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# Density of $C_{-4}$ -critical signed graphs

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## Abstract

A signed bipartite (simple) graph  $(G, \sigma)$  is said to be  $C_{-4}$ -critical if it admits no homomorphism to  $C_{-4}$  (a negative 4-cycle) but every proper subgraph of it does. In this work, first of all we show that the notion of 4-coloring of graphs and signed graphs is captured, through simple graph operations, by the notion of homomorphism to  $C_{-4}$ . In particular, the 4-color theorem is equivalent to: Given a planar graph  $G$ , the signed bipartite graph obtained from  $G$  by replacing each edge with a negative path of length 2 maps to  $C_{-4}$ .

We prove that, except for one particular signed bipartite graph on 7 vertices and 9 edges, any  $C_{-4}$ -critical signed graph on  $n$  vertices must have at least  $\lceil \frac{4n}{3} \rceil$  edges, and that this bound or  $\lceil \frac{4n}{3} \rceil + 1$  is attained for each value of  $n \geq 9$ . As an application, we conclude that all signed bipartite planar graphs of negative girth at least 8 map to  $C_{-4}$ . Furthermore, we show that there exists an example of a signed bipartite planar graph of girth 6 which does not map to  $C_{-4}$ , showing that 8 is the best possible and disproving a conjecture of Naserasr, Rollova and Sopena, in extension of the above mentioned restatement of the 4CT.

## 1 Introduction

A *homomorphism* of a graph  $G$  to a graph  $H$  is a mapping of the vertices of  $G$  to the vertices of  $H$  such that adjacencies are preserved. The theory of graph homomorphism is a natural extension of the notion of *proper coloring* where a proper  $k$ -coloring (of a graph  $G$ ) can be viewed as a homomorphism (of  $G$ ) to  $K_k$ . One of the key notions in the study of proper coloring is the concept of  *$k$ -critical* graphs, which is defined to be a graph of chromatic number  $k$  all whose proper subgraphs are  $(k - 1)$ -colorable. An extension of the notion to homomorphism was proposed in 1980's by Catlin [3], but the concept was not drawn much attention until recently. Given a graph  $H$ , a graph  $G$  is said to be  *$H$ -critical*, if  $G$  does not admit a homomorphism to  $H$  but every proper subgraph of it does.

Next to the complete graphs, the most studied graphs in the theory of homomorphism are odd cycles. It is a folklore fact that the  $C_{2k+1}$ -coloring problem captures the  $(2k + 1)$ -coloring problem via a basic graph operation: Given a graph

$G$ , let  $T'_{2k-1}(G)$  be obtained from  $G$  by subdividing each edge into path of length  $2k-1$ . Then  $T'_{2k-1}(G)$  admits a homomorphism to  $C_{2k+1}$  if and only if  $G$  is properly  $(2k+1)$ -colorable (see [7]).

One of the key directions of the study of  $k$ -critical graphs is to bound below the number of edges as a function of  $k$  and  $n$  (the number of vertices). Kostochka and Yancy gave a nearly tight lower bound in [10], almost settling a conjecture of Gallai. Observing that being a 4-critical graph is the same as being  $C_3$ -critical, it follows from the special case presented in [9] that any  $C_3$ -critical graph on  $n$  vertices has at least  $\lceil \frac{5n-2}{3} \rceil$  edges. Their approach is extended to the study of  $C_5$ -critical graphs in [6] and to  $C_7$ -critical graphs in [17]. In [6], it is proved that any  $C_5$ -critical graph on  $n$  vertices has at least  $\frac{5n-2}{4}$  edges and they conjecture that the bound can be improved to  $\frac{14n-9}{11}$ . Similarly, in [17], it is proved that any  $C_7$ -critical graph on  $n$  vertices has at least  $\frac{17n-2}{15}$  edges and they conjecture that the bound can be improved to  $\frac{27n-20}{23}$ .

In this work, based on recent development of the theory of homomorphisms of signed graphs, we show that by replacing odd cycles with negative cycles, we can fill the parity gap in this study. Then focusing on  $C_{-4}$ -critical signed graphs, we show that any such a signed graph on  $n$  vertices must have at least  $\lceil \frac{4n}{3} \rceil$  edges with a sole exception of a signed bipartite graph on 7 vertices which has only 9 edges.

In the next section, we present the necessary terminology, and the relation between coloring of graphs and homomorphisms of signed graphs to negative cycles. In Section 3, we prove our main result which is on the minimum number of edges of  $C_{-4}$ -critical signed graphs. In Section 4 we introduce some techniques to build  $C_{-4}$ -critical signed graphs of low edge-density, using which we conclude tightness of our bound. Then in Section 5, we consider applications to planar case and relation to a bipartite analogue of Jaeger-Zhang conjecture, and discuss further direction of study.

## 2 Signed graphs and homomorphisms

A *signed graph*  $(G, \sigma)$  is a graph together with an assignment, called *signature*,  $\sigma$  of signs (i.e.  $+$  or  $-$ ) to the edges of  $G$ . When the signature is not of high importance, we may write  $\hat{G}$  in place of  $(G, \sigma)$ . When all edges are positive (resp. negative) we write  $(G, +)$  (respectively  $(G, -)$ ). For drawing a signed graph, we use solid or blue lines to represent positive edges and dashed or red lines to represent negative edges. For underlying graphs (with no signature) we use gray color. Given a signed graph  $(H, \sigma')$ , it is said to be an (*induced*) *subgraph* of  $(G, \sigma)$  if  $H$  is an (induced) subgraph of  $G$  and  $\sigma'$  is a signature on  $H$  such that for every  $e \in E(H)$ :  $\sigma'(e) = \sigma(e)$ . For simplicity, we may write  $(H, \sigma)$  in place of  $(H, \sigma')$ .

A *switching* of a signed graph  $(G, \sigma)$  at a vertex  $x$  is to switch the signs of all the edges incident to  $x$ . A *switching* of  $(G, \sigma)$  is a collection of switchings at each of the elements of a given set  $X$  of vertices. That is equivalent to switching the signs of all edges in the edge-cut  $(X, V \setminus X)$ . Two signatures  $\sigma_1$  and  $\sigma_2$  on a graph  $G$  are said to be *equivalent* if one can be obtained from the other by a switching, in which case we say  $(G, \sigma_1)$  is *switching equivalent* to  $(G, \sigma_2)$ .

The sign of a structure in  $(G, \sigma)$  is the product of the signs of the edges in the given structure, counting multiplicity. The sign of some structures, such as a cycle or a closed walk, is invariant under a switching, while for some other structures, such as a path, the sign of it may change (e.g., if a switching is done in one of

the two ends of a path). Thus we may relax or restrict our use accordingly. For example, when speaking of sign of a cycle, we may refer to any equivalent signature, but when speaking of sign of a path, we are restricted to the signature in hand.

In particular, any signed graph on  $C_l$  with an even number of negative edges will simply be referred to as  $C_{+l}$  and when there are an odd number of negative edges it will be denoted by  $C_{-l}$ . As  $C_{-4}$  is the primary subject of this work, we will use the labeling of Figure 1 when referring to this signed graph on its own, but as a subgraph of another signed graph it will have an induced labeling.

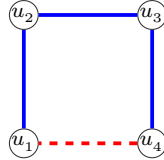


Figure 1:  $C_{-4}$

One of the preliminary facts in the study of signed graphs is that two signatures on a graph  $G$  are equivalent if and only if they induce a same set of negative cycles (see [19]). Thus, when a class of switching equivalent signed graphs on a graph  $G$  is to be considered, one may refer to a partition of cycles, or, more generally, closed walks, of  $G$  into two sets of positive and negative (see [15] for more). Thus we have two natural definitions of homomorphisms of signed graphs.

A (switching) *homomorphism* of a signed graph  $(G, \sigma)$  to a signed graph  $(H, \pi)$ , is a mapping of  $V(G)$  and  $E(G)$  respectively to  $V(H)$  and  $E(H)$  that preserves the adjacencies, the incidences and the signs of closed walks. When there exists a homomorphism of  $(G, \sigma)$  to  $(H, \pi)$ , we may write  $(G, \sigma) \rightarrow (H, \pi)$ . We may also, equivalently, say  $(G, \sigma)$  is  $(H, \pi)$ -colorable.

An *edge-sign preserving homomorphism* of a signed graph  $(G, \sigma)$  to a signed graph  $(H, \pi)$ , is a mapping of  $V(G)$  and  $E(G)$  respectively to  $V(H)$  and  $E(H)$  that preserves the adjacencies, the incidences and the signs of edges. When there exists an edge-sign preserving homomorphism of  $(G, \sigma)$  to  $(H, \pi)$ , we may write  $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$ .

We note that an edge-sign preserving homomorphism is equivalent to what is known as the homomorphism of 2-edge-colored graphs in the literature.

The two notions of homomorphisms are connected through the following observation:

**Observation 2.1.** *Given signed graphs  $(G, \sigma)$  and  $(H, \pi)$ , we have  $(G, \sigma) \rightarrow (H, \pi)$  if and only if there exists an equivalent signature  $\sigma'$  of  $\sigma$  such that  $(G, \sigma') \xrightarrow{s.p.} (H, \pi)$ .*

While closely related, the two notions are also fundamentally different. In particular, for our main target  $C_{-4}$ , while deciding if a signed graph  $(G, \sigma)$  admits a homomorphism to it is an NP-complete problem [5], the analogue edge-sign preserving problem becomes polynomial time through a duality presented in Theorem 3.1.

In practice, we will take the condition of Observation 2.1 as the definition. Thus a homomorphism  $\phi$  of  $(G, \sigma)$  to  $(H, \pi)$  consists of three parts:  $\phi_1 : V(G) \rightarrow \{+, -\}$ , which decides for each vertex  $v$  whether a switching is done at  $v$ ,  $\phi_2 : V(G) \rightarrow V(H)$  which decides to which vertex of  $H$  the vertex  $v$  is mapped to, and  $\phi_3 : E(G) \rightarrow E(H)$  which decides the image of each edge. However, as we will consider only simple graphs in this work,  $\phi_3$  is induced by  $\phi_2$  and, therefore, the mapping  $\phi$  is

composed of  $\phi_1$  and  $\phi_2$ , i.e.  $\phi = (\phi_1, \phi_2)$ . We note that since switching at  $X$  is the same as switching at  $V \setminus X$ , the two mappings  $(\phi_1, \phi_2)$  and  $(-\phi_1, \phi_2)$  are identical.

Each of the notions leads to a corresponding notion of *isomorphism* that is a homomorphism  $\phi$  where  $\phi_2$  and  $\phi_3$  are one-to-one and onto. This, furthermore, leads to two notions of automorphism. Thus, for example, the signed graph  $C_{-4}$ , as a 2-edge-colored graph, has only one non-trivial automorphism which is  $u_1 \leftrightarrow u_4$ ,  $u_2 \leftrightarrow u_3$ . Whereas, it is both vertex-transitive and edge-transitive with respect to the notion of (switching) homomorphism. It would be clear from the context which notion of isomorphism or automorphism we refer to. Following this notion of isomorphism, if  $(G_1, \sigma_1)$  is a subgraph of  $(G, \sigma')$  where  $\sigma'$  is equivalent to  $\sigma$ , then we may refer to  $(G_1, \sigma_1)$  as a subgraph of  $(G, \sigma)$  as well.

Given a signed graph  $(G, \sigma)$  and an element  $ij \in \mathbb{Z}_2^2$ , we define  $g_{ij}(G, \sigma)$  to be the length of a shortest closed walk  $W$  whose number of negative edges modulo 2 is  $i$  and whose length modulo 2 is  $j$ . When there exists no such a closed walk, we define  $g_{ij}(G, \sigma) = \infty$ . By the definition of homomorphisms of signed graphs, we have the following no-homomorphism lemma.

**Lemma 2.2.** [The no-homomorphism lemma] *If  $(G, \sigma) \rightarrow (H, \pi)$ , then*

$$g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$$

for each  $ij \in \mathbb{Z}_2^2$ .

We note that, algorithmically, it is not difficult to determine  $g_{ij}(G, \sigma)$ , we refer to [15] and [4] for more on this.

We may now define the main notion of study in this work.

## 2.1 $(H, \pi)$ -critical signed graphs

Given a signed graph  $(H, \pi)$ , a signed graph  $(G, \sigma)$  is said to be  $(H, \pi)$ -critical if the followings are satisfied:

- $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$  for  $ij \in \mathbb{Z}_2^2$ , (condition of no-homomorphism lemma),
- $(G, \sigma) \not\rightarrow (H, \pi)$ ,
- $(G', \sigma) \rightarrow (H, \pi)$  for every proper subgraph  $G'$  of  $G$ .

The notion captures and extends the notion of  $k$ -critical graphs as follows: a graph  $G$  is  $k$ -critical if the signed graph  $(G, -)$  is  $(K_{k-1}, -)$ -critical, here the condition of no-homomorphism lemma implies that  $G$  has no loop. The notion of  $H$ -critical graph is also captured by viewing  $H$  as the signed graph  $(H, -)$  but with a minor revision. If  $G$  is an  $H$ -critical graph in the sense of [3] and it has an odd cycle  $C_{2k+1}$ , where  $\text{odd-girth}(H) > 2k + 1$ , then  $G$  is the odd cycle  $C_{2k+1}$ . Our first condition then eliminates these trivial cases.

For the particular case when  $(H, \pi) = C_{-l}$ , we identify two cases based on the parity of  $l$ :

- $l = 2k + 1$ . In this case, in order for  $(G, \sigma)$  to satisfy the conditions of no-homomorphism lemma, in particular, we must have  $(G, -)$  switching equivalent to  $(G, \sigma)$ . After a switching of  $(G, \sigma)$  to  $(G, -)$  and  $C_{-l}$  to  $(C_{2k+1}, -)$ , the problem is reduced to the study of  $C_{2k+1}$ -critical graphs (of odd-girth at least  $2k + 1$ ).

- $l = 2k$ . In this case, in order for  $(G, \sigma)$  to satisfy the conditions of no-homomorphism lemma,  $G$  must, in particular, be bipartite. This is the case of main interest in this work.

We note that in the first case, to determine if  $(G, \sigma)$  is switching equivalent to  $(G, -)$  can be done in polynomial time and quite efficiently, but to determine if  $G \rightarrow C_{2k+1}$  is an NP-complete problem. In contrast, in the second case, to find an equivalent signature under which we can map  $(G, \sigma)$  to  $C_{-l}$  is the hard part, and given a fixed signature, we can determine, in polynomial time, if there exists an edge-sign preserving homomorphism (see Theorem 3.1).

## 2.2 $k$ -coloring and $C_{-l}$ -coloring

Given a signed graph  $(G, \sigma)$ , we define  $T_l(G, \sigma)$  to be the signed graph  $(G_l, \pi)$  where  $G_l$  is obtained from  $G$  by subdividing each edge so to become a path of length  $l$  and  $\pi$  is an assignment of signs on the edges of  $G_l$  so that the sign of the  $u - v$  path, corresponding to the edge  $uv \in E(G)$ , is the same as  $-\sigma(uv)$ .

The following lemma then shows the importance of the study of  $C_{-l}$ -coloring.

**Lemma 2.3.** *A graph  $G$  is  $k$ -colorable if and only if  $T_{k-2}(G, +)$  is  $C_{-k}$ -colorable.*

*Proof.* We consider two cases based on the parity of  $k$ . If  $k$  is an odd number, then in  $T_{k-2}(G, +)$  a cycle is negative if and only if it is of an odd length. Thus, this signed graph is switching equivalent to  $(G_{k-2}, +)$ . Then, the problem of mapping  $T_{k-2}(G, +)$  to  $C_{-k}$  is reduced to a graph homomorphism problem of mapping  $G_{k-2}$  to  $C_k$ . The equivalence then can be easily checked and we refer to [7] for a proof.

We now assume that  $k = 2l$  is an even number, in which case  $T_{k-2}(G, +)$  is a signed bipartite graph.

We first show that if  $T_{k-2}(G, +) \rightarrow C_{-k}$ , then  $G$  is  $k$ -colorable. Since this can be done independently on each connected component of  $G$ , we may assume  $G$  is connected. Observe that, as a signed graph equipped with switching,  $C_{-k}$  is both vertex-transitive and edge-transitive. Let  $x_1, x_2, \dots, x_{2l}$  be the vertices of  $C_{-k}$  in the cyclic order. Then  $X_1 = \{x_1, x_3, \dots, x_{2l-1}\}$  and  $X_2 = \{x_2, x_4, \dots, x_{2l}\}$  is the bipartition of the underlying  $2l$ -cycle. Let  $\phi$  be a homomorphism of  $T_{k-2}(G, +)$  to  $C_{-k}$ . Observe that as  $G$ , and, therefore,  $T_{k-2}(G, +)$  is connected, the mapping  $\phi$  preserves the bipartition of  $T_{k-2}(G, +)$ . Thus we may assume, without loss of generality, that the vertices of  $T_{k-2}(G, +)$  which correspond to the vertices of  $G$  map to the vertices in  $X_1$ . Furthermore, recall that the homomorphism  $\phi$  consists of two components  $\phi_1 : V(T_{k-2}(G, +)) \rightarrow \{+, -\}$ , and  $\phi_2 : V(T_{k-2}(G, +)) \rightarrow X_1 \cup X_2$ . Thus restriction of  $\phi$  onto  $V(G)$  is a mapping to the set  $\{+, -\} \times X_1$  which is of order  $2l$ . We claim that  $\phi$  is a proper coloring of  $G$ . That is simply because if  $\phi$  maps two adjacent vertices to a same element of  $\{+, -\} \times X_1$ , the negative  $(k-2)$ -path that is connecting them in  $T_{k-2}(G, +)$  is mapped to a negative closed walks of length at most  $k-2$ , but that contradicts the no-homomorphism lemma.

The converse then is easier. Assume  $\chi(G) \leq 2l$  and let  $\psi$  be a  $2l$ -coloring of  $G$  where  $\{+, -\} \times X_1$  is the color set. We claim that  $\psi$  can be extended as a homomorphism of  $T_{k-2}(G, +)$  to  $C_{-k}$ . For any edge  $uv$  in  $G$ , noting that  $\psi(u) = \psi(v)$  is not possible because  $\psi$  is a proper coloring, we consider two possibilities:

- $(\psi_1(u), \psi_2(u)) = (-\psi_1(v), \psi_2(v))$ . The mapping  $\psi$  then has applied a switching only in one end of the  $u - v$  path, and thus switches it to a positive (even)

path. After identifying its end points the resulting positive even cycle can be mapped to just an edge of any sign.

- $\psi_2(u) \neq \psi_2(v)$ . Then  $\psi_2(u)$  and  $\psi_2(v)$  split  $C_{-k}$  into two paths of even length, one positive and one negative, each of which is of length at most  $k - 2$ . The  $u - v$  path then can be mapped to the path of the same sign where the sign is taken after applying possible switching by  $\psi_1$  at its end points.

□

**Corollary 2.4.** *A graph  $G$  is 4-colorable if and only if  $T_2(G, +)$  maps to  $C_{-4}$ .*

In particular, the four-color theorem can be restated as:

**Theorem 2.5.** [The 4CT theorem restated] *For any planar graph  $G$ , the signed bipartite planar graph  $T_2(G, +)$  maps to  $C_{-4}$ .*

Observing that, for a graph  $G$ , the shortest (negative) cycle in  $T_2(G, +)$  is of length at least 6, (corresponding to a triangle of  $G$ ), and introducing the bipartite analogue of Jaeger-Zhang conjecture, Naserasr, Rollova and Sopena [14] conjectured that any signed bipartite planar graph whose shortest negative girth is 6 maps to  $C_{-4}$ . In section 5, we disprove this conjecture. However, as an application of our work we prove that if the condition on negative girth is increased to 8, then the result holds.

### 3 $C_{-4}$ -critical signed graphs

It follows from Corollary 2.4 that the  $C_{-4}$ -coloring problem is an NP-complete problem (see [1], and [2], for more on this subject). However, when edge-sign preserving homomorphisms are considered, we have a simple duality theorem, given in [4], based on Figure 2, that makes it rather easy to determine the existence of an edge-sign preserving homomorphism to  $C_{-4}$ . This duality notion will be used in our proofs.

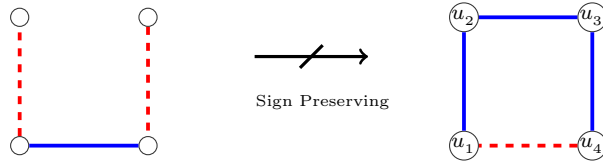


Figure 2:  $C_{-4}$  and its edge-sign preserving dual

**Theorem 3.1.** [4] *Given a signed bipartite graph  $(G, \sigma)$ , we have  $(G, \sigma) \xrightarrow{s.p.} C_{-4}$  if and only if  $(P_3, \pi) \xrightarrow{s.p.} (G, \sigma)$  where  $(P_3, \pi)$  is the signed path of length 3 given in Figure 2.*

Combined with Observation 2.1, this theorem says that in order to map a signed bipartite graph  $(G, \sigma)$  to  $C_{-4}$  it is necessary and sufficient to find a switching  $\sigma'$  of  $\sigma$  where no positive edge is incident with a negative edge at each end of it.

It can be easily verified that any signed bipartite graph with at most two vertices on one of the two parts maps to  $C_{-4}$ . Thus the first example of  $C_{-4}$ -critical signed

graph must have at least six vertices. Let  $\Gamma$  be the signed graph obtained from  $K_4$  by subdividing two parallel edges, each once, with a signature assigned in such a way that each triangle of the  $K_4$  become a negative 4-cycle. It can easily be verified that  $\Gamma$  is an example of a  $C_{-4}$ -critical signed graph on six vertices, in fact, up to switching, it is the only  $C_{-4}$ -critical signed graph on six vertices. We further note that  $\Gamma$  has  $8 = \frac{4}{3} \times 6$  edges.

An example of higher interest, which is also a signed graph on a subdivision of  $K_4$ , is the signed graph  $\hat{W}$  of Figure 3 which is depicted in two different ways. This signed graph is proved in [4] to have smallest maximum average degree among all signed bipartite graphs that does not map to  $C_{-4}$ , that is an average degree of  $\frac{18}{7}$ . Using the extended notion of critical signed graphs we introduced here, we will prove  $\hat{W}$  to be the sole exception among the signed bipartite graphs of average degree less than  $\frac{8}{3}$ .



Figure 3:  $C_{-4}$ -critical signed graph  $\hat{W}$  depicted in two ways

We give two different proofs for the fact that  $\hat{W}$  does not map to  $C_{-4}$ . Each proof takes advantage of one of the presentations in Figure 3, and leads to different development of ideas.

**Proposition 3.2.** *The signed graph  $\hat{W}$  of Figure 3 does not map to  $C_{-4}$ . Moreover, up to a switching equivalence, this is the only signature on this graph with this property.*

*Proof.* Based on the presentation on the left side, if  $\hat{W}$  maps to  $C_{-4}$ , then the outer 6-cycle, as it is a negative cycle, must map surjectively to  $C_{-4}$ , but then  $v_0$  must be identified with one of  $v_2, v_4, v_6$ , thus creating a negative cycle of length 2 and, therefore, contradicting the no-homomorphism lemma.

The equivalence class of signatures on this graph is determined by the signs of its three facial 4-cycles as depicted in the left side of the figure. If one of these facial 4-cycles is positive, then degree 2 vertex on this face can be mapped to  $v_0$ , after a switching if needed. The resulting image then easily maps to  $C_{-4}$ .  $\square$

**An alternative proof.** Based on the presentation on the right side, observe that each pair among  $y_1, y_2, y_3$  is connected by a positive 2-path (through  $x_1$ ) and by a negative 2-path. Thus identifying any two of them would create  $C_{-2}$ . In other words, in any homomorphic image of  $\hat{W}$  which is a signed simple graph, the vertices  $y_1, y_2$  and  $y_3$  must have distinct images.

Automorphisms of  $\hat{W}$  split its vertices to three orbits:  $\{v_0\}$ ,  $\{v_1, v_3, v_5\}$  and  $\{v_2, v_4, v_6\}$  and split its edges to two orbits: those incident to  $v_0$  and those on the outer 6-cycle. We need to consider two signed graphs obtained from  $\hat{W}$  by subdividing one of its edges twice and then assigning a signature on the edges of this path so that the sign of the path is the same as the sign of the edge it has



replaced. Since there are two orbits of the edges on  $\hat{W}$ , essentially we have only two signed graphs obtained in this way. Presentations of these two signed graphs, each after a switching, are given in Figures 4 and 5. The signed graph of Figure 4,  $\Omega_1$ , is obtained from  $\hat{W}$  by subdividing the edge  $x_1y_3$  twice (where all three edges are assigned positive signs) and then switching at the vertex set  $\{x_2, x_3, y_3\}$ . The signed graph of Figure 5,  $\Omega_2$ , is obtained from  $\hat{W}$  by subdividing the edge  $x_4y_1$  twice (where all three edges are assigned positive signs) and then switching at the vertex set  $\{x_2, x_4, y_2\}$ .

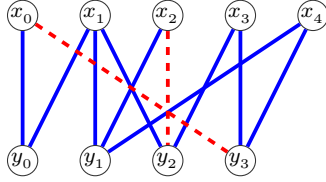


Figure 4:  $\Omega_1$

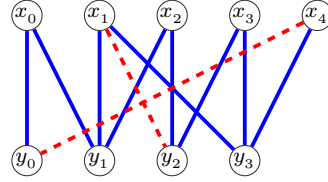


Figure 5:  $\Omega_2$

It is easily observed that each of the two signed graphs with the signature presented in the Figures 4 and 5 satisfies the conditions of Theorem 3.1, and, therefore, each of them maps to  $C_{-4}$ . In the next two lemmas, we show that one cannot make either of these two signed graphs  $C_{-4}$ -critical by only adding a vertex of degree 2.

**Lemma 3.3.** *Let  $\Omega_1$  be the signed graph of Figure 4. If we add a vertex  $v$  to one part of  $\Omega_1$  and connect it with two vertices in the other part (with any signature), the resulting signed graph admits a homomorphism to  $C_{-4}$ .*

*Proof.* Let  $\Omega_1$  be the signed bipartite graph of Figure 4 consisting of a bipartition  $(X, Y)$  where  $X = \{x_0, x_1, x_2, x_3, x_4\}$  and  $Y = \{y_0, y_1, y_2, y_3\}$ .

If the two edges incident to the new vertex  $v$  are of a same sign, by switching at that new vertex, if needed, we consider them both positive. The result is a signature satisfying Theorem 3.1, which implies that  $\Omega_1$  maps to  $C_{-4}$ . So we assume that the two edges incident to  $v$  are of different signs and consider two possibilities depending on to which part the vertex  $v$  belongs to:

**Case 1.**  $v$  is added to the  $X$  part.

If  $v$  is adjacent to  $y_3$ , then, by switching at  $v$ , if necessary, we assume that  $vy_3$  is negative. If the other edge, which is positive, is  $vy_2$ , then we switch at  $x_2$ . In all the cases (when  $v$  is adjacent to  $y_3$ ), we find a signature satisfying the conditions of Theorem 3.1. If  $v$  is not adjacent to  $y_3$  but  $v$  is adjacent to  $y_2$ , then we consider  $vy_2$  to be negative and we are done. If  $v$  is adjacent to both of  $y_0$  and  $y_1$ , then we take  $vy_1$  as a negative edge and we are done after a switching at  $x_2$ .

**Case 2.**  $v$  is added to the  $Y$  part.

If  $v$  is adjacent to one of  $x_2$  and  $x_0$  (or both), we switch at one of  $x_2$  and  $x_0$ . Then by a switching at  $v$  (if needed) we have both edges incident to  $v$  of positive sign. Otherwise we may switch at  $v$  such that the negative edge at  $v$  is either  $vx_4$  or  $vx_1$ . If the former happens, then we switch at  $x_0$ , else we switch at  $x_2$ . In all cases, we find a signature satisfying the condition of Theorem 3.1.  $\square$

**Lemma 3.4.** *Let  $\Omega_2$  be the signed graph of Figure 5. If we add a vertex  $v$  to one part of  $\Omega_2$  and connect it with two vertices in the other part (with any signature), the resulting signed graph either contains  $\hat{W}$  and maps to it or admits a homomorphism to  $C_{-4}$ .*

*Proof.* Let  $\Omega_2$  be the signed bipartite graph of Figure 5 consisting of a bipartition  $(X, Y)$  where  $X = \{x_0, x_1, x_2, x_3, x_4\}$  and  $Y = \{y_0, y_1, y_2, y_3\}$ .

As in the previous lemma, we can assume that considering the two edges incident with  $v$ , one is negative and the other is positive. Again, we consider two cases depending on to which part  $v$  belongs to.

**Case 1.**  $v$  is added to the  $X$  part.

If  $v$  is adjacent to  $y_2$ , then, by a switching at  $v$  if needed, we may assume  $vy_2$  is negative. Then the only possible problem against Theorem 3.1 is by  $vy_0$ , which can be taken care of by switching at  $y_0$ . Else, if  $v$  is adjacent to  $y_0$ , then by considering  $vy_0$  as the negative edge incident to  $v$ , and then by switching at  $y_0$ , the resulting signed graph satisfies the condition of Theorem 3.1. Finally, if  $v$  is adjacent to both  $y_1$  and  $y_3$ , then the subgraph induced by  $x_1, x_2, x_3, y_1, y_2, y_3$  and  $v$  is isomorphic to  $\hat{W}$ . It is then easy to map the remaining vertices to  $\hat{W}$ .

**Case 2.**  $v$  is added to the  $Y$  part.

If  $v$  is adjacent to  $x_1$ , then we choose  $vx_1$  to be negative. The only obstacle against Theorem 3.1 then can come from the  $vx_4$  edge, but we can switch at  $y_0$  to resolve this issue. Else, if  $v$  is adjacent to  $x_4$ , then we choose  $vx_4$  to be negative, then we already have a signature satisfying conditions of Theorem 3.1. Finally, if  $v$  is adjacent to both  $x_2$  and  $x_3$ , then the subgraph induced by  $x_1, x_2, x_3, y_1, y_2, y_3$  and  $v$  is isomorphic to  $\hat{W}$ . It is then easy to map the remaining vertices to  $\hat{W}$ .  $\square$

Some general structural properties of a  $C_{-4}$ -critical signed graph are as follows.

**Lemma 3.5.** *Every  $C_{-4}$ -critical signed graph is 2-connected.*

*Proof.* This is an easy consequence of the fact that  $C_{-4}$  is vertex transitive and we leave the details as an exercise.  $\square$

We say a path  $P$  of length  $k$  in  $\hat{G}$  is a  $k$ -thread if all of its  $k - 1$  internal vertices are of degree 2 in  $G$ . It is easily observed that the maximum length of a thread in an  $(H, \pi)$ -critical graph is bounded by a function of  $(H, \pi)$ . For  $C_{-4}$ -critical signed graphs, we have:

**Lemma 3.6.** *A  $C_{-4}$ -critical signed graph  $\hat{G}$  does not contain a 3-thread.*

*Proof.* Assume to the contrary that  $G$  has a 3-thread  $P = x_0x_1x_2x_3$ . Recall that a  $C_{-4}$ -critical signed graph is bipartite. As  $x_0$  and  $x_3$  are connected by a path of length 3, they are in different parts of  $G$ . Since  $\hat{G}$  is  $C_{-4}$ -critical, the signed graph  $\hat{G}' = \hat{G} - \{x_1, x_2\}$  maps to  $C_{-4}$ . Let  $\varphi$  be such a mapping. Observe that, by Lemma 3.5,  $\hat{G}'$  is connected, thus  $\varphi$  preserves the bipartition of  $G'$ . In particular  $\varphi(x_0)$  and  $\varphi(x_3)$  are in two different parts of  $C_{-4}$  and thus adjacent. We note that  $\varphi$  has possibly applied switchings on some vertices of  $G'$ , working with the resulting signature obtained from the same switching on  $\hat{G}$ , we let  $\hat{P}$  be the signed graph induced on  $P$ . If  $\hat{P}$  has a same sign as the edge  $\varphi(x_0)\varphi(x_3)$ , then  $\varphi$  can be extended by mapping  $\hat{P}$  to this edge as well. Otherwise,  $\varphi$  can be extended by mapping  $\hat{P}$  to the rest of the  $C_{-4}$  (that is  $C_{-4} - \varphi(x_0)\varphi(x_3)$ ).  $\square$

In this lemma, length 3 for a forbidden thread is the best one can do. We have already seen examples of  $C_{-4}$ -critical signed graphs with vertices of degree 2 that correspond to 2-threads. However, we may still apply some restriction on such threads:

**Observation 3.7.** *Given a  $C_{-4}$ -critical signed graph, a vertex of degree 2 cannot be on a  $C_{+4}$ .*

We are now ready to state and prove our main result on the structure of  $C_{-4}$ -critical signed graphs.

### 3.1 Edge-density of $C_{-4}$ -critical signed graphs

We will use the notion of potential developed in [9] and then further used in [6] and [17] to prove the following.

**Theorem 3.8.** *If  $\hat{G}$  is a  $C_{-4}$ -critical signed graph which is not isomorphic to  $\hat{W}$ , then*

$$|E(G)| \geq \frac{4|V(G)|}{3}.$$

Thus the natural *potential* function of graphs we may work with is:

$$p(G) = 4|V(G)| - 3|E(G)|.$$

We note that potential of a signed graph is the potential of its underlying graph.

**Observation 3.9.** *We have  $p(K_1) = 4$ ,  $p(K_2) = 5$ ,  $p(P_3) = 6$  and  $p(C_4) = 4$ . Thus any signed bipartite graph on at most 4 vertices has potential at least 4.*

In the rest of this section, we let  $\hat{G} = (G, \sigma)$  be a minimum counterexample to Theorem 3.8. That is to say,  $\hat{G}$  is a  $C_{-4}$ -critical signed graph which is not isomorphic to  $\hat{W}$ , it satisfies  $p(\hat{G}) \geq 1$ , and that for any signed graph  $\hat{H}$ ,  $\hat{H} \neq \hat{W}$ , with  $|V(\hat{H})| < |V(\hat{G})|$  satisfying  $p(\hat{H}) \geq 1$ ,  $\hat{H}$  admits a homomorphism to  $C_{-4}$ .

Given a signed graph  $\hat{H}$ , we denote a signed graph obtained from  $\hat{H}$  by adding a new vertex and joining it to two vertices of  $\hat{H}$  (of arbitrary choices of signs) by  $P_2(\hat{H})$ . In the following lemma, we list the plausible potential of the subgraphs of the minimum counterexample  $\hat{G}$ .

**Lemma 3.10.** *Let  $\hat{G} = (G, \sigma)$  be a minimum counterexample of Theorem 3.8 and let  $\hat{H}$  be a subgraph of  $\hat{G}$ . Then*

1.  $p(\hat{H}) \geq 1$  if  $\hat{G} = \hat{H}$ ;
2.  $p(\hat{H}) \geq 3$  if  $\hat{G} = P_2(\hat{H})$ ;
3.  $p(\hat{H}) \geq 4$  otherwise.

*Proof.* The first claim is our assumption on  $\hat{G}$ . If  $\hat{G} = P_2(\hat{H})$ , then  $p(\hat{G}) = p(\hat{H}) + 4 \times 1 - 3 \times 2$ , and then, since  $p(\hat{G}) \geq 1$ , we have  $p(\hat{H}) \geq 3$ . We now prove that for any other subgraph of  $\hat{G}$ ,  $p(\hat{H}) \geq 4$ .

Suppose to the contrary that  $\hat{G}$  contains a proper subgraph  $\hat{H}$  which does not satisfy  $\hat{G} = P_2(\hat{H})$ , and satisfies  $p(\hat{H}) \leq 3$ . Among all such subgraphs, let  $\hat{H}$  be chosen so that  $|V(\hat{H})| + |E(\hat{H})|$  is maximized. As adding an edge to a graph only decreases the potential, the assumption of the maximality implies that  $\hat{H}$  is an induced subgraph of  $\hat{G}$ .

By Observation 3.9,  $|V(\hat{H})| \geq 5$ . As  $\hat{G}$  is  $C_{-4}$ -critical and  $\hat{H}$  is a proper subgraph, there is a homomorphism  $\varphi$  of  $\hat{H}$  to  $C_{-4}$ . Since  $C_{-4}$  is vertex transitive, we may assume that  $\varphi$  preserves the bipartition of  $\hat{H}$  induced by bipartition of  $\hat{G}$ , this is automatic if  $H$  is connected, but important if  $H$  is not connected.

Observe that the mapping  $\varphi$  may have applied switching on some vertices of  $\hat{H}$ . Applying switching on the same set of vertices of  $\hat{G}$ , we get a switching equivalent signed graph. For simplicity, and without loss of generality, we may assume that  $\hat{G}$  was given with this signature already. In other words, we may assume, without loss of generality, that  $\varphi_1(x) = +$  for every vertex  $x$  of  $\hat{H}$  (recall that  $\varphi = (\varphi_1, \varphi_2)$ ).

Define  $\hat{G}_1$  to be a signed (multi)graph obtained from  $\hat{G}$  by first identifying vertices of  $\hat{H}$  which are mapped to a same vertex of  $C_{-4}$  under  $\varphi$ , and then identifying all parallel edges of a same sign. Observe that  $\hat{G}_1$  is a homomorphic image of  $\hat{G}$  and that  $\varphi(\hat{H})$  is (isomorphic to) the image of  $\hat{H}$  in this mapping. As in the mapping of  $\hat{G}$  to  $\hat{G}_1$ , the bipartition is preserved,  $\hat{G}_1$  is bipartite. Since homomorphism is an associative relation, and since  $\hat{G} \not\rightarrow C_{-4}$ , we have  $\hat{G}_1 \not\rightarrow C_{-4}$ . This can only be for one of the two reasons: Either  $\hat{G}_1$  contains a  $C_{-2}$ , or  $\hat{G}_1$  contains a  $C_{-4}$ -critical subgraph. We consider the two cases separately:

**Case 1.**  $\hat{G}_1$  contains a  $C_{-2}$ .

It implies that  $\hat{G}$  contains a negative path  $\hat{P}$  of length 2 with both endpoints in  $\hat{H}$  and with its internal vertex in  $V(\hat{G}) \setminus V(\hat{H})$ . We have that

$$p(\hat{H} + \hat{P}) = p(\hat{H}) + 4 \times 1 - 3 \times 2 = p(\hat{H}) - 2 < p(\hat{H}). \quad (1)$$

Recall that  $\hat{H}$  is a maximum proper subgraph satisfying that  $\hat{G} \neq P_2(\hat{H})$  and  $p(\hat{H}) \leq 3$ . Noting that  $\hat{H} \subsetneq \hat{H} + \hat{P}$  and  $\hat{H} + \hat{P}$  is a subgraph of  $\hat{G}$ , and by the maximality of  $\hat{H}$ , there are two possibilities: either  $\hat{H} + \hat{P}$  is not a proper subgraph of  $\hat{G}$ , i.e.,  $\hat{G} = \hat{H} + \hat{P}$ , or  $\hat{G} = P_2(\hat{H} + \hat{P})$ . The former case is impossible as  $\hat{G} \neq P_2(\hat{H})$ . So  $\hat{G} = P_2(\hat{H} + \hat{P})$  and then

$$p(\hat{H} + \hat{P}) = p(\hat{G}) - 4 \times 1 + 3 \times 2 \geq 1 - 4 + 6 = 3 \geq p(\hat{H}), \quad (2)$$

which is in contradiction with (1).

**Case 2.**  $\hat{G}_1$  contains a  $C_{-4}$ -critical subgraph  $\hat{G}_2$ .

We classify the vertices of  $\hat{G}_2$  into two parts: vertices that are also vertices of  $\hat{G}$ , that is  $A = V(\hat{G}_2) \setminus \varphi(\hat{H})$ , and vertices that are the image of vertices of  $\hat{H}$  under the mapping  $\varphi : \hat{H} \rightarrow C_{-4}$ , let  $X_1$  be this set of vertices. The subgraph induced by  $X_1$  will be denoted by  $\hat{X}$ , in other words  $\hat{X} = \varphi(\hat{H}) \cap \hat{G}_2$ . Observe that since  $\hat{G}_2 \not\rightarrow C_{-4}$  and  $\varphi(\hat{H}) \subset C_{-4}$ ,  $A \neq \emptyset$ .

Since  $|V(\hat{H})| \geq 5$ , and  $\varphi$  is a mapping of  $\hat{H}$  to  $C_{-4}$ , at least two vertices are identified and, therefore,  $|V(\hat{G}_2)| \leq |V(\hat{G}_1)| < |V(\hat{G})|$ . As  $\hat{G}$  is a minimum counterexample, we have either  $p(\hat{G}_2) \leq 0$  or  $\hat{G}_2 = \hat{W}$ . Since  $p(\hat{W}) = 1$ , in all cases we have  $p(\hat{G}_2) \leq 1$ .

We now define a subgraph  $\hat{G}_3$  of  $\hat{G}$  as follows: Vertices of  $\hat{G}_3$  are those vertices of  $\hat{G}$  that are either also a vertex of  $\hat{G}_2$  or a vertex of  $\hat{H}$ . In other words,  $V(\hat{G}_3) = \{V(\hat{G}_2) \cup V(\hat{H})\} \setminus V(\hat{X})$ . To give the edge set of  $\hat{G}_3$ , we first choose a set  $E'$  of the edges of  $\hat{G}$  as follows: If a vertex  $u \in V(\hat{G}_2) \setminus X_1$  is adjacent to a vertex  $v \in X_1$ , then we choose a vertex  $v' \in V(\hat{H})$  such that first of all  $\varphi(v') = v$ , second of all  $uv' \in E(G)$ . From the construction of  $\hat{G}_1$ , it is clear that there is such a vertex  $v'$ . We note that, there might be more choices for  $v'$ , among which we select exactly one at random, and then let  $uv'$  be an edge in  $E'$ . The edge set of  $\hat{G}_3$  is then defined to be the set of edges of  $\hat{G}$  that are either induced by  $A$ , or by  $V(\hat{H})$  or edges in  $E'$ , with induced signature from the fixed signature of  $\hat{G}$  that was modified by  $\varphi$ . In other words,  $E(\hat{G}_3) = E(\hat{G}_2 - \hat{X}) + E(\hat{H}) + E'$ . Since each connection between the vertices of  $\hat{X}$  and  $\hat{G}_2 - \hat{X}$  has a unique corresponding edge in  $E'$ , it follows that  $|E(\hat{G}_3)| = |E(\hat{G}_2)| - |E(\hat{X})| + |E(\hat{H})|$  and, therefore,  $p(\hat{G}_3) = p(\hat{G}_2) - p(\hat{X}) + p(\hat{H})$ .

Since  $\hat{G}$  and  $\hat{G}_2$  are both  $C_{-4}$ -critical signed graphs,  $\hat{G}_2$  is not a subgraph of  $\hat{G}$ , that is to say  $\hat{X} \neq \emptyset$ . As  $\hat{X}$  is a subgraph of  $C_{-4}$ , by Observation 3.9,  $p(\hat{X}) \geq 4$ . Then we obtain that

$$p(\hat{G}_3) = p(\hat{G}_2) - p(\hat{X}) + p(\hat{H}) \leq 1 - 4 + p(\hat{H}) = p(\hat{H}) - 3 < p(\hat{H}). \quad (3)$$

Since  $\hat{G}_3$  is a subgraph of  $\hat{G}$  and  $\hat{H} \subsetneq \hat{G}_3$  (because  $A \neq \emptyset$ ), by the maximality of  $\hat{H}$ , either  $\hat{G} = \hat{G}_3$  or  $\hat{G} = P_2(\hat{G}_3)$ . If  $\hat{G} = \hat{G}_3$ , then  $p(\hat{G}_3) \geq 1$ ; if  $\hat{G} = P_2(\hat{G}_3)$ , then  $p(\hat{G}_3) \geq 3$ . But

$$p(\hat{H}) = p(\hat{X}) + p(\hat{G}_3) - p(\hat{G}_2) \geq 4 + 1 - 1 = 4, \quad (4)$$

contradicting that  $p(\hat{H}) \leq 3$ .  $\square$

Towards proving our claim, next we show that the underlying graph  $G$  of the minimum counterexample  $\hat{G}$  does not contain two 4-cycles sharing edges.

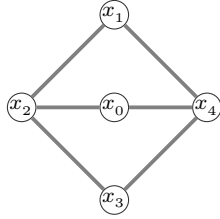


Figure 6:  $\Theta_1$

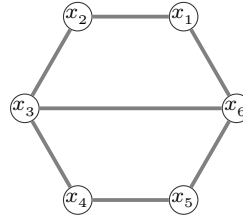


Figure 7:  $\Theta_2$

**Claim 3.11.** *Given a minimum counterexample  $\hat{G}$  to Theorem 3.8, the underlying graph  $G$  does not contain the graph  $\Theta_1$  of Figure 6 as a subgraph.*

*Proof.* By contradiction, assume  $\Theta_1$  is a subgraph of  $G$  and let  $x_0, x_1, \dots, x_4$  be the labeling of its vertices in  $G$  as well. Observe that  $p(\Theta_1) = 2$ . Thus, by Lemma 3.10,  $G = (\Theta_1, \sigma)$  for some signature  $\sigma$ . Noting that there are three 4-cycles in  $\Theta_1$ , of which at least one must be a positive 4-cycle. By Observation 3.7 and as  $d(x_0) = d(x_1) = d(x_3) = 2$ , no signature on  $\Theta_1$  would result in a  $C_{-4}$ -critical signed graph.  $\square$

**Claim 3.12.** *Given a minimum counterexample  $\hat{G}$  to Theorem 3.8, the underlying graph  $G$  does not contain the graph  $\Theta_2$  of Figure 7 as a subgraph.*

*Proof.* By contradiction, assume  $\Theta_2$  is a subgraph of  $G$  and let  $x_1, x_2, \dots, x_6$  be its vertices in  $G$  as well. Observe that  $p(\Theta_2) = 3$ , thus, by Lemma 3.10, either  $G$  has only six vertices, or it has seven vertices and  $G = P_2(\Theta_2)$ . By Lemma 3.6, no signature on  $\Theta_2$  would result in a  $C_{-4}$ -critical signed graph. Adding another edge, then we will have a graph on  $\frac{4}{3} \times 6 = 8$  edges, thus this cannot form a counterexample. We note that after adding an edge one may assign a signature to get the only  $C_{-4}$ -critical signed graph on six vertices  $\Gamma$ .

Thus we must have  $G = P_2(\Theta_2)$ . Let  $w$  be the added vertex. Then up to symmetries of  $\Theta_2$ , we have two choices for the two neighbors of  $w$ : I.  $w$  is adjacent to  $x_1$  and  $x_5$ , the resulting graph  $\Theta'_2$  is illustrated in Figure 8. II.  $w$  is adjacent to  $x_2$  and  $x_6$ , the resulting graph  $\Theta''_2$  is illustrated in Figure 9.

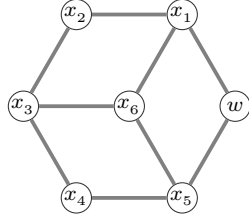


Figure 8:  $\Theta'_2$

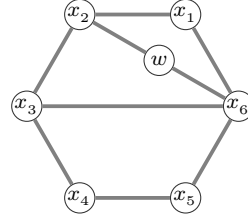


Figure 9:  $\Theta''_2$

**Case I.** Note that  $\Theta'_2$  is the same as the underlying graph of  $\hat{W}$  of Proposition 3.2. In that proposition we have shown that, up to switching, only one signature can make  $\Theta'_2$  a  $C_{-4}$ -critical signed graph. However, we have assumed in the statement of our Theorem that  $\hat{G}$  is not switching equivalent to  $\hat{W}$ .

**Case II.** We claim that no signature on  $\Theta''_2$  can make it a  $C_{-4}$ -critical signed graph. That is because, otherwise, of the three 4-cycles induced by  $x_1, x_2, x_3, x_6, w$  at least one would be a positive 4-cycle, thus contradicting Observation 3.7.  $\square$

In the next Lemma, we imply further structure on the neighborhood of a 2-thread.

**Lemma 3.13.** *Let  $vv_1u$  be a 2-thread in  $\hat{G}$ . Suppose that  $v$  is a 3-vertex and let  $v_2, v_3$  be the other two neighbors of  $v$ . Then the path  $v_2vv_3$  must be contained in a negative 4-cycle in  $\hat{G}$ .*

*Proof.* Suppose to the contrary that the path  $v_2vv_3$  is not contained in a negative 4-cycle. If needed, by switching at  $v_2$  or  $v_3$ , we may assume that both  $vv_2$  and  $vv_3$  are of a positive sign. Then by identifying  $v_2$  and  $v_3$  to a new vertex  $v_0$ , we get a homomorphic image  $\hat{G}_1$  of  $\hat{G}$ . Observe that by our assumption,  $\hat{G}_1$  does not contain a  $C_{-2}$ . As  $\hat{G}$  does not map to  $C_{-4}$ , its homomorphic image,  $\hat{G}_1$ , does not map either. Since  $g_{ij}(\hat{G}_1) \geq g_{ij}(C_{-4})$ , there must be a  $C_{-4}$ -critical subgraph  $\hat{G}_2$  of  $\hat{G}_1$ . Observe that, by Lemmas 3.6 and 3.5, none of the vertices  $v$  and  $v_1$  is a vertex of  $\hat{G}_2$ . On the other hand,  $v_0 \in V(\hat{G}_2)$ , as otherwise  $\hat{G}_2$  is a proper subgraph of  $\hat{G}$  which does not map to  $C_{-4}$ , contradicting the fact that  $\hat{G}$  is  $C_{-4}$ -critical.

Since  $\hat{G}$  is a minimum counterexample to the Theorem, and that  $|V(\hat{G}_2)| < |V(\hat{G})|$ , we have either  $p(\hat{G}_2) \leq 0$  or  $\hat{G}_2 = \hat{W}$  in which case  $p(\hat{G}_2) = 1$ .

Let  $\hat{G}_3$  be the signed graph obtained from  $\hat{G}_2$  by splitting  $v_0$  back to  $v_2$  and  $v_3$ , adding the vertex  $v$  and adding the positive edges  $vv_2$  and  $vv_3$  back. Note that  $\hat{G}_3$  is a subgraph of  $\hat{G}$ . We observe that

$$p(\hat{G}_3) = p(\hat{G}_2) + 4 \times 2 - 3 \times 2 = p(\hat{G}_2) + 2 \leq 3. \quad (5)$$

Furthermore, the equality is only possible if  $\hat{G}_2 = \hat{W}$ . As  $v_1 \notin V(\hat{G}_3)$ , we know that  $\hat{G}_3 \neq \hat{G}$ . By Lemma 3.10, we must have  $p(\hat{G}_3) = 3$  and that  $\hat{G} = P_2(\hat{G}_3)$ . And since equality in (5) must hold, we also have  $\hat{G}_2 = \hat{W}$ .

As  $\hat{G}_2 = \hat{W}$ , vertices of  $\hat{G}_2$  are of degree 2 or 3, and, thus, the splitting operation on  $v_0$ , that we considered in order to build  $\hat{G}_3$ , is the same as subdividing one of its edges twice. As there are only two types of edges in  $\hat{W}$ , the subdivided result,  $\hat{G}_3$ , is one of the two graphs: either  $\Omega_1$  of Figure 4, or  $\Omega_2$  of Figure 5. However, we have already seen in Lemma 3.3 and Lemma 3.4 that neither  $P_2(\Omega_1)$  nor  $P_2(\Omega_2)$  can be a  $C_{-4}$ -critical signed graph.  $\square$

By combining Lemma 3.13 with Claims 3.11 and 3.12, we have our main forbidden configuration as follow:

**Corollary 3.14.** *A vertex of degree 3 in the minimum counterexample  $\hat{G}$  does not have two neighbors of degree 2.*

We are now ready to employ the discharging technique to prove Theorem 3.8.

*Proof.* (Of Theorem 3.8) Applying discharging technique, we assign an initial charge of  $c(v) = d(v)$  to each vertex of  $G$ . Observe that the total charge is  $2|E(G)|$ . We apply the following discharging rule:

“Every 2-vertex receives a charge of  $\frac{1}{3}$  from each of its neighbors.”

After this procedure, since there is no 3-thread in  $G$ , each 2-vertex  $v$  receives a total of  $\frac{2}{3}$  from its two neighbors and thus  $c'(v) = 2 + \frac{2}{3} = \frac{8}{3}$ . Each 3-vertex  $u$  has at most one neighbor of degree 2, so  $c'(u) \geq 3 - \frac{1}{3} = \frac{8}{3}$ . Each vertex  $w$  of degree at least 4 has charge  $c'(w) \geq d(w) - \frac{d(w)}{3} = \frac{2d(w)}{3} \geq \frac{8}{3}$ . Thus the total charge is at least  $\frac{8|V(G)|}{3}$ . That contradicts the assumption that  $p(\hat{G}) = 4|V(G)| - 3|E(G)| \geq 1$ .  $\square$

Applying this result in terms of maximum average degree of the (underlying) graph, denoted  $mad(G)$ , we have the following.

**Corollary 3.15.** *Given a signed bipartite (simple) graph  $\hat{G}$ , if  $mad(G) < \frac{8}{3}$  and  $\hat{G}$  does not contain  $\hat{W}$  as a subgraph (with no equivalent signature), then  $\hat{G} \rightarrow C_{-4}$ .*

## 4 Constructions of (sparse) $C_{-4}$ -critical signed graphs

We have already seen that if  $\chi(G) \geq k+1$ , then  $T_{k-2}(G)$  does not map to  $C_{-k}$ . It is easy to verify that, furthermore, if  $G$  is  $(k+1)$ -critical, then  $T_{k-2}(G)$  is a  $C_{-k}$ -critical signed graph.

We may extend this construction through an extension of notion of proper coloring of graphs to proper coloring of signed graphs introduced by Zaslavsky in [18]. More precisely:

**Definition 4.1.** *Given the set  $X_k = \{\pm 1, \pm 2, \dots, \pm k\}$ , a signed multigraph  $(G, \sigma)$  is said to be  $X_k$ -colorable if there exists an assignment  $c : V(G) \rightarrow X_k$  such that for each edge  $xy$  of  $G$ ,*

$$c(x) \neq \sigma(xy)c(y).$$

Using this notion, Lemma 2.3 can be extended to the following theorem. A proof of this theorem is also obtained by revising the proof of Lemma 2.3 given in Section 2.2 and we leave the details to the reader.

**Theorem 4.2.** *A signed multigraph  $\hat{G}$  admits an  $X_k$ -coloring if and only if  $T_{k-2}(\hat{G}) \rightarrow C_{-2k}$ .*

Given a graph  $G$ , let  $\tilde{G}$  be the signed multigraph obtained from  $G$  by replacing each edge of  $G$  with a pair of edges: one of the positive sign, another of the negative sign. It is easily observed that:

**Observation 4.3.** *A graph  $G$  is  $k$ -colorable if and only if the signed multigraph  $\tilde{G}$  is  $X_k$ -colorable.*

Combining Theorem 4.2 and Observation 4.3, we have the following.

**Theorem 4.4.** *Given a graph  $G$ , we have  $\chi(G) \leq k$  if and only if  $T_{k-2}(\tilde{G}) \rightarrow C_{-2k}$ . Moreover,  $G$  is  $(k+1)$ -critical if and only if  $T_{k-2}(\tilde{G})$  is  $C_{-2k}$ -critical.*

This theorem further emphasizes on the importance of the study of  $C_{-2k}$ -critical graphs, and, more generally, the homomorphisms of signed bipartite graphs. The operation  $T'_{2k-1}$  applied on graphs (as defined in the introduction) connects  $(2k+1)$ -coloring problem of graphs to  $C_{2k+1}$ -coloring problem of graphs. Thus only odd values of the chromatic number are captured by  $C_{2k+1}$ -coloring problem. The operation  $T_{k-2}$ , when applied on signed multigraphs  $\tilde{G}$ , connects the  $k$ -coloring problem of  $G$  to  $C_{-2k}$ -coloring problem of signed graphs. Thus  $C_{-2k}$ -coloring problem captures  $k$ -coloring problem for all the values of  $k$ . We note that  $T_2(\tilde{G})$  is the same as  $S(G)$  defined in [14] and refer to this reference for more on the importance of the homomorphisms of signed bipartite graphs.

By Theorem 4.4 and noting that odd cycles are the only 3-critical graphs, we have  $T_2(\tilde{C}_{2k+1})$  as an example of  $C_{-4}$ -critical signed graph for each value of  $k$ . See Figures 10 and 11 for  $T_2(\tilde{C}_3)$  and  $T_2(\tilde{C}_5)$ . The signed bipartite graph  $\hat{G}_{2k+1} = T_2(\tilde{C}_{2k+1})$  has  $6k+3$  vertices and  $8k+4$  edges. Thus  $T_2(\tilde{C}_{2k+1})$  is an example of  $C_{-4}$ -critical signed graphs for which the bound of Theorem 3.8 is tight.

Let  $\hat{G}'_{2k+1}$  be the signed (bipartite) graph obtained from  $\hat{G}_{2k+1}$  by identifying two vertices of degree 2 which are at distance two and their common neighbor is adjacent to both with positive edges, see Figure 12 for an illustration of  $\hat{G}'_5$ . Observe that  $\hat{G}'_3$  contains  $\hat{W}$  as a proper subgraph and, thus, is not  $C_{-4}$ -critical. For  $k \geq 2$ ,  $\hat{G}'_{2k+1}$  does not map to  $C_{-4}$  because it is a homomorphic image of  $\hat{G}_{2k+1}$ . Moreover, it has average degree of  $\frac{8|V(\hat{G}'_{2k+1})|+2}{3|V(\hat{G}'_{2k+1})|}$ , it does not contain  $\hat{W}$  as a subgraph and any proper subgraph of it has average degree strictly less than  $\frac{8}{3}$ . Thus, by Corollary 3.15, it is a  $C_{-4}$ -critical signed graph for which the bound of Theorem 3.8 is tight. Further identification of vertices of degree 2 would lead to other examples for which the bound of Theorem 3.8 is either tight or nearly tight.

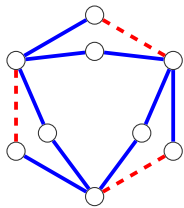


Figure 10:  $T_2(\tilde{C}_3)$

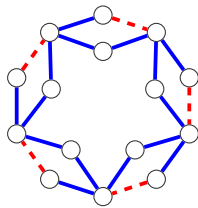


Figure 11:  $T_2(\tilde{C}_5)$

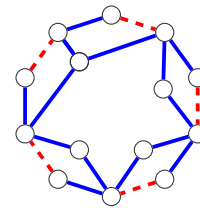


Figure 12:  $\hat{G}'_5$

Another method of building  $C_{-4}$ -critical signed graphs is as follows. Let  $\hat{G}_1$  and  $\hat{G}_2$  be two  $C_{-4}$ -critical signed graphs each with a vertex of degree 2. Suppose  $u$  is a vertex of degree 2 in  $\hat{G}_1$  with  $u_1$  and  $u_2$  as its neighbors, and  $v$  is a vertex of degree 2 in  $\hat{G}_2$  with  $v_1$  and  $v_2$  as its neighbors. As  $\hat{G}_1$  is a  $C_{-4}$ -critical signed graph,  $\hat{G}_1 - u$  maps to  $C_{-4}$ . But any such a mapping must map  $u_1$  and  $u_2$  to a same vertex of  $C_{-4}$  and must have applied a switching on  $\hat{G}_1 - u$  so that with the same switching on  $\hat{G}_1$ , the path  $u_1uu_2$  is negative. We consider  $\hat{G}_1$  with this signature and do the same on  $\hat{G}_2$ . We then build a signed graph  $\mathcal{F}(\hat{G}_1, \hat{G}_2) = \hat{G}$  from disjoint union of  $\hat{G}_1$  and



$\hat{G}_2$  by deleting  $u$  and  $v$ , and adding a positive edge  $u_1v_1$  and a negative edge  $u_2v_2$ . We leave it to the reader to verify that the result is a  $C_{-4}$ -critical signed graph. In Figure 13, we have depicted the signed graph obtained from this operation on two disjoint copies of  $\hat{W}$ . We note that this is an example of a  $C_{-4}$ -critical signed graph on 12 vertices for which the bound of Theorem 3.8 is tight. One may note that, furthermore, the same technique can be applied to build a new  $C_{-2k}$ -critical signed graph from two  $C_{-2k}$ -critical signed graphs each having a vertex of degree 2. Moreover, towards building a  $C_{-2k}$ -critical signed graph of lower edge-density, instead of connecting  $u_i v_i$  directly, one may use paths of length  $k - 1$ , one of the positive sign, one of the negative sign.

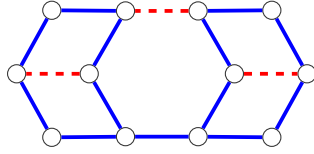


Figure 13:  $\mathcal{F}(\hat{W}, \hat{W})$

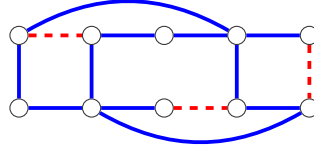


Figure 14:  $\mathcal{H}(\Gamma, \Gamma)$

**Analogue of Hajós construction.** The Hajós construction of  $k$ -critical graphs can be adapted to build  $C_{-4}$ -critical signed graphs from two given  $C_{-4}$ -critical signed graphs. The general case for will be addressed in a forthcoming work. Let  $\hat{G}_1$  be a  $C_{-4}$ -critical signed graph and let  $u_1v_1$  be a positive edge of  $\hat{G}_1$ . Then  $\hat{G}_1 - x_1y_1$  admits a homomorphism  $\phi$  to  $C_{-4}$ . Since  $C_{-4}$  is vertex transitive, and since  $(-\phi_1, \phi_2)$  is the same as  $(\phi_1, \phi_2)$ , we may consider only the mappings for which  $\phi_1(x_1) = +$  (referring the labeling of Figure 1). Then we must have  $\phi_1(y_1) = -$  as otherwise,  $\phi$  is also a mapping of  $\hat{G}_1$  to  $C_{-4}$ . Furthermore, for any other edges  $e$ , if we take a mapping  $\phi'$  of  $\hat{G}_1 - e$  satisfying  $\phi'(x_1) = (+, u_2)$ , then we must have  $\phi(y_1) = +$ . Similarly, consider a  $C_{-4}$ -critical signed graph  $\hat{G}_2$  with a negative edge  $x_2y_2$ . Then by a similar argument, for any mapping  $\psi$  of  $\hat{G}_2 - x_2y_2$  for which  $\psi(x_2) = (+, u_2)$ , we must have  $\psi(y_2) = +$ .

We now build a new  $C_{-4}$ -critical signed graph  $\hat{H} = \mathcal{H}(\hat{G}_1, \hat{G}_2)$  as follows:  $\hat{H}$  is obtained from vertex disjoint copies of  $\hat{G}_1$  and  $\hat{G}_2$  by identifying  $x_1$  with  $x_2$  to get a vertex  $x$  and  $y_1$  with  $y_2$  to get a vertex  $y$ . We observe that if there exists a homomorphism  $\varphi$  of  $\hat{H}$  to  $C_{-4}$ , then, by symmetries, we may assume  $\varphi(x) = (+, u_2)$ . Then the restriction on  $\hat{G}_1$  implies  $\varphi_1(y) = -$  and the restriction on  $\hat{G}_2$  implies  $\varphi_1(y) = +$ , a contradiction, implying that  $\hat{H}$  does not map to  $C_{-4}$ . Removing an edge from one part of  $\hat{H}$  then leads in mappings of the two different parts that can be merged together, which shows that  $\hat{H}$  is  $C_{-4}$ -critical. An example of this construction, using two disjoint copies of the unique  $C_{-4}$ -critical signed graph  $\Gamma$  on six vertices (see Section 3) is given in Figure 14.

The signed graph  $\mathcal{H}(\hat{G}_1, \hat{G}_1)$  has  $|V(\hat{G}_1)| + |V(\hat{G}_1)| - 2$  vertices. Using the techniques mentioned above one can easily build  $C_{-4}$ -critical signed graphs of orders 9, 10, 11, 12. Then applying Hajós construction to a previously built  $C_{-4}$ -critical signed graph and  $\Gamma$  (on 6 vertices), one can build a  $C_{-4}$ -critical signed graphs on any number  $n$  of vertices for  $n \geq 9$ .

Given positive integers  $k$  and  $n$  ( $n \geq k + 2$ ), let  $f(n, k)$  be the minimum number of edges of a  $k$ -critical graph on  $n$  vertices. We refer to [10] for almost precise value of  $f(n, k)$  and for historical background on the study of this function. We similarly may define  $g(n, k)$  to be the minimum number of edges of a  $C_{-k}$ -critical signed graph

on  $n$  vertices. As noted above,  $g(n, 4)$  is well-defined for  $n \geq 9$ . It can be similarly shown that  $g(n, k)$  is well-defined for  $n \geq N_k$  where  $N_k$  is an integer depending on  $k$  only.

Lemma 2.3 and Theorem 4.4 imply the following relations between  $f(n, k)$  and  $g(n, k)$ .

- By Lemma 2.3,

$$g(n + (k - 3)f(n, k), k) \leq (k - 2)f(n, k). \quad (6)$$

- By Theorem 4.4,

$$g(n + 2(l - 1)f(n, l), 2l) \leq 2(l - 1)f(n, l). \quad (7)$$

Authors of [6] and [17] suggest that for  $k = 5$  and  $k = 7$  the inequality (6) is almost tight. Our work here shows that for  $C_{-4}$ -critical signed graphs the inequality of (7) provides a tight bound. For  $k = 6$ , the two inequalities provide similar bounds where the only difference is in the constant (in the favor of inequality of (6)). For other values of  $k = 2l$  the inequality of (6) provides a better bound than (7) and it is tempting to suggest that (6) gives a nearly tight bound for  $g(n, k)$  for  $k \geq 5$ . A point of hesitation here is that, while the notion of  $k$ -critical graphs is widely studied and the value and behavior of  $f(n, k)$  are almost determined, the notion of critical signed graphs, aside from its relation to  $(2k + 1)$ -critical graphs (with no sign), is a new notion and hardly anything is known about it. More precisely, we may define an  $X_k$ -critical signed graph to be a signed graph which does not admit an  $X_k$ -coloring but every proper subgraph of it does. What can then be said about the minimum number of edges of an  $X_k$ -critical signed graph? Constructions other than  $\hat{G}$  may provide better bounds on  $g(n, 2k)$  by Theorem 4.2.

## 5 Application to signed bipartite planar graphs

Introducing a bipartite analogue of Jaeger-Zhang conjecture, it was conjectured in [14] that every signed bipartite planar graph, whose shortest negative cycles are of length at least  $4k - 2$ , maps to  $C_{-2k}$ . In support of the conjecture, the claim is proved, in [4], for a weaker condition when a shortest negative cycle is of length at least  $8k - 2$ . Here we use the folding Lemma of [13] to prove that every signed bipartite planar graph whose shortest negative cycle is of length at least 8 maps to  $C_{-4}$  and we show that this bound is tight, thus disproving the exact claim of the conjecture for the case  $k = 2$ .

**Lemma 5.1.** [13] *Given a signed bipartite planar graph  $(G, \sigma)$  with an embedding on the plane, if the length of shortest negative cycles of  $(G, \sigma)$  is at least  $2k$  and a face  $F$  of it is not a negative cycle of length  $2k$ , then there is a homomorphic image of  $(G, \sigma)$  which identifies two vertices at distance 2 of  $F$  and such that its shortest negative cycles are also of length at least  $2k$ .*

Observe that this identification preserves both planarity and bipartiteness. Thus, repeatedly applying the lemma, we get a homomorphic image where all faces are negative cycles of length  $2k$ . Taking  $k = 4$ , starting from a signed bipartite planar graph whose shortest negative cycles are of length at least 8, we get a homomorphic image  $\hat{G}$  with a planar embedding where all faces are (negative) 8-cycles. Applying

the Euler formula on this graph, we have  $|E(G)| \leq \frac{4}{3}(|V(G)| - 2)$ . By taking  $\hat{G}$  to be a smallest signed bipartite planar graph which does not map to  $C_{-4}$  and whose shortest negative cycle is of length 8, we conclude that on the one hand  $\hat{G}$  must be  $C_{-4}$ -critical, and thus, by Theorem 3.8, has at least  $\frac{4}{3}|V(G)|$  edges, but on the other hand, by the argument above, it has at most  $\frac{4}{3}(|V(G)| - 2)$  edges. This contradiction is a proof that:

**Theorem 5.2.** *Any signed bipartite planar graph whose shortest negative cycle is of length at least 8 maps to  $C_{-4}$ .*

We now claim that the condition of shortest negative cycles being of length at least 8 in this theorem is tight.

For this, it would be enough to build a signed planar (simple) graph  $(G, \sigma)$  which is not  $\{\pm 1, \pm 2\}$ -colorable. Then, by Theorem 4.2,  $T_2(G, \sigma)$  is a signed bipartite planar graph which does not map to  $C_{-4}$ . Furthermore, that  $G$  is simple implies that  $T_2(G, \sigma)$  has no cycle of length smaller than 6.

That every signed planar graph is  $\{\pm 1, \pm 2\}$ -colorable was conjectured in [11]. This conjecture was disproved in [8], we refer to [12] for a direct proof. Thus we have:

**Theorem 5.3.** *There exists a bipartite planar graph  $G$  of girth 6 with a signature  $\sigma$  such that  $(G, \sigma) \not\rightarrow C_{-4}$ .*

Smallest examples built in this way, which we have so far, has 150 vertices. However, such examples have extra property that vertices on one part of the (bipartite) graph are all of degree 2. Perhaps simpler examples can be built which do not satisfy this property.

It is proved in [5] that  $C_{-4}$ -coloring problem even when restricted to the class of signed (bipartite) planar graphs remains an NP-complete problem. Thus, one does not expect to find an efficient classification of signed bipartite planar graphs which map to  $C_{-4}$ . However, some strong sufficient conditions could be provided. One such a condition is a based on the restatement of the four-color theorem given in Theorem 2.5. Another is Theorem 5.2 of this work that shows no negative cycle of length 2, 4, 6 is a sufficient condition. As a generalization of Theorem 2.5 (the four-color theorem) which also captures essential cases of Theorem 5.2, we may propose the following:

**Conjecture 5.4.** *Let  $G$  be a bipartite planar graph of girth at least 6. Let  $\sigma$  be a signature on  $G$  such that in  $(G, \sigma)$  all 6-cycles are of a same sign. Then  $(G, \sigma) \rightarrow C_{-4}$ .*

We note that, while one may use Lemma 5.1 to reduce facial 4-cycles of a signed graph which is the subject of Theorem 5.2, there could be separating 4-cycles in a signed bipartite planar graph to which this theorem may apply. Therefore, the conjecture does not capture all cases to which Theorem 5.2 applies.

As a final remark, we would like to point out that some of the results in this work can be restated using the language of the circular coloring of signed graphs which is recently developed in [16].

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