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DEGREE GROWTH OF POLYNOMIAL AUTOMORPHISMS AND BIRATIONAL MAPS: SOME EXAMPLES

JULIE DÉSERTI

ABSTRACT. We provide the existence of new degree growths in the context of polynomial automorphisms of $\mathbb{C}^k$: if k is an integer ≥ 3, then for any $\ell \leq \left[\frac{k-1}{2}\right]$ there exist polynomial automorphisms f of $\mathbb{C}^k$ such that $\deg f^n \sim n^\ell$. We also give counter-examples in dimension $k \geq 3$ to some classical properties satisfied by polynomial automorphisms of $\mathbb{C}^2$.

We provide the existence of new degree growths in the context of birational maps of $\mathbb{P}^k_{\mathbb{C}}$: assume $k \geq 3$; forall $0 \leq \ell \leq k$ there exist birational maps ϕ of $\mathbb{P}^k_{\mathbb{C}}$ such that $\deg \phi^n \sim n^\ell$.

1. INTRODUCTION

Let f be a polynomial automorphism of $\mathbb{C}^2$, then either $(\deg f^n)_{n\in\mathbb{N}}$ is bounded, or $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially. In higher dimensions there are intermediate growths:

Theorem A. *Let k be an integer ≥ 3. For any $\ell \leq \left[\frac{k-1}{2}\right]$ there exist polynomial automorphisms f of $\mathbb{C}^k$ such that*

$$\deg f^n \sim n^\ell.$$

The group of polynomial automorphisms of $\mathbb{C}^2$ has a structure of amalgamated product ([6]); using this rigidity a lot of properties of polynomial automorphisms of $\mathbb{C}^2$ have been established. All these properties show a dichotomy; up to conjugacy there are two types of polynomial automorphisms of $\mathbb{C}^2$: the elementary ones and the Hénon ones. Furthermore if f and g are two polynomial automorphisms of $\mathbb{C}^2$ and H_∞ denotes the line at infinity (we view $\mathbb{C}^2$ in $\mathbb{P}^2_{\mathbb{C}}$), then

- $(\mathcal{P}_1)$: f is algebraically stable if and only if $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially;
- $(\mathcal{P}_2)$: for any $n \geq 1$ the equality $\deg f^n = (\deg f)^n$ holds if and only if f does not preserve a fibration in hyperplanes;
- $(\mathcal{P}_3)$: the sequences $(\deg f^n)_{n\in\mathbb{N}}$ and $(\deg g^n)_{n\in\mathbb{N}}$ have the same growth if and only if the configurations of $\big(f(H_\infty), f^{-1}(H_\infty), \mathrm{Ind}(f), \mathrm{Ind}(f^{-1})\big)$ and $\big(g(H_\infty), g^{-1}(H_\infty), \mathrm{Ind}(g), \mathrm{Ind}(g^{-1})\big)$ are the same;
- $(\mathcal{P}_4)$: $\deg f^2 = (\deg f)^2$ if and only if $\deg f^n = (\deg f)^n$ for any $n \in \mathbb{N}$ (*see* [5, Proposition 3]).

Note that in $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_4)$ one can add "if and only if f is of Hénon type".

Fact B. *We give counter-examples to Properties $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$ in dimension ≥ 3.*

Nevertheless one can prove a similar result to property $\mathcal{P}_4$:

Proposition C. *Let f be a polynomial automorphism of $\mathbb{C}^k$. Then $\deg f^i = (\deg f)^i$ for $1 \leq i \leq k$ if and only if $\deg f^n = (\deg f)^n$ for any $n \geq 1$.*

If ϕ is a birational self map of $\mathbb{P}^2_{\mathbb{C}}$, then $(\deg \phi^n)_{n\in\mathbb{N}}$ is either bounded, or grows linearly, or grows quadratically, or grows exponentially ([3]). In this context there exist other types of growth in higher

dimension. Lin has studied the degree growth of monomial maps of $\mathbb{P}^k_{\mathbb{C}}$ (*see* [8]); he proves in particular that if A is a $k\times k$ integer matrix with nonzero determinant, then there exist two constants $\alpha\geq\beta\geq 0$ and a unique integer $0\leq\ell\leq k-1$ such that for any $n\in\mathbb{N}$

$$\beta\rho(A)^{\ell}n^{\ell}\leq\deg\phi_A^n\leq\alpha\rho(A)^{\ell}n^{\ell}$$

where $\rho(A)$ denotes the spectral radius of A and ϕ_A the monomial map associated to A. Does there exist in dimension k birational maps of $\mathbb{P}^k_{\mathbb{C}}$ with growth n^{ℓ} with $\ell>k-1$?

Theorem D. *Assume $k\geq 3$; forall $0\leq\ell\leq k$ there exist birational maps ϕ of $\mathbb{P}^k_{\mathbb{C}}$ such that $\deg\phi^n\sim n^{\ell}$.*

Note that there exists a birational self map f of $\mathbb{P}^3_{\mathbb{C}}$ such that $\deg f^n\sim n^4$ (*see* [10]).

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2. RECALLS, DEFINITIONS, NOTATIONS

2.1. **The group of polynomial automorphisms of $\mathbb{C}^k$.** A ***polynomial automorphism*** f of $\mathbb{C}^k$ is a polynomial map of the type

$$f:\mathbb{C}^k\to\mathbb{C}^k,\qquad (z_0,z_1,\dots,z_{k-1})\mapsto\big(f_0(z_0,z_1,\dots,z_{k-1}),f_1(z_0,z_1,\dots,z_{k-1}),\dots,f_{k-1}(z_0,z_1,\dots,z_{k-1})\big)$$

that is bijective. The set of polynomial automorphisms of $\mathbb{C}^k$ form a group denoted $\mathrm{Aut}(\mathbb{C}^k)$.

The automorphisms of $\mathbb{C}^k$ of the form $(f_0,f_1,\dots,f_{k-1})$ where f_i depends only on z_i, z_{i+1}, …, z_{k-1} are called ***elementary automorphisms*** and form a subgroup E_k of $\mathrm{Aut}(\mathbb{C}^k)$. Moreover we have the inclusions

$$\mathrm{GL}(\mathbb{C}^k)\subset\mathrm{Aff}_k\subset\mathrm{Aut}(\mathbb{C}^k)$$

where Aff_k denotes the ***group of affine maps***

$$f:(z_0,z_1,\dots,z_{k-1})\mapsto\big(f_0(z_0,z_1,\dots,z_{k-1}),f_1(z_0,z_1,\dots,z_{k-1}),\dots,f_{k-1}(z_0,z_1,\dots,z_{k-1})\big)$$

with f_i affine; Aff_k is the semi-direct product of $\mathrm{GL}(\mathbb{C}^k)$ with the commutative subgroups of translations. The subgroup $\mathrm{Tame}_k\subset\mathrm{Aut}(\mathbb{C}^k)$ generated by E_k and Aff_k is called the ***group of tame automorphisms***. If $k=2$ one has:

Theorem 2.1 ([6])**.** *In dimension 2 the group of tame automorphisms coincides with the whole group of polynomial automorphism; more precisely*

$$\mathrm{Aut}(\mathbb{C}^2)=\mathrm{Aff}_2*_{\mathrm{Aff}_2\cap\mathrm{E}_2}\mathrm{E}_2.$$

But this is not the case in higher dimension: $\mathrm{Tame}_3\subsetneq\mathrm{Aut}(\mathbb{C}^3)$ (*see* [9]).

Another important result in dimension 2 is the following:

Theorem 2.2 ([4])**.** *Let f be an element of $\mathrm{Aut}(\mathbb{C}^2)$. Then, up to conjugacy,*

– *either f belongs to* E_2,
– *or f can be written as*

$$\varphi_\ell \circ \varphi_{\ell-1} \circ \dots \varphi_1$$

where $\varphi_i \colon (z_0, z_1) \mapsto (z_1, P_i(z_1) - \delta_i z_0)$, $\delta_i \in \mathbb{C}^*$, $P_i \in \mathbb{C}[z_1]$, $\deg P_i \geq 2$.

We denote by $\mathcal{H}$ the set of polynomial automorphisms of $\mathbb{C}^2$ can be written up to conjugacy as

$$\varphi_\ell \circ \varphi_{\ell-1} \circ \dots \varphi_1$$

where $\varphi_i \colon (z_0, z_1) \mapsto (z_1, P_i(z_1) - \delta_i z_0)$, $\delta_i \in \mathbb{C}^*$, $P_i \in \mathbb{C}[z_1]$, $\deg P_i \geq 2$. The elements of $\mathcal{H}$ are ***of Hénon type***.

From now one we will denote $f = (f_0, f_1, \dots, f_{k-1})$ instead of

$$f \colon (z_0, z_1, \dots, z_{k-1}) \mapsto \big(f_0(z_0, z_1, \dots, z_{k-1}), f_1(z_0, z_1, \dots, z_{k-1}), \dots, f_{k-1}(z_0, z_1, \dots, z_{k-1})\big).$$

The ***algebraic degree*** $\deg f$ of $f = (f_0, f_1, \dots, f_{k-1}) \in \mathrm{Aut}(\mathbb{C}^k)$ is $\max(\deg f_0, \deg f_1, \dots, \deg f_{k-1})$.

2.2. **The Cremona group.** A ***rational self map*** f of $\mathbb{P}^k_{\mathbb{C}}$ can be written

$$\big(z_0 : z_1 : \dots : z_k\big) \dashrightarrow \big(f_0(z_0, z_1, \dots, z_k) : f_1(z_0, z_1, \dots, z_k) : \dots : f_k(z_0, z_1, \dots, z_k)\big)$$

where the f_i's are homogeneous polynomials of the same degree ≥ 1 and without common factor of positive degree. The ***degree*** of f is the degree of the f_i. If there exists a rational self map g of $\mathbb{P}^k_{\mathbb{C}}$ such that $fg = gf = \mathrm{id}$ we say that the rational self map f of $\mathbb{P}^k_{\mathbb{C}}$ is ***birational***. The set of birational self maps of $\mathbb{P}^k_{\mathbb{C}}$ form a group denoted $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ and called the ***Cremona group***. Of course $\mathrm{Aut}(\mathbb{C}^k)$ is a subgroup of $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$. An other natural subgroup of $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ is the group $\mathrm{Aut}(\mathbb{P}^k_{\mathbb{C}}) \simeq \mathrm{PGL}(k+1; \mathbb{C})$ of automorphisms of $\mathbb{P}^k_{\mathbb{C}}$.

The ***indeterminacy set*** $\mathrm{Ind}(f)$ of f is the set of the common zeros of the f_i's. The ***exceptional set*** $\mathrm{Exc}(f)$ of f is the (finite) union of subvarieties M_i of $\mathbb{P}^k_{\mathbb{C}}$ such that f is not injective on any open subset of M_i.

2.3. **A little bit of dynamics.** Let f be a polynomial automorphism of $\mathbb{C}^k$. One can see f as a birational self map of $\mathbb{P}^k_{\mathbb{C}}$ still denoted f. We will say that f is ***algebraically stable*** if for any $n > 0$

$$f^n\big(\{z_k = 0\} \smallsetminus \mathrm{Ind}(f^n)\big)$$

is not contained in $\mathrm{Ind}(f)$. This is equivalent to the fact that $(\deg f)^n = \deg f^n$ for any $n > 0$. For instance elements of $\mathcal{H}$ are algebraically stable.

Remark 2.3. Note that in dimension 2 one usually says that f is algebraically stable if for any $n > 0$

$$f^n\big(\{z_2 = 0\} \smallsetminus \mathrm{Ind}(f^n)\big) \cap \mathrm{Ind}(f) = \emptyset.$$

Be careful this is not equivalent in higher dimension: consider for instance

$$f = \big(5z_0^2 + z_2^2 + 6z_0z_2 + z_1, z_2^2 + z_0, z_2\big);$$

then
– on the one hand $(-1 : 0 : 1 : 0)$ belongs to $\{z_3 = 0\} \smallsetminus \mathrm{Ind}(f)$ and $f(-1 : 0 : 1 : 0) = (0 : 1 : 0 : 0) \in \mathrm{Ind}(f) = \{(0 : 1 : 0 : 0)\}$.
– on the other hand for any $n \geq 1$ $\deg f^n = (\deg f)^n$.

The algebraic degree of a birational map f of $\mathbb{P}^n_{\mathbb{C}}$ (resp. a polynomial automorphism of $\mathbb{C}^k$) is not a dynamical invariant so we introduce the ***dynamical degree***

$$\lambda(f) = \lim_{n \to +\infty} (\deg f^n)^{1/n}$$

which is a dynamical invariant. In other words if $f \in \mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ (resp. $\mathrm{Aut}(\mathbb{C}^k)$), then for any $g \in \mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ (resp. $g \in \mathrm{Aut}(\mathbb{C}^k)$) one has $\lambda(f) = \lambda(gfg^{-1})$. For any element f in $\mathcal{H}$ the algebraic and dynamical degrees coincide; more precisely if

$$f = \varphi_\ell \circ \varphi_{\ell-1} \circ \ldots \circ \varphi_1$$

where $\varphi_i = (z_1, P_i(z_1) - \delta_i z_0)$, $\delta_i \in \mathbb{C}^*$, $P_i \in \mathbb{C}[z_1]$, $\deg P_i \geq 2$ one has ([4])

$$\lambda(f) = \prod_{i=1}^{\ell} \deg \varphi_i \geq 2.$$

A polynomial automorphism f is in E_k if and only if for any $n \geq 1$ the equality $\deg f = \deg f^n$ holds hence $\lambda(f) = 1$. In other words a polynomial automorphism f of $\mathbb{C}^2$ belongs to $\mathcal{H}$ if and only if $\lambda(f) > 1$. There is an other characterization of the automorphisms of Hénon type:

Theorem 2.4 ([7]). *The centralizer of f in $\mathrm{Aut}(\mathbb{C}^2)$, that is $\{g \in \mathrm{Aut}(\mathbb{C}^2) \mid fg = gf\}$, is countable if and only if f belongs to $\mathcal{H}$.*

3. Automorphisms with polynomial growths

3.1. **The growths of a polynomial automorphisms and its inverse.** If f is a polynomial automorphism of $\mathbb{C}^k$, then $(\deg f, \deg f^{-1})$ is the bidegree of f. There is a relationship between $\deg f$ and $\deg f^{-1}$ (*see* [1]):

$$\begin{cases} \deg f^{-1} \leq (\deg f)^{k-1} \\ \deg f \leq (\deg f^{-1})^{k-1} \end{cases} \tag{3.1}$$

As a result if f is a polynomial automorphism of $\mathbb{C}^k$, the degree growths of f and f^{-1} are linked:

Proposition 3.1. *Let f be a polynomial automorphism of $\mathbb{C}^k$.*

- *The sequence $(\deg f^n)_{n\in\mathbb{N}}$ is bounded if and only if the sequence $(\deg f^{-n})_{n\in\mathbb{N}}$ is bounded.*
- *The sequence $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially if and only if $(\deg f^{-n})_{n\in\mathbb{N}}$ grows exponentially.*
- *If $\deg f^n \simeq n^p$ and $\deg f^{-n} \simeq n^q$ for some integers $p, q \geq 1$, then*

$$(p,q) \in \left\{ \left(\left[\frac{q+1}{k} \right], q \right), \ldots, (kq, q) \right\}.$$

Remark 3.2. When we write "The sequence $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially if and only if $(\deg f^{-n})_{n\in\mathbb{N}}$ grows exponentially" it does not mean that $(\deg f^n)_{n\in\mathbb{N}}$ and $(\deg f^{-n})_{n\in\mathbb{N}}$ have exactly the same behavior: the polynomial automorphism of $\mathbb{C}^3$ given by $f = (z_0^2 + z_1 + z_2, z_0^2 + z_1, z_0)$ satisfies forall $n \geq 1$

$$\begin{cases} \deg f^n = 2^n \\ \deg f^{-n} = 2^{[\frac{n+1}{2}]} \end{cases}$$

3.2. **Examples of polynomial automorphisms with new polynomial growths.** Let us now give examples of polynomial automorphisms with polynomial growths.

Lemma 3.3. *Let us consider the polynomial automorphism of $\mathbb{C}^3$ given by*

$$f = \left(z_1 + z_0 z_2^d, z_0, z_2\right)$$

where $d \geq 1$. One has $\deg f^n = dn + 1$ and $\deg f^n = \deg f^{-n}$ for any $n \geq 1$.

Proof. Assume $n \geq 1$. Set $f^n = (f_{0,n}, f_{1,n}, z_2)$ and $\delta_n = \deg f^n$. Note that $\delta_1 = d + 1$ and since

$$f^n = f f^{n-1} = (f_{1,n-1} + f_{0,n-1} z_2^d, f_{0,n-1}, z_2)$$

one has

$$\delta_n = \max\left(\deg f_{1,n-1}, \deg f_{0,n-1} + d, \deg f_{0,n-1}, 1\right).$$

But $f_{1,n-1} = f_{0,n-2}$ and $d \geq 1$ so $\delta_n = \deg f_{0,n-1} + d = \delta_{n-1} + d$. □

Similarly one can prove:

Lemma 3.4. *Let us consider the polynomial automorphism of $\mathbb{C}^5$ given by*

$$g = \left(z_1 + z_0 z_2^d, z_0, z_2, z_4 + z_0^p z_3, z_3\right)$$

where $p \geq d \geq 1$. One has for any $n \geq 1$

$$\deg g^n = \frac{pd}{2} n^2 + \frac{p(2-d)}{2} n + 1$$

and $\deg g^n = \deg g^{-n}$ for any $n \geq 1$.

Let us consider the polynomial automorphism of $\mathbb{C}^7$ given by

$$h = \left(z_1 + z_0 z_2^d, z_0, z_2, z_4 + z_0^p z_3, z_3, z_6 + z_3^\ell z_5, z_5\right)$$

where $\ell \geq p \geq d \geq 1$. For any $n \geq 1$

$$\deg h^n = 1 + \ell\left(1 - \frac{p}{2} + \frac{pd}{3}\right) n + \frac{\ell p(1-d)}{2} n^2 + \frac{\ell pd}{6} n^3$$

and $\deg h^n = \deg h^{-n}$.

So Lemma 3.3 gives an example f of polynomial automorphism of $\mathbb{C}^3$ with linear growth; from f one gets a polynomial automorphism g of $\mathbb{C}^5$ such that $\deg g^n \sim n^2$, and from g one gets a polynomial automorphism h of $\mathbb{C}^7$ such that $\deg h^n \sim n^3$ (Lemma 3.4). By repeating this process one gets the following statement:

Proposition 3.5. *There exist polynomial automorphisms f of $\mathbb{C}^{2k+1}$, $k \geq 2$, such that*

$$\deg f^n \sim n^k.$$

Theorem A follows from Proposition 3.5.

3.3. **A consequence.** Come back to $f = (z_1 + z_0 z_2^d, z_0, z_2)$ and consider the birational map of $\mathbb{P}^4_{\mathbb{C}}$ given by $F = (f, z_0^p z_3)$, that is $F = (z_1 + z_0 z_2^d, z_0, z_2, z_0^p z_3)$. Assume that $d \leq p$. Then one can prove by induction that for any $n \geq 1$

$$\deg F^n = \frac{pd}{2} n^2 + \frac{p(2-d)}{2} n + 1.$$

Let us now define the birational map F of $\mathbb{P}^5_{\mathbb{C}}$ by

$$G = (z_1 + z_0 z_2^d, z_0, z_2, z_0^p z_3, z_3^\ell z_4);$$

for any $n \geq 1$ one has

$$\deg G^n = \frac{\ell pd}{6} n^3 + \left(1 - \frac{3d}{4}\right) \ell p n^2 + \left(\frac{13}{12} pd - 2p + 1\right) \ell n - \frac{\ell pd}{2} + \ell p + 1.$$

Repeating this process one gets

Proposition 3.6. *If $k \geq 3$, then for any $0 \leq \ell \leq k-2$ there exist birational maps F of $\mathbb{P}^k_{\mathbb{C}}$ such that*

$$\deg F^n \sim n^\ell.$$

4. Properties $(\mathcal{P}_i)$

Note that Theorem 2.4 can be also stated as follows: the centralizer of a polyonomial automorphism f of $\mathbb{C}^2$ is countable if and only if $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially. This property is not true in higher dimension: there exist polynomial automorphisms of $\mathbb{C}^3$ with uncountable centralizer and exponential degree growth ([2]). Let us now that this is also the case for other properties, and in particular for $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, $(\mathcal{P}_3)$, $(\mathcal{P}_4)$.

4.1. **Property** $(\mathcal{P}_1)$**.** Property $(\mathcal{P}_1)$ is not satisfied in dimension 3:

Proposition 4.1. *If*
$$f=\big(z_2,(z_2^2+z_0)^2+z_2^2+z_0+z_1,z_2^2+z_0\big)$$
then
- $\mathrm{Ind}(f)=\{z_2=z_3=0\}$ *and* $z_3=0$ *is blown down by* f *onto* $(0:1:0:0)\in\mathrm{Ind}(f)$*,*
- *for all* $n\geq 1$ *one has* $\deg f^n=2^{n+1}$*.*

4.2. **Property** $(\mathcal{P}_2)$**.** Property $(\mathcal{P}_2)$ does not hold in higher dimension:

Proposition 4.2. *The polynomial automorphism* f *of* $\mathbb{C}^3$ *given by*
$$\big(z_0^2+z_1,z_0,z_2+1\big)$$
preserves the fibration $z_2=cst$ *and for all* $n\geq 1$ *the equality* $\deg f^n=2^n$ *holds.*

The polynomial automorphism g *of* $\mathbb{C}^3$ *given by*
$$\big(z_1^2+z_0z_1+z_2,z_1+1,z_0\big)$$
preserves the fibration $z_1=cst$ *and for all* $n\geq 1$ *the equality* $\deg g^n=n+1$ *holds.*

4.3. **Property** $(\mathcal{P}_3)$**.** Property $(\mathcal{P}_3)$ is not satisfied in dimension 3.

Proposition 4.3. *Let us consider the polynomial automorphisms* f *and* g *of* $\mathbb{C}^3$ *defined by*
$$f=\big(z_0+z_1+z_2,z_0^2+z_0+z_1,z_0\big),\qquad\qquad g=\big(z_1^2+z_0+z_1+z_2,z_1,z_0\big).$$
For all $n\geq 1$ *one has*
$$\begin{cases}\deg f^{2n}=2^{n+1},\ \deg f^{2n+1}=2^{n+1}\\ \deg f^{-n}=2^n\\ \deg g^n=\deg g^{-n}=2\end{cases}$$

The automorphism f *sends* $z_3=0$ *onto* $(0:1:0:0)$ *and* f^{-1} *sends* $z_3=0$ *onto* $(0:1:1:0)$*. Furthermore* $\mathrm{Ind}(f)=\{z_0=z_3=0\}$ *and* $\mathrm{Ind}(f^{-1})=\{z_2=z_3=0\}$*.*

The automorphism g *sends* $z_3=0$ *onto* $(1:0:0:0)$ *and* g^{-1} *sends* $z_3=0$ *onto* $(0:0:1:0)$*. Besides* $\mathrm{Ind}(g)=\{z_1=z_3=0\}$ *and* $\mathrm{Ind}(g^{-1})=\{z_2=z_3=0\}$*.*

4.4. **Property** $(\mathcal{P}_4)$**.** In [5] Furter proves that if f is a polynomial automorphism of $\mathbb{C}^2$ then $\deg f^2=(\deg f)^2$ if and only if $\deg f^n=(\deg f)^n$ for all $n\in\mathbb{N}$. This property does not hold in higher dimension: consider for instance the polynomial automorphism f given by
$$f=(z_1^2+z_5,z_5^2+z_4,z_2,z_1,z_0,z_4^2+z_3).$$
One can check that $\deg f=2$, $\deg f^2=4$, $\deg f^3=8$ but $\deg f^4=8$.

Let f be a polynomial automorphism of $\mathbb{C}^k$. For any integer $n\geq 0$ set $\Omega_n=f^n\big((z_{k-1}=0)\smallsetminus\mathrm{Ind}(f^n)\big)$; note that $\Omega_n\subset(z_{k-1}=0)$ for any n. We say that f ***is not algebraically stable after*** ℓ ***steps*** if ℓ is the smallest integer such that $\Omega_\ell\subset\mathrm{Ind}(f)$.

Let us first remark that if $\Omega_1\cap\mathrm{Ind}(f)=\emptyset$, then $\Omega_2=f(\Omega_1)\subseteq\Omega_1$ so $\Omega_2\cap\mathrm{Ind}(f)=\emptyset$. By induction one gets for any $n\geq 1$ that $\Omega_n\cap\mathrm{Ind}(f)=\emptyset$ and $\deg f^n=(\deg f)^n$, *i.e.* f is algebraically stable.

Let us now assume that $\Omega_1\cap\mathrm{Ind}(f)\neq\emptyset$. Then:

(1) Either $\Omega_1\subset\mathrm{Ind}(f)$, that is f is not algebraically stable after 1 step.

(2) Or $\Omega_1\not\subset\mathrm{Ind}(f)$ hence $\Omega_2=f\big(\Omega_1\smallsetminus\mathrm{Ind}(f)\big)\subseteq\Omega_1$.
   - Either $\dim\Omega_2=\dim\Omega_1$, so $\Omega_2=\Omega_1$ and then $\Omega_n=\Omega_1$ for any n; in particular $\Omega_n\not\subset\mathrm{Ind}(f)$ for any n and f is algebraically stable.

– Or $\dim\Omega_2 < \dim\Omega_1$, then either $\Omega_2 \subset \mathrm{Ind}(f)$ and f is not algebraically stable after 2 steps or $\Omega_2 \not\subset \mathrm{Ind}(f)$ and we come back to the previous alternative, that is either $\dim\Omega_3 = \dim\Omega_2$ or $\dim\Omega_3 < \dim\Omega_2$. Since for any n one has $0 \leq \dim\Omega_n \leq k-1$ one gets that either f is algebraically stable, or f is not algebraically stable after at most $k-1$ steps.

Hence one can state:

Proposition 4.4. *Let f be a polynomial automorphism of $\mathbb{C}^k$. Either f is algebraically stable, or f is not algebraically stable after ℓ steps, with $\ell \leq k-1$.*

In other words $\deg f^i = (\deg f)^i$ *for* $1 \leq i \leq k$ *if and only if* $\deg f^n = (\deg f)^n$ *for any* $n \geq 1$.

5. Birational maps with new polynomial growths

Let us first recall that

Lemma 5.1 ([3])**.** *The birational self map φ of $\mathbb{P}^2_{\mathbb{C}}$ given in the affine chart $z_2 = 1$ by*

$$\varphi(z_0, z_1) = \left(z_1 + \frac{2}{3}, z_0 \frac{z_1 - \frac{1}{3}}{z_1 + 1}\right)$$

satisfies $\deg \varphi^n \sim n^2$.

Denote φ^n by $\left(\frac{P_n}{Q_n}, \frac{R_n}{S_n}\right)$ where P_n, Q_n, R_n and S_n denote some elements of $\mathbb{C}[z_0, z_1]$ without common factor. Set $p_n = \deg P_n$, $q_n = \deg Q_n$, $r_n = \deg R_n$ and $s_n = \deg S_n$. The following equalities hold (by iteration)

$$\begin{cases} p_n = s_{n-1} + 1 \\ q_n = s_{n-1} \\ r_n = s_n + 1 \\ \deg \varphi^n = s_{n-1} + s_n + 1 \end{cases}$$

Let us now consider the birational self map of $\mathbb{P}^3_{\mathbb{C}}$ given in the affine chart $z_3 = 1$ by

$$\Psi_3(z_0, z_1, z_2) = \left(z_1 + \frac{2}{3}, z_0 \frac{z_1 - \frac{1}{3}}{z_1 + 1}, z_0 z_2\right).$$

One can also check that

$$\Psi_3 = \left(\frac{P_n}{Q_n}, \frac{R_n}{S_n}, \frac{U_n}{V_n}\right)$$

where $U_n = z_0 z_2 P_1 P_2 \ldots P_{n-1}$ and $V_n = Q_1 Q_2 \ldots Q_{n-1}$ have no common factor. Since $s_i \sim i^2$ one gets:

Lemma 5.2. *The birational self map Ψ_3 of $\mathbb{P}^3_{\mathbb{C}}$ satisfies*

$$\deg \Psi_3 \sim n^3.$$

Let us now consider the birational self map of $\mathbb{P}^4_{\mathbb{C}}$ defined in the affine chart $z_4 = 1$ by

$$\Psi_4(z_0, z_1, z_2, z_3) = \left(z_1 + \frac{2}{3}, z_0 \frac{z_1 - \frac{1}{3}}{z_1 + 1}, z_0 z_2, z_2 z_3\right).$$

One can check that $\Psi_4 = \left(\frac{P_n}{Q_n}, \frac{R_n}{S_n}, \frac{U_n}{V_n}, \frac{W_n}{X_n}\right)$ where $W_n = W_1 U_1 U_2 \ldots U_{n-1}$ and $X_n = X_1 V_1 V_2 \ldots V_{n-1}$ have no common factor. Since $\deg U_n \sim n^3$ and $\deg V_n \sim n^3$ one has:

Lemma 5.3. *The birational self map Ψ_4 of $\mathbb{P}^4_{\mathbb{C}}$ satisfies*

$$\deg \Psi_4 \sim n^4.$$

By repeating this process one gets:

Theorem 5.4. *Let k be an integer ≥ 3. There exist birational maps ϕ of $\mathbb{P}^k_{\mathbb{C}}$ such that*

$$\deg \phi^n \sim n^k.$$

This statement and Lin's result ([8]) imply Theorem D.

REFERENCES

[1] H. Bass. The Jacobian conjecture and inverse degrees. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 65–75. Birkhäuser, Boston, Mass., 1983.

[2] C. Bisi. On commuting polynomial automorphisms of $\mathbb{C}^k$, $k \geq 3$. *Math. Z.*, 258(4):875–891, 2008.

[3] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.

[4] S. Friedland and J. Milnor. Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Systems*, 9(1):67–99, 1989.

[5] J.-P. Furter. On the degree of iterates of automorphisms of the affine plane. *Manuscripta Math.*, 98(2):183–193, 1999.

[6] H. W. E. Jung. Über ganze birationale Transformationen der Ebene. *J. Reine Angew. Math.*, 184:161–174, 1942.

[7] S. Lamy. L'alternative de Tits pour $\mathrm{Aut}[\mathbb{C}^2]$. *J. Algebra*, 239(2):413–437, 2001.

[8] J.-L. Lin. Algebraic stability and degree growth of monomial maps. *Math. Z.*, 271(1-2):293–311, 2012.

[9] I. P. Shestakov and U. U. Umirbaev. The tame and the wild automorphisms of polynomial rings in three variables. *J. Amer. Math. Soc.*, 17(1):197–227, 2004.

[10] C. Urech. Remarks on the degree growth of birational transformations, `arXiv:1606.04822`. 2016.

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