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SOME PROPERTIES OF THE GROUP OF BIRATIONAL MAPS GENERATED BY THE AUTOMORPHISMS OF $\mathbb{P}^n_{\mathbb{C}}$ AND THE STANDARD INVOLUTION

by

Julie Déserti

Abstract. — We give some properties of the subgroup $G_n(\mathbb{C})$ of the group of birational self-maps of $\mathbb{P}^n_{\mathbb{C}}$ generated by the standard involution and the group of automorphisms of $\mathbb{P}^n_{\mathbb{C}}$. We prove that there is no nontrivial finite-dimensional linear representation of $G_n(\mathbb{C})$. We also establish that $G_n(\mathbb{C})$ is perfect, and that $G_n(\mathbb{C})$ equipped with the Zariski topology is simple. Furthermore if φ is an automorphism of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, then up to birational conjugacy, and up to the action of a field automorphism $\varphi_{|G_n(\mathbb{C})}$ is trivial.

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1. Introduction

The group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ of birational self-maps of $\mathbb{P}^2_{\mathbb{C}}$, also called the Cremona group of rank 2, has been the object of a lot of studies. For finite subgroups let us mention for example [**3**, **27**, **9**]; other subgroups have been dealt with ([**21**, **23**]), and some group properties have been established ([**21**, **22**, **19**, **14**, **10**, **12**, **5**, **6**, **4**]). One can also find a lot of properties between algebraic geometry and dynamics ([**26**, **13**, **7**]). The Cremona group in higher dimension is far less well known; let us mention some references about finite subgroups ([**42**, **43**, **41**, **40**, **33**]), about algebraic subgroups of maximal rank ([**18**, **51**, **47**, **49**, **48**]), about other subgroups ([**38**, **39**]), about (abstract) homomorphisms from $\mathrm{PGL}(r+1;\mathbb{C})$ to the group $\mathrm{Bir}(M)$ where M denotes a complex projective variety ([**11**]), and about maps of small bidegree ([**35**, **36**, **34**, **29**, **24**]).

In this article we consider the subgroup of birational self-maps of $\mathbb{P}^n_{\mathbb{C}}$ introduced by Coble in [**15**]

$$G_n(\mathbb{C}) = \langle \sigma_n, \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})\rangle$$

where σ_n denotes the involution

$$(z_0 : z_1 : \ldots : z_n) \dashrightarrow \left(\prod_{\substack{i=0\\ i\neq 0}}^{n} z_i : \prod_{\substack{i=0\\ i\neq 1}}^{n} z_i : \ldots : \prod_{\substack{i=0\\ i\neq n}}^{n} z_i \right).$$

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Hudson also deals with this group ([29]):

> "For a general space transformation, there is nothing to answer either to a plane characteristic or Noether theorem. There is however a group of transformations, called punctual because each is determined by a set of points, which are defined to satisfy an analogue of Noether theorem, and possess characteristics, and for which we can set up parallels to a good deal of the plane theory."

Note that the maps of $G_3(\mathbb{C})$ are in fact not so "punctual" ([8, §8]). It follows from Noether theorem ([1, 44]) that $G_2(\mathbb{C})$ coincides with $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$; it is not the case in higher dimension where $G_n(\mathbb{C})$ is a strict subgroup of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ (*see* [29, 35]). However the following theorems show that $G_n(\mathbb{C})$ shares good properties with $G_2(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.

In [14] we proved that for any integer $n \geq 2$ the group $\mathrm{Bir}(\mathbb{P}^n_{\Bbbk})$, where $\Bbbk$ denotes an algebraically closed field, is not linear; we obtain a similar statement for $G_n(\Bbbk)$, $n \geq 2$:

Theorem A. — *If $\Bbbk$ is an algebraically closed field, there is no nontrivial finite-dimensional linear representation of $G_n(\Bbbk)$ over any field.*

The group $G_n(\mathbb{C})$ contains some "big" subgroups:

Proposition B. —
- *The group of polynomial automorphisms of $\mathbb{C}^n$ generated by the affine automorphisms and the Jonquières ones is a subgroup of $G_n(\mathbb{C})$.*
- *If $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_k$ are some generic automorphisms of $\mathbb{P}^n_{\mathbb{C}}$, then $\langle \mathfrak{g}_0\sigma_n, \mathfrak{g}_1\sigma_n, \ldots, \mathfrak{g}_k\sigma_n \rangle \subset G_n(\mathbb{C})$ is a free subgroup of $G_n(\mathbb{C})$.*

Remark 1.1. — For the meaning of "generic" see the proof of Proposition 4.3.

In [14] we establish that $G_2(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is perfect, *i.e.* $[G_2(\mathbb{C}), G_2(\mathbb{C})] = G_2(\mathbb{C})$; the same holds for any n:

Theorem C. — *If $\Bbbk$ is an algebraically closed field, $G_n(\Bbbk)$ is perfect.*

In [21] we determine the automorphisms group of $G_2(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$; in higher dimensions we have a similar description. Before giving a precise result, let us introduce some notation: the group of the field automorphisms acts on $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$: if f is an element of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, and κ is a field automorphism we denote by ${}^{\kappa}f$ the element obtained by letting κ acting on f.

Theorem D. — *Let φ be an automorphism of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$. There exist κ an automorphism of the field $\mathbb{C}$, and ψ a birational map of $\mathbb{P}^n_{\mathbb{C}}$ such that*

$$\varphi(f) = {}^{\kappa}(\psi f \psi^{-1}) \qquad \forall\, f \in G_n(\mathbb{C}).$$

The question "is the Cremona group simple ?" is a very old one; Cantat and Lamy recently gave a negative answer in dimension 2 (*see* [12]). One can consider the same question when $G_2(\Bbbk)$ is equipped with the Zariski topology ($\Bbbk$ denotes here an algebraically closed field) ; Blanc looked at it, and obtained a positive answer ([5]). What about $G_n(\Bbbk)$?

Proposition E. — *If $\Bbbk$ is an algebraically closed field, the group $G_n(\Bbbk)$, equipped with the Zariski topology, is simple.*

Organisation of the article. — We first recall a result of Pan about the set of group generators of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, $n \geq 3$ (*see* §2); we then note that as soon as $n \geq 3$, there are birational maps of degree $n = \deg \sigma_n$ that do not belong to $G_n(\mathbb{C})$. In §3 we prove Theorem A, and in §4 Proposition B. Let us remark that the fact that the group of tame automorphisms is contained in $G_n(\mathbb{C})$ implies that $G_n(\mathbb{C})$ contains maps of any degree, it was not obvious *a priori*. In §5 we study the normal subgroup in $G_n(\mathbb{C})$ generated by σ_n (resp. by an automorphism of $\mathbb{P}^n_{\mathbb{C}}$); it allows us to establish Theorem C. We finish §5 with the proofs of Theorem D, and Proposition E.

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2. About the set of group generators of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, $n \geq 3$

2.1. Some definitions. — A ***polynomial automorphism*** φ of $\mathbb{C}^n$ is a map $\mathbb{C}^n \to \mathbb{C}^n$ of the type

$$(z_0, z_1, \ldots, z_{n-1}) \mapsto \big(\varphi_0(z_0, z_1, \ldots, z_{n-1}), \varphi_1(z_0, z_1, \ldots, z_{n-1}), \ldots, \varphi_{n-1}(z_0, z_1, \ldots, z_{n-1})\big),$$

with $\varphi_i \in \mathbb{C}[z_0, z_1, \ldots, z_{n-1}]$, that is bijective; we denote φ by $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_{n-1})$. A ***rational self-map*** $\phi\colon \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$ is given by

$$(z_0 : z_1 : \ldots : z_n) \dashrightarrow \big(\phi_0(z_0, z_1, \ldots, z_n) : \phi_1(z_0, z_1, \ldots, z_n) : \ldots : \phi_n(z_0, z_1, \ldots, z_n)\big)$$

where the ϕ_i are homogeneous polynomials of the same positive degree, and without common factor of positive degree. Let us denote by $\mathbb{C}[z_0, z_1, \ldots, z_n]_d$ the set of homogeneous polynomials in z_0, z_1, …, z_n of degree d. The ***degree*** of ϕ is by definition the degree of the ϕ_i. A ***birational self-map*** of $\mathbb{P}^n_{\mathbb{C}}$ is a rational self-map that admits a rational inverse. The set of polynomial automorphisms of $\mathbb{C}^n$ (resp. birational self-maps of $\mathbb{P}^n_{\mathbb{C}}$) form a group denoted $\mathrm{Aut}(\mathbb{C}^n)$ (resp. $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$).

2.2. A result of Pan. — Let us recall a construction of Pan ([**35**]) which, given a birational self-map of $\mathbb{P}^n_{\mathbb{C}}$, allows one to construct a birational self-map of $\mathbb{P}^{n+1}_{\mathbb{C}}$. Let $P \in \mathbb{C}[z_0, z_1, \ldots, z_n]_d$, $Q \in \mathbb{C}[z_0, z_1, \ldots, z_n]_\ell$, and let R_0, R_1, …, $R_{n-1} \in \mathbb{C}[z_0, z_1, \ldots, z_{n-1}]_{d-\ell}$ be some homogeneous polynomials. Denote by $\Psi_{P,Q,R}\colon \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$ and $\widetilde{\Psi}\colon \mathbb{P}^{n-1}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{n-1}_{\mathbb{C}}$ the rational maps defined by

$$\Psi_{P,Q,R} = \big(QR_0 : QR_1 : \ldots : QR_{n-1} : P\big) \qquad \& \qquad \widetilde{\Psi}_R = \big(R_0 : R_1 : \ldots : R_{n-1}\big).$$

Lemma 2.1 **([35])**. — *Let d, ℓ be some integers such that $d \geq \ell + 1 \geq 2$. Take Q in $\mathbb{C}[z_0, z_1, \ldots, z_n]_\ell$, and P in $\mathbb{C}[z_0, z_1, \ldots, z_n]_d$ without common factors. Let R_1, …, R_n be some elements of $\mathbb{C}[z_0, z_1, \ldots, z_{n-1}]_{d-\ell}$. Assume that*

$$P = z_n P_{d-1} + P_d \qquad\qquad Q = z_n Q_{\ell-1} + Q_\ell$$

with P_{d-1}, P_d, $Q_{\ell-1}$, $Q_\ell \in \mathbb{C}[z_0, z_1, \ldots, z_{n-1}]$ of degree $d-1$, resp. d, resp. $\ell - 1$, resp. ℓ and such that $(P_{d-1}, Q_{\ell-1}) \neq (0, 0)$.

The map $\Psi_{P,Q,R}$ is birational if and only if $\widetilde{\Psi}_R$ is birational.

Let us give the motivation of this construction:

Theorem 2.2 **([29, 35])**. — *Any set of group generators of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, $n \geq 3$, contains uncountably many non-linear maps.*

We will give an idea of the proof of this statement.

Lemma 2.3 **([35])**. — *Let $n \geq 3$. Let $\mathcal{S}$ be an hypersurface of $\mathbb{P}^n_{\mathbb{C}}$ of degree $\ell \geq 1$ having a point p of multiplicity $\geq \ell - 1$.*

Then there exists a birational self-map of $\mathbb{P}^n_{\mathbb{C}}$ of degree $d \geq \ell + 1$ that blows down $\mathcal{S}$ onto a point.

Proof. — One can assume without loss of generality that $p = (0:0:\ldots:0:1)$. Denote by $q' = 0$ the equation of $\mathcal{S}$, and take a generic plane passing through p given by the equation $h = 0$. Finally choose $P = z_n P_{d-1} + P_d$ such that

- $P_{d-1} \neq 0$;
- $\mathrm{pgcd}\,(P, hq') = 1$.

Now set $Q = h^{d-\ell-1} q'$, $R_i = z_i$, and conclude with Lemma 2.1. □

Proof of Theorem 2.2. — Let us consider the family of hypersurfaces given by $q(z_1, z_2, z_3) = 0$ where $q = 0$ defines a smooth curve $\mathcal{C}_q$ of degree ℓ on $\{z_0 = z_4 = z_5 = \ldots = z_n = 0\}$. Let us note that $q = 0$ is birationally equivalent to $\mathbb{P}^{n-2}_{\mathbb{C}} \times \mathcal{C}_q$. Furthermore $q = 0$ and $q' = 0$ are birationally equivalent if and only if $\mathcal{C}_q$ and $\mathcal{C}_{q'}$ are isomorphic. Note that for $\ell = 2$ the set of isomorphism classes of smooth cubics is a 1-parameter family, and that according to Lemma 2.3 for any $\mathcal{C}_q$ there exists a birational self-map of $\mathbb{P}^n_{\mathbb{C}}$ that blows down $\mathcal{C}_q$ onto a point. Hence any set of group generators of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, $n \geq 3$, has to contain uncountably many non-linear maps. □

One can take $d = \ell + 1$ in Lemma 2.3. In particular

Corollary 2.4. — *As soon as $n \geq 3$, there are birational maps of degree $n = \deg \sigma_n$ that do not belong to $G_n(\mathbb{C})$.*

Remark 2.5. — The maps $\Psi_{P,Q,R}$ that are birational form a subgroup of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ denoted by $\mathrm{J}_0(1; \mathbb{P}^n_{\mathbb{C}})$, and studied in [37] : in particular $\mathrm{J}_0(1; \mathbb{P}^3_{\mathbb{C}})$ inherits the property of Theorem 2.2.

2.3. A first remark. — Let ϕ be a birational map of $\mathbb{P}^3_{\mathbb{C}}$. A ***regular resolution*** of ϕ is a morphism $\pi \colon Z \to \mathbb{P}^3_{\mathbb{C}}$ which is a sequence of blow-ups

$$\pi = \pi_1 \circ \ldots \circ \pi_r$$

along smooth irreducible centers, such that

- $\phi \circ \pi \colon Z \to \mathbb{P}^3_{\mathbb{C}}$ is a birational morphism,
- and each center B_i of the blow-up $\pi_i \colon Z_i \to Z_{i-1}$ is contained in the base locus of the induced map $Z_{i-1} \dashrightarrow \mathbb{P}^3_{\mathbb{C}}$.

It follows from Hironaka that such a resolution always exists. If B is a smooth irreducible center of a blow-up in a smooth projective complex variety of dimension 3, then B is either a point, or a smooth curve. We define the genus of B as follows: it is 0 if B is a point, the usual genus otherwise. Frumkin defines the ***genus*** of ϕ to be the maximum of the genera of the centers of the blow-ups in the resolution of ϕ (*see* [28]), and shows that this definition does not depend on the choice of the regular resolution. In [32] an other definition of the genus of a birational map is given. Let us recall that if E is an irreducible divisor contracted by a birational map between smooth projective complex varieties of dimension 3, then E is birational to $\mathbb{P}^1_{\mathbb{C}} \times \mathcal{C}$, where $\mathcal{C}$ denotes a smooth curve

([32]). The genus of a birational map ϕ of $\mathbb{P}^3_{\mathbb{C}}$ is the maximum of the genera of the irreducible divisors in $\mathbb{P}^3_{\mathbb{C}}$ contracted by ϕ. Lamy proves that these two definitions of genus agree ([32]).

Let ϕ be in $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$, and let $\mathcal{H}$ be an irreducible hypersurface of $\mathbb{P}^3_{\mathbb{C}}$. We say that $\mathcal{H}$ is ϕ-***exceptional*** if ϕ is not injective on any open subset of $\mathcal{H}$ (or equivalently if there is an open subset of $\mathcal{H}$ which is mapped into a subset of codimension ≥ 2 by ϕ). Let $\phi_1, \ldots, \phi_k$ be in $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$, and let $\phi = \phi_k \circ \ldots \circ \phi_1$. Let $\mathcal{H}$ be an irreducible hypersurface of $\mathbb{P}^3_{\mathbb{C}}$. If $\mathcal{H}$ is ϕ-exceptional, then there exists $1 \leq i \leq k$ and a ϕ_i-exceptional hypersurface $\mathcal{H}_i$ such that

– $\phi_{i-1} \circ \ldots \circ \phi_1$ realizes a birational isomorphism from $\mathcal{H}$ to $\mathcal{H}_i$;
– ϕ_i contracts $\mathcal{H}_i$.

In particular one has the following statement.

Proposition 2.6. — *The group $G_3(\mathbb{C})$ is contained in the subgroup of birational self-maps of $\mathbb{P}^3_{\mathbb{C}}$ of genus* 0.

3. Non-linearity of $G_n(\mathbb{C})$

If V is a finite dimensional vector space over $\mathbb{C}$ there is no faithful linear representation $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}}) \to \mathrm{GL}(V)$ (*see* [14, Proposition 5.1]). The proof of this statement is based on the following Lemma due to Birkhoff ([2, Lemma 1]): if $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are three elements of $\mathrm{GL}(n;\mathbb{C})$ such that

$$[\mathfrak{a},\mathfrak{b}] = \mathfrak{c}, \quad [\mathfrak{a},\mathfrak{c}] = [\mathfrak{b},\mathfrak{c}] = \mathrm{id}, \quad \mathfrak{c}^p = \mathrm{id} \text{ for some } p \text{ prime}$$

then $p \leq n$. Assume that there exists an injective homomorphism ρ from $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ to $\mathrm{GL}(n;\mathbb{C})$. For any $p > n$ prime consider in the affine chart $z_2 = 1$ the maps

$$(\exp(2\mathbf{i}\pi/p)z_0, z_1), \qquad (z_0, z_0 z_1), \qquad (z_0, \exp(-2\mathbf{i}\pi/p)z_1).$$

The image by ρ of these maps satisfy Birkhoff Lemma so $p \leq n$: contradiction. In any dimension we have the same property: $G_n(\mathbb{C})$ is not linear, *i.e.* if V is a finite dimensional vector space over $\mathbb{C}$ there is no faithful linear representation $G_n(\mathbb{C}) \to \mathrm{GL}(V)$. Actually $G_n(\mathbb{C})$ satisfies a more precise property due to Cornulier in dimension 2 (*see* [16]):

Proposition 3.1. — *The group $G_n(\mathbb{C})$ has no non-trivial finite dimensional representation.*

Lemma 3.2. — *The map $\varsigma = \big(z_0 z_{n-1} : z_1 z_{n-1} : \ldots : z_{n-2} z_{n-1} : z_{n-1} z_n : z_n^2\big)$ belongs to $G_n(\mathbb{C})$.*

Proof. — We have $\varsigma = \mathfrak{a}_1 \sigma_n \mathfrak{a}_2 \sigma_n \mathfrak{a}_3$ where

$$\mathfrak{a}_1 = \big(z_2 - z_1 : z_3 - z_1 : \ldots : z_n - z_1 : z_1 : z_1 - z_0\big),$$
$$\mathfrak{a}_2 = \big(z_{n-1} + z_n : z_n : z_0 : z_1 : \ldots : z_{n-2}\big),$$
$$\mathfrak{a}_3 = \big(z_0 + z_n : z_1 + z_n : \ldots : z_{n-2} + z_n : z_{n-1} - z_n : z_n\big).$$

□

Proof of Proposition 3.1. — We adapt the proof of [16].

Let us now work in the affine chart $z_n = 1$. By Lemma 3.2 in $G_n(\mathbb{C})$ there is a natural copy of $H = (\mathbb{C}^*)^n \rtimes \mathbb{Z}$; indeed $\langle \varsigma = \big(z_0 z_{n-1}, z_1 z_{n-1}, \ldots, z_{n-2} z_{n-1}, z_{n-1}\big)\rangle \simeq \mathbb{Z}$ acts on $\big\{(\alpha_0 z_0, \alpha_1 z_1, \ldots, \alpha_{n-1} z_{n-1}) \,|\, \alpha_i \in \mathbb{C}^*\big\} \simeq (\mathbb{C}^*)^n$ and H is the group of maps

$$\big\{(\alpha_0 z_0 z_{n-1}^k, \alpha_1 z_1 z_{n-1}^k, \ldots, \alpha_{n-2} z_{n-2} z_{n-1}^k, \alpha_{n-1} z_{n-1}) \,|\, \alpha_i \in \mathbb{C}^*, k \in \mathbb{Z}\big\}.$$

Consider any linear representation $\rho\colon H \to \mathrm{GL}(k;\mathbb{C})$. If p is prime, and if ξ_p is a primitive p-root of unity, set

$$\mathfrak{g}_p = (\xi_p z_0, \xi_p z_1, \dots, \xi_p z_{n-1}), \quad \mathfrak{h}_p = (\xi_p z_0, \xi_p z_1, \dots, \xi_p z_{n-2}, z_{n-1}).$$

Then $\mathfrak{h}_p = [\varsigma, \mathfrak{g}_p]$ commutes with both ϕ and $\mathfrak{g}_p$. By [2, Lemma 1] if $\rho(\mathfrak{g}_p) \neq 1$, then $k \geq p$.

Picking p to be greater than k, this shows that if we have an arbitrary representation $f\colon G_n(\mathbb{C}) \to \mathrm{GL}(k;\mathbb{C})$, the restriction $f_{|\mathrm{PGL}(n+1;\mathbb{C})}$ is not faithful. Since $\mathrm{PGL}(n+1;\mathbb{C})$ is simple, this implies that f is trivial on $\mathrm{PGL}(n+1;\mathbb{C})$. We conclude by using the fact that the two involutions $-\mathrm{id}$ and σ_n are conjugate via the map ψ given by

$$\left(\frac{z_0+1}{z_0-1}, \frac{z_1+1}{z_1-1}, \dots, \frac{z_{n-1}+1}{z_{n-1}-1}\right)$$

and $\psi = \mathfrak{a}_1 \sigma_n \mathfrak{a}_2$ where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ denote the two following automorphisms of $\mathbb{P}^n_{\mathbb{C}}$

$$\mathfrak{a}_1 = \big(z_0+1, z_1+1, \dots, z_{n-1}+1\big),$$

$$\mathfrak{a}_2 = \left(\frac{z_0-1}{2}, \frac{z_1-1}{2}, \dots, \frac{z_{n-1}-1}{2}\right).$$

□

Remark 3.3. — Proposition 3.1 is also true for $G_n(\Bbbk)$ where $\Bbbk$ is an algebraically closed field.

4. Subgroups of $G_n(\mathbb{C})$

4.1. The tame automorphisms. — The automorphisms of $\mathbb{C}^n$ written in the form $(\phi_0, \phi_1, \dots, \phi_{n-1})$ where

$$\phi_i = \phi_i(z_i, z_{i+1}, \dots, z_{n-1})$$

depends only on z_i, z_{i+1}, …, z_{n-1} form the ***Jonquières subgroup*** $\mathrm{J}_n \subset \mathrm{Aut}(\mathbb{C}^n)$. A polynomial automorphism $(\phi_0, \phi_1, \dots, \phi_{n-1})$ where all the ϕ_i are linear is ***an affine transformation***. Denote by Aff_n the ***group of affine transformations***; Aff_n is the semi-direct product of $\mathrm{GL}(n;\mathbb{C})$ with the commutative unipotent subgroup of translations. We have the following inclusions

$$\mathrm{GL}(n;\mathbb{C}) \subset \mathrm{Aff}_n \subset \mathrm{Aut}(\mathbb{C}^n).$$

The subgroup $\mathrm{Tame}_n \subset \mathrm{Aut}(\mathbb{C}^n)$ generated by J_n and Aff_n is called the ***group of tame automorphisms***. For $n = 2$ one has $\mathrm{Tame}_2 = \mathrm{Aut}(\mathbb{C}^2)$, this follows from the fact that $\mathrm{Aut}(\mathbb{C}^2) = \mathrm{J}_2 *_{\mathrm{J}_2 \cap \mathrm{Aff}_2} \mathrm{Aff}_2$ (*see* [30]). The group Tame_3 does not coincide with $\mathrm{Aut}(\mathbb{C}^3)$: the Nagata automorphism is not tame ([46]). Derksen gives a set of generators of Tame_n (*see* [50] for a proof):

Theorem 4.1. — *Let $n \geq 3$ be a natural integer. The group Tame_n is generated by Aff_n, and the Jonquières map $\big(z_0 + z_1^2, z_1, z_2, \dots, z_{n-1}\big)$.*

Proposition 4.2. — *The group $G_n(\mathbb{C})$ contains the group of tame polynomial automorphisms of $\mathbb{C}^n$.*

Proof. — The inclusion $\mathrm{Aff}_n \subset \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ is obvious; according to Theorem 4.1 we thus just have to prove that $\big(z_0 + z_1^2, z_1, z_2, \dots, z_{n-1}\big)$ belongs to $G_n(\mathbb{C})$. But

$$\big(z_0 z_n + z_1^2 : z_1 z_n : z_2 z_n : \dots : z_{n-1} z_n : z_n^2\big) = \mathfrak{g}_1 \sigma_n \mathfrak{g}_2 \sigma_n \mathfrak{g}_3 \sigma_n \mathfrak{g}_2 \sigma_n \mathfrak{g}_4$$

where

$$\begin{aligned}
\mathfrak{g}_1 &= \big(z_2 - z_1 + z_0 : 2z_1 - z_0 : z_3 : z_4 : \ldots : z_n : z_1 - z_0\big),\\
\mathfrak{g}_2 &= \big(z_0 + z_2 : z_0 : z_1 : z_3 : z_4 : \ldots : z_n\big),\\
\mathfrak{g}_3 &= \big(-z_1 : z_0 + z_2 - 3z_1 : z_0 : z_3 : z_4 : \ldots : z_n\big),\\
\mathfrak{g}_4 &= \big(z_1 - z_n : -2z_n - z_0 : 2z_n - z_1 : -z_2 : -z_3 : \ldots : -z_{n-1}\big).
\end{aligned}$$

□

4.2. Free groups and $G_n(\mathbb{C})$. — Following the idea of [14, Proposition 5.7] we prove that:

Proposition 4.3. — *Let $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_k$ be some generic elements of* $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$. *The group generated by $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_k$, and σ_n is the free product*

$$\overbrace{\mathbb{Z} * \ldots * \mathbb{Z}}^{k+1} * (\mathbb{Z}/2\mathbb{Z}),$$

the $\mathfrak{g}_i$'s and σ_n being the generators for the factors of this free product.

In particular the subgroup $\langle \mathfrak{g}_0\sigma_n, \mathfrak{g}_1\sigma_n, \ldots, \mathfrak{g}_k\sigma_n\rangle$ of $G_n(\mathbb{C})$ is a free group.

Remark 4.4. — The meaning of "generic" is explained in the proof below.

Proof. — Let us show the statement for $k = 0$ (in the general case it is sufficient to replace the free product $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$).

If $\langle \mathfrak{g}, \sigma_n\rangle$ is not isomorphic to $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, then there exists a word $M_{\mathfrak{g}}$ in $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ such that $M_{\mathfrak{g}}(\mathfrak{g}, \sigma_n) = \mathrm{id}$. Note that the set of words $M_{\mathfrak{g}}$ is countable, and that for a given word M the set

$$R_M = \big\{\mathfrak{g} \,\big|\, M(\mathfrak{g}, \sigma_n) = \mathrm{id}\big\}$$

is algebraic in $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$. Consider an automorphism $\mathfrak{g}$ written in the following form

$$\big(\alpha z_0 + \beta z_1 : \gamma z_0 + \delta z_1 : z_2 : z_3 : \ldots : z_n\big)$$

where $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{PGL}(2;\mathbb{C})$. Since the pencil $z_0 = tz_1$ is invariant by both σ_n and $\mathfrak{g}$, one inherits a linear representation

$$\langle \mathfrak{g}, \sigma_n\rangle \to \mathrm{PGL}(2;\mathbb{C})$$

defined by

$$\mathfrak{g} \colon t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \sigma_n \colon t \mapsto \frac{1}{t}.$$

But the group generated by $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is generically isomorphic to $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (*see* [17]). Hence the complements R_M^C are dense open subsets, and their intersection is dense by Baire property. □

5. Some algebraic properties of $G_n(\mathbb{C})$

5.1. The group $G_n(\mathbb{C})$ is perfect. — If G is a group, and if g is an element of G, we denote by

$$\mathrm{N}(g;\mathrm{G}) = \langle fgf^{-1} \mid f \in \mathrm{G}\rangle.$$

the normal subgroup generated by g in G.

Proposition 5.1. — *The following assertions hold:*

1. $\mathrm{N}(\mathfrak{g};\mathrm{PGL}(n+1;\mathbb{C})) = \mathrm{PGL}(n+1;\mathbb{C})$ *for any* $\mathfrak{g} \in \mathrm{PGL}(n+1;\mathbb{C}) \smallsetminus \{\mathrm{id}\}$*;*
2. $\mathrm{N}(\sigma_n; G_n(\mathbb{C})) = G_n(\mathbb{C})$*;*
3. $\mathrm{N}(\mathfrak{g}; G_n(\mathbb{C})) = G_n(\mathbb{C})$ *for any* $\mathfrak{g} \in \mathrm{PGL}(n+1;\mathbb{C}) \smallsetminus \{\mathrm{id}\}$.

Proof. — Let us work in the affine chart $z_n = 1$.

1. Since $\mathrm{PGL}(n+1;\mathbb{C})$ is simple one has the first assertion.
2. Let ϕ be in $G_n(\mathbb{C})$; there exist $\mathfrak{g}_0$, $\mathfrak{g}_1$, …, $\mathfrak{g}_k$ in $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ such that
$$\phi = (\mathfrak{g}_0)\,\sigma_n\,\mathfrak{g}_1\,\sigma_n \ldots \sigma_n\,\mathfrak{g}_k\,(\sigma_n).$$
As $\mathrm{PGL}(n+1;\mathbb{C})$ is simple
$$\mathrm{N}(-\mathrm{id};\mathrm{PGL}(n+1;\mathbb{C})) = \mathrm{PGL}(n+1;\mathbb{C}),$$
and for any $0 \leq i \leq k$ there exist $\mathfrak{f}_{i,0}$, $\mathfrak{f}_{i,1}$, …, $\mathfrak{f}_{i,\ell_i}$ in $\mathrm{PGL}(n+1;\mathbb{C})$ such that
$$\mathfrak{g}_i = \mathfrak{f}_{i,0}\big(-\mathrm{id}\big)\mathfrak{f}_{i,0}^{-1}\,\mathfrak{f}_{i,1}\big(-\mathrm{id}\big)\mathfrak{f}_{i,1}^{-1} \ldots \mathfrak{f}_{i,\ell_i}\big(-\mathrm{id}\big)\mathfrak{f}_{i,\ell_i}^{-1}.$$
We conclude by using the fact that $-\mathrm{id}$ and σ_n are conjugate via an element of $G_n(\mathbb{C})$ (*see* the proof of Proposition 3.1).
3. Fix $\mathfrak{g}$ in $\mathrm{PGL}(n+1;\mathbb{C}) \smallsetminus \{\mathrm{id}\}$. Since $\mathrm{N}(\mathfrak{g};\mathrm{PGL}(n+1;\mathbb{C})) = \mathrm{PGL}(n+1;\mathbb{C})$, the involution $-\mathrm{id}$ can be written as a composition of some conjugates of $\mathfrak{g}$. The maps $-\mathrm{id}$ and σ_n being conjugate one has
$$\sigma_n = (f_0\mathfrak{g}f_0^{-1})\,(f_1\mathfrak{g}f_1^{-1}) \ldots (f_\ell\mathfrak{g}f_\ell^{-1})$$
for some f_i in $G_n(\mathbb{C})$. So $\mathrm{N}(\sigma_n; G_n(\mathbb{C})) \subset \mathrm{N}(\mathfrak{g}; G_n(\mathbb{C}))$, and one concludes with the second assertion.

□

Corollary 5.2. — *The group $G_n(\mathbb{C})$ satisfies the following properties:*

1. $G_n(\mathbb{C})$ *is perfect,* i.e. $[G_n(\mathbb{C}), G_n(\mathbb{C})] = G_n(\mathbb{C})$*;*
2. *for any* ϕ *in* $G_n(\mathbb{C})$ *there exist* $\mathfrak{g}_0$, $\mathfrak{g}_1$, …, $\mathfrak{g}_k$ *automorphisms of* $\mathbb{P}^n_{\mathbb{C}}$ *such that*
$$\phi = (\mathfrak{g}_0\sigma_n\mathfrak{g}_0^{-1})(\mathfrak{g}_1\sigma_n\mathfrak{g}_1^{-1}) \ldots (\mathfrak{g}_k\sigma_n\mathfrak{g}_k^{-1})$$

Proof. — 1. The third assertion of Proposition 5.1 implies that any element of $G_n(\mathbb{C})$ can be written as a composition of some conjugates of

$$\mathfrak{t} = \big(z_0 : z_1 + z_n : z_2 + z_n : \ldots : z_{n-1} + z_n : z_n\big).$$

As

$$\mathfrak{t} = \Big[\big(z_0 : 3z_1 : 3z_2 : \ldots : 3z_{n-1} : z_n\big), \big(2z_0 : z_1 + z_n : z_2 + z_n : \ldots : z_{n-1} + z_n : 2z_n\big)\Big],$$

the group $G_n(\mathbb{C})$ is perfect.

2. For any α_0, α_1, …, α_n in $\mathbb{C}^*$ set $\mathfrak{d}(\alpha_0, \alpha_1, \ldots, \alpha_n) = (\alpha_0 z_0 : \alpha_1 z_1 : \ldots : \alpha_n z_n)$, and let us define H as follows:

$$\mathrm{H} = \Big\{\mathfrak{g}_0 \sigma_n \mathfrak{g}_0^{-1} \mathfrak{g}_1 \sigma_n \mathfrak{g}_1^{-1} \ldots \mathfrak{g}_\ell \sigma_n \mathfrak{g}_\ell^{-1} \,|\, \mathfrak{g}_i \in \mathrm{PGL}(n+1;\mathbb{C}),\, \ell \in \mathbb{N}\Big\}.$$

The second assertion of the Corollary is then equivalent to $\mathrm{H} = G_n(\mathbb{C})$. Let us remark that H is a group that contains σ_n, and that $\mathrm{PGL}(n+1;\mathbb{C})$ acts by conjugacy on it. One can check that

$$\mathfrak{d}_\alpha \sigma_n \mathfrak{d}_\alpha^{-1} = \mathfrak{d}_\alpha^2 \sigma_n = \sigma_n \mathfrak{d}_\alpha^{-2}. \tag{5.1}$$

Hence for each $\mathfrak{g}$ in $\mathrm{PGL}(n+1;\mathbb{C})$ we have $\mathfrak{g}\mathfrak{d}_\alpha \sigma_n \mathfrak{d}_\alpha^{-1} \mathfrak{g}^{-1} = (\mathfrak{g}\mathfrak{d}_\alpha^2 \mathfrak{g}^{-1})(\mathfrak{g}\sigma_n \mathfrak{g}^{-1})$, so $\mathfrak{g}\mathfrak{d}_\alpha^2 \mathfrak{g}^{-1}$ belongs to H. Since any automorphism of $\mathbb{P}^n_{\mathbb{C}}$ can be written as a product of diagonalizable matrices, $\mathrm{PGL}(n+1;\mathbb{C}) \subset \mathrm{H}$.

□

5.2. On the restriction of automorphisms of the group birational maps to $G_n(\mathbb{C})$. — If M is a projective variety defined over a field $\Bbbk \subset \mathbb{C}$ the group $\mathrm{Aut}_{\Bbbk}(\mathbb{C})$ of automorphisms of the field extension $\mathbb{C}/\Bbbk$ acts on $M(\mathbb{C})$, and on both $\mathrm{Aut}(M)$ and $\mathrm{Bir}(M)$ as follows

$${}^{\kappa}\psi(p) = (\kappa\psi\kappa^{-1})(p) \tag{5.2}$$

for any κ in $\mathrm{Aut}_{\Bbbk}(\mathbb{C})$, any ψ in $\mathrm{Bir}(M)$, and any point p in $M(\mathbb{C})$ for with both sides of (5.2) are well defined. Hence $\mathrm{Aut}_{\Bbbk}(\mathbb{C})$ acts by automorphisms on $\mathrm{Bir}(M)$. If $\kappa\colon \mathbb{C} \to \mathbb{C}$ is a morphism field, this contruction gives an injective morphism

$$\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}}) \to \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}}) \qquad \mathfrak{g} \mapsto \mathfrak{g}^{\vee}.$$

Indeed, write $\mathbb{C}$ as the algebraic closure of a purely transcendental extension $\mathbb{Q}(x_i, i \in I)$ of $\mathbb{Q}$; if $f\colon I \to I$ is an injective map, then there exists a field morphism

$$\kappa\colon \mathbb{C} \to \mathbb{C} \qquad x_i \mapsto x_{f(i)}.$$

Note that such a morphism is surjective if and only if f is onto.

In 2006, using the structure of amalgamated product of $\mathrm{Aut}(\mathbb{C}^2)$, the automorphisms of this group have been described:

Theorem 5.3 **([20])**. — *Let φ be an automorphism of $\mathrm{Aut}(\mathbb{C}^2)$. There exist a polynomial automorphism ψ of $\mathbb{C}^2$, and a field automorphism κ such that*

$$\varphi(f) = {}^{\kappa}(\psi f \psi^{-1}) \qquad \forall\, f \in \mathrm{Aut}(\mathbb{C}^2).$$

Then, in 2011, Kraft and Stampfli show that every automorphism of $\mathrm{Aut}(\mathbb{C}^n)$ is inner up to field automorphisms when restricted to the group Tame_n:

Theorem 5.4 **([31])**. — *Let φ be an automorphism of $\mathrm{Aut}(\mathbb{C}^n)$. There exist a polynomial automorphism ψ of $\mathbb{C}^n$, and a field automorphism κ such that*

$$\varphi(f) = {}^{\kappa}(\psi f \psi^{-1}) \qquad \forall\, f \in \mathrm{Tame}_n.$$

Even if $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ hasn't the same structure as $\mathrm{Aut}(\mathbb{C}^2)$ (*see* Appendix of [12]) the automorphisms group of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ can be described, and a similar result as Theorem 5.3 is obtained ([21]). There is no such result in higher dimension; nevertheless in [11] Cantat classifies all (abstract) homomorphisms from $\mathrm{PGL}(k+1;\mathbb{C})$ to the group $\mathrm{Bir}(M)$ of birational maps of a complex projective variety M, provided $k \geq \dim_{\mathbb{C}} M$. Before recalling his statement let us introduce some notation. Given $\mathfrak{g}$ in $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}}) = \mathrm{PGL}(n+1;\mathbb{C})$ we denote by ${}^{\mathrm{t}}\mathfrak{g}$ the linear transpose of $\mathfrak{g}$. The involution

$$\mathfrak{g} \mapsto \mathfrak{g}^{\vee} = ({}^{\mathrm{t}}\mathfrak{g})^{-1}$$

determines an exterior and algebraic automorphism of the group $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ (*see* [25]).

Theorem 5.5 **([11])**. — *Let M be a smooth, connected, complex projective variety, and let n be its dimension. Let k be a positive integer, and let $\rho\colon \mathrm{Aut}(\mathbb{P}^k_{\mathbb{C}}) \to \mathrm{Bir}(M)$ be an injective morphism of groups. Then $n \geq k$, and if $n = k$ there exists a field morphism $\kappa\colon \mathbb{C} \to \mathbb{C}$, and a birational map $\psi\colon M \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$ such that either*

$$\psi\rho(\mathfrak{g})\psi^{-1} = {}^{\kappa}\mathfrak{g} \qquad \forall\, \mathfrak{g} \in \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$$

or

$$\psi\rho(\mathfrak{g})\psi^{-1} = ({}^{\kappa}\mathfrak{g})^{\vee} \qquad \forall\, \mathfrak{g} \in \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}});$$

in particular M is rational. Moreover, κ is an automorphism of $\mathbb{C}$ if ρ is an isomorphism.

Let us give the proof of Theorem D:

Theorem 5.6. — *Let φ be an automorphism of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$. There exists a birational map ψ of $\mathbb{P}^n_{\mathbb{C}}$, and a field automorphism κ such that*

$$\varphi(g) = {}^{\kappa}(\psi g \psi^{-1}) \qquad \forall\, g \in G_n(\mathbb{C}).$$

Proof. — Let us consider $\varphi \in \mathrm{Aut}(\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}}))$. Theorem 5.5 implies that up to birational conjugacy and up the action of a field automorphism

$$\begin{cases} \text{either } \varphi(\mathfrak{g}) = \mathfrak{g} & \forall\, \mathfrak{g} \in \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}}) \\ \text{or } \varphi(\mathfrak{g}) = \mathfrak{g}^{\vee} & \forall\, \mathfrak{g} \in \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}}). \end{cases} \tag{5.3}$$

In other words up to birational conjugacy and up to the action of a field automorphism one cas assume that either $\varphi_{|\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})}\colon \mathfrak{g} \mapsto \mathfrak{g}$, or $\varphi_{|\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})}\colon \mathfrak{g} \mapsto \mathfrak{g}^{\vee}$. Now determine $\varphi(\sigma_n)$. Let us work in the affine chart $z_n = 1$. For $0 \leq i \leq n-2$ denote by τ_i the automorphism of $\mathbb{P}^n_{\mathbb{C}}$ that permutes z_i and z_{n-1}

$$\tau_i = \big(z_0, z_1, \ldots, z_{i-1}, z_{n-1}, z_{i+1}, z_{i+2}, \ldots, z_{n-2}, z_i\big).$$

Let η be given by

$$\eta = \left(z_0, z_1, \ldots, z_{n-2}, \frac{1}{z_{n-1}}\right).$$

One has

$$\sigma_n = \big(\tau_0\eta\tau_0\big)\big(\tau_1\eta\tau_1\big)\ldots\big(\tau_{n-2}\eta\tau_{n-2}\big)\eta$$

so

$$\varphi(\sigma_n) = \big(\varphi(\tau_0)\varphi(\eta)\varphi(\tau_0)\big)\big(\varphi(\tau_1)\varphi(\eta)\varphi(\tau_1)\big)\dots\big(\varphi(\tau_{n-2})\varphi(\eta)\varphi(\tau_{n-2})\big)\varphi(\eta).$$

Since any τ_i belongs to $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ one can, thanks to (5.3), compute $\varphi(\tau_i)$, and one gets: $\varphi(\tau_i) = \tau_i$.

Let us now focus on $\varphi(\eta)$. We will distinguish the two cases of (5.3). Assume that $\varphi_{|\mathrm{PGL}(n+1;\mathbb{C})} = \mathrm{id}$. For any $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ in $(\mathbb{C}^*)^n$ set

$$\mathfrak{d}_\alpha = (\alpha_0 z_0, \alpha_1 z_1, \dots, \alpha_{n-1} z_{n-1});$$

the involution η satisfies for any $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in (\mathbb{C}^*)^n$

$$\mathfrak{d}_\beta \eta = \eta \mathfrak{d}_\alpha$$

where $\beta = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}^{-1})$. Hence $\varphi(\eta) = \left(\pm z_0, \pm z_1, \dots, \pm z_{n-2}, \frac{\alpha}{z_{n-1}}\right)$ for $\alpha \in \mathbb{C}^*$. As η commutes with

$$\mathfrak{t} = \big(z_0 + 1, z_1 + 1, \dots, z_{n-2} + 1, z_{n-1}\big),$$

the image $\varphi(\eta)$ of η commutes to $\varphi(\mathfrak{t}) = \mathfrak{t}$. Therefore

$$\varphi(\eta) = \left(z_0, z_1, \dots, z_{n-2}, \frac{\alpha}{z_{n-1}}\right).$$

If

$$\mathfrak{h}_n = \left(\frac{z_0}{z_0 - 1}, \frac{z_0 - z_1}{z_0 - 1}, \frac{z_0 - z_2}{z_0 - 1}, \dots, \frac{z_0 - z_{n-1}}{z_0 - 1}\right)$$

then $\varphi(\mathfrak{h}_n) = \mathfrak{h}_n$, and $(\mathfrak{h}_n \sigma_n)^3 = \mathrm{id}$ implies that $\varphi(\sigma_n) = \sigma_n$. If $\varphi_{|\mathrm{PGL}(n+1;\mathbb{C})}$ coincides with $\mathfrak{g} \mapsto \mathfrak{g}^\vee$, a similar argument yields $\big(\varphi(\mathfrak{h}_n)\varphi(\sigma_n)\big)^3 \neq \mathrm{id}$. □

5.3. Simplicity of $G_n(\mathbb{C})$. — An ***algebraic family*** of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is the data of a rational map

$$\phi\colon M \times \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}},$$

where M is a $\mathbb{C}$-variety, defined on a dense open subset $\mathcal{U}$ such that

– for any $m \in M$ the intersection $\mathcal{U}_m = \mathcal{U} \cap (\{m\} \times \mathbb{P}^n_{\mathbb{C}})$ is a dense open subset of $\{m\} \times \mathbb{P}^n_{\mathbb{C}}$,
– and the restriction of $\mathrm{id} \times \phi$ to $\mathcal{U}$ is an isomorphism of $\mathcal{U}$ on a dense open subset of $M \times \mathbb{P}^n_{\mathbb{C}}$.

For any $m \in M$ the birational map $z \dashrightarrow \phi(m,z)$ represents an element ϕ_m in $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$; the map

$$M \to \mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}}), \qquad m \mapsto \phi_m$$

is called ***morphism*** from M to $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$. These notions yield the natural Zariski topology on $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, introduced by Demazure ([18]) and Serre ([45]): the subset Ω of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is ***closed*** if for any $\mathbb{C}$-variety M, and any morphism $M \to \mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ the preimage of Ω in M is closed. Note that in restriction to $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ one obtains the usual Zariski topology of the algebraic group $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}}) = \mathrm{PGL}(n+1;\mathbb{C})$.

Let us recall the following statement:

Proposition 5.7 **([5])**. — *Let $n \geq 2$. Let* H *be a non-trivial, normal, and closed subgroup of* $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$. *Then* H *contains* $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ *and* $\mathrm{PSL}\big(2;\mathbb{C}(z_0, z_1, \dots, z_{n-2})\big)$.

In our context we have a similar statement:

Proposition 5.8. — *Let $n \geq 2$. Let* H *be a non-trivial, normal, and closed subgroup of $G_n(\mathbb{C})$. Then* H *contains* $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ *and* σ_n.

Proof. — A similar argument as in [5] allows us to prove that $\mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$ is contained in H.
The fact $-\mathrm{id}$ and σ_n are conjugate in $G_n(\mathbb{C})$ (*see* Proof of Proposition 3.1) yields the conclusion. □

The proof of Proposition E follows from Proposition 5.8 and Corollary 5.2.

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