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**SOME PROPERTIES OF THE GROUP OF BIRATIONAL MAPS
GENERATED BY THE AUTOMORPHISMS OF $\mathbb{P}_{\mathbb{C}}^n$
AND THE STANDARD INVOLUTION**

by

Julie Déserti

Abstract. — We give some properties of the subgroup $G_n(\mathbb{C})$ of the group of birational self-maps of $\mathbb{P}_{\mathbb{C}}^n$ generated by the standard involution and the group of automorphisms of $\mathbb{P}_{\mathbb{C}}^n$. We prove that there is no nontrivial finite-dimensional linear representation of $G_n(\mathbb{C})$. We also establish that $G_n(\mathbb{C})$ is perfect, and that $G_n(\mathbb{C})$ equipped with the Zariski topology is simple. Furthermore if φ is an automorphism of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, then up to birational conjugacy, and up to the action of a field automorphism $\varphi|_{G_n(\mathbb{C})}$ is trivial.

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1. Introduction

The group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ of birational self-maps of $\mathbb{P}_{\mathbb{C}}^2$, also called the Cremona group of rank 2, has been the object of a lot of studies. For finite subgroups let us mention for example [3, 27, 9]; other subgroups have been dealt with ([21, 23]), and some group properties have been established ([21, 22, 19, 14, 10, 12, 5, 6, 4]). One can also find a lot of properties between algebraic geometry and dynamics ([26, 13, 7]). The Cremona group in higher dimension is far less well known; let us mention some references about finite subgroups ([42, 43, 41, 40, 33]), about algebraic subgroups of maximal rank ([18, 51, 47, 49, 48]), about other subgroups ([38, 39]), about (abstract) homomorphisms from $\text{PGL}(r+1; \mathbb{C})$ to the group $\text{Bir}(M)$ where M denotes a complex projective variety ([11]), and about maps of small bidegree ([35, 36, 34, 29, 24]).

In this article we consider the subgroup of birational self-maps of $\mathbb{P}_{\mathbb{C}}^n$ introduced by Coble in [15]

$$G_n(\mathbb{C}) = \langle \sigma_n, \text{Aut}(\mathbb{P}_{\mathbb{C}}^n) \rangle$$

where σ_n denotes the involution

$$(z_0 : z_1 : \dots : z_n) \mapsto \left(\prod_{\substack{i=0 \\ i \neq 0}}^n z_i : \prod_{\substack{i=0 \\ i \neq 1}}^n z_i : \dots : \prod_{\substack{i=0 \\ i \neq n}}^n z_i \right).$$

Hudson also deals with this group ([29]):

"For a general space transformation, there is nothing to answer either to a plane characteristic or Noether theorem. There is however a group of transformations, called punctual because each is determined by a set of points, which are defined to satisfy an analogue of Noether theorem, and possess characteristics, and for which we can set up parallels to a good deal of the plane theory."

Note that the maps of $G_3(\mathbb{C})$ are in fact not so "punctual" ([8, §8]). It follows from Noether theorem ([1, 44]) that $G_2(\mathbb{C})$ coincides with $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$; it is not the case in higher dimension where $G_n(\mathbb{C})$ is a strict subgroup of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ (see [29, 35]). However the following theorems show that $G_n(\mathbb{C})$ shares good properties with $G_2(\mathbb{C}) = \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$.

In [14] we proved that for any integer $n \geq 2$ the group $\text{Bir}(\mathbb{P}_{\mathbb{k}}^n)$, where \mathbb{k} denotes an algebraically closed field, is not linear; we obtain a similar statement for $G_n(\mathbb{k})$, $n \geq 2$:

Theorem A. — *If \mathbb{k} is an algebraically closed field, there is no nontrivial finite-dimensional linear representation of $G_n(\mathbb{k})$ over any field.*

The group $G_n(\mathbb{C})$ contains some "big" subgroups:

Proposition B. — *The group of polynomial automorphisms of \mathbb{C}^n generated by the affine automorphisms and the Jonquières ones is a subgroup of $G_n(\mathbb{C})$.*

– *If $\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_k$ are some generic automorphisms of $\mathbb{P}_{\mathbb{C}}^n$, then $\langle \mathfrak{g}_0\sigma_n, \mathfrak{g}_1\sigma_n, \dots, \mathfrak{g}_k\sigma_n \rangle \subset G_n(\mathbb{C})$ is a free subgroup of $G_n(\mathbb{C})$.*

Remark 1.1. — For the meaning of "generic" see the proof of Proposition 4.3.

In [14] we establish that $G_2(\mathbb{C}) = \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is perfect, i.e. $[G_2(\mathbb{C}), G_2(\mathbb{C})] = G_2(\mathbb{C})$; the same holds for any n :

Theorem C. — *If \mathbb{k} is an algebraically closed field, $G_n(\mathbb{k})$ is perfect.*

In [21] we determine the automorphisms group of $G_2(\mathbb{C}) = \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$; in higher dimensions we have a similar description. Before giving a precise result, let us introduce some notation: the group of the field automorphisms acts on $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$: if f is an element of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, and κ is a field automorphism we denote by ${}^{\kappa}f$ the element obtained by letting κ acting on f .

Theorem D. — *Let φ be an automorphism of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$. There exist κ an automorphism of the field \mathbb{C} , and ψ a birational map of $\mathbb{P}_{\mathbb{C}}^n$ such that*

$$\varphi(f) = {}^{\kappa}(\psi f \psi^{-1}) \quad \forall f \in G_n(\mathbb{C}).$$

The question "is the Cremona group simple?" is a very old one; Cantat and Lamy recently gave a negative answer in dimension 2 (see [12]). One can consider the same question when $G_2(\mathbb{k})$ is equipped with the Zariski topology (\mathbb{k} denotes here an algebraically closed field); Blanc looked at it, and obtained a positive answer ([5]). What about $G_n(\mathbb{k})$?

Proposition E. — *If \mathbb{k} is an algebraically closed field, the group $G_n(\mathbb{k})$, equipped with the Zariski topology, is simple.*

Organisation of the article. — We first recall a result of Pan about the set of group generators of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, $n \geq 3$ (see §2); we then note that as soon as $n \geq 3$, there are birational maps of degree $n = \deg \sigma_n$ that do not belong to $G_n(\mathbb{C})$. In §3 we prove Theorem A, and in §4 Proposition B. Let us remark that the fact that the group of tame automorphisms is contained in $G_n(\mathbb{C})$ implies that $G_n(\mathbb{C})$ contains maps of any degree, it was not obvious *a priori*. In §5 we study the normal subgroup in $G_n(\mathbb{C})$ generated by σ_n (resp. by an automorphism of $\mathbb{P}_{\mathbb{C}}^n$); it allows us to establish Theorem C. We finish §5 with the proofs of Theorem D, and Proposition E.

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2. About the set of group generators of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, $n \geq 3$

2.1. Some definitions. — A *polynomial automorphism* φ of \mathbb{C}^n is a map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ of the type

$$(z_0, z_1, \dots, z_{n-1}) \mapsto (\varphi_0(z_0, z_1, \dots, z_{n-1}), \varphi_1(z_0, z_1, \dots, z_{n-1}), \dots, \varphi_{n-1}(z_0, z_1, \dots, z_{n-1})),$$

with $\varphi_i \in \mathbb{C}[z_0, z_1, \dots, z_{n-1}]$, that is bijective; we denote φ by $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_{n-1})$. A *rational self-map* $\phi: \mathbb{P}_{\mathbb{C}}^n \dashrightarrow \mathbb{P}_{\mathbb{C}}^n$ is given by

$$(z_0 : z_1 : \dots : z_n) \dashrightarrow (\phi_0(z_0, z_1, \dots, z_n) : \phi_1(z_0, z_1, \dots, z_n) : \dots : \phi_n(z_0, z_1, \dots, z_n))$$

where the ϕ_i are homogeneous polynomials of the same positive degree, and without common factor of positive degree. Let us denote by $\mathbb{C}[z_0, z_1, \dots, z_n]_d$ the set of homogeneous polynomials in z_0, z_1, \dots, z_n of degree d . The *degree* of ϕ is by definition the degree of the ϕ_i . A *birational self-map* of $\mathbb{P}_{\mathbb{C}}^n$ is a rational self-map that admits a rational inverse. The set of polynomial automorphisms of \mathbb{C}^n (resp. birational self-maps of $\mathbb{P}_{\mathbb{C}}^n$) form a group denoted $\text{Aut}(\mathbb{C}^n)$ (resp. $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$).

2.2. A result of Pan. — Let us recall a construction of Pan ([35]) which, given a birational self-map of $\mathbb{P}_{\mathbb{C}}^n$, allows one to construct a birational self-map of $\mathbb{P}_{\mathbb{C}}^{n+1}$. Let $P \in \mathbb{C}[z_0, z_1, \dots, z_n]_d$, $Q \in \mathbb{C}[z_0, z_1, \dots, z_n]_\ell$, and let $R_0, R_1, \dots, R_{n-1} \in \mathbb{C}[z_0, z_1, \dots, z_{n-1}]_{d-\ell}$ be some homogeneous polynomials. Denote by $\Psi_{P,Q,R}: \mathbb{P}_{\mathbb{C}}^n \dashrightarrow \mathbb{P}_{\mathbb{C}}^n$ and $\tilde{\Psi}: \mathbb{P}_{\mathbb{C}}^{n-1} \dashrightarrow \mathbb{P}_{\mathbb{C}}^{n-1}$ the rational maps defined by

$$\Psi_{P,Q,R} = (QR_0 : QR_1 : \dots : QR_{n-1} : P) \quad \& \quad \tilde{\Psi}_R = (R_0 : R_1 : \dots : R_{n-1}).$$

Lemma 2.1 ([35]). — Let d, ℓ be some integers such that $d \geq \ell + 1 \geq 2$. Take Q in $\mathbb{C}[z_0, z_1, \dots, z_n]_\ell$, and P in $\mathbb{C}[z_0, z_1, \dots, z_n]_d$ without common factors. Let R_1, \dots, R_n be some elements of $\mathbb{C}[z_0, z_1, \dots, z_{n-1}]_{d-\ell}$. Assume that

$$P = z_n P_{d-1} + P_d \quad Q = z_n Q_{\ell-1} + Q_\ell$$

with $P_{d-1}, P_d, Q_{\ell-1}, Q_\ell \in \mathbb{C}[z_0, z_1, \dots, z_{n-1}]$ of degree $d-1$, resp. d , resp. $\ell-1$, resp. ℓ and such that $(P_{d-1}, Q_{\ell-1}) \neq (0, 0)$.

The map $\Psi_{P,Q,R}$ is birational if and only if $\tilde{\Psi}_R$ is birational.

Let us give the motivation of this construction:

Theorem 2.2 ([29, 35]). — Any set of group generators of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, $n \geq 3$, contains uncountably many non-linear maps.

We will give an idea of the proof of this statement.

Lemma 2.3 ([35]). — *Let $n \geq 3$. Let S be an hypersurface of $\mathbb{P}_{\mathbb{C}}^n$ of degree $\ell \geq 1$ having a point p of multiplicity $\geq \ell - 1$.*

Then there exists a birational self-map of $\mathbb{P}_{\mathbb{C}}^n$ of degree $d \geq \ell + 1$ that blows down S onto a point.

Proof. — One can assume without loss of generality that $p = (0 : 0 : \dots : 0 : 1)$. Denote by $q' = 0$ the equation of S , and take a generic plane passing through p given by the equation $h = 0$. Finally choose $P = z_n P_{d-1} + P_d$ such that

- $P_{d-1} \neq 0$;
- $\text{pgcd}(P, hq') = 1$.

Now set $Q = h^{d-\ell-1} q'$, $R_i = z_i$, and conclude with Lemma 2.1. \square

Proof of Theorem 2.2. — Let us consider the family of hypersurfaces given by $q(z_1, z_2, z_3) = 0$ where $q = 0$ defines a smooth curve C_q of degree ℓ on $\{z_0 = z_4 = z_5 = \dots = z_n = 0\}$. Let us note that $q = 0$ is birationally equivalent to $\mathbb{P}_{\mathbb{C}}^{n-2} \times C_q$. Furthermore $q = 0$ and $q' = 0$ are birationally equivalent if and only if C_q and $C_{q'}$ are isomorphic. Note that for $\ell = 2$ the set of isomorphism classes of smooth cubics is a 1-parameter family, and that according to Lemma 2.3 for any C_q there exists a birational self-map of $\mathbb{P}_{\mathbb{C}}^n$ that blows down C_q onto a point. Hence any set of group generators of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, $n \geq 3$, has to contain uncountably many non-linear maps. \square

One can take $d = \ell + 1$ in Lemma 2.3. In particular

Corollary 2.4. — *As soon as $n \geq 3$, there are birational maps of degree $n = \deg \sigma_n$ that do not belong to $G_n(\mathbb{C})$.*

Remark 2.5. — The maps $\Psi_{P,Q,R}$ that are birational form a subgroup of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ denoted by $J_0(1; \mathbb{P}_{\mathbb{C}}^n)$, and studied in [37] : in particular $J_0(1; \mathbb{P}_{\mathbb{C}}^3)$ inherits the property of Theorem 2.2.

2.3. A first remark. — Let ϕ be a birational map of $\mathbb{P}_{\mathbb{C}}^3$. A **regular resolution** of ϕ is a morphism $\pi : Z \rightarrow \mathbb{P}_{\mathbb{C}}^3$ which is a sequence of blow-ups

$$\pi = \pi_1 \circ \dots \circ \pi_r$$

along smooth irreducible centers, such that

- $\phi \circ \pi : Z \rightarrow \mathbb{P}_{\mathbb{C}}^3$ is a birational morphism,
- and each center B_i of the blow-up $\pi_i : Z_i \rightarrow Z_{i-1}$ is contained in the base locus of the induced map $Z_{i-1} \dashrightarrow \mathbb{P}_{\mathbb{C}}^3$.

It follows from Hironaka that such a resolution always exists. If B is a smooth irreducible center of a blow-up in a smooth projective complex variety of dimension 3, then B is either a point, or a smooth curve. We define the genus of B as follows: it is 0 if B is a point, the usual genus otherwise. Frumkin defines the **genus** of ϕ to be the maximum of the genera of the centers of the blow-ups in the resolution of ϕ (see [28]), and shows that this definition does not depend on the choice of the regular resolution. In [32] an other definition of the genus of a birational map is given. Let us recall that if E is an irreducible divisor contracted by a birational map between smooth projective complex varieties of dimension 3, then E is birational to $\mathbb{P}_{\mathbb{C}}^1 \times C$, where C denotes a smooth curve

([32]). The genus of a birational map ϕ of $\mathbb{P}_{\mathbb{C}}^3$ is the maximum of the genera of the irreducible divisors in $\mathbb{P}_{\mathbb{C}}^3$ contracted by ϕ . Lamy proves that these two definitions of genus agree ([32]).

Let ϕ be in $\text{Bir}(\mathbb{P}_{\mathbb{C}}^3)$, and let \mathcal{H} be an irreducible hypersurface of $\mathbb{P}_{\mathbb{C}}^3$. We say that \mathcal{H} is **ϕ -exceptional** if ϕ is not injective on any open subset of \mathcal{H} (or equivalently if there is an open subset of \mathcal{H} which is mapped into a subset of codimension ≥ 2 by ϕ). Let ϕ_1, \dots, ϕ_k be in $\text{Bir}(\mathbb{P}_{\mathbb{C}}^3)$, and let $\phi = \phi_k \circ \dots \circ \phi_1$. Let \mathcal{H} be an irreducible hypersurface of $\mathbb{P}_{\mathbb{C}}^3$. If \mathcal{H} is ϕ -exceptional, then there exists $1 \leq i \leq k$ and a ϕ_i -exceptional hypersurface \mathcal{H}_i such that

- $\phi_{i-1} \circ \dots \circ \phi_1$ realizes a birational isomorphism from \mathcal{H} to \mathcal{H}_i ;
- ϕ_i contracts \mathcal{H}_i .

In particular one has the following statement.

Proposition 2.6. — *The group $G_3(\mathbb{C})$ is contained in the subgroup of birational self-maps of $\mathbb{P}_{\mathbb{C}}^3$ of genus 0.*

3. Non-linearity of $G_n(\mathbb{C})$

If V is a finite dimensional vector space over \mathbb{C} there is no faithful linear representation $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n) \rightarrow \text{GL}(V)$ (see [14, Proposition 5.1]). The proof of this statement is based on the following Lemma due to Birkhoff ([2, Lemma 1]): if \mathfrak{a} , \mathfrak{b} and \mathfrak{c} are three elements of $\text{GL}(n; \mathbb{C})$ such that

$$[\mathfrak{a}, \mathfrak{b}] = \mathfrak{c}, \quad [\mathfrak{a}, \mathfrak{c}] = [\mathfrak{b}, \mathfrak{c}] = \text{id}, \quad \mathfrak{c}^p = \text{id} \text{ for some } p \text{ prime}$$

then $p \leq n$. Assume that there exists an injective homomorphism ρ from $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ to $\text{GL}(n; \mathbb{C})$. For any $p > n$ prime consider in the affine chart $z_2 = 1$ the maps

$$(\exp(2i\pi/p)z_0, z_1), \quad (z_0, z_0 z_1), \quad (z_0, \exp(-2i\pi/p)z_1).$$

The image by ρ of these maps satisfy Birkhoff Lemma so $p \leq n$: contradiction. In any dimension we have the same property: $G_n(\mathbb{C})$ is not linear, *i.e.* if V is a finite dimensional vector space over \mathbb{C} there is no faithful linear representation $G_n(\mathbb{C}) \rightarrow \text{GL}(V)$. Actually $G_n(\mathbb{C})$ satisfies a more precise property due to Cornulier in dimension 2 (see [16]):

Proposition 3.1. — *The group $G_n(\mathbb{C})$ has no non-trivial finite dimensional representation.*

Lemma 3.2. — *The map $\varsigma = (z_0 z_{n-1} : z_1 z_{n-1} : \dots : z_{n-2} z_{n-1} : z_{n-1} z_n : z_n^2)$ belongs to $G_n(\mathbb{C})$.*

Proof. — We have $\varsigma = \mathfrak{a}_1 \sigma_n \mathfrak{a}_2 \sigma_n \mathfrak{a}_3$ where

$$\begin{aligned} \mathfrak{a}_1 &= (z_2 - z_1 : z_3 - z_1 : \dots : z_n - z_1 : z_1 : z_1 - z_0), \\ \mathfrak{a}_2 &= (z_{n-1} + z_n : z_n : z_0 : z_1 : \dots : z_{n-2}), \\ \mathfrak{a}_3 &= (z_0 + z_n : z_1 + z_n : \dots : z_{n-2} + z_n : z_{n-1} - z_n : z_n). \end{aligned}$$

□

Proof of Proposition 3.1. — We adapt the proof of [16].

Let us now work in the affine chart $z_n = 1$. By Lemma 3.2 in $G_n(\mathbb{C})$ there is a natural copy of $H = (\mathbb{C}^*)^n \rtimes \mathbb{Z}$; indeed $\langle \varsigma = (z_0 z_{n-1}, z_1 z_{n-1}, \dots, z_{n-2} z_{n-1}, z_{n-1}) \rangle \simeq \mathbb{Z}$ acts on $\{(\alpha_0 z_0, \alpha_1 z_1, \dots, \alpha_{n-1} z_{n-1}) \mid \alpha_i \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^n$ and H is the group of maps

$$\{(\alpha_0 z_0 z_{n-1}^k, \alpha_1 z_1 z_{n-1}^k, \dots, \alpha_{n-2} z_{n-2} z_{n-1}^k, \alpha_{n-1} z_{n-1}) \mid \alpha_i \in \mathbb{C}^*, k \in \mathbb{Z}\}.$$

Consider any linear representation $\rho: H \rightarrow \mathrm{GL}(k; \mathbb{C})$. If p is prime, and if ξ_p is a primitive p -root of unity, set

$$\mathfrak{g}_p = (\xi_p z_0, \xi_p z_1, \dots, \xi_p z_{n-1}), \quad \mathfrak{h}_p = (\xi_p z_0, \xi_p z_1, \dots, \xi_p z_{n-2}, z_{n-1}).$$

Then $\mathfrak{h}_p = [\zeta, \mathfrak{g}_p]$ commutes with both ϕ and \mathfrak{g}_p . By [2, Lemma 1] if $\rho(\mathfrak{g}_p) \neq 1$, then $k \geq p$.

Picking p to be greater than k , this shows that if we have an arbitrary representation $f: G_n(\mathbb{C}) \rightarrow \mathrm{GL}(k; \mathbb{C})$, the restriction $f|_{\mathrm{PGL}(n+1; \mathbb{C})}$ is not faithful. Since $\mathrm{PGL}(n+1; \mathbb{C})$ is simple, this implies that f is trivial on $\mathrm{PGL}(n+1; \mathbb{C})$. We conclude by using the fact that the two involutions $-\mathrm{id}$ and σ_n are conjugate via the map ψ given by

$$\left(\frac{z_0+1}{z_0-1}, \frac{z_1+1}{z_1-1}, \dots, \frac{z_{n-1}+1}{z_{n-1}-1} \right)$$

and $\psi = \alpha_1 \sigma_n \alpha_2$ where α_1 and α_2 denote the two following automorphisms of $\mathbb{P}_{\mathbb{C}}^n$

$$\begin{aligned} \alpha_1 &= (z_0 + 1, z_1 + 1, \dots, z_{n-1} + 1), \\ \alpha_2 &= \left(\frac{z_0-1}{2}, \frac{z_1-1}{2}, \dots, \frac{z_{n-1}-1}{2} \right). \end{aligned}$$

□

Remark 3.3. — Proposition 3.1 is also true for $G_n(\mathbb{k})$ where \mathbb{k} is an algebraically closed field.

4. Subgroups of $G_n(\mathbb{C})$

4.1. The tame automorphisms. — The automorphisms of \mathbb{C}^n written in the form $(\phi_0, \phi_1, \dots, \phi_{n-1})$ where

$$\phi_i = \phi_i(z_i, z_{i+1}, \dots, z_{n-1})$$

depends only on $z_i, z_{i+1}, \dots, z_{n-1}$ form the **Jonquières subgroup** $J_n \subset \mathrm{Aut}(\mathbb{C}^n)$. A polynomial automorphism $(\phi_0, \phi_1, \dots, \phi_{n-1})$ where all the ϕ_i are linear is an **affine transformation**. Denote by Aff_n the **group of affine transformations**; Aff_n is the semi-direct product of $\mathrm{GL}(n; \mathbb{C})$ with the commutative unipotent subgroup of translations. We have the following inclusions

$$\mathrm{GL}(n; \mathbb{C}) \subset \mathrm{Aff}_n \subset \mathrm{Aut}(\mathbb{C}^n).$$

The subgroup $\mathrm{Tame}_n \subset \mathrm{Aut}(\mathbb{C}^n)$ generated by J_n and Aff_n is called the **group of tame automorphisms**. For $n = 2$ one has $\mathrm{Tame}_2 = \mathrm{Aut}(\mathbb{C}^2)$, this follows from the fact that $\mathrm{Aut}(\mathbb{C}^2) = J_2 *_{J_2 \cap \mathrm{Aff}_2} \mathrm{Aff}_2$ (see [30]). The group Tame_3 does not coincide with $\mathrm{Aut}(\mathbb{C}^3)$: the Nagata automorphism is not tame ([46]). Derksen gives a set of generators of Tame_n (see [50] for a proof):

Theorem 4.1. — *Let $n \geq 3$ be a natural integer. The group Tame_n is generated by Aff_n , and the Jonquières map $(z_0 + z_1^2, z_1, z_2, \dots, z_{n-1})$.*

Proposition 4.2. — *The group $G_n(\mathbb{C})$ contains the group of tame polynomial automorphisms of \mathbb{C}^n .*

Proof. — The inclusion $\mathrm{Aff}_n \subset \mathrm{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ is obvious; according to Theorem 4.1 we thus just have to prove that $(z_0 + z_1^2, z_1, z_2, \dots, z_{n-1})$ belongs to $G_n(\mathbb{C})$. But

$$(z_0 z_n + z_1^2 : z_1 z_n : z_2 z_n : \dots : z_{n-1} z_n : z_n^2) = \mathfrak{g}_1 \sigma_n \mathfrak{g}_2 \sigma_n \mathfrak{g}_3 \sigma_n \mathfrak{g}_2 \sigma_n \mathfrak{g}_4$$

where

$$\begin{aligned}\mathfrak{g}_1 &= (z_2 - z_1 + z_0 : 2z_1 - z_0 : z_3 : z_4 : \dots : z_n : z_1 - z_0), \\ \mathfrak{g}_2 &= (z_0 + z_2 : z_0 : z_1 : z_3 : z_4 : \dots : z_n), \\ \mathfrak{g}_3 &= (-z_1 : z_0 + z_2 - 3z_1 : z_0 : z_3 : z_4 : \dots : z_n), \\ \mathfrak{g}_4 &= (z_1 - z_n : -2z_n - z_0 : 2z_n - z_1 : -z_2 : -z_3 : \dots : -z_{n-1}).\end{aligned}$$

□

4.2. Free groups and $G_n(\mathbb{C})$. — Following the idea of [14, Proposition 5.7] we prove that:

Proposition 4.3. — *Let $\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_k$ be some generic elements of $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$. The group generated by $\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_k$, and σ_n is the free product*

$$\overbrace{\mathbb{Z} * \dots * \mathbb{Z}}^{k+1} * (\mathbb{Z}/2\mathbb{Z}),$$

the \mathfrak{g}_i 's and σ_n being the generators for the factors of this free product.

In particular the subgroup $\langle \mathfrak{g}_0\sigma_n, \mathfrak{g}_1\sigma_n, \dots, \mathfrak{g}_k\sigma_n \rangle$ of $G_n(\mathbb{C})$ is a free group.

Remark 4.4. — The meaning of "generic" is explained in the proof below.

Proof. — Let us show the statement for $k = 0$ (in the general case it is sufficient to replace the free product $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$).

If $\langle \mathfrak{g}, \sigma_n \rangle$ is not isomorphic to $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, then there exists a word $M_{\mathfrak{g}}$ in $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ such that $M_{\mathfrak{g}}(\mathfrak{g}, \sigma_n) = \text{id}$. Note that the set of words $M_{\mathfrak{g}}$ is countable, and that for a given word M the set

$$R_M = \{ \mathfrak{g} \mid M(\mathfrak{g}, \sigma_n) = \text{id} \}$$

is algebraic in $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$. Consider an automorphism \mathfrak{g} written in the following form

$$(\alpha z_0 + \beta z_1 : \gamma z_0 + \delta z_1 : z_2 : z_3 : \dots : z_n)$$

where $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{PGL}(2; \mathbb{C})$. Since the pencil $z_0 = tz_1$ is invariant by both σ_n and \mathfrak{g} , one inherits a linear representation

$$\langle \mathfrak{g}, \sigma_n \rangle \rightarrow \text{PGL}(2; \mathbb{C})$$

defined by

$$\mathfrak{g} : t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \sigma_n : t \mapsto \frac{1}{t}.$$

But the group generated by $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is generically isomorphic to $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (see [17]). Hence the complements R_M^C are dense open subsets, and their intersection is dense by Baire property. □

5. Some algebraic properties of $G_n(\mathbb{C})$

5.1. The group $G_n(\mathbb{C})$ is perfect. — If G is a group, and if g is an element of G , we denote by

$$N(g; G) = \langle fgf^{-1} \mid f \in G \rangle.$$

the normal subgroup generated by g in G .

Proposition 5.1. — *The following assertions hold:*

1. $N(\mathfrak{g}; \mathrm{PGL}(n+1; \mathbb{C})) = \mathrm{PGL}(n+1; \mathbb{C})$ for any $\mathfrak{g} \in \mathrm{PGL}(n+1; \mathbb{C}) \setminus \{\mathrm{id}\}$;
2. $N(\sigma_n; G_n(\mathbb{C})) = G_n(\mathbb{C})$;
3. $N(\mathfrak{g}; G_n(\mathbb{C})) = G_n(\mathbb{C})$ for any $\mathfrak{g} \in \mathrm{PGL}(n+1; \mathbb{C}) \setminus \{\mathrm{id}\}$.

Proof. — Let us work in the affine chart $z_n = 1$.

1. Since $\mathrm{PGL}(n+1; \mathbb{C})$ is simple one has the first assertion.
2. Let ϕ be in $G_n(\mathbb{C})$; there exist $\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_k$ in $\mathrm{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ such that

$$\phi = (\mathfrak{g}_0) \sigma_n \mathfrak{g}_1 \sigma_n \dots \sigma_n \mathfrak{g}_k (\sigma_n).$$

As $\mathrm{PGL}(n+1; \mathbb{C})$ is simple

$$N(-\mathrm{id}; \mathrm{PGL}(n+1; \mathbb{C})) = \mathrm{PGL}(n+1; \mathbb{C}),$$

and for any $0 \leq i \leq k$ there exist $\mathfrak{f}_{i,0}, \mathfrak{f}_{i,1}, \dots, \mathfrak{f}_{i,\ell_i}$ in $\mathrm{PGL}(n+1; \mathbb{C})$ such that

$$\mathfrak{g}_i = \mathfrak{f}_{i,0} (-\mathrm{id}) \mathfrak{f}_{i,0}^{-1} \mathfrak{f}_{i,1} (-\mathrm{id}) \mathfrak{f}_{i,1}^{-1} \dots \mathfrak{f}_{i,\ell_i} (-\mathrm{id}) \mathfrak{f}_{i,\ell_i}^{-1}.$$

We conclude by using the fact that $-\mathrm{id}$ and σ_n are conjugate via an element of $G_n(\mathbb{C})$ (see the proof of Proposition 3.1).

3. Fix \mathfrak{g} in $\mathrm{PGL}(n+1; \mathbb{C}) \setminus \{\mathrm{id}\}$. Since $N(\mathfrak{g}; \mathrm{PGL}(n+1; \mathbb{C})) = \mathrm{PGL}(n+1; \mathbb{C})$, the involution $-\mathrm{id}$ can be written as a composition of some conjugates of \mathfrak{g} . The maps $-\mathrm{id}$ and σ_n being conjugate one has

$$\sigma_n = (f_0 \mathfrak{g} f_0^{-1}) (f_1 \mathfrak{g} f_1^{-1}) \dots (f_\ell \mathfrak{g} f_\ell^{-1})$$

for some f_i in $G_n(\mathbb{C})$. So $N(\sigma_n; G_n(\mathbb{C})) \subset N(\mathfrak{g}; G_n(\mathbb{C}))$, and one concludes with the second assertion. □

Corollary 5.2. — *The group $G_n(\mathbb{C})$ satisfies the following properties:*

1. $G_n(\mathbb{C})$ is perfect, i.e. $[G_n(\mathbb{C}), G_n(\mathbb{C})] = G_n(\mathbb{C})$;
2. for any ϕ in $G_n(\mathbb{C})$ there exist $\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_k$ automorphisms of $\mathbb{P}_{\mathbb{C}}^n$ such that

$$\phi = (\mathfrak{g}_0 \sigma_n \mathfrak{g}_0^{-1}) (\mathfrak{g}_1 \sigma_n \mathfrak{g}_1^{-1}) \dots (\mathfrak{g}_k \sigma_n \mathfrak{g}_k^{-1})$$

Proof. — 1. The third assertion of Proposition 5.1 implies that any element of $G_n(\mathbb{C})$ can be written as a composition of some conjugates of

$$t = (z_0 : z_1 + z_n : z_2 + z_n : \dots : z_{n-1} + z_n : z_n).$$

As

$$t = \left[(z_0 : 3z_1 : 3z_2 : \dots : 3z_{n-1} : z_n), (2z_0 : z_1 + z_n : z_2 + z_n : \dots : z_{n-1} + z_n : 2z_n) \right],$$

the group $G_n(\mathbb{C})$ is perfect.

2. For any $\alpha_0, \alpha_1, \dots, \alpha_n$ in \mathbb{C}^* set $\mathfrak{d}(\alpha_0, \alpha_1, \dots, \alpha_n) = (\alpha_0 z_0 : \alpha_1 z_1 : \dots : \alpha_n z_n)$, and let us define H as follows:

$$H = \left\{ g_0 \sigma_n g_0^{-1} g_1 \sigma_n g_1^{-1} \dots g_\ell \sigma_n g_\ell^{-1} \mid g_i \in \text{PGL}(n+1; \mathbb{C}), \ell \in \mathbb{N} \right\}.$$

The second assertion of the Corollary is then equivalent to $H = G_n(\mathbb{C})$. Let us remark that H is a group that contains σ_n , and that $\text{PGL}(n+1; \mathbb{C})$ acts by conjugacy on it. One can check that

$$\mathfrak{d}_\alpha \sigma_n \mathfrak{d}_\alpha^{-1} = \mathfrak{d}_\alpha^2 \sigma_n = \sigma_n \mathfrak{d}_\alpha^{-2}. \quad (5.1)$$

Hence for each g in $\text{PGL}(n+1; \mathbb{C})$ we have $g \mathfrak{d}_\alpha \sigma_n \mathfrak{d}_\alpha^{-1} g^{-1} = (g \mathfrak{d}_\alpha^2 g^{-1})(g \sigma_n g^{-1})$, so $g \mathfrak{d}_\alpha^2 g^{-1}$ belongs to H. Since any automorphism of $\mathbb{P}_{\mathbb{C}}^n$ can be written as a product of diagonalizable matrices, $\text{PGL}(n+1; \mathbb{C}) \subset H$. □

5.2. On the restriction of automorphisms of the group birational maps to $G_n(\mathbb{C})$. — If M is a projective variety defined over a field $\mathbb{k} \subset \mathbb{C}$ the group $\text{Aut}_{\mathbb{k}}(\mathbb{C})$ of automorphisms of the field extension \mathbb{C}/\mathbb{k} acts on $M(\mathbb{C})$, and on both $\text{Aut}(M)$ and $\text{Bir}(M)$ as follows

$$\kappa \psi(p) = (\kappa \psi \kappa^{-1})(p) \quad (5.2)$$

for any κ in $\text{Aut}_{\mathbb{k}}(\mathbb{C})$, any ψ in $\text{Bir}(M)$, and any point p in $M(\mathbb{C})$ for with both sides of (5.2) are well defined. Hence $\text{Aut}_{\mathbb{k}}(\mathbb{C})$ acts by automorphisms on $\text{Bir}(M)$. If $\kappa: \mathbb{C} \rightarrow \mathbb{C}$ is a morphism field, this construction gives an injective morphism

$$\text{Aut}(\mathbb{P}_{\mathbb{C}}^n) \rightarrow \text{Aut}(\mathbb{P}_{\mathbb{C}}^n) \quad g \mapsto g^\vee.$$

Indeed, write \mathbb{C} as the algebraic closure of a purely transcendental extension $\mathbb{Q}(x_i, i \in I)$ of \mathbb{Q} ; if $f: I \rightarrow I$ is an injective map, then there exists a field morphism

$$\kappa: \mathbb{C} \rightarrow \mathbb{C} \quad x_i \mapsto x_{f(i)}.$$

Note that such a morphism is surjective if and only if f is onto.

In 2006, using the structure of amalgamated product of $\text{Aut}(\mathbb{C}^2)$, the automorphisms of this group have been described:

Theorem 5.3 ([20]). — *Let φ be an automorphism of $\text{Aut}(\mathbb{C}^2)$. There exist a polynomial automorphism ψ of \mathbb{C}^2 , and a field automorphism κ such that*

$$\varphi(f) = \kappa(\psi f \psi^{-1}) \quad \forall f \in \text{Aut}(\mathbb{C}^2).$$

Then, in 2011, Kraft and Stampfli show that every automorphism of $\text{Aut}(\mathbb{C}^n)$ is inner up to field automorphisms when restricted to the group Tame_n :

Theorem 5.4 ([31]). — *Let φ be an automorphism of $\text{Aut}(\mathbb{C}^n)$. There exist a polynomial automorphism ψ of \mathbb{C}^n , and a field automorphism κ such that*

$$\varphi(f) = \kappa(\psi f \psi^{-1}) \quad \forall f \in \text{Tame}_n.$$

Even if $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ hasn't the same structure as $\text{Aut}(\mathbb{C}^2)$ (see Appendix of [12]) the automorphisms group of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ can be described, and a similar result as Theorem 5.3 is obtained ([21]). There is no such result in higher dimension; nevertheless in [11] Cantat classifies all (abstract) homomorphisms from $\text{PGL}(k+1; \mathbb{C})$ to the group $\text{Bir}(M)$ of birational maps of a complex projective variety M , provided $k \geq \dim_{\mathbb{C}} M$. Before recalling his statement let us introduce some notation. Given \mathfrak{g} in $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n) = \text{PGL}(n+1; \mathbb{C})$ we denote by ${}^t\mathfrak{g}$ the linear transpose of \mathfrak{g} . The involution

$$\mathfrak{g} \mapsto \mathfrak{g}^{\vee} = ({}^t\mathfrak{g})^{-1}$$

determines an exterior and algebraic automorphism of the group $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ (see [25]).

Theorem 5.5 ([11]). — *Let M be a smooth, connected, complex projective variety, and let n be its dimension. Let k be a positive integer, and let $\rho: \text{Aut}(\mathbb{P}_{\mathbb{C}}^k) \rightarrow \text{Bir}(M)$ be an injective morphism of groups. Then $n \geq k$, and if $n = k$ there exists a field morphism $\kappa: \mathbb{C} \rightarrow \mathbb{C}$, and a birational map $\psi: M \dashrightarrow \mathbb{P}_{\mathbb{C}}^n$ such that either*

$$\psi \rho(\mathfrak{g}) \psi^{-1} = \kappa \mathfrak{g} \quad \forall \mathfrak{g} \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$$

or

$$\psi \rho(\mathfrak{g}) \psi^{-1} = (\kappa \mathfrak{g})^{\vee} \quad \forall \mathfrak{g} \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^n);$$

in particular M is rational. Moreover, κ is an automorphism of \mathbb{C} if ρ is an isomorphism.

Let us give the proof of Theorem D:

Theorem 5.6. — *Let φ be an automorphism of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$. There exists a birational map ψ of $\mathbb{P}_{\mathbb{C}}^n$, and a field automorphism κ such that*

$$\varphi(g) = \kappa(\psi g \psi^{-1}) \quad \forall g \in G_n(\mathbb{C}).$$

Proof. — Let us consider $\varphi \in \text{Aut}(\text{Bir}(\mathbb{P}_{\mathbb{C}}^n))$. Theorem 5.5 implies that up to birational conjugacy and up the action of a field automorphism

$$\left\{ \begin{array}{l} \text{either } \varphi(\mathfrak{g}) = \mathfrak{g} \quad \forall \mathfrak{g} \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^n) \\ \text{or } \varphi(\mathfrak{g}) = \mathfrak{g}^{\vee} \quad \forall \mathfrak{g} \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^n). \end{array} \right. \quad (5.3)$$

In other words up to birational conjugacy and up to the action of a field automorphism one can assume that either $\varphi|_{\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)}: \mathfrak{g} \mapsto \mathfrak{g}$, or $\varphi|_{\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)}: \mathfrak{g} \mapsto \mathfrak{g}^{\vee}$. Now determine $\varphi(\sigma_n)$. Let us work in the affine chart $z_n = 1$. For $0 \leq i \leq n-2$ denote by τ_i the automorphism of $\mathbb{P}_{\mathbb{C}}^n$ that permutes z_i and z_{n-1}

$$\tau_i = (z_0, z_1, \dots, z_{i-1}, z_{n-1}, z_{i+1}, z_{i+2}, \dots, z_{n-2}, z_i).$$

Let η be given by

$$\eta = \left(z_0, z_1, \dots, z_{n-2}, \frac{1}{z_{n-1}} \right).$$

One has

$$\sigma_n = (\tau_0 \eta \tau_0) (\tau_1 \eta \tau_1) \dots (\tau_{n-2} \eta \tau_{n-2}) \eta$$

so

$$\varphi(\sigma_n) = (\varphi(\tau_0)\varphi(\eta)\varphi(\tau_0))(\varphi(\tau_1)\varphi(\eta)\varphi(\tau_1)) \dots (\varphi(\tau_{n-2})\varphi(\eta)\varphi(\tau_{n-2}))\varphi(\eta).$$

Since any τ_i belongs to $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ one can, thanks to (5.3), compute $\varphi(\tau_i)$, and one gets: $\varphi(\tau_i) = \tau_i$.

Let us now focus on $\varphi(\eta)$. We will distinguish the two cases of (5.3). Assume that $\varphi|_{\text{PGL}(n+1;\mathbb{C})} = \text{id}$. For any $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ in $(\mathbb{C}^*)^n$ set

$$\partial_\alpha = (\alpha_0 z_0, \alpha_1 z_1, \dots, \alpha_{n-1} z_{n-1});$$

the involution η satisfies for any $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in (\mathbb{C}^*)^n$

$$\partial_\beta \eta = \eta \partial_\alpha$$

where $\beta = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}^{-1})$. Hence $\varphi(\eta) = \left(\pm z_0, \pm z_1, \dots, \pm z_{n-2}, \frac{\alpha}{z_{n-1}} \right)$ for $\alpha \in \mathbb{C}^*$. As η commutes with

$$\mathfrak{t} = (z_0 + 1, z_1 + 1, \dots, z_{n-2} + 1, z_{n-1}),$$

the image $\varphi(\eta)$ of η commutes to $\varphi(\mathfrak{t}) = \mathfrak{t}$. Therefore

$$\varphi(\eta) = \left(z_0, z_1, \dots, z_{n-2}, \frac{\alpha}{z_{n-1}} \right).$$

If

$$\mathfrak{h}_n = \left(\frac{z_0}{z_0 - 1}, \frac{z_0 - z_1}{z_0 - 1}, \frac{z_0 - z_2}{z_0 - 1}, \dots, \frac{z_0 - z_{n-1}}{z_0 - 1} \right)$$

then $\varphi(\mathfrak{h}_n) = \mathfrak{h}_n$, and $(\mathfrak{h}_n \sigma_n)^3 = \text{id}$ implies that $\varphi(\sigma_n) = \sigma_n$. If $\varphi|_{\text{PGL}(n+1;\mathbb{C})}$ coincides with $\mathfrak{g} \mapsto \mathfrak{g}^\vee$, a similar argument yields $(\varphi(\mathfrak{h}_n)\varphi(\sigma_n))^3 \neq \text{id}$. \square

5.3. Simplicity of $G_n(\mathbb{C})$. — An *algebraic family* of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ is the data of a rational map

$$\phi: M \times \mathbb{P}_{\mathbb{C}}^n \dashrightarrow \mathbb{P}_{\mathbb{C}}^n,$$

where M is a \mathbb{C} -variety, defined on a dense open subset \mathcal{U} such that

- for any $m \in M$ the intersection $\mathcal{U}_m = \mathcal{U} \cap (\{m\} \times \mathbb{P}_{\mathbb{C}}^n)$ is a dense open subset of $\{m\} \times \mathbb{P}_{\mathbb{C}}^n$,
- and the restriction of $\text{id} \times \phi$ to \mathcal{U} is an isomorphism of \mathcal{U} on a dense open subset of $M \times \mathbb{P}_{\mathbb{C}}^n$.

For any $m \in M$ the birational map $z \dashrightarrow \phi(m, z)$ represents an element ϕ_m in $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$; the map

$$M \rightarrow \text{Bir}(\mathbb{P}_{\mathbb{C}}^n), \quad m \mapsto \phi_m$$

is called *morphism* from M to $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$. These notions yield the natural Zariski topology on $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, introduced by Demazure ([18]) and Serre ([45]): the subset Ω of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ is *closed* if for any \mathbb{C} -variety M , and any morphism $M \rightarrow \text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ the preimage of Ω in M is closed. Note that in restriction to $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ one obtains the usual Zariski topology of the algebraic group $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n) = \text{PGL}(n+1;\mathbb{C})$.

Let us recall the following statement:

Proposition 5.7 ([5]). — *Let $n \geq 2$. Let H be a non-trivial, normal, and closed subgroup of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$. Then H contains $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ and $\text{PSL}(2; \mathbb{C}(z_0, z_1, \dots, z_{n-2}))$.*

In our context we have a similar statement:

Proposition 5.8. — *Let $n \geq 2$. Let H be a non-trivial, normal, and closed subgroup of $G_n(\mathbb{C})$. Then H contains $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ and σ_n .*

Proof. — A similar argument as in [5] allows us to prove that $\text{Aut}(\mathbb{P}_{\mathbb{C}}^n)$ is contained in H.

The fact $-\text{id}$ and σ_n are conjugate in $G_n(\mathbb{C})$ (see Proof of Proposition 3.1) yields the conclusion. \square

The proof of Proposition E follows from Proposition 5.8 and Corollary 5.2.

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