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## DEGREE GROWTH OF POLYNOMIAL AUTOMORPHISMS AND BIRATIONAL MAPS: SOME EXAMPLES

JULIE DÉSERTI

**ABSTRACT.** We provide the existence of new degree growths in the context of polynomial automorphisms of  $\mathbb{C}^k$ : if  $k$  is an integer  $\geq 3$ , then for any  $\ell \leq \lfloor \frac{k-1}{2} \rfloor$  there exist polynomial automorphisms  $f$  of  $\mathbb{C}^k$  such that  $\deg f^n \sim n^\ell$ . We also give counter-examples in dimension  $k \geq 3$  to some classical properties satisfied by polynomial automorphisms of  $\mathbb{C}^2$ .

We provide the existence of new degree growths in the context of birational maps of  $\mathbb{P}_{\mathbb{C}}^k$ : assume  $k \geq 3$ ; for all  $0 \leq \ell \leq k$  there exist birational maps  $\phi$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that  $\deg \phi^n \sim n^\ell$ .

### 1. INTRODUCTION

Let  $f$  be a polynomial automorphism of  $\mathbb{C}^2$ , then either  $(\deg f^n)_{n \in \mathbb{N}}$  is bounded, or  $(\deg f^n)_{n \in \mathbb{N}}$  grows exponentially. In higher dimensions there are intermediate growths:

**Theorem A.** *Let  $k$  be an integer  $\geq 3$ . For any  $\ell \leq \lfloor \frac{k-1}{2} \rfloor$  there exist polynomial automorphisms  $f$  of  $\mathbb{C}^k$  such that*

$$\deg f^n \sim n^\ell.$$

The group of polynomial automorphisms of  $\mathbb{C}^2$  has a structure of amalgamated product ([6]); using this rigidity a lot of properties of polynomial automorphisms of  $\mathbb{C}^2$  have been established. All these properties show a dichotomy; up to conjugacy there are two types of polynomial automorphisms of  $\mathbb{C}^2$ : the elementary ones and the Hénon ones. Furthermore if  $f$  and  $g$  are two polynomial automorphisms of  $\mathbb{C}^2$  and  $H_\infty$  denotes the line at infinity (we view  $\mathbb{C}^2$  in  $\mathbb{P}_{\mathbb{C}}^2$ ), then

- ( $\mathcal{P}_1$ ) :  $f$  is algebraically stable if and only if  $(\deg f^n)_{n \in \mathbb{N}}$  grows exponentially;
- ( $\mathcal{P}_2$ ) : for any  $n \geq 1$  the equality  $\deg f^n = (\deg f)^n$  holds if and only if  $f$  does not preserve a fibration in hyperplanes;
- ( $\mathcal{P}_3$ ) : the sequences  $(\deg f^n)_{n \in \mathbb{N}}$  and  $(\deg g^n)_{n \in \mathbb{N}}$  have the same growth if and only if the configurations of  $(f(H_\infty), f^{-1}(H_\infty), \text{Ind}(f), \text{Ind}(f^{-1}))$  and  $(g(H_\infty), g^{-1}(H_\infty), \text{Ind}(g), \text{Ind}(g^{-1}))$  are the same;
- ( $\mathcal{P}_4$ ) :  $\deg f^2 = (\deg f)^2$  if and only if  $\deg f^n = (\deg f)^n$  for any  $n \in \mathbb{N}$  (see [5, Proposition 3]).

Note that in ( $\mathcal{P}_1$ ), ( $\mathcal{P}_2$ ) and ( $\mathcal{P}_4$ ) one can add "if and only if  $f$  is of Hénon type".

**Fact B.** *We give counter-examples to Properties ( $\mathcal{P}_1$ ), ( $\mathcal{P}_2$ ), ( $\mathcal{P}_3$ ) and ( $\mathcal{P}_4$ ) in dimension  $\geq 3$ .*

Nevertheless one can prove a similar result to property  $\mathcal{P}_4$ :

**Proposition C.** *Let  $f$  be a polynomial automorphism of  $\mathbb{C}^k$ . Then  $\deg f^i = (\deg f)^i$  for  $1 \leq i \leq k$  if and only if  $\deg f^n = (\deg f)^n$  for any  $n \geq 1$ .*

If  $\phi$  is a birational self map of  $\mathbb{P}_{\mathbb{C}}^2$ , then  $(\deg \phi^n)_{n \in \mathbb{N}}$  is either bounded, or grows linearly, or grows quadratically, or grows exponentially ([3]). In this context there exist other types of growth in higher

dimension. Lin has studied the degree growth of monomial maps of  $\mathbb{P}_{\mathbb{C}}^k$  (see [8]); he proves in particular that if  $A$  is a  $k \times k$  integer matrix with nonzero determinant, then there exist two constants  $\alpha \geq \beta \geq 0$  and a unique integer  $0 \leq \ell \leq k - 1$  such that for any  $n \in \mathbb{N}$

$$\beta \rho(A)^\ell n^\ell \leq \deg \phi_A^n \leq \alpha \rho(A)^\ell n^\ell$$

where  $\rho(A)$  denotes the spectral radius of  $A$  and  $\phi_A$  the monomial map associated to  $A$ . Does there exist in dimension  $k$  birational maps of  $\mathbb{P}_{\mathbb{C}}^k$  with growth  $n^\ell$  with  $\ell > k - 1$ ?

**Theorem D.** *Assume  $k \geq 3$ ; for all  $0 \leq \ell \leq k$  there exist birational maps  $\phi$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that  $\deg \phi^n \sim n^\ell$ .*

Note that there exists a birational self map  $f$  of  $\mathbb{P}_{\mathbb{C}}^3$  such that  $\deg f^n \sim n^4$  (see [10]).

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## 2. RECALLS, DEFINITIONS, NOTATIONS

**2.1. The group of polynomial automorphisms of  $\mathbb{C}^k$ .** A *polynomial automorphism*  $f$  of  $\mathbb{C}^k$  is a polynomial map of the type

$$f: \mathbb{C}^k \rightarrow \mathbb{C}^k, \quad (z_0, z_1, \dots, z_{k-1}) \mapsto (f_0(z_0, z_1, \dots, z_{k-1}), f_1(z_0, z_1, \dots, z_{k-1}), \dots, f_{k-1}(z_0, z_1, \dots, z_{k-1}))$$

that is bijective. The set of polynomial automorphisms of  $\mathbb{C}^k$  form a group denoted  $\text{Aut}(\mathbb{C}^k)$ .

The automorphisms of  $\mathbb{C}^k$  of the form  $(f_0, f_1, \dots, f_{k-1})$  where  $f_i$  depends only on  $z_i, z_{i+1}, \dots, z_{k-1}$  are called *elementary automorphisms* and form a subgroup  $E_k$  of  $\text{Aut}(\mathbb{C}^k)$ . Moreover we have the inclusions

$$\text{GL}(\mathbb{C}^k) \subset \text{Aff}_k \subset \text{Aut}(\mathbb{C}^k)$$

where  $\text{Aff}_k$  denotes the *group of affine maps*

$$f: (z_0, z_1, \dots, z_{k-1}) \mapsto (f_0(z_0, z_1, \dots, z_{k-1}), f_1(z_0, z_1, \dots, z_{k-1}), \dots, f_{k-1}(z_0, z_1, \dots, z_{k-1}))$$

with  $f_i$  affine;  $\text{Aff}_k$  is the semi-direct product of  $\text{GL}(\mathbb{C}^k)$  with the commutative subgroups of translations. The subgroup  $\text{Tame}_k \subset \text{Aut}(\mathbb{C}^k)$  generated by  $E_k$  and  $\text{Aff}_k$  is called the *group of tame automorphisms*. If  $k = 2$  one has:

**Theorem 2.1** ([6]). *In dimension 2 the group of tame automorphisms coincides with the whole group of polynomial automorphism; more precisely*

$$\text{Aut}(\mathbb{C}^2) = \text{Aff}_2 *_{\text{Aff}_2 \cap E_2} E_2.$$

But this is not the case in higher dimension:  $\text{Tame}_3 \subsetneq \text{Aut}(\mathbb{C}^3)$  (see [9]).

Another important result in dimension 2 is the following:

**Theorem 2.2** ([4]). *Let  $f$  be an element of  $\text{Aut}(\mathbb{C}^2)$ . Then, up to conjugacy,*

- either  $f$  belongs to  $E_2$ ,
- or  $f$  can be written as

$$\varphi_\ell \circ \varphi_{\ell-1} \circ \dots \circ \varphi_1$$

where  $\varphi_i: (z_0, z_1) \mapsto (z_1, P_i(z_1) - \delta_i z_0)$ ,  $\delta_i \in \mathbb{C}^*$ ,  $P_i \in \mathbb{C}[z_1]$ ,  $\deg P_i \geq 2$ .

We denote by  $\mathcal{H}$  the set of polynomial automorphisms of  $\mathbb{C}^2$  can be written up to conjugacy as

$$\varphi_\ell \circ \varphi_{\ell-1} \circ \dots \circ \varphi_1$$

where  $\varphi_i: (z_0, z_1) \mapsto (z_1, P_i(z_1) - \delta_i z_0)$ ,  $\delta_i \in \mathbb{C}^*$ ,  $P_i \in \mathbb{C}[z_1]$ ,  $\deg P_i \geq 2$ . The elements of  $\mathcal{H}$  are of **Hénon type**.

From now on we will denote  $f = (f_0, f_1, \dots, f_{k-1})$  instead of

$$f: (z_0, z_1, \dots, z_{k-1}) \mapsto (f_0(z_0, z_1, \dots, z_{k-1}), f_1(z_0, z_1, \dots, z_{k-1}), \dots, f_{k-1}(z_0, z_1, \dots, z_{k-1})).$$

The **algebraic degree**  $\deg f$  of  $f = (f_0, f_1, \dots, f_{k-1}) \in \text{Aut}(\mathbb{C}^k)$  is  $\max(\deg f_0, \deg f_1, \dots, \deg f_{k-1})$ .

**2.2. The Cremona group.** A **rational self map**  $f$  of  $\mathbb{P}_{\mathbb{C}}^k$  can be written

$$(z_0 : z_1 : \dots : z_k) \mapsto (f_0(z_0, z_1, \dots, z_k) : f_1(z_0, z_1, \dots, z_k) : \dots : f_k(z_0, z_1, \dots, z_k))$$

where the  $f_i$ 's are homogeneous polynomials of the same degree  $\geq 1$  and without common factor of positive degree. The **degree** of  $f$  is the degree of the  $f_i$ . If there exists a rational self map  $g$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that  $fg = gf = \text{id}$  we say that the rational self map  $f$  of  $\mathbb{P}_{\mathbb{C}}^k$  is **birational**. The set of birational self maps of  $\mathbb{P}_{\mathbb{C}}^k$  form a group denoted  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^k)$  and called the **Cremona group**. Of course  $\text{Aut}(\mathbb{C}^k)$  is a subgroup of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^k)$ . An other natural subgroup of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^k)$  is the group  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^k) \simeq \text{PGL}(k+1; \mathbb{C})$  of automorphisms of  $\mathbb{P}_{\mathbb{C}}^k$ .

The **indeterminacy set**  $\text{Ind}(f)$  of  $f$  is the set of the common zeros of the  $f_i$ 's. The **exceptional set**  $\text{Exc}(f)$  of  $f$  is the (finite) union of subvarieties  $M_i$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that  $f$  is not injective on any open subset of  $M_i$ .

**2.3. A little bit of dynamics.** Let  $f$  be a polynomial automorphism of  $\mathbb{C}^k$ . One can see  $f$  as a birational self map of  $\mathbb{P}_{\mathbb{C}}^k$  still denoted  $f$ . We will say that  $f$  is **algebraically stable** if for any  $n > 0$

$$f^n(\{z_k = 0\} \setminus \text{Ind}(f^n))$$

is not contained in  $\text{Ind}(f)$ . This is equivalent to the fact that  $(\deg f)^n = \deg f^n$  for any  $n > 0$ . For instance elements of  $\mathcal{H}$  are algebraically stable.

**Remark 2.3.** Note that in dimension 2 one usually says that  $f$  is algebraically stable if for any  $n > 0$

$$f^n(\{z_2 = 0\} \setminus \text{Ind}(f^n)) \cap \text{Ind}(f) = \emptyset.$$

Be careful this is not equivalent in higher dimension: consider for instance

$$f = (5z_0^2 + z_2^2 + 6z_0z_2 + z_1, z_2^2 + z_0, z_2);$$

then

- on the one hand  $(-1 : 0 : 1 : 0)$  belongs to  $\{z_3 = 0\} \setminus \text{Ind}(f)$  and  $f(-1 : 0 : 1 : 0) = (0 : 1 : 0 : 0) \in \text{Ind}(f) = \{(0 : 1 : 0 : 0)\}$ .
- on the other hand for any  $n \geq 1$   $\deg f^n = (\deg f)^n$ .

The algebraic degree of a birational map  $f$  of  $\mathbb{P}_{\mathbb{C}}^n$  (resp. a polynomial automorphism of  $\mathbb{C}^k$ ) is not a dynamical invariant so we introduce the **dynamical degree**

$$\lambda(f) = \lim_{n \rightarrow +\infty} (\deg f^n)^{1/n}$$

which is a dynamical invariant. In other words if  $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^k)$  (resp.  $\text{Aut}(\mathbb{C}^k)$ ), then for any  $g \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^k)$  (resp.  $g \in \text{Aut}(\mathbb{C}^k)$ ) one has  $\lambda(f) = \lambda(gfg^{-1})$ . For any element  $f$  in  $\mathcal{H}$  the algebraic and dynamical degrees coincide; more precisely if

$$f = \varphi_{\ell} \circ \varphi_{\ell-1} \circ \dots \circ \varphi_1$$

where  $\varphi_i = (z_1, P_i(z_1) - \delta_i z_0)$ ,  $\delta_i \in \mathbb{C}^*$ ,  $P_i \in \mathbb{C}[z_1]$ ,  $\deg P_i \geq 2$  one has ([4])

$$\lambda(f) = \prod_{i=1}^{\ell} \deg \varphi_i \geq 2.$$

A polynomial automorphism  $f$  is in  $E_k$  if and only if for any  $n \geq 1$  the equality  $\deg f = \deg f^n$  holds hence  $\lambda(f) = 1$ . In other words a polynomial automorphism  $f$  of  $\mathbb{C}^2$  belongs to  $\mathcal{H}$  if and only if  $\lambda(f) > 1$ . There is an other characterization of the automorphisms of Hénon type:

**Theorem 2.4** ([7]). *The centralizer of  $f$  in  $\text{Aut}(\mathbb{C}^2)$ , that is  $\{g \in \text{Aut}(\mathbb{C}^2) \mid fg = gf\}$ , is countable if and only if  $f$  belongs to  $\mathcal{H}$ .*

### 3. AUTOMORPHISMS WITH POLYNOMIAL GROWTHS

**3.1. The growths of a polynomial automorphisms and its inverse.** If  $f$  is a polynomial automorphism of  $\mathbb{C}^k$ , then  $(\deg f, \deg f^{-1})$  is the bidegree of  $f$ . There is a relationship between  $\deg f$  and  $\deg f^{-1}$  (see [1]):

$$\begin{cases} \deg f^{-1} \leq (\deg f)^{k-1} \\ \deg f \leq (\deg f^{-1})^{k-1} \end{cases} \quad (3.1)$$

As a result if  $f$  is a polynomial automorphism of  $\mathbb{C}^k$ , the degree growths of  $f$  and  $f^{-1}$  are linked:

**Proposition 3.1.** *Let  $f$  be a polynomial automorphism of  $\mathbb{C}^k$ .*

- *The sequence  $(\deg f^n)_{n \in \mathbb{N}}$  is bounded if and only if the sequence  $(\deg f^{-n})_{n \in \mathbb{N}}$  is bounded.*
- *The sequence  $(\deg f^n)_{n \in \mathbb{N}}$  grows exponentially if and only if  $(\deg f^{-n})_{n \in \mathbb{N}}$  grows exponentially.*
- *If  $\deg f^n \simeq n^p$  and  $\deg f^{-n} \simeq n^q$  for some integers  $p, q \geq 1$ , then*

$$(p, q) \in \left\{ \left( \left[ \frac{q+1}{k} \right], q \right), \dots, (kq, q) \right\}.$$

**Remark 3.2.** When we write "The sequence  $(\deg f^n)_{n \in \mathbb{N}}$  grows exponentially if and only if  $(\deg f^{-n})_{n \in \mathbb{N}}$  grows exponentially" it does not mean that  $(\deg f^n)_{n \in \mathbb{N}}$  and  $(\deg f^{-n})_{n \in \mathbb{N}}$  have exactly the same behavior: the polynomial automorphism of  $\mathbb{C}^3$  given by  $f = (z_0^2 + z_1 + z_2, z_0^2 + z_1, z_0)$  satisfies for all  $n \geq 1$

$$\begin{cases} \deg f^n = 2^n \\ \deg f^{-n} = 2^{\lfloor \frac{n+1}{2} \rfloor} \end{cases}$$

**3.2. Examples of polynomial automorphisms with new polynomial growths.** Let us now give examples of polynomial automorphisms with polynomial growths.

**Lemma 3.3.** *Let us consider the polynomial automorphism of  $\mathbb{C}^3$  given by*

$$f = (z_1 + z_0 z_2^d, z_0, z_2)$$

where  $d \geq 1$ . One has  $\deg f^n = dn + 1$  and  $\deg f^n = \deg f^{-n}$  for any  $n \geq 1$ .

*Proof.* Assume  $n \geq 1$ . Set  $f^n = (f_{0,n}, f_{1,n}, z_2)$  and  $\delta_n = \deg f^n$ . Note that  $\delta_1 = d + 1$  and since

$$f^n = f f^{n-1} = (f_{1,n-1} + f_{0,n-1} z_2^d, f_{0,n-1}, z_2)$$

one has

$$\delta_n = \max(\deg f_{1,n-1}, \deg f_{0,n-1} + d, \deg f_{0,n-1}, 1).$$

But  $f_{1,n-1} = f_{0,n-2}$  and  $d \geq 1$  so  $\delta_n = \deg f_{0,n-1} + d = \delta_{n-1} + d$ . □

Similarly one can prove:

**Lemma 3.4.** *Let us consider the polynomial automorphism of  $\mathbb{C}^5$  given by*

$$g = (z_1 + z_0 z_2^d, z_0, z_2, z_4 + z_0^p z_3, z_3)$$

where  $p \geq d \geq 1$ . One has for any  $n \geq 1$

$$\deg g^n = \frac{pd}{2} n^2 + \frac{p(2-d)}{2} n + 1$$

and  $\deg g^n = \deg g^{-n}$  for any  $n \geq 1$ .

Let us consider the polynomial automorphism of  $\mathbb{C}^7$  given by

$$h = (z_1 + z_0 z_2^d, z_0, z_2, z_4 + z_0^p z_3, z_3, z_6 + z_3^\ell z_5, z_5)$$

where  $\ell \geq p \geq d \geq 1$ . For any  $n \geq 1$

$$\deg h^n = 1 + \ell \left( 1 - \frac{p}{2} + \frac{pd}{3} \right) n + \frac{\ell p(1-d)}{2} n^2 + \frac{\ell pd}{6} n^3$$

and  $\deg h^n = \deg h^{-n}$ .

So Lemma 3.3 gives an example  $f$  of polynomial automorphism of  $\mathbb{C}^3$  with linear growth; from  $f$  one gets a polynomial automorphism  $g$  of  $\mathbb{C}^5$  such that  $\deg g^n \sim n^2$ , and from  $g$  one gets a polynomial automorphism  $h$  of  $\mathbb{C}^7$  such that  $\deg h^n \sim n^3$  (Lemma 3.4). By repeating this process one gets the following statement:

**Proposition 3.5.** *There exist polynomial automorphisms  $f$  of  $\mathbb{C}^{2k+1}$ ,  $k \geq 2$ , such that*

$$\deg f^n \sim n^k.$$

Theorem A follows from Proposition 3.5.

**3.3. A consequence.** Come back to  $f = (z_1 + z_0 z_2^d, z_0, z_2)$  and consider the birational map of  $\mathbb{P}_{\mathbb{C}}^4$  given by  $F = (f, z_0^p z_3)$ , that is  $F = (z_1 + z_0 z_2^d, z_0, z_2, z_0^p z_3)$ . Assume that  $d \leq p$ . Then one can prove by induction that for any  $n \geq 1$

$$\deg F^n = \frac{pd}{2} n^2 + \frac{p(2-d)}{2} n + 1.$$

Let us now define the birational map  $F$  of  $\mathbb{P}_{\mathbb{C}}^5$  by

$$G = (z_1 + z_0 z_2^d, z_0, z_2, z_0^p z_3, z_3^\ell z_4);$$

for any  $n \geq 1$  one has

$$\deg G^n = \frac{\ell pd}{6} n^3 + \left( 1 - \frac{3d}{4} \right) \ell p n^2 + \left( \frac{13}{12} pd - 2p + 1 \right) \ell n - \frac{\ell pd}{2} + \ell p + 1.$$

Repeating this process one gets

**Proposition 3.6.** *If  $k \geq 3$ , then for any  $0 \leq \ell \leq k - 2$  there exist birational maps  $F$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that*

$$\deg F^n \sim n^\ell.$$

#### 4. PROPERTIES $(\mathcal{P}_i)$

Note that Theorem 2.4 can be also stated as follows: the centralizer of a polynomial automorphism  $f$  of  $\mathbb{C}^2$  is countable if and only if  $(\deg f^n)_{n \in \mathbb{N}}$  grows exponentially. This property is not true in higher dimension: there exist polynomial automorphisms of  $\mathbb{C}^3$  with uncountable centralizer and exponential degree growth ([2]). Let us now that this is also the case for other properties, and in particular for  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ ,  $(\mathcal{P}_3)$ ,  $(\mathcal{P}_4)$ .

4.1. **Property** ( $\mathcal{P}_1$ ). Property ( $\mathcal{P}_1$ ) is not satisfied in dimension 3:

**Proposition 4.1.** *If*

$$f = (z_2, (z_2^2 + z_0)^2 + z_2^2 + z_0 + z_1, z_2^2 + z_0)$$

then

- $\text{Ind}(f) = \{z_2 = z_3 = 0\}$  and  $z_3 = 0$  is blown down by  $f$  onto  $(0 : 1 : 0 : 0) \in \text{Ind}(f)$ ,
- for all  $n \geq 1$  one has  $\deg f^n = 2^{n+1}$ .

4.2. **Property** ( $\mathcal{P}_2$ ). Property ( $\mathcal{P}_2$ ) does not hold in higher dimension:

**Proposition 4.2.** *The polynomial automorphism  $f$  of  $\mathbb{C}^3$  given by*

$$(z_0^2 + z_1, z_0, z_2 + 1)$$

preserves the fibration  $z_2 = \text{cst}$  and for all  $n \geq 1$  the equality  $\deg f^n = 2^n$  holds.

*The polynomial automorphism  $g$  of  $\mathbb{C}^3$  given by*

$$(z_1^2 + z_0 z_1 + z_2, z_1 + 1, z_0)$$

preserves the fibration  $z_1 = \text{cst}$  and for all  $n \geq 1$  the equality  $\deg g^n = n + 1$  holds.

4.3. **Property** ( $\mathcal{P}_3$ ). Property ( $\mathcal{P}_3$ ) is not satisfied in dimension 3.

**Proposition 4.3.** *Let us consider the polynomial automorphisms  $f$  and  $g$  of  $\mathbb{C}^3$  defined by*

$$f = (z_0 + z_1 + z_2, z_0^2 + z_0 + z_1, z_0), \quad g = (z_1^2 + z_0 + z_1 + z_2, z_1, z_0).$$

For all  $n \geq 1$  one has

$$\begin{cases} \deg f^{2n} = 2^{n+1}, \deg f^{2n+1} = 2^{n+1} \\ \deg f^{-n} = 2^n \\ \deg g^n = \deg g^{-n} = 2 \end{cases}$$

*The automorphism  $f$  sends  $z_3 = 0$  onto  $(0 : 1 : 0 : 0)$  and  $f^{-1}$  sends  $z_3 = 0$  onto  $(0 : 1 : 1 : 0)$ . Furthermore  $\text{Ind}(f) = \{z_0 = z_3 = 0\}$  and  $\text{Ind}(f^{-1}) = \{z_2 = z_3 = 0\}$ .*

*The automorphism  $g$  sends  $z_3 = 0$  onto  $(1 : 0 : 0 : 0)$  and  $g^{-1}$  sends  $z_3 = 0$  onto  $(0 : 0 : 1 : 0)$ . Besides  $\text{Ind}(g) = \{z_1 = z_3 = 0\}$  and  $\text{Ind}(g^{-1}) = \{z_2 = z_3 = 0\}$ .*

4.4. **Property** ( $\mathcal{P}_4$ ). In [5] Furter proves that if  $f$  is a polynomial automorphism of  $\mathbb{C}^2$  then  $\deg f^2 = (\deg f)^2$  if and only if  $\deg f^n = (\deg f)^n$  for all  $n \in \mathbb{N}$ . This property does not hold in higher dimension: consider for instance the polynomial automorphism  $f$  given by

$$f = (z_1^2 + z_5, z_5^2 + z_4, z_2, z_1, z_0, z_4^2 + z_3).$$

One can check that  $\deg f = 2$ ,  $\deg f^2 = 4$ ,  $\deg f^3 = 8$  but  $\deg f^4 = 8$ .

Let  $f$  be a polynomial automorphism of  $\mathbb{C}^k$ . For any integer  $n \geq 0$  set  $\Omega_n = f^n((z_{k-1} = 0) \setminus \text{Ind}(f^n))$ ; note that  $\Omega_n \subset (z_{k-1} = 0)$  for any  $n$ . We say that  $f$  is **not algebraically stable after  $\ell$  steps** if  $\ell$  is the smallest integer such that  $\Omega_\ell \subset \text{Ind}(f)$ .

Let us first remark that if  $\Omega_1 \cap \text{Ind}(f) = \emptyset$ , then  $\Omega_2 = f(\Omega_1) \subseteq \Omega_1$  so  $\Omega_2 \cap \text{Ind}(f) = \emptyset$ . By induction one gets for any  $n \geq 1$  that  $\Omega_n \cap \text{Ind}(f) = \emptyset$  and  $\deg f^n = (\deg f)^n$ , i.e.  $f$  is algebraically stable.

Let us now assume that  $\Omega_1 \cap \text{Ind}(f) \neq \emptyset$ . Then:

- (1) Either  $\Omega_1 \subset \text{Ind}(f)$ , that is  $f$  is not algebraically stable after 1 step.
- (2) Or  $\Omega_1 \not\subset \text{Ind}(f)$  hence  $\Omega_2 = f(\Omega_1 \setminus \text{Ind}(f)) \subseteq \Omega_1$ .
  - Either  $\dim \Omega_2 = \dim \Omega_1$ , so  $\Omega_2 = \Omega_1$  and then  $\Omega_n = \Omega_1$  for any  $n$ ; in particular  $\Omega_n \not\subset \text{Ind}(f)$  for any  $n$  and  $f$  is algebraically stable.

- Or  $\dim \Omega_2 < \dim \Omega_1$ , then either  $\Omega_2 \subset \text{Ind}(f)$  and  $f$  is not algebraically stable after 2 steps or  $\Omega_2 \not\subset \text{Ind}(f)$  and we come back to the previous alternative, that is either  $\dim \Omega_3 = \dim \Omega_2$  or  $\dim \Omega_3 < \dim \Omega_2$ . Since for any  $n$  one has  $0 \leq \dim \Omega_n \leq k-1$  one gets that either  $f$  is algebraically stable, or  $f$  is not algebraically stable after at most  $k-1$  steps.

Hence one can state:

**Proposition 4.4.** *Let  $f$  be a polynomial automorphism of  $\mathbb{C}^k$ . Either  $f$  is algebraically stable, or  $f$  is not algebraically stable after  $\ell$  steps, with  $\ell \leq k-1$ .*

*In other words  $\deg f^i = (\deg f)^i$  for  $1 \leq i \leq k$  if and only if  $\deg f^n = (\deg f)^n$  for any  $n \geq 1$ .*

## 5. BIRATIONAL MAPS WITH NEW POLYNOMIAL GROWTHS

Let us first recall that

**Lemma 5.1** ([3]). *The birational self map  $\varphi$  of  $\mathbb{P}_{\mathbb{C}}^2$  given in the affine chart  $z_2 = 1$  by*

$$\varphi(z_0, z_1) = \left( z_1 + \frac{2}{3}, z_0 \frac{z_1 - \frac{1}{3}}{z_1 + 1} \right)$$

*satisfies  $\deg \varphi^n \sim n^2$ .*

Denote  $\varphi^n$  by  $\left( \frac{P_n}{Q_n}, \frac{R_n}{S_n} \right)$  where  $P_n, Q_n, R_n$  and  $S_n$  denote some elements of  $\mathbb{C}[z_0, z_1]$  without common factor. Set  $p_n = \deg P_n, q_n = \deg Q_n, r_n = \deg R_n$  and  $s_n = \deg S_n$ . The following equalities hold (by iteration)

$$\begin{cases} p_n = s_{n-1} + 1 \\ q_n = s_{n-1} \\ r_n = s_n + 1 \\ \deg \varphi^n = s_{n-1} + s_n + 1 \end{cases}$$

Let us now consider the birational self map of  $\mathbb{P}_{\mathbb{C}}^3$  given in the affine chart  $z_3 = 1$  by

$$\Psi_3(z_0, z_1, z_2) = \left( z_1 + \frac{2}{3}, z_0 \frac{z_1 - \frac{1}{3}}{z_1 + 1}, z_0 z_2 \right).$$

One can also check that

$$\Psi_3 = \left( \frac{P_n}{Q_n}, \frac{R_n}{S_n}, \frac{U_n}{V_n} \right)$$

where  $U_n = z_0 z_2 P_1 P_2 \dots P_{n-1}$  and  $V_n = Q_1 Q_2 \dots Q_{n-1}$  have no common factor. Since  $s_i \sim i^2$  one gets:

**Lemma 5.2.** *The birational self map  $\Psi_3$  of  $\mathbb{P}_{\mathbb{C}}^3$  satisfies*

$$\deg \Psi_3 \sim n^3.$$

Let us now consider the birational self map of  $\mathbb{P}_{\mathbb{C}}^4$  defined in the affine chart  $z_4 = 1$  by

$$\Psi_4(z_0, z_1, z_2, z_3) = \left( z_1 + \frac{2}{3}, z_0 \frac{z_1 - \frac{1}{3}}{z_1 + 1}, z_0 z_2, z_2 z_3 \right).$$

One can check that  $\Psi_4 = \left( \frac{P_n}{Q_n}, \frac{R_n}{S_n}, \frac{U_n}{V_n}, \frac{W_n}{X_n} \right)$  where  $W_n = W_1 U_1 U_2 \dots U_{n-1}$  and  $X_n = X_1 V_1 V_2 \dots V_{n-1}$  have no common factor. Since  $\deg U_n \sim n^3$  and  $\deg V_n \sim n^3$  one has:

**Lemma 5.3.** *The birational self map  $\Psi_4$  of  $\mathbb{P}_{\mathbb{C}}^4$  satisfies*

$$\deg \Psi_4 \sim n^4.$$



By repeating this process one gets:

**Theorem 5.4.** *Let  $k$  be an integer  $\geq 3$ . There exist birational maps  $\phi$  of  $\mathbb{P}_{\mathbb{C}}^k$  such that*

$$\deg \phi^n \sim n^k.$$

This statement and Lin's result ([8]) imply Theorem D.

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