1_ϕ is generically locally complete intersection. So we have the scheme equality $F^1_\phi=C_2$. Thus $\mathrm{h}^0(\mathcal{I}_{C_2}(4H))=\mathrm{h}^0(\mathcal{I}_{F^1_\phi}(4H))\geq 4$. Therefore from the liaison exact sequence, the dualizing sheaf $\omega_{C_1}=\mathcal{H}om(\mathcal{O}_{C_1},\mathcal{O}_{C_1\cup C_2}(4H))$ has at least two independent sections. In other words the arithmetic genus of C_1 is greater or equal to 2. By Castelnuovo inequality (cf. [10, Th. 3.3]) the arithmetic genus of a non degenerate locally Cohen-Macaulay space curve of degree 4 is at most 1. Hence C_1 is a plane rational quartic. By liaison, C_2

is a complete intersection $(3,4)$. Moreover it gives the equality $\mathscr{H}om(\mathcal{O}_{C_1}, \mathcal{O}_{C_1\cup C_2}(4H)) = \mathcal{O}_{C_1}(H)$, so the restriction of ϕ to C_1 is a pencil of sections of $\mathcal{O}_{C_1}(H)$. But ϕ sends C_1 birationally to a line of $\mathbb{P}'_3$, hence this pencil must be the lines through a triple point p of the plane curve C_1. As S has a finite singular locus, the point p is independent of the choice of l' and it is a triple point for all the quartics in the linear system $|\mathscr{I}_{F_\phi}(4H)|$. In conclusion F_ϕ is a monoidal complete intersection $(3,4)$ with p as immerged point. Besides ϕ belongs to $\mathcal{J}_{4,4}$ because the irreducibility of $S = S_d$ implies $\gcd(S_{d-1}, S_d) = 1$ and $P_{d-1}Q_{d-1} \neq Q_{d-2}P_d$. □

Corollary 3.6. *The closure of $\mathcal{J}_{4,4}$ is an irreducible component of* $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$.

3.2. **The ruled family $\mathcal{R}_{d,d}$.** In this paragraph we will prove as a direct application of Lemma 3.10 the following

Proposition 3.7. *Consider a line δ in $\mathbb{P}_3$ and an integer d with $d \geq 2$. Let Δ_1, …, Δ_{d-1} be $d-1$ general disjoint lines that intersect δ in length one, and let p_1, …, p_{d-1} be $(d-1)$ general points in $\mathbb{P}_3$. Then the linear system $\left|\mathscr{I}_\delta^{d-1} \cap \mathscr{I}_{\Delta_1} \cap \ldots \cap \mathscr{I}_{\Delta_{d-1}} \cap \mathscr{I}_{p_1} \cap \ldots \cap \mathscr{I}_{p_{d-1}}(d \cdot H)\right|$ gives a birational map*

$$\mathbb{P}_3 \dashrightarrow \left|\mathscr{I}_\delta^{d-1} \cap \mathscr{I}_{\Delta_1} \cap \ldots \cap \mathscr{I}_{\Delta_{d-1}} \cap \mathscr{I}_{p_1} \cap \ldots \cap \mathscr{I}_{p_{d-1}}(d \cdot H)\right|^\vee.$$

Moreover the base scheme is defined by the ideal $\mathscr{I}_\delta^{d-1} \cap \mathscr{I}_{\Delta_1} \cap \ldots \cap \mathscr{I}_{\Delta_{d-1}} \cap \mathscr{I}_{p_1} \cap \ldots \cap \mathscr{I}_{p_{d-1}}$ and has degree $\frac{(d+2)(d-1)}{2}$.

Definition 3.8. Birational maps of Proposition 3.7 composed with an isomorphism

$$\left|\mathscr{I}_\delta^{d-1} \cap \mathscr{I}_{\Delta_1} \cap \ldots \cap \mathscr{I}_{\Delta_{d-1}} \cap \mathscr{I}_{p_1} \cap \ldots \cap \mathscr{I}_{p_{d-1}}(d \cdot H)\right|^\vee \simeq \mathbb{P}'_3$$

form the family $\mathcal{R}_{d,d} \subset \mathrm{Bir}_{d,d}(\mathbb{P}_3, \mathbb{P}'_3)$.

Remark 3.9. Note that surfaces of degree d in such linear systems have a line of multiplicity $d-1$, so they are ruled. For $d > 3$ there may exist ruled surfaces of degree d without a line of multiplicity $d-1$. Nevertheless we keep the terminology "ruled" for $\mathcal{R}_{d,d}$ to emphasize the analogy with the traditional naming convention when $d = 3$.

Lemma 3.10. *With notation of Proposition 3.7, the image of $\mathbb{P}_3$ by the linear system*

$$\left|\mathscr{I}_\delta^{d-1} \cap \mathscr{I}_{\Delta_1} \cap \ldots \cap \mathscr{I}_{\Delta_{d-1}}(d \cdot H)\right|$$

is a non degenerate 3-dimensional variety X_d of degree d of $\mathbb{P}_{d+2}$.

Remark 3.11. Let us note that X_d is of minimal degree (*see* [6]).

Proof of Lemma 3.10. Let us choose coordinates such that the ideal of δ in $\mathbb{P}_3$ is $\mathscr{I}_\delta = (z_0, z_1)$. Let $\mathscr{I}_{\Delta_j}$ be (ℓ_j, ℓ'_j) with

— $\gcd(\ell_j, \ell'_k) = 1$ for $1 \leq k, j < d$;
— $\gcd(\ell_j, \ell_k) = 1$, $\gcd(\ell'_j, \ell'_k) = 1$, for $1 \leq k, j < d, k \neq j$;
— $\ell_j \in \mathscr{I}_\delta$, $\gcd(\ell_j, z_0) = 1$;
— $\ell'_j \notin \mathscr{I}_\delta$.

Set $\mathcal{F}_0 = \mathscr{I}_\delta^{d-1} \cap \mathscr{I}_{\Delta_1} \cap \ldots \cap \mathscr{I}_{\Delta_{d-1}}$. The ideal $\mathcal{F}_0$ is defined by the $(d-1)\times(d-1)$ minors of the matrix

$$M = \begin{pmatrix} z_0\ell'_1 & z_0\ell'_2 & \ldots & z_0\ell'_{d-1} \\ \ell_1 & 0 & \ldots & 0 \\ 0 & \ell_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \ell_{d-1} \end{pmatrix}.$$

Let $\widetilde{\mathbb{P}}_3 \subset \mathbb{P}_1 \times \mathbb{P}_3$ be the blow up of $\mathbb{P}_3$ at δ. Denote by s (resp. H) the hyperplane class of $\mathbb{P}_1$ (resp. $\mathbb{P}_3$) and by $\widetilde{M}$ the strict transform of the entries of M. The resolution of the ideal of the proper transform of $\cup_{i=1}^{d-1}\Delta_i$ is

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1}(-s) \xrightarrow{\widetilde{M}} \mathcal{O}_{\widetilde{\mathbb{P}}_3}(H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1} \to \mathscr{I}_Z((d-1)s+H) \to 0. \tag{3.1}$$

This surjection gives a rational map

$$\widetilde{\mathbb{P}}_3 \dashrightarrow X_d \subset \mathrm{Proj}(Sym(\mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(H))) \subset \widetilde{\mathbb{P}}_3 \times \mathbb{P}_{d+2}$$

with $\mathbb{P}_{d+2} = |\mathscr{I}_Z((d-1)s+H)|^\vee$. Denote by H' the hyperplane class of $\mathbb{P}_{d+2}$, so the class of X_d in $\mathrm{Proj}(Sym(\mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(H)))$ is $(H'+s)^{d-1}$ with the relation $H'^d = H'^{d-1}\cdot H$ ([7, Example 8.3.4]). Since we have the equality

$$(H'+s)^{d-1}\cdot H'^3 = d,$$

the image of $\widetilde{\mathbb{P}}_3$ by $|\mathscr{I}_Z((d-1)s+H)|$ is of degree d in $\mathbb{P}_{d+2}$. Moreover, from resolution (3.1), it is not in a hyperplane of $\mathbb{P}_{d+2}$. □

Proof of Proposition 3.7. With notation of Lemma 3.10 choose $d-1$ points $(p_i)_{1\le i<d}$ in general position in X_d. They span in $\mathbb{P}_{d+2}$ a projective space π_{d-2} of dimension $d-2$. The projection from π_{d-2} restricts to a birational map from X_d to a 3-dimensional projective space. To see that the intersection $\pi_{d-2}\cap X_d$ is the reduced scheme defined by the $(p_i)_{1\le i<d}$, let us consider an additional general point p_d of X_d and denote by π_{d-1} the projective space spanned by the $(p_i)_{1\le i\le d}$. The threefold X_d is non degenerate of degree d in $\mathbb{P}_{d+2}$, so $X_d\cap\pi_{d-1}$ is the intersection of X_d by a general linear space of codimension three, hence it is just the d points $(p_i)_{1\le i\le d}$. In conclusion $\pi_{d-2}\cap X_d$ is the reduced scheme defined by the $(p_i)_{1\le i<d}$. Moreover we can assume that these points are in the locus where $\mathbb{P}_3 \dashrightarrow X_d$ is an isomorphism. As a result the base scheme in $\mathbb{P}_3$ of the composition $\mathbb{P}_3 \dashrightarrow X_d$ with the linear projection from π_{d-2} is defined by the ideal $\mathscr{I}_\delta^{d-1} \cap \mathscr{I}_{\Delta_1}\cap\ldots\cap\mathscr{I}_{\Delta_{d-1}}\cap\mathscr{I}_{p_1}\cap\ldots\cap\mathscr{I}_{p_{d-1}}$, it thus has degree $\frac{d(d-1)}{2}+(d-1)$. □

Corollary 3.12. *The family $\mathcal{R}_{d,d} \subset \mathrm{Bir}_{d,d}(\mathbb{P}_3,\mathbb{P}'_3)$ is irreducible of dimension*

$$4+3(d-1)+3(d-1)+15 = 6d+13.$$

Proposition 3.13. *The closure of $\mathcal{R}_{4,4}$ is an irreducible component of* $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$.

Proof. Let $\mathcal{B}$ be an irreducible component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ containing $\mathcal{R}_{4,4}$. Let ϕ be a general element of $\mathcal{B}$, and let ψ be a general element of $\mathcal{R}_{4,4}$. According to Proposition 3.7 the scheme F^1_ψ has degree 9 and ψ is in $\mathcal{B}$, thus by Corollary 2.7 the set $\alpha^{-1}(]-\infty,9])$ is non empty and open in $\mathcal{B}$ so the general element ϕ satisfies $\deg(F^1_\phi)\le 9$. With Notation 3.4 we have $\deg C_2 = 12$ so F^1_ϕ has

a component that is not locally complete intersection. Let H be a general hyperplane of $\mathbb{P}_3$, and let $\mathscr{S}$ be the set of points p such that $F^1_\phi \cap H$ is not locally complete intersection at p. Denote by Z (resp. Y) the union of the irreducible components of $F^1_\phi \cap H$ (resp. $C_2 \cap H$) supported on points p of $\mathscr{S}$. We can now bound the degree of Z. By Corollary 2.5 and Chevalley's semi-continuity theorem there is an open subset U of $\alpha^{-1}(]-\infty,9])$ (hence of $\mathcal{B}$) such that $\psi \in U$ and $\pi_U\colon F_U \cdot H \to U$ is finite (with notation of Def. 2.4). By Lemma 2.8 the set

$$\Sigma = \{(u,p) \in F_U \cdot H \subset U \times H,\ \mathrm{rank}((\Omega_{\pi_U})_{(u,p)}) \geq 2\}$$

is closed in $F_U \cdot H$. As a consequence $\pi_U\colon \Sigma \to U$ is also finite. Moreover it is onto because $U \subset \alpha^{-1}(]-\infty,9])$ so any element of U must have a base scheme that is not locally complete intersection. But the base scheme of ψ have three smooth points so the degree of the fiber Σ_ψ is at most $9-3=6$. Lemma 2.6 implies the inequality $\deg(\Sigma_\phi) \leq 6$ for the general element ϕ of U, hence of $\mathcal{B}$. But $Z \subset \Sigma_\phi$ hence $\deg(Z) \leq 6$ and $\deg(Y) - \deg(Z) = \deg(C_2) - \deg(F^1_\phi) \geq 3$. The cardinal of $\mathscr{S}$ is at most 2 because $\deg(Z) \leq 6$ and schemes of length at most 2 are locally complete intersection. We thus have the following possibilities:

— if $\mathscr{S} = \{p_1, p_2\}$, $p_1 \neq p_2$, then $\mathscr{I}_Z = \mathscr{I}^2_{p_1} \cap \mathscr{I}^2_{p_2}$ because $\deg(Z) \leq 6$. Therefore each component of Y has length 4 and $\deg(Y) - \deg(Z) = 8-6$ can't be 3.

— if $\mathscr{S} = \{p\}$ with $\mathscr{I}_p = (x,y)$, then we have the following analytic classification of all non locally complete intersection ideals of $\mathbb{C}\{x,y\}$ of colength at most 6 (*see* [3, §4.2])

length(Z)	ideal of Z	length of a general complete intersection Y
$n+1$	$(y^2, xy, x^n), 2 \leq n \leq 5$	$n+2$
$n+2$	$(y^2 + x^{n-1}, x^2y, x^n)$, $3 \leq n \leq 4$	$n+3$
$n+2$	(y^2, x^2y, x^n), $3 \leq n \leq 4$	$2n$
6	$(y(y+x), x^2y, x^4)$	7
6	(x^3, x^2y, xy^2, y^3)	9

and note that the only solution to $\deg(Y) - \deg(Z) \geq 3$, $\deg(Z) \leq 6$ is the last case.

Hence F^1_ϕ has a triple line Δ and a smooth curve of degree 3 denoted by $\mathcal{K}$. Any bisecant line to $\mathcal{K}$ that intersects Δ must be in all the quartics defining ϕ so it is in F^1_ϕ. As a result $\mathcal{K}$ is a union of 3 disjoint lines that intersect Δ, and ϕ belongs to $\mathcal{R}_{4,4}$. □

4. The determinantal family $\mathcal{D}_{4,4}$

Notation 4.1. Let $\mathbb{P}_1$, $\mathbb{P}_3$, $\mathbb{P}'_3$ be three complex projective spaces of dimension 1, 3, 3; denote by s, H, H' their hyperplane class and by L, A, A' the following vector spaces

$$L = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1}(s)), \quad A = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(H)), \quad A' = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(H')).$$

4.1. Construction of $\mathcal{D}_{4,4}$.

4.1.1. *Description via* $\widetilde{\mathbb{P}}_3$. Let X be a complete intersection in $\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3$ given by the vanishing of a general section of the bundle

$$\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}'_3}(s+H) \oplus \mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}'_3}(s+H') \oplus \mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}'_3}(H+H') \oplus \mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}'_3}(s+H+H')$$

Lemma 4.2. *The projections from X to $\mathbb{P}_3$ and $\mathbb{P}'_3$ are birational.*

Proof. It is an immediate computation from the class of

$$X \sim (s+H)\cdot(s+H')\cdot(H+H')\cdot(s+H+H')$$

so $X\cdot H^3 = s\cdot H^3\cdot H'^3 = X\cdot H'^3$. □

Notation 4.3. Let n_0 (resp. n_1) be a non zero section of $\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}'_3}(s+H)$ (resp. $\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}'_3}(s+H')$) vanishing on X. The section n_0 defines in $\mathbb{P}_1\times\mathbb{P}_3$ a divisor $\widetilde{\mathbb{P}}_3$ isomorphic to the blow up of $\mathbb{P}_3$ along a line Δ.

Lemma 4.4. *The complete intersection X is the blow up of $\widetilde{\mathbb{P}}_3$ along the curve Γ of ideal $\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}$ with resolution defined by a general map $\widetilde{G}$*

$$0\to \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s)\oplus\mathcal{O}_{\widetilde{\mathbb{P}}_3}(-H)\oplus\mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s-H)\xrightarrow{\widetilde{G}} A'\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_3}\to\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H)\to 0. \tag{4.1}$$

Moreover,

— *the curve Γ is smooth irreducible of genus 5 and $|\mathcal{O}_\Gamma(s)|$ is a g^1_3,*

— *the sheaf $\mathcal{O}_\Gamma(H)$ has degree 8.*

Proof. The resolution of $\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H)$ claimed in (4.1) is just the direct image by the first projection of the resolution of $\mathcal{O}_X(H')$ as $\mathcal{O}_{\widetilde{\mathbb{P}}_3\times\mathbb{P}'_3}$-module. So the map $\widetilde{G}$ is general. Since the bundle $A'\otimes(\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s)\oplus\mathcal{O}_{\widetilde{\mathbb{P}}_3}(H)\oplus\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s+H))$ is globally generated the curve Γ is smooth because it is the degeneracy locus of the general map $\widetilde{G}$ ([2, § 4.1], *see* [16, Th. 1]). Moreover, the resolution (4.1) gives the vanishing $\mathrm{h}^1(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3})=0$. As a consequence $\mathrm{h}^0(\mathcal{O}_\Gamma)=1$ and Γ is connected and smooth so irreducible. The dualizing sheaf of $\widetilde{\mathbb{P}}_3$ is $\omega_{\widetilde{\mathbb{P}}_3}=\mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s-3H)$ so $\mathscr{E}xt^1(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H),\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s-H))=\omega_\Gamma$. Besides the functor $\mathscr{H}om(\cdot,\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s-H))$ applied to sequence (4.1) gives the following exact sequence

$$0\to\mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s-3H)\to A'^{\vee}\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s-H)\to\mathcal{O}_{\widetilde{\mathbb{P}}_3}(2s-H)\oplus\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s)\oplus\mathcal{O}_{\widetilde{\mathbb{P}}_3}(2s)\to\omega_\Gamma\to 0. \tag{4.2}$$

As a result $\mathrm{h}^0(\omega_\Gamma)=5$ and Γ has genus 5. The class of Γ is obtained from the degeneracy locus formula for $\widetilde{G}$ by computing the coefficient of t^2 in the serie $\frac{1}{(1-s\cdot t)(1-H\cdot t)(1-(s+H)\cdot t)}$. Therefore in the Chow ring of $\widetilde{\mathbb{P}}_3$ one has

$$\Gamma\sim 5s\cdot H+3H^2.$$

Hence $\Gamma\cdot H$ has degree 8, $\Gamma\cdot s$ has degree 3 and $|\mathcal{O}_\Gamma(s)|$ is a g^1_3. □

4.1.2. *Description in $\mathbb{P}_3$.* Let G be the composition $\widetilde{G}\circ\begin{pmatrix}(E_\Delta)&0&0\\0&1&0\\0&0&(E_\Delta)\end{pmatrix}$ where (E_Δ) is an equation of the exceptional divisor E_Δ of $\widetilde{\mathbb{P}}_3$.

Proposition 4.5. *Let $\mathscr{I}_{\overline{\Gamma}}$ be the ideal of the projection $\overline{\Gamma}$ of Γ in $\mathbb{P}_3$, and let $\mathscr{I}_\Delta$ be the ideal of the line Δ defined in Notation 4.3. Then*

— *the line Δ is 5-secant to $\overline{\Gamma}$,*

— *one has the following exact sequence*

$$0 \to \mathcal{O}^2_{\widetilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\mathbb{P}_3}(-2H) \xrightarrow{G} A' \otimes \mathcal{O}_{\mathbb{P}_3} \to (\mathscr{I}^2_\Delta \cap \mathscr{I}_{\overline{\Gamma}})(4H) \to 0. \tag{4.3}$$

— *the linear system* $|(\mathscr{I}^2_\Delta \cap \mathscr{I}_{\overline{\Gamma}})(4H)|$ *gives a birational map from* $\mathbb{P}_3$ *to* $\mathbb{P}'_3$ *that factors through* X.

Proof. The intersection of $\Delta \cap \overline{\Gamma}$ is computed in $\widetilde{\mathbb{P}}_3$ from Lemma 4.4 by $\Gamma \cdot E_\Delta = \Gamma \cdot H - \Gamma \cdot s = 8-3$. From Lemma 4.4 one gets the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathcal{O}^2_{\widetilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-2H) \arrow[r, "G"] \arrow[d] & A' \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_3} \arrow[r] \arrow[d, equal] & \operatorname{coker}(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s-H) \arrow[r, "\widetilde{G}"] \arrow[d] & A' \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_3} \arrow[r] & \mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H) \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}_{E_\Delta}(-s) \oplus \mathcal{O}_{E_\Delta}(-s-H) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

so $\operatorname{coker}(G)$ is in the extension

$$0 \to \mathcal{O}_{E_\Delta}(-s) \oplus \mathcal{O}_{E_\Delta}(-s-H) \to \operatorname{coker}(G) \to \mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H) \to 0$$

and we have the isomorphism $\rho_*(\operatorname{coker}(G)) = \rho_*(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H))$ where ρ is the projection from $\widetilde{\mathbb{P}}_3$ to $\mathbb{P}_3$. This gives the resolution (4.3) after applying ρ to the exact sequence

$$0 \to \mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H) \to \mathcal{O}_{\widetilde{\mathbb{P}}_3}(4H) \to \mathcal{O}_{2E_\Delta \cup \Gamma}(4H) \to 0$$

□

In the next section we will prove that any trigonal curve of genus 5 embedded in $\mathbb{P}_3$ by a general line bundle of degree 8 is the curve $\overline{\Gamma}$ for some choice of $X \subset \mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3$.

Remark 4.6. The inverse of this rational map is obtained by the same construction from n_1 and $\mathbb{P}'_3$ (Notation 4.3). Hence the inverse map is also given by quartic polynomials.

So let us introduce the following

Definition 4.7. The birational maps constructed in Proposition 4.5 will be called ***determinantal quarto-quartic birational map***. They form an irreducible family denoted by $\mathcal{D}_{4,4}$.

Proposition 4.8. *The family* $\mathcal{D}_{4,4} \subset \mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ *is irreducible of dimension* 46.

Remark 4.9. In Corollary 4.15 we will prove that this family turns out to be an irreducible component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}_3')$.

Proof of Proposition 4.8. First let us detail the choices made to get X. The choice of a complete intersection of this type is equivalent to the choices of:

i) $\overline{n_0}\in\mathbb{P}\left(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}_3'}(s+H))\right)$,

ii) $\overline{n_1}\in\mathbb{P}\left(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}_3'}(s+H'))\right)$,

iii) $\overline{n_2}\in\mathbb{P}\left(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}_3'}(H+H'))\right)$,

iv) $\overline{n_3}\in\mathbb{P}\left(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_3\times\mathbb{P}_3'}(s+H+H'))/(\overline{n_0}\cdot A'+\overline{n_1}\cdot A+\overline{n_2}\cdot L)\right)$.

So the choice of X is made in an irreducible variety of dimension $7+7+15+21=50$. Let $I_2\subset\mathbb{P}_3\times\mathbb{P}_3'$ be the divisor defined by $\overline{n_2}$. Pushing down $\mathcal{O}_X(s)$ to I_2 one gets the following resolution of the ideal $\mathcal{G}_X$ of the graph of the birational transformation constructed from X

$$0\to L^\vee\otimes\mathcal{O}_{I_2}\xrightarrow{\begin{pmatrix}n_0\\n_1\\n_3\end{pmatrix}}\mathcal{O}_{I_2}(H)\oplus\mathcal{O}_{I_2}(H')\oplus\mathcal{O}_{I_2}(H+H')\to\mathcal{G}_X(2H+2H')\to 0$$

so $\dim(\mathcal{D}_{4,4})=50-\dim(GL(L^\vee))=46$. □

In § 4.3 we provide an explicit construction of this map and its inverse and give more geometric properties.

4.2. **Trigonal curves of genus 5.** In the previous section we obtained a trigonal curve of genus 5 naturally embedded in $\widetilde{\mathbb{P}}_3$. To understand how this construction is general, we start in this section with an abstract trigonal curve of genus 5, then we choose some line bundle and explain how to obtain the previous construction. Notation of the various line bundles introduced here will turn out to be compatible with notation in the previous section.

4.2.1. *Model in* $\widetilde{\mathbb{P}}_2$. The following result is detailed in [1, Chap. 5]. Let Γ be any trigonal curve of genus 5, let ω_Γ be its canonical bundle, and denote by $\mathcal{O}_\Gamma(s)$ the line bundle of degree 3 generated by two sections. Without any extra choice we have the following embedding of Γ:

— The linear system $|\omega_\Gamma(-s)|$ sends Γ to a plane curve of degree 5. This plane curve has a unique singular point p_0 (a node or an ordinary cusp).

— Let $\widetilde{\mathbb{P}}_2$ be the blow up of this plane at p_0, denote by E_0 the exceptional divisor and by h the hyperplane class of this plane:

$$|\mathcal{O}_{\widetilde{\mathbb{P}}_2}(h)|\colon\widetilde{\mathbb{P}}_2\longrightarrow\mathbb{P}_2=|\omega_\Gamma(-s)|^\vee$$

4.2.2. *Embeddings of degree* 8 *in* $\mathbb{P}_3$. With notation of section 4.2.1. Let $O_\Gamma(H)$ be a general line bundle of degree 8 on Γ. One has $\mathrm{h}^0(O_\Gamma(H-s)) = 1$. The vanishing locus of this section defines in Γ a unique effective divisor D'_5 of degree 5 such that $O_\Gamma(H) = O_\Gamma(D'_5+s)$. Moreover $\mathrm{h}^0(O_\Gamma(2h-D'_5))$ is also 1 so there is a unique effective divisor D_5 on Γ such that $O_\Gamma(2h) = O_\Gamma(D_5+D'_5)$. Note that both D_5 and D'_5 have degree 5. So the picture in $\mathbb{P}_2$ looks like this. The curve Γ is sent via $|O_\Gamma(h)|$ to a quintic curve singular at p_0. This curve contains two divisors D_5, D'_5 of degree 5 such that $D_5+D'_5$ is the complete intersection of the plane quintic with a conic. From the general assumptions on H (hence on D'_5) the conic is smooth, it doesn't contain the point p_0 and both D_5 and D'_5 are supported by five distinct points. Let $p_1, p_2, \ldots, p_5$ be the five points of D_5, and let S_3 be the blow up of $\mathbb{P}_2$ at the six points $p_0, p_1, \ldots, p_5$. Let $(E_i)_{0\le i\le 5}$ be the corresponding exceptional divisors. Remark that $O_\Gamma(H)$ is the restriction to Γ of $O_{S_3}(3h-\sum_{i=0}^5 E_i)$; the linear system $|O_\Gamma(H)|$ embeds Γ on the cubic surface S_3 in $\mathbb{P}_3$. Denote by $\overline{\Gamma}$ this curve of degree 8 in $\mathbb{P}_3$ and keep the notation H for the class $3h-\sum_{i=0}^5 E_i$ on S_3. We can summarize these data in the following diagram:

$$\begin{tikzcd}
 & \Gamma \arrow[r, "|O_\Gamma(H)|", "\sim"'] \arrow[dl, "|O_\Gamma(s)|"'] \arrow[d] & \overline{\Gamma} & \subset & S_3 \arrow[d, equal] & \subset & \mathbb{P}_3 \\
\mathbb{P}_1 & \widetilde{\mathbb{P}}_2 = \widetilde{\mathbb{P}_2(p_0)} \arrow[l] \arrow[d] & & & \widetilde{\mathbb{P}_2(p_0,p_1,\ldots,p_5)} \arrow[ll] & & \\
 & |O_\Gamma(h)|^\vee = \mathbb{P}_2 & & & & &
\end{tikzcd}$$

Remark 4.10. Any irreducible smooth curve of degree 8 and genus 5 in $\mathbb{P}_3$ with a 5-secant line Δ is trigonal.

Proof. Indeed, the planes containing Δ give a base point free linear system of degree 3. □

Notation 4.11. Let Δ be the line of S_3 equivalent to $2h-\sum_{i=1}^5 E_i$. One has $\Delta\cap\overline{\Gamma} = D'_5$ and Δ is the unique line 5-secant to $\overline{\Gamma}$. Denote by $\widetilde{\mathbb{P}}_3$ the blow up of $\mathbb{P}_3$ in Δ and by $\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}$ the ideal of the proper transform of $\overline{\Gamma}$.

Keeping notation s, H for the pull back of the hyperplane classes of $\mathbb{P}_1$ and $\mathbb{P}_3$ on $\widetilde{\mathbb{P}}_3$ one gets the following

Lemma 4.12. *We have an exact sequence*

$$0 \longrightarrow O_{\widetilde{\mathbb{P}}_3}(s) \longrightarrow \mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H) \longrightarrow O_{S_3}(H-E_0) \longrightarrow 0.$$

Hence $\mathrm{h}^0(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H)) = 4$ *and the sheaf* $\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H)$ *is globally generated.*

Proof. As $\Delta\subset S_3$ the ideal of S_3 in $\widetilde{\mathbb{P}}_3$ is $\mathscr{I}_{S_3|\widetilde{\mathbb{P}}_3} = O_{\widetilde{\mathbb{P}}_3}(-s-2H)$. The class of $\overline{\Gamma}$ in S_3 is $5h-2E_0-\sum_{i=1}^5 E_i$ and $s\sim H-\Delta\sim h-E_0$; the ideal of $\overline{\Gamma}$ in S_3 is thus $\mathscr{I}_{\Gamma|S_3} = O_{S_3}(-H-2s-E_0)$. Hence the following exact sequence of ideals

$$0 \longrightarrow \mathscr{I}_{S_3|\widetilde{\mathbb{P}}_3} \longrightarrow \mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3} \longrightarrow \mathscr{I}_{\Gamma|S_3} \longrightarrow 0$$

twisted by $2s+2H$ gives the exact sequence of the statement. To conclude just remark that $\mathrm{h}^1(\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s))=0$ and that both $\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s)$ and $\mathcal{O}_{S_3}(H-E_0)$ have two sections and are globally generated. □

Proposition 4.13. *Let A' be the 4-dimensional vector space $\mathrm{H}^0(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H))$; let us consider $\mathscr{N}$ defined by the following exact sequence*

$$0\longrightarrow \mathscr{N}\longrightarrow A'\otimes \mathcal{O}_{\widetilde{\mathbb{P}}_3}\longrightarrow \mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+2H)\longrightarrow 0.$$

We have

$$\mathscr{N}\simeq \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s)\oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-H)\oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s-H).$$

Proof. First remark that $\mathscr{N}$ is locally free. Indeed, the sheaf $\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}$ is the ideal of a 1-dimensional scheme of $\widetilde{\mathbb{P}}_3$ without embedded nor isolated points. As a consequence for all closed point x of $\widetilde{\mathbb{P}}_3$ we have the vanishing of the following localizations

$$\forall i\geq 2,\quad \mathscr{E}xt^i(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3,x},\mathcal{O}_{\widetilde{\mathbb{P}}_3,x})=0$$

so

$$\forall i\geq 1,\quad \mathscr{E}xt^i(\mathscr{N}_x,\mathcal{O}_{\widetilde{\mathbb{P}}_3,x})=0$$

and $\mathscr{N}$ is locally free. Now construct an injection from $\mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s)\oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-H)\oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s-H)$ to $\mathscr{N}$.

— Since $\mathrm{h}^0(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(3s+2H))=\mathrm{h}^0(\mathcal{O}_{\widetilde{\mathbb{P}}_3}(3s)+\mathrm{h}^0(\mathcal{O}_{S_3}(2H-E_0-\Delta))=7$ and $\mathrm{h}^0(A'\otimes \mathcal{O}_{\widetilde{\mathbb{P}}_3}(s))=8$ we have $\mathrm{h}^0(\mathscr{N}(s))\geq 1$.

— $\mathrm{h}^0(\mathscr{N}(H))\geq 2$ because $\mathrm{h}^0(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s+3H))=\mathrm{h}^0(\mathcal{O}_{\widetilde{\mathbb{P}}_3}(s+H))+\mathrm{h}^0(\mathcal{O}_{S_3}(2H-E_0))=14$ and $\mathrm{h}^0(A'\otimes \mathcal{O}_{\widetilde{\mathbb{P}}_3}(H))=16$.

— As $\mathrm{h}^0(\mathscr{I}_{\Gamma|\widetilde{\mathbb{P}}_3}(3s+3H))=\mathrm{h}^0(\mathcal{O}_{\widetilde{\mathbb{P}}_3}(2s+H))+\mathrm{h}^0(\mathcal{O}_{S_3}(3H-E_0-\Delta))=21$ and $\mathrm{h}^0(A'\otimes \mathcal{O}_{\widetilde{\mathbb{P}}_3}(s+H))=28$ we have $\mathrm{h}^0(\mathscr{N}(s+H))\geq 7$.

We are thus able to find a non zero section σ_0 of $\mathscr{N}(s)$ then a section σ_1 of $\mathscr{N}(H)$ independent with $E_\Delta\cdot\sigma_0$ (with notation of § 4.1.2), then a section σ_2 of $\mathscr{N}(s+H)$ not in $A\otimes\sigma_0\oplus L\otimes\sigma_1$ (Notation 4.1). As a result σ_0, σ_1 and σ_2 give an injection of $\mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s)\oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-H)\oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(-s-H)$ into $\mathscr{N}$. Moreover these two vector bundles have the same first Chern class, so it is an isomorphism. □

In other words, Proposition 4.13 gives the existence of the map $\widetilde{G}$ from Lemma 4.4 for any trigonal curve Γ of genus 5, therefore we have

Corollary 4.14. *Let $\overline{\Gamma}$ be a trigonal curve of genus 5 embedded in $\mathbb{P}_3$ by a general linear system of degree 8. The quartics containing $\overline{\Gamma}$ and singular along its unique 5-secant line give an element of $\mathcal{D}_{4,4}$.*

Corollary 4.15. *The closure of $\mathcal{D}_{4,4}$ is an irreducible component of* $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$.

Proof. From Corollary 4.8 the family $\mathcal{D}_{4,4}$ is already irreducible. Choose an irreducible component $\mathcal{B}$ of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ containing $\mathcal{D}_{4,4}$. Let ϕ be a general element of $\mathcal{B}$, and let ψ be a general element of $\mathcal{D}_{4,4}$. With notation of §2 the degree of the 1-dimensional scheme F_ψ is 11 and ψ is in

$\mathcal{B}$. So by Corollary 2.7 for the family $\mathcal{B}$, the degree of the 1-dimensional scheme F_ϕ is at most 11 because ϕ is general in $\mathcal{B}$. For a general hyperplane H' of $\mathbb{P}'_3$ the quartic surface $\phi^{-1}(H')$ is thus not normal. Indeed, we proved in Lemma 3.5 that if $\phi^{-1}(H')$ is normal then the schemes F^1_ϕ and C_2 are equal and $\deg F_\phi = 12$. With Definition 2.4 we have $\dim(F^\sigma_\phi) = 1$, hence $\deg(F^\sigma_\phi) \geq 1$. But with notation of Corollary 2.7 we have $\beta(\phi) = 1, \beta(\psi) = 1$ and $\eta(\psi) = 1$. As a consequence by semi-continuity of η we have $\deg(F^\sigma_\phi) = 1$ for the general element ϕ of $\mathcal{B}$. There is thus a line Δ in $\mathbb{P}_3$ such that $\phi^{-1}(H')$ is singular on Δ. Denote again by $\widetilde{\mathbb{P}}_3$ the blow up of $\mathbb{P}_3$ in Δ, then ϕ factors through a rational map $\widetilde{\phi}\colon \widetilde{\mathbb{P}}_3 \dashrightarrow \mathbb{P}'_3$ via a linear subsystem W_ϕ of $|O_{\widetilde{\mathbb{P}}_3}(2s+2H)|$. Now adapt Notation 3.4 to geometric liaison in $\widetilde{\mathbb{P}}_3$ instead of $\mathbb{P}_3$. Let ℓ' be a general line of $\mathbb{P}'_3$, set $\widetilde{C}_1 = \widetilde{\phi}_*^{-1}(\ell')$, and let $\widetilde{C}_2$ be the residual of $\widetilde{C}_1$ in the complete intersection $\widetilde{\phi}^{-1}(\ell')$. The curves $\widetilde{C}_1$ and $\widetilde{C}_2$ are geometrically linked by two hypersurfaces of class $2s+2H$. The liaison exact sequence gives

$$0 \longrightarrow \mathscr{I}_{\widetilde{C}_1\cup\widetilde{C}_2}(2s+2H) \longrightarrow \mathscr{I}_{\widetilde{C}_2}(2s+2H) \longrightarrow \omega_{\widetilde{C}_1}(H-s) \longrightarrow 0.$$

By Lemma 2.8 the curves C_1, $\widetilde{C}_1$ and $\widetilde{C}_2$ are smooth because it is already true for $\widetilde{\psi}$. Thus $\widetilde{C}_2$ is the base scheme of $\widetilde{\phi}$ and all the elements of W_ϕ vanish on $\widetilde{C}_2$. Hence $\mathrm{h}^0(\mathscr{I}_{\widetilde{C}_2}(2s+2H)) \geq 4$ so we have $\mathrm{h}^0(\omega_{\widetilde{C}_1}(H-s)) \geq 2$ on the smooth rational curve $\widetilde{C}_1$. Hence $O_{\widetilde{C}_1}(H-s)$ has degree at least 3. The class of $\widetilde{C}_1$ in the Chow ring of $\widetilde{\mathbb{P}}_3$ is thus $\widetilde{C}_1 \sim (4-a).H^2 + aH\cdot s$ with $a \geq 3$. But $\widetilde{C}_1 \cdot s = 4-a \geq 1$ because C_1 is irreducible of degree 4 and not in a plane containing Δ. So $a = 3$ and the class of $\widetilde{C}_2$ is $3H^2 + 5H\cdot s$. As a consequence $|O_{\widetilde{C}_2}(s)|$ is a g^1_3. The restrictions to $\widetilde{C}_1$ and $\widetilde{C}_2$ of the exact sequences

$$0 \longrightarrow \omega_{\widetilde{C}_{3-i}} \longrightarrow \omega_{\widetilde{C}_1\cup\widetilde{C}_2} \longrightarrow \omega_{\widetilde{C}_1\cup\widetilde{C}_2} \otimes O_{\widetilde{C}_i} \longrightarrow 0 \qquad i \in \{1,2\}$$

show that $\widetilde{C}_1 \cap \widetilde{C}_2$ has length 9 and that $\widetilde{C}_2$ has genus 5. As a result the base locus of $\widetilde{\phi}$ is a trigonal curve of genus 5, so ϕ belongs to $\mathcal{D}_{4,4}$ by Corollary 4.14. □

4.3. **Explicit construction of the birational map and its inverse.** In the next paragraphs we want to expose properties similar to the usual properties of the classical cubo-cubic transformation such as explicit computation of the inverse and isomorphism between the base loci ([5, §7.2.2]).

4.3.1. *Explicit construction of* $\overline{\tau}\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$. In the proof of Proposition 4.8 we detailed the choices needed to construct the complete intersection $X \subset \mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3$. Let us replace them by the choice of some linear maps.

Definition 4.16. With Notation 4.1, choose general linear maps N_0, N_1, M, T between the following vector spaces

$$N_0\colon L^\vee \longrightarrow A, \quad N_1\colon L^\vee \longrightarrow A', \quad M\colon A^\vee \longrightarrow A', \quad T\colon \begin{array}{ccc} L^\vee & \longrightarrow & \mathrm{Hom}(A^\vee, A') \\ \lambda & \longmapsto & T_\lambda \end{array}.$$

For any $\lambda \in L^\vee$ the image of λ by T will be denoted by T_λ. Choose also generators for $\wedge^2 L, \wedge^4 A, \wedge^4 A'$ and denote the corresponding isomorphisms by

$$B\colon L \xrightarrow{\sim} L^\vee, \quad \alpha\colon \wedge^3 A \xrightarrow{\sim} A^\vee, \quad \alpha'\colon \wedge^3 A' \xrightarrow{\sim} A'^\vee.$$

The duality bracket between a vector space and its dual will be denoted by $< \cdot, \cdot >$.

For any z in $A^\vee$ let us define the elements g_0, g_1, g_2 of A' by

$$g_0 = N_1 \circ B \circ {}^{\mathrm{t}}N_0(z), \quad g_1 = M(z), \quad g_2 = T_{B\circ {}^{\mathrm{t}}N_0(z)}(z).$$

Now consider the quartic map

$$\begin{array}{rccc} \tau\colon & A^\vee & \longrightarrow & A'^\vee \\ & z & \longmapsto & \alpha'(g_0 \wedge g_1 \wedge g_2) \end{array}$$

and also the rational map $\overline{\tau}\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$ induced by τ.

Example 4.17. Take basis for L, A, A' and choose the identity matrix for M. With

$$N_0 = N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, U_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, U_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

define $T\colon (\lambda_0, \lambda_1) \mapsto \lambda_0 U_0 + \lambda_1 U_1$. Then we have

$$G = \begin{pmatrix} -z_1 & z_0 & -z_1^2 + z_0 z_3 \\ z_0 & z_1 & z_0^2 - z_1 z_2 \\ 0 & z_2 & z_0 z_1 - z_1 z_3 \\ 0 & z_3 & -z_0 z_1 + z_0 z_2 \end{pmatrix}.$$

Denote by $\Delta_k(G)$ the determinant of the matrix G with its k-th line removed. The map $\overline{\tau}$ defined by

$$(z_0 : z_1 : z_2 : z_3) \dashrightarrow (\Delta_1(G) : -\Delta_2(G) : \Delta_3(G) : -\Delta_4(G))$$

is birational (see § 4.3.2) and is in the closure of $\mathcal{D}_{4,4}$.

4.3.2. *Explicit construction of* $\overline{\tau}^{-1}$. For any y in $A'^\vee$ denote by g'_0, g'_1, g'_2 the following elements of A

$$g'_0 = -N_0 \circ B \circ {}^{\mathrm{t}}N_1(y), \quad g'_1 = {}^{\mathrm{t}}M(y), \quad g'_2 = {}^{\mathrm{t}}(T_{B\circ {}^{\mathrm{t}}N_1(y)})(y).$$

Now consider the quartic map

$$\begin{array}{rccc} \tau'\colon & A'^\vee & \longrightarrow & A^\vee \\ & y & \longmapsto & \alpha(g'_0 \wedge g'_1 \wedge g'_2) \end{array}$$

and also the rational map $\overline{\tau'}\colon \mathbb{P}'_3 \dashrightarrow \mathbb{P}_3$ induced by τ'.

Lemma 4.18. *For $z \in A^\vee$ and $y \in A'^\vee$ we have*

$$y \wedge \tau(z) = 0 \iff \forall i \in \{0,1,2\}, < y, g_i >= 0 \iff z \wedge \tau'(y) = 0.$$

Proof. By definition of α and α' we have

$$\begin{array}{rcl} y \wedge \tau(z) = 0 & \iff & \forall i \in \{0,1,2\} < y, g_i >= 0, \\ z \wedge \tau'(y) = 0 & \iff & \forall i \in \{0,1,2\} < z, g'_i >= 0. \end{array}$$

Now remark that $< y, g_0 >= 0 \iff < {}^{\mathrm{t}}N_1(y), B \circ {}^{\mathrm{t}}N_0(z) >= 0 \iff < g'_0, z >= 0$. Note that it is also equivalent to the proportionality of ${}^{\mathrm{t}}N_1(y)$ and ${}^{\mathrm{t}}N_0(z)$. Furthermore remind that T is general in

$\mathrm{Hom}(L^\vee, \mathrm{Hom}(A^\vee, A'))$, so it is injective. Thus we have $< y, g_0 >= 0$ if and only if $T_{B\circ {}^tN_1(y)}$ and $T_{B\circ {}^tN_0(z)}$ are proportional in $\mathrm{Hom}(A^\vee, A')$. Therefore we have

$$\left\{ \begin{array}{l} < y, g_0 >= 0 \\ < y, g_2 >= 0 \end{array} \right. \iff \left\{ \begin{array}{l} T_{B\circ {}^tN_1(y)} \wedge T_{B\circ {}^tN_0(z)} = 0 \\ < y, T_{B\circ {}^tN_1(y)}(z) >= 0 \end{array} \right. \iff \left\{ \begin{array}{l} < g_0', z >= 0 \\ < g_2', z >= 0 \end{array} \right. .$$

We achieve the proof because $< y, g_1 >=< g_1', z >$. □

Corollary 4.19. *For $(\lambda, z) \in L^\vee \times A^\vee$, consider the elements of $A'^\vee$ given by*

$$\widetilde{g_0} = N_1(\lambda),\ \widetilde{g_1} = M(z),\ \widetilde{g_2} = T_\lambda(z).$$

Thus $\overline{\tau'} \circ \overline{\tau} = \mathrm{id}_{\mathbb{P}_3}$ and $\overline{\tau}$ is the element of $\mathcal{D}_{4,4}$ constructed with

$$\widetilde{G}_{(\lambda,z)} = (\widetilde{g_0}, \widetilde{g_1}, \widetilde{g_2}) \text{ and } G_z = (g_0, g_1, g_2).$$

Proof. Lemma 4.18 directly implies that $\overline{\tau'} \circ \overline{\tau}$ is the identity map of $\mathbb{P}_3$. Remind that for $(\lambda, z) \in L^\vee \times A^\vee$ the corresponding point of $\mathbb{P}_1 \times \mathbb{P}_3$ is in $\widetilde{\mathbb{P}}_3$ if and only if $< \lambda, {}^tN_0(z) > = 0$. But it is equivalent to the proportionality of λ and $B \circ {}^tN_0(z)$. So for all $y \in A'^\vee$ and $(\overline{\lambda}, \overline{z}) \in \widetilde{\mathbb{P}}_3$, we have $y \wedge \tau(z) = 0$ if and only if $< y, \widetilde{g_i} >= 0$ for any $i \in \{0, 1, 2\}$. Thus $\overline{\tau}$ is the element of $\mathcal{D}_{4,4}$ constructed from $\widetilde{G}$. □

4.3.3. *The isomorphism between Γ and Γ'.*

Proposition 4.20. *Let $\overline{\tau}\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}_3'$ be a element of $\mathcal{D}_{4,4}$. Let Γ (resp. Γ') be the trigonal curve of genus 5 defining $\overline{\tau}$ (resp. $\overline{\tau}^{-1}$). Then Γ' is isomorphic to Γ and the embedding of Γ in $\mathbb{P}_3'$ is given by*

$$\mathcal{O}_\Gamma(H') = \omega_\Gamma^{\otimes 2}(-H).$$

We will prove this result after the construction in Lemma 4.23 of two exact sequences over $\widetilde{\mathbb{P}}_2$.

Notation 4.21. In the context of § 4.2.1 we have

$$\widetilde{\mathbb{P}}_2 \subset \mathbb{P}_1 \times \mathbb{P}_2,\ U = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_2}(h)),\ L = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1}(s)).$$

The equation of $\widetilde{\mathbb{P}}_2$ in $\mathbb{P}_1 \times \mathbb{P}_2$ gives a linear map

$$N_2\colon L^\vee \longrightarrow U.$$

Let $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-s) \xrightarrow{\Lambda} L^\vee \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}$ be the pull back of the tautological injection on $\mathbb{P}_1$, and let Q, Q' be the cokernels of $N_0 \circ \Lambda$ and $N_1 \circ \Lambda$. Hence we have the exact sequences defining π and π'

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-s) \xrightarrow{N_0\circ\Lambda} A \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2} \xrightarrow{\pi} Q \to 0,$$

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-s) \xrightarrow{N_1\circ\Lambda} A' \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2} \xrightarrow{\pi'} Q' \to 0.$$

Remark 4.22. Both Q and Q' are isomorphic to $\mathcal{O}_{\widetilde{\mathbb{P}}_2}^2 \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2}(s)$, but we prefer to not choose one of these isomorphisms to avoid confusions.

— Choose a linear map $\gamma\colon U^\vee \to \mathbb{C}$ such that the map

$$\begin{array}{rcl} U^\vee & \to & \mathbb{C}\oplus L \\ u & \mapsto & (\gamma(u), {}^t N_2(u)) \end{array}$$

has rank 3.

— Define

$$\widetilde{R}\colon \begin{array}{rcl} U^\vee & \longrightarrow & \mathrm{Hom}(A^\vee, A') \\ u & \longmapsto & \widetilde{R}_u = \gamma(u)\cdot M + T_{B\circ {}^t N_2(u)} \end{array}$$

and denote by $\widetilde{R}_u$ the image of u by $\widetilde{R}$.

— Set $R = \pi'\circ \widetilde{R}\circ {}^t\pi$.

We have the commutative diagram

$$\begin{tikzcd} A^\vee(-h) \arrow[r, "\widetilde{R}"] & A'\otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2} \arrow[d, "\pi'"] \\ Q^\vee(-h) \arrow[u, hook, "{}^t\pi"] \arrow[r, "R"] & Q' \end{tikzcd}$$

Lemma 4.23. *We have the following exact sequences of $\mathcal{O}_{\widetilde{\mathbb{P}}_2}$-modules*

$$0 \longrightarrow Q^\vee(-h) \xrightarrow{R} Q' \longrightarrow \mathcal{O}_\Gamma(H') \longrightarrow 0,$$

$$0 \longrightarrow Q'^\vee(-h) \xrightarrow{{}^tR} Q \longrightarrow \mathcal{O}_\Gamma(H) \longrightarrow 0.$$

Proof. Remark first that the class in $\mathrm{Pic}(\widetilde{\mathbb{P}}_2)$ of the determinant of R is $2s+3h$ and that it is also the class of Γ in $\widetilde{\mathbb{P}}_2$. Nevertheless we need to explicitly relate points p of $\widetilde{\mathbb{P}}_2$ such that R_p is not injective to points p of $\widetilde{\mathbb{P}}_3$ such that $\widetilde{G}_p$ is not injective. So consider (λ, u) in $L^\vee\times U^\vee$ such that the corresponding point $(\overline{\lambda},\overline{u})$ of $\mathbb{P}_1\times\mathbb{P}_2$ is in $\widetilde{\mathbb{P}}_2$. In other words we have

$$<\lambda, {}^tN_2(u)>=0.$$

The map $R_{(\overline{\lambda},\overline{u})}$ is not injective if and only if there exists z in the image of ${}^t\pi_{(\overline{\lambda},\overline{u})}$ such that $\widetilde{R}_u(z)$ and $N_1(\lambda)$ are proportional. Therefore

$$(R_{(\overline{\lambda},\overline{u})} \text{ not injective }) \iff (\exists z\in A^\vee-\{0\},\ <z, N_0(\lambda)>=0 \text{ and } \widetilde{R}_u(z)\wedge N_1(\lambda)=0).$$

By definition $\widetilde{R}_u(z)=\gamma(u)\cdot M(z)+T_\lambda(z)$ so $\widetilde{R}_u(z)\wedge N_1(\lambda)=0$ implies that $N_1(\lambda), M(z), T_\lambda(z)$ are linearly dependent elements of A'. Now remind that $<z, N_0(\lambda)>=0$ is just the equation of $\widetilde{\mathbb{P}}_3$ in $\mathbb{P}_1\times\mathbb{P}_3$, we thus have

$$\Big((\overline{\lambda},\overline{u})\in\widetilde{\mathbb{P}}_2 \text{ and } R_{(\overline{\lambda},\overline{u})} \text{ not injective }\Big) \Rightarrow \Big(\exists\,\overline{z}\in\mathbb{P}_3 \text{ such that } (\overline{\lambda},\overline{z})\in\widetilde{\mathbb{P}}_3 \text{ and } \widetilde{G}_{(\overline{\lambda},\overline{z})} \text{ not injective }\Big).$$

Conversely, if $(\overline{\lambda},\overline{z})\in\widetilde{\mathbb{P}}_3$ and $N_1(\lambda), M(z), T_\lambda(z)$ are linearly dependent then there exists by definition of γ an element $u\in U^\vee-\{0\}$ such that $(\gamma(u)\cdot M(z)+T_\lambda(z))\wedge N_1(\lambda)=0$ and $<\lambda, {}^tN_2(u)>=0$. As a result

$$\Big((\overline{\lambda},\overline{z})\in\widetilde{\mathbb{P}}_3 \text{ and } \widetilde{G}_{(\overline{\lambda},\overline{z})} \text{ not injective }\Big) \Rightarrow \Big(\exists\,\overline{u}\in\mathbb{P}_2 \text{ such that } (\overline{\lambda},\overline{u})\in\widetilde{\mathbb{P}}_2 \text{ and } R_{(\overline{\lambda},\overline{u})} \text{ not injective }\Big).$$

These two implications show that the cokernel of ${}^tR$ is $O_\Gamma(H)$. The statement follows by exchanging $\mathbb{P}_3$ and $\mathbb{P}'_3$. □

Proof of Proposition 4.20. Lemma 4.23 implies

$$O_\Gamma(H') = \mathscr{E}xt^1(O_\Gamma(H), O_{\widetilde{\mathbb{P}}_2}(-h)) = \omega_\Gamma \otimes O_{\widetilde{\mathbb{P}}_2}(s+h) = \omega_\Gamma^{\otimes 2}(-H).$$

□

4.4. **Geometric description of the P-locus.**

4.4.1. *The ruled surface S_9.* Remind from § 4.1.1 that $\overline{\tau}' \colon \mathbb{P}'_3 \dashrightarrow \mathbb{P}_3$ factors through $X \to \widetilde{\mathbb{P}}_3$. Therefore if N_Γ denotes the normal bundle of Γ in $\widetilde{\mathbb{P}}_3$, Sequence (4.1) shows that $\overline{\tau}'$ contracts the ruled surface $\widetilde{S}'_9 = \mathrm{Proj}(Sym(N_\Gamma^\vee(2s+2H)))$ to Γ. Moreover, the exact sequence of Lemma 4.12 restricted to Γ gives

$$0 \longrightarrow O_\Gamma(s) \longrightarrow N_\Gamma^\vee(2s+2H) \longrightarrow O_\Gamma(H-E_0) \longrightarrow 0.$$

Sections of $O_\Gamma(s)$ are thus hyperplane sections of $\mathrm{Proj}(Sym(N_\Gamma^\vee(2s+2H)))$ and each of them contains three rules of this surface. So the projection of this surface in $\mathbb{P}'_3$ is the union of the "triangles" with vertices given by the g^1_3. In a symmetric way, the P-locus of $\overline{\tau}$ contains a ruled surface $S_9 \subset \mathbb{P}_3$. The basis of S_9 is isomorphic to Γ and its rules are the lines of $\mathbb{P}_3$ that cut an element of the g^1_3 in length 2. In other words, S_9 is the union of the bi-secant lines of the intersection of $\overline{\Gamma} \subset \mathbb{P}_3$ with any plane containing the 5-secant Δ. The smooth model $\widetilde{S}_9$ of S_9 is an extension of $O_\Gamma(H'-E_0)$ by $O_\Gamma(s)$; hence S_9 has degree 9 and $\overline{\tau}(S_9)$ is the curve Γ embedded in $\mathbb{P}'_3$ by $O_\Gamma(H') = \omega_\Gamma^{\otimes 2}(-H)$ (Proposition 4.20). It is easy to explain geometrically why S_9 is contracted by $\overline{\tau}$. Indeed, by the previous description, the rules of Γ intersect the scheme defined by the ideal $\mathscr{I}_\Delta^2 \cap \mathscr{I}_{\overline{\Gamma}}$ in length 4, so these lines are contracted by $|\mathscr{I}_\Delta^2 \cap \mathscr{I}_{\overline{\Gamma}}(4H)|$.

4.4.2. *The cubic surface S_3.* It is interesting to understand geometrically why the cubic surface S_3 defined in § 4.2.2 is also contracted by $\overline{\tau}$ because it can't be explained anymore by 4-secant lines. This time, consider the planes of $\mathbb{P}_3$ containing the line $E_0 \subset S_3$. The general element of the linear system $|O_{S_3}(H-E_0)|$ gives a conic of $\mathbb{P}_3$ intersecting Δ in a point and $\overline{\Gamma}$ in six points. Any of these conics thus cuts the scheme defined by $\mathscr{I}_\Delta^2 \cap \mathscr{I}_{\overline{\Gamma}}$ in length at least 8. So these conics are contracted by $|\mathscr{I}_\Delta^2 \cap \mathscr{I}_{\overline{\Gamma}}(4H)|$ and $\overline{\tau}(S_3)$ is the line 5-secant to the image of Γ in $\mathbb{P}'_3$. We have achieved the description of the P-locus of $\overline{\tau}$ with the following

Corollary 4.24. *The jacobian of $\overline{\tau}$ is the product of equations of S_9 and S_3.*

Proof. Indeed, the jacobian of $\overline{\tau}$ has degree 12 and vanishes on S_9 and S_3. □

5. On the components of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$

In the previous sections we found elements ϕ of $\mathrm{Bir}_{4,4}(\mathbb{P}_3,\mathbb{P}'_3)$ such that for a general hyperplane H' of $\mathbb{P}'_3$, the quartic surface $\phi^{-1}(H')$ has the following properties:

— $\phi \in \mathcal{J}_{4,4} \iff \phi^{-1}(H')$ is normal (Lemma 3.5) $\iff$ the genus of ϕ is 3.

— If ϕ belongs to $\mathcal{D}_{4,4}$, then $\phi^{-1}(H')$ has a double line; furthermore $\phi^{-1}(H')$ has a double line if and only if the genus of ϕ is 2.

— The singular locus of $\phi^{-1}(H')$ cannot contain two disjoint lines or a reduced cubic curve (Corollary 2.2).

— If ϕ is in $\mathcal{R}_{4,4}$, then $\phi^{-1}(H')$ has a triple line and the genus of ϕ is 0.

So it remains to provide a genus one example or to know if $\phi^{-1}(H')$ can be singular along a conic. This is achieved with the following example.

5.1. The family $\mathcal{C}_{4,4}$.

Proposition 5.1. *Let p be a point of $\mathbb{P}_3$. Choose some coordinates such that the ideal of p in $\mathbb{P}_3$ is $\mathscr{I}_p = (z_0,z_1,z_2)$. Take Q_1, Q_2, f general in the following spaces*

$$Q_1 \in \mathrm{H}^0(\mathscr{I}_p(2H)),\ Q_2 \in \mathrm{H}^0(\mathscr{I}_p^2(2H)),\ f \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(H))$$

and a general point p_1 of $\mathbb{P}_3$. Then

— *The ideal*

$$\mathscr{I} = (f,Q_1)^2 \cap (Q_1,Q_2) \cap \mathscr{I}_p^2 \cap \mathscr{I}_{p_1}$$

gives a 3-dimensional linear system $|\mathscr{I}(4H)|$.

— *Any isomorphism $|\mathscr{I}(4H)|^\vee \simeq \mathbb{P}'_3$ defines a $(4,4)$ birational map ϕ from $\mathbb{P}_3$ into $\mathbb{P}'_3$ such that $\phi^{-1}(H')$ is a quartic with a double conic.*

— *The base scheme F_ϕ of ϕ has degree 10. The subscheme F_ϕ^1 is the union of a multiple structure on a conic and a quartic curve singular at p.*

Proof. First consider the ideal

$$\mathcal{G} = (f,Q_1)^2 \cap (Q_1,Q_2) \cap \mathscr{I}_p^2.$$

In other words we have

$$\mathcal{G} = (f^2Q_2, Q_1^2, fQ_1z_0, fQ_1z_1, fQ_1z_2).$$

From the inclusion $\mathrm{H}^0(\mathscr{I}_p^2(2H)) \subset \mathbb{C}[z_0,z_1,z_2]$ we consider Q_2 as a homogeneous polynomial of degree 2 in three variables. Therefore the image of the rational map

$$\begin{array}{llll} |\mathcal{G}(4H)|: & \mathbb{P}_3 & \dashrightarrow & \mathbb{P}_4 \\ & (z_0:z_1:z_2:z_3) & \mapsto & (Z_0:Z_1:Z_2:Z_3:Z_4) = (fQ_1z_0 : fQ_1z_1 : fQ_1z_2 : f^2Q_2 : Q_1^2) \end{array}$$

is the quadric of $\mathbb{P}_4$ of equation $Q_2(Z_0,Z_1,Z_2) - Z_3Z_4$ because we have by homogeneity of Q_2

$$Q_2(fQ_1z_0, fQ_1z_1, fQ_1z_2) = (fQ_1)^2 Q_2(z_0,z_1,z_2).$$

We have $\mathscr{I} = \mathcal{G} \cap \mathscr{I}_{p_1}$ where $\mathscr{I}_{p_1}$ is the ideal of a general point p_1 of $\mathbb{P}_3$. Hence the map ϕ factors through the projection from a point of a quadric of $\mathbb{P}_4$ so it is birational with base scheme defined by $\mathscr{I}$. Let us compute the degree of ϕ^{-1}. Let H' be a general hyperplane of $\mathbb{P}'_3$, then $\phi^{-1}(H')$ is a quartic with a double conic. With Notation 3.4 the curve C_2 is the union of a component

supported on the conic of ideal (f, Q_1) and the singular quartic curve of ideal (Q_1, Q_2). The multiple structure defined by C_2 on the conic has degree 8, as a result $\deg C_2 = 12$ and $\deg C_1 = 4$, then $\phi \in \mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$. On another hand, the multiple structure defined by $\mathscr{I}$ on the conic has degree 6, so $\deg F_\phi = 10$. □

Definition 5.2. Birational maps described in Proposition 5.1 form the family $\mathcal{C}_{4,4}$.

Corollary 5.3. *The family* $\mathcal{C}_{4,4} \subset \mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ *is irreducible of dimension* 37.

Proof. The ideal $\mathscr{I}$ is determined by the singular quartic curve and the conic, but these two curves must be on a unique quadric defined by Q_1. Thus to construct $\mathscr{I}$ we need a general quadric, then a singular biquadratic on this quadric, a general plane and the isolated point p_1. It gives an irreducible parameter space of dimension $9+7+3+3$ to obtain $\mathscr{I}$, so

$$\dim(\mathcal{C}_{4,4}) = 22 + \dim(\mathrm{PGL}_4) = 37.$$

□

5.2. **Proof of Theorem B.** From Proposition 3.13, Corollaries 3.6, 4.15 and 5.3 one has:

— the closure of $\mathcal{J}_{4,4}$ is an irreducible component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$,
— the closure of $\mathcal{R}_{4,4}$ is an irreducible component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$,
— the closure of $\mathcal{D}_{4,4}$ is an irreducible component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$,
— the family $\mathcal{C}_{4,4}$ is irreducible.

Note that our definitions of $\mathcal{R}_{4,4}$, $\mathcal{D}_{4,4}$, $\mathcal{C}_{4,4}$ contain general position assumptions so we were able to compute degrees and dimensions for any element of these families (Proposition 3.7, Proposition 4.5, Proposition 5.1). With notation of Definition 2.4 we have: If ϕ is any element of $\mathcal{J}_{4,4}$, then by Definition 3.1 $\deg F^1_\phi = 12$, but this degree is 10 for an element of $\mathcal{C}_{4,4}$. So $\mathcal{C}_{4,4}$ is not in the closure of $\mathcal{J}_{4,4}$ by Corollary 2.7. Moreover $\dim F^\sigma_\phi = 0$ but it is 1 for any element of $\mathcal{C}_{4,4}$ hence by Corollary 2.7 $\mathcal{J}_{4,4}$ is not in the closure of $\mathcal{C}_{4,4}$.

— If ϕ is any element of $\mathcal{D}_{4,4}$, then $\deg F_\phi = 11$, $\dim F^\sigma_\phi = 1$ and $\deg F^\sigma_\phi = 1$.
— If ϕ is any element of $\mathcal{C}_{4,4}$, then $\deg F_\phi = 10$, $\dim F^\sigma_\phi = 1$ and $\deg F^\sigma_\phi = 2$.
— If ϕ is any element of $\mathcal{R}_{4,4}$, then $\deg F_\phi = 9$, $\dim F^\sigma_\phi = 1$ and $\deg F^\sigma_\phi = 3$.

By Corollary 2.7, since the map α and η cannot decrease by specialization one gets that none of the family $\mathcal{R}_{4,4}$, $\mathcal{C}_{4,4}$, $\mathcal{D}_{4,4}$, $\mathcal{J}_{4,4}$ is in the closure of another one. Let H (resp. H') be a general plane of $\mathbb{P}_3$ (resp. $\mathbb{P}'_3$). The quartic surface $\phi^{-1}(H')$ has the following property

ϕ general in	$\mathcal{R}_{4,4}$	$\mathcal{C}_{4,4}$	$\mathcal{D}_{4,4}$	$\mathcal{J}_{4,4}$
$\phi^{-1}(H')$	has a triple line	has a double conic	has a double line	is normal

It gives the genus of $H \cap \phi^{-1}(H')$. □

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E-mail address: `deserti@math.univ-paris-diderot.fr`

E-mail address: `frederic.han@imj-prg.fr`

UNIVERSITÉ PARIS DIDEROT, SORBONNE PARIS CITÉ, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, UMR 7586, CNRS, SORBONNE UNIVERSITÉS, UPMC UNIV PARIS 06, F-75013 PARIS, FRANCE.