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JONQUIÈRES MAPS AND $SL(2; \mathbb{C})$ -COCYCLES

by

Julie DÉSERTI

Abstract. — We start the study of the family of birational maps $(f_{\alpha,\beta})$ of $\mathbb{P}_{\mathbb{C}}^2$ in [12]. For " (α, β) well chosen" of modulus 1 the centraliser of $f_{\alpha,\beta}$ is trivial, the topological entropy of $f_{\alpha,\beta}$ is 0, there exist two domains of linearisation : in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. On $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ any $f_{\alpha,\beta}$ can be viewed as a cocycle ; using recent results about $SL(2; \mathbb{C})$ -cocycles ([1]) we determine the LYAPUNOV exponent of the cocycle associated to $f_{\alpha,\beta}$.

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Introduction

In this article we deal with a family of birational maps $(f_{\alpha,\beta})$ given by

$$f_{\alpha,\beta} : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2 \quad (x : y : z) \dashrightarrow ((\alpha x + y)z : \beta y(x + z) : z(x + z))$$

where α, β denote two complex numbers with modulus 1, case where we know almost nothing about the dynamics. Let us consider the set Ω of pairs of complex numbers of modulus 1 that satisfy diophantine condition. The family $(f_{\alpha,\beta})$ satisfies the following properties ([12]):

- for $(\alpha, \beta) \in \Omega$ the centraliser of $f_{\alpha,\beta}$, that is the set of birational maps of $\mathbb{P}_{\mathbb{C}}^2$ that commutes with $f_{\alpha,\beta}$, is isomorphic to \mathbb{Z} ;
- the topological entropy of $f_{\alpha,\beta}$ is 0;
- rotation domains of ranks 1 and 2 coexist: there is a domain of linearisation where the orbit of a generic point under $f_{\alpha,\beta}$ is a torus, and there is an other domain of linearisation where the orbit of a generic point under $f_{\alpha,\beta}^2$ is a circle.

We can also see $f_{\alpha,\beta}$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ (since all the computations of [12] have been done in an affine chart they may all be carried on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$); the sets $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{S}_{\rho}^1$, where $\mathbb{S}_{\rho}^1 = \{y \in \mathbb{C} \mid |y| = \rho\}$, are invariant.

Let us define $A_n^{\alpha,\rho} : \mathbb{S}_\rho^1 \rightarrow \mathbf{M}(2; \mathbb{C})$ given in terms of $A^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y \\ 1 & 1 \end{bmatrix}$ by

$$A_n^{\alpha,\rho}(\cdot) = A^{\alpha,\rho}(\beta^n \cdot) A^{\alpha,\rho}(\beta^{n-1} \cdot) \dots A^{\alpha,\rho}(\beta \cdot) A^{\alpha,\rho}(\cdot).$$

To compute $f_{\alpha,\beta}^n(x,y)$ is equivalent to compute $A_n^{\alpha,\rho}(y)$ as soon as $f_{\alpha,\beta}^k(x,y) \neq (-1, \alpha)$ for any $1 \leq k \leq n$.

Using [1] we are able to determine the LYAPUNOV exponent of the cocycle $(A^{\alpha,\rho}, \beta)$:

Theorem A. — *The LYAPUNOV exponent of $(A^{\alpha,\rho}, \beta)$ is*

- *positive as soon as $\rho > 1$;*
- *zero as soon as $\rho \leq 1$.*

More precisely $f_{\alpha,\beta}$ is semi conjugate to $(\frac{\alpha x + y^2}{x+1}, \beta^{1/2} y)$ and the LYAPUNOV exponent of the cocycle $(B^{\alpha,\rho}, \beta^{1/2})$, where

$$B^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix},$$

is equal to $\max(0, \ln \rho)$.

In the next section we introduce the family $(f_{\alpha,\beta})$ and its properties (§1). Then we deal with the recent works of AVILA on $\mathrm{SL}(2; \mathbb{C})$ -cocycles. In the last section we give the proof of Theorem A (see §2). Let us explain the sketch of it. We associate to $(B^{\alpha,\rho}, \beta^{1/2})$ a cocycle $(\tilde{B}^{\alpha,\rho}, \beta^{1/2})$ that belongs to $\mathrm{SL}(2; \mathbb{C})$. We first determine

$$\lim_{\rho \rightarrow 0} L(\tilde{B}^{\alpha,\rho}, \beta^{1/2}),$$

and then

$$\lim_{\rho \rightarrow +\infty} L(\tilde{B}^{\alpha,\rho}, \beta^{1/2})$$

where $L(C, \gamma)$ denotes the LYAPUNOV exponent of the $\mathrm{SL}(2; \mathbb{C})$ -cocycle (C, γ) . In both cases, we get 0. Using [1, Theorem 5] we obtain that $L(\tilde{B}^{\alpha,\rho}, \beta^{1/2})$ vanishes everywhere; it allows us to determine $L(A^{\alpha,\rho}, \beta)$ since

$$L(B^{\alpha,\rho}(y), \beta^{1/2}) = L(\tilde{B}^{\alpha,\rho}(y), \beta^{1/2}) + \max(0, \ln \rho),$$

and since $(A^{\alpha,\rho}, \beta)$ and $(\beta^{1/2}, B^{\alpha,\rho})$ are conjugate.

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1. Some properties of the family $(f_{\alpha, \beta})$

A *rational map* ϕ from $\mathbb{P}_{\mathbb{C}}^2$ into itself is a map of the form

$$(x : y : z) \dashrightarrow (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)),$$

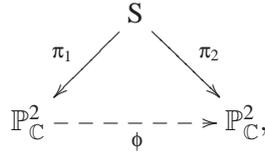
where the ϕ_i 's are some homogeneous polynomials of the same degree without common factor; ϕ is *birational* if it admits an inverse of the same type. We will denote by $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ the group of birational maps of $\mathbb{P}_{\mathbb{C}}^2$, also called the **CREMONA group**. The *degree* of ϕ , denoted $\deg \phi$, is the degree of the ϕ_i 's. The degree is not a birational invariant: $\deg \psi \phi \psi^{-1} \neq \deg \phi$ for generic birational maps ϕ and ψ . The *first dynamical degree* of ϕ given by

$$\lambda(\phi) = \lim_{n \rightarrow +\infty} (\deg \phi^n)^{1/n},$$

is a birational invariant; it is strongly related to the topological entropy $h_{\text{top}}(\phi)$ of ϕ (see [17, 20])

$$h_{\text{top}}(\phi) \leq \log \lambda(\phi) \tag{1.1}$$

Any birational map ϕ admits a resolution



where $\pi_1, \pi_2 : S \rightarrow \mathbb{P}_{\mathbb{C}}^2$ are sequences of blow-ups (see [3] for example). The resolution is *minimal* if and only if no (-1) -curve of S is contracted by both π_1 and π_2 . The *base-points* of ϕ are the points blown-up in π_1 , which can be points of $\mathbb{P}_{\mathbb{C}}^2$ or infinitely near points. We denote by $b(\phi)$ the number of such points, which is also equal to the difference of the ranks of $\text{Pic}(S)$ and $\text{Pic}(\mathbb{P}_{\mathbb{C}}^2)$, and thus equals to $b(\phi^{-1})$. The *dynamical number of base-points of* ϕ introduced in [8] is by definition

$$\mu(\phi) = \lim_{n \rightarrow +\infty} \frac{b(\phi^n)}{n};$$

it is a real positive number that satisfies $\mu(\phi^n) = |n\mu(\phi)|$ for any $n \in \mathbb{Z}$, $\mu(\psi \phi \psi^{-1}) = \mu(\phi)$, and allows us to give a characterization of birational maps conjugate to automorphisms:

Theorem 1.1 ([8]). — *Let S be a smooth projective surface; the birational map $\phi \in \text{Bir}(S)$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(\phi) = 0$.*

The behavior of $\phi \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is strongly related to the behavior of $(\deg \phi^n)_{n \in \mathbb{N}}$ (see [16, 15, 8]); up to birational conjugacy exactly one of the following holds:

1. the sequence $(\deg \phi^n)_{n \in \mathbb{N}}$ is bounded and either ϕ is of finite order, or ϕ is an automorphism of $\mathbb{P}_{\mathbb{C}}^2$;
2. there exists an integer k such that

$$\lim_{n \rightarrow +\infty} \frac{\deg \phi^n}{n} = k^2 \frac{\mu(\phi)}{2}$$

and ϕ is not an automorphism;

3. there exists an integer $k \geq 3$ such that

$$\lim_{n \rightarrow +\infty} \frac{\deg \phi^n}{n^2} = k^2 \frac{\kappa(\phi)}{9}$$

where $\kappa(\phi) \in \mathbb{Q}$ is a birational invariant, and ϕ is an automorphism;

4. the sequence $(\deg \phi^n)_{n \in \mathbb{N}}$ grows exponentially (see [15] for more precise dynamical properties).

In the first three cases $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$. In case 2. (resp. 3.) the map ϕ preserves a unique fibration which is rational (resp. elliptic).

In case 1. (resp. 2., resp. 3, resp. 4) we say that ϕ is *elliptic* (resp. a **JONQUIÈRES twist**, resp. an **HALPHEN twist**, resp. *hyperbolic*).

Let us give some examples. Let

$$\phi(x, y) = \left(\frac{a(y)x + b(y)}{c(y)x + d(y)}, \frac{\alpha y + \beta}{\gamma y + \delta} \right)$$

be an element of the **JONQUIÈRES group** $\mathrm{PGL}(2; \mathbb{C}(y)) \rtimes \mathrm{PGL}(2; \mathbb{C})$; either ϕ is elliptic (for instance $\phi: (x : y : z) \dashrightarrow (yz : xz : xy)$), or ϕ is a JONQUIÈRES twist (for example $\phi: (x : y : z) \dashrightarrow (xz : xy : z^2)$ for which the unique invariant fibration is $y/z = \text{constant}$). The map

$$\phi: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2 \quad (x : y : z) \dashrightarrow ((2y + z)(y + z) : x(2y - z) : 2z(y + z))$$

is an HALPHEN twist ([15, Proposition 9.5]). HÉNON automorphisms give by homogenization examples of hyperbolic maps.

Clearly elliptic birational maps have a poor dynamical behavior contrary to hyperbolic ones. The study of automorphisms of positive entropy is strongly related with birational maps of $\mathbb{P}_{\mathbb{C}}^2$:

Theorem 1.2 ([9]). — *Let S be a compact complex surface that carries an automorphism ϕ of positive topological entropy.*

- *Either the KODAIRA dimension of S is zero and ϕ is conjugate to an automorphism on the unique minimal model of S that necessarily is a torus, or a K3 surface or an ENRIQUES surface;*
- *or the surface S is a non-minimal rational one, isomorphic to $\mathbb{P}_{\mathbb{C}}^2$ blown up at n points, $n \geq 10$, and ϕ is conjugate to a birational map of $\mathbb{P}_{\mathbb{C}}^2$.*

This yields many examples of hyperbolic birational maps for which we can establish a lot of dynamical properties ([18, 4, 5, 6, 7, 14, 13]).

Another way to measure chaos is to look at the size of centralisers. Let us give two examples. The polynomial automorphisms of \mathbb{C}^2 having rich dynamics are HÉNON maps; furthermore a polynomial automorphism of \mathbb{C}^2 is a HÉNON one if and only if its centraliser is countable. Let us now consider rational maps on \mathbb{S}^1 ; if the centraliser of such maps is not trivial⁽¹⁾, then the JULIA set is "special". The centraliser of an elliptic birational map of infinite order is uncountable ([8]). The centralisers of HALPHEN twists are described in [16]. The centraliser of a hyperbolic map is countable ([10]). In [11] we end the story by studying centralisers of JONQUIÈRES twists. If the

1. The centraliser of a map ϕ is trivial if it coincides with the iterates of ϕ .

fibration is fiberwise invariant, then the centraliser is uncountable ; but if it isn't, then generically the centraliser is isomorphic to \mathbb{Z} . We don't know a lot about dynamics of these maps, in this article we will thus focus on a family of such maps. We consider the JONQUIÈRES maps

$$f_{\alpha, \beta}: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2 \quad (x : y : z) \dashrightarrow ((\alpha x + y)z : \beta y(x + z) : z(x + z))$$

where α, β denote two complex numbers with modulus 1. The base-points of $f_{\alpha, \beta}$ are

$$(1 : 0 : 0), \quad (0 : 1 : 0), \quad (-1 : \alpha : 1).$$

Any $f_{\alpha, \beta}$ preserves a rational fibration (the fibration $y = \text{constant}$ in the affine chart $z = 1$). Each element of the family $(f_{\alpha, \beta})$ has first dynamical degree 1 hence topological entropy zero (1.1); more precisely one has ([8, Example 4.3])

$$\mu(f_{\alpha, \beta}) = \frac{1}{2}$$

so $f_{\alpha, \beta}$ is not conjugate to an automorphism (Theorem 1.1). The centralizer of $f_{\alpha, \beta}$ is isomorphic to \mathbb{Z} (see [12, Theorem 1.6]). The idea of the proof is the following: the point $p = (1 : \alpha : 1)$ is blown-up onto a fiber of the fibration $y = \text{constant}$. Let ψ be an element of

$$\text{Cent}(f_{\alpha, \beta}) = \{g \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2) \mid g \circ f_{\alpha, \beta} = f_{\alpha, \beta} \circ g\};$$

since ψ blows down a finite number of curves there exists a positive integer k (chosen minimal) such that $f_{\alpha, \beta}^k(p)$ is not blown down by ψ . Replacing ψ by $\tilde{\psi} = \psi \circ f_{\alpha, \beta}^{k-1}$ one gets that $\tilde{\psi}(p)$ is an indeterminacy point of $f_{\alpha, \beta}$. In other words $\tilde{\psi}$ permutes the indeterminacy points of $f_{\alpha, \beta}$. A more precise study allows us to establish that p is fixed by $\tilde{\psi}$. The pair (α, β) being in Ω , the closure of the negative orbit of p under the action of $f_{\alpha, \beta}$ is ZARISKI dense; since $\tilde{\psi}$ fixes any element of the orbit of p one obtains $\tilde{\psi} = \text{id}$.

Let us recall that if ψ is an automorphism on a compact complex manifold M , the **FATOU set** $\mathcal{F}(\psi)$ of ψ is the set of points that have a neighborhood \mathcal{U} such that $\{f_{\psi}^n \mid n \in \mathbb{N}\}$ is a normal family. Set

$$\mathcal{G}(\mathcal{U}) = \{\phi: \mathcal{U} \rightarrow \overline{\mathcal{U}} \mid \phi = \lim_{n_j \rightarrow +\infty} \psi^{n_j}\};$$

we say that \mathcal{U} is a **rotation domain** if $\mathcal{G}(\mathcal{U})$ is a subgroup of $\text{Aut}(\mathcal{U})$. An equivalent definition is the following: a component \mathcal{U} of $\mathcal{F}(\psi)$ which is invariant by ψ is a rotation domain if $\psi|_{\mathcal{U}}$ is conjugate to a linear rotation. If \mathcal{U} is a rotation domain, $\mathcal{G}(\mathcal{U})$ is a compact LIE group, and the action of $\mathcal{G}(\mathcal{U})$ on \mathcal{U} is analytic real. Since $\mathcal{G}(\mathcal{U})$ is a compact, infinite, abelian LIE group, the connected component of the identity of $\mathcal{G}(\mathcal{U})$ is a torus of dimension $0 \leq d \leq \dim_{\mathbb{C}} M$. The integer d is the **rank of the rotation domain**. The rank coincides with the dimension of the closure of a generic orbit of a point in \mathcal{U} .

We can also see $f_{\alpha, \beta}$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ and that is what we will do in the sequel (since all the computations of [12] have been done in an affine chart they may all be carried on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$); the sets $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{S}_{\rho}^1$ are invariant. In [12] we show that there are two rotation domains for $f_{\alpha, \beta}^2$, one of

rank 1, and the other one of rank 2⁽²⁾; in the first case we give here a more precise statement than in [12]:

Theorem 1.3. — Assume that (α, β) belongs to Ω .

There exists a strictly positive real number r such that $f_{\alpha, \beta}$ is conjugate to $(\alpha x, \beta y)$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{D}(0, r)$ where $\mathbb{D}(0, r)$ denotes the disk centered at the origin with radius r .

There exists a strictly positive real number \tilde{r} such that $f_{\alpha, \beta}^2$ is conjugate to $\left(\frac{x}{\beta}, \frac{z}{\beta^2}\right)$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{D}(0, \tilde{r})$.

Remark 1.4. — The point $(\alpha - 1, 0)$ is also a fixed point of $f_{\alpha, \beta}$ where the behavior of $f_{\alpha, \beta}$ is the same as near $(0, 0)$.

Proof. — The first assertion is proved in [12].

Let us consider the map $\Psi(x, z) = \left(\frac{a(z)x+b(z)}{c(z)x+1}, z\right)$. The equation

$$\Psi^{-1} f_{\alpha, \beta}^2 \Psi = \left(\frac{x}{\beta}, \frac{z}{\beta^2}\right)$$

yields

$$\begin{aligned} & \beta a(\beta^{-2}z)c(z) + \beta a(\beta^{-2}z)a(z) - c(\beta^{-2}z)a(z) + \alpha a(\beta^{-2}z)a(z) \\ & + z(\alpha^2 a(\beta^{-2}z)c(z) - \alpha c(\beta^{-2}z)c(z) - c(\beta^{-2}z)c(z) - c(\beta^{-2}z)a(z)) = 0, \end{aligned} \quad (1.2)$$

$$\begin{aligned} & \beta a(\beta^{-2}z) - \beta a(z) + z(\alpha^2 a(\beta^{-2}z) - \alpha \beta c(z) - \beta c(z) - \beta a(z) - \alpha c(\beta^{-2}z) - c(\beta^{-2}z)) \\ & + \beta(\alpha + \beta)a(z)b(\beta^{-2}z) + (\alpha + \beta)b(z)a(\beta^{-2}z) + \beta^2 b(\beta^{-2}z)c(z) - b(z)c(\beta^{-2}z) \\ & + z(\alpha^2 \beta b(\beta^{-2}z)c(z) - b(z)c(\beta^{-2}z)) = 0 \end{aligned} \quad (1.3)$$

and

$$(\alpha + 1)z + b(z) - \beta b(\beta^{-2}z) - \alpha^2 z b(\beta^{-2}z) + z b(z) - (\alpha + \beta)b(\beta^{-2}z)b(z) = 0 \quad (1.4)$$

Let us set

$$a(z) = \sum_{i \geq 0} a_i z^i, \quad b(z) = \sum_{i \geq 0} b_i z^i, \quad c(z) = \sum_{i \geq 0} c_i z^i.$$

We easily get $a_0 = 1 - \beta$, $b_0 = 0$ and $c_0 = \alpha + \beta$.

Relation (1.4) implies that

$$b_1 = \frac{\beta(1 + \alpha)}{1 - \beta} \quad \& \quad \beta b_v (1 - \beta^{1-2v}) + F_i(b_i | i < v) = 0 \quad \forall v > 1,$$

(1.3) yields

$$a_v (\beta^{1-2v} - \beta) + b_v \left((\alpha + \beta)a_0 (1 + \beta^{1-2v}) + c_0 (\beta^{2-2v} - 1) \right) + G_i(a_i, b_i, c_i | i < v) = 0$$

and (1.2) to

$$c_v a_0 (\beta - \beta^{-2v}) + a_v \left((\alpha + \beta)a_0 (1 + \beta^{-2v}) + c_0 (\beta^{1-2v} - 1) \right) + H_i(a_i, b_i, c_i | i < v) = 0$$

2. There already exists an example of automorphism of positive entropy with rotation domains of rank 1 and 2 (see [5]), but $f_{\alpha, \beta}$ is not conjugate to an automorphism on a rational surface.

where the F_i 's, G_i 's and H_i 's denote universal polynomials; this allows to compute b_v , a_v and c_v . Thus we get a formal conjugacy of $f_{\alpha, \beta}^2$ to its linear part. Since this linear part satisfies a Rüssmann condition (see [19, Theorem 2.1] condition (2)), according to [19, Theorem 2.1] any formal linearizing map conjugating $f_{\alpha, \beta}^2$ to its linear part is convergent on a polydisc. \square

2. About $SL(2; \mathbb{C})$ -cocycles

A (one-frequency, analytic) *quasiperiodic* $SL(2; \mathbb{C})$ -cocycle is a pair (A, β) , where $\beta \in \mathbb{R}$ and

$$A: \mathbb{S}_1^1 \rightarrow SL(2; \mathbb{C})$$

is analytic, and defines a linear skew product acting on $\mathbb{C}^2 \times \mathbb{S}_1^1$ by

$$(x, y) \mapsto (A(y) \cdot x, \beta y).$$

The iterates of the cocycle are given by $(A_n, n\beta)$ where A_n is given by

$$A_n(y) = A(\beta^{n-1}y) \dots A(y) \quad n \geq 1, \quad A_0(y) = \text{id}, \quad A_{-n}(y) = A_n(\beta^{-n}y)^{-1}.$$

The **LYAPUNOV exponent** $L(A, \beta)$ of a quasiperiodic $SL(2; \mathbb{C})$ -cocycle (A, β) is given by

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{S}_1^1} \ln \|A_n(y)\| \, dy.$$

A quasiperiodic $SL(2; \mathbb{C})$ -cocycle (A, β) is *uniformly hyperbolic* if there exist analytic functions

$$u, s: \mathbb{S}_1^1 \rightarrow \mathbb{P}_{\mathbb{C}}^2,$$

called the *unstable and stable directions*, and $n \geq 1$ such that for any $y \in \mathbb{S}_1^1$,

$$A(y) \cdot u(y) = u(\beta y) \quad A(y) \cdot s(y) = s(\beta y),$$

and for any unit vector $x \in s(y)$ (resp. $x \in u(y)$) we have $\|A_n(y) \cdot x\| < 1$ (resp. $\|A_n(y) \cdot x\| > 1$). The unstable and stable directions are uniquely characterized by those properties, and clearly $u(y) \neq s(y)$ for any $y \in \mathbb{S}_1^1$. If (A, β) is uniformly hyperbolic, then $L(A, \beta) > 0$. Let us denote by

$$\mathcal{UH} \subset C^\omega(SL(2; \mathbb{C}), \mathbb{S}_1^1)$$

the set of A such that (A, β) is uniformly hyperbolic. Uniform hyperbolicity is a stable property: \mathcal{UH} is open, and $A \mapsto L(A, \beta)$ is analytic over \mathcal{UH} (regularity properties of the LYAPUNOV exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables).

Definition. — Let (A, β) be a quasiperiodic $SL(2; \mathbb{C})$ -cocycle. If $L(A, \beta) > 0$ but $(A, \beta) \notin \mathcal{UH}$, then (A, β) is *nonuniformly hyperbolic*.

If $A \in C^\omega(SL(2; \mathbb{C}), \mathbb{S}_1^1)$ admits a holomorphic extension to $|\text{Im } y| < \delta$ then for $|\varepsilon| < \delta$ we can define $A_\varepsilon \in C^\omega(SL(2; \mathbb{C}), \mathbb{S}_1^1)$ by

$$A_\varepsilon(y) = A(y + i\varepsilon).$$

The LYAPUNOV exponent $L(A_\varepsilon, \beta)$ is a convex function of ε . We can thus introduce the following notion. The *acceleration* of a quasiperiodic $SL(2; \mathbb{C})$ -cocycle (A, β) is given by

$$\omega(A, \beta) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi\varepsilon} (L(A_\varepsilon, \beta) - L(A, \beta)).$$

Remark 2.1. — The convexity of the LYAPUNOV exponent in function of ε implies that the acceleration is decreasing.

Since the LYAPUNOV exponent is a convex and continuous function the acceleration is an upper semi-continuous function in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathrm{SL}(2; \mathbb{C}), \mathbb{S}_1^1)$. The acceleration is quantized:

Theorem 2.2 ([1]). — *If (A, β) is a $\mathrm{SL}(2; \mathbb{C})$ -cocycle with $\beta \in \mathbb{R} \setminus \mathbb{Q}$, then $\omega(A, \beta)$ is always an integer.*

A direct consequence is the following:

Corollary 2.3. — *The function $\varepsilon \mapsto L(A_\varepsilon, \beta)$ is a piecewise affine function of ε .*

It is thus natural to introduce the notion of regularity. A cocycle

$$(A, \beta) \in C^\omega(\mathrm{SL}(2; \mathbb{C}), \mathbb{S}_1^1) \times \mathbb{R} \setminus \mathbb{Q}$$

is **regular** if $L(A_\varepsilon, \beta)$ is affine for ε in a neighborhood of 0. In other words (A, β) is regular if the equality

$$L(A_\varepsilon, \beta) - L(A, \beta) = 2\pi\varepsilon\omega(A, \beta)$$

holds for all ε small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near (A, β) . It is an open condition in $C^\omega(\mathrm{SL}(2; \mathbb{C}), \mathbb{S}_1^1) \times \mathbb{R} \setminus \mathbb{Q}$. The following statement gives a characterization of the dynamics of regular cocycles with positive LYAPUNOV exponent:

Theorem 2.4 ([1]). — *Let (A, β) be a $\mathrm{SL}(2; \mathbb{C})$ -cocycle with $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $L(A, \beta) > 0$; then (A, β) is regular if and only if (A, β) is \mathcal{UH} .*

One striking consequence is the following:

Corollary 2.5 ([1]). — *For any (A, β) in $C^\omega(\mathrm{SL}(2; \mathbb{C}), \mathbb{S}_1^1) \times \mathbb{R} \setminus \mathbb{Q}$ there exists ε_0 such that*

- $L(A_\varepsilon, \beta) = 0$ (and $\omega(A, \beta) = 0$) for every $0 < \varepsilon < \varepsilon_0$,
- or $(A_\varepsilon, \beta) \in \mathcal{UH}$ for every $0 < \varepsilon < \varepsilon_0$.

Remark 2.6. — Let us mention that there is a link between $\mathrm{SL}(2; \mathbb{C})$ -cocycles and SCHRÖDINGER operators (see [1] for more details).

3. Proof of Theorem A

Suppose that $\rho \neq 1$, and let us consider the cocycle $(B^{\alpha, \rho}, \beta^{1/2})$ where

$$B^{\alpha, \rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix}.$$

Since

$$\left(\frac{\alpha x + y}{x + 1}, \beta y \right) (x, y^2) = (x, y^2) \left(\frac{\alpha x + y^2}{x + 1}, \beta^{1/2} y \right)$$

the cocycles $(A^{\alpha, \rho}, \beta)$ and $(B^{\alpha, \rho}, \beta^{1/2})$ have the same behavior. Using two different arguments of monodromy (one for $\rho < 1$, and the other one for $\rho > 1$) we see that there is a continuous determination for the square root of $\det B^{\alpha, \rho}(y) = \alpha - y^2$. Let us set

$$\tilde{B}^{\alpha, \rho}(y) = \frac{1}{\sqrt{\alpha - y^2}} B^{\alpha, \rho}(y) \in SL(2; \mathbb{C})$$

that is thus defined on two different domains of analyticity.

According to Theorem 1.3 one has $L(\tilde{B}^{\alpha, \rho}, \beta^{1/2}) = 0$ when ρ is close to both 0 and ∞ .

Assume that $L(\tilde{B}^{\alpha, \rho}, \beta^{1/2})$ is non constant. When $\tilde{B}^{\alpha, \rho}$ is holomorphic, so in particular when $\rho < 1$ and $\rho > 1$, the acceleration is decreasing (Remark 2.1); furthermore the acceleration is positive for $\rho < 1$ and negative for $\rho > 1$ (because L is continuous). Theorem 2.2 thus implies

$$\omega(\tilde{B}^{\alpha, 1^+}, \beta^{1/2}) - \omega(\tilde{B}^{\alpha, 1^-}, \beta^{1/2}) \leq -2.$$

By definition of $\tilde{B}^{\alpha, \rho}$ we have

$$\begin{aligned} L(\tilde{B}^{\alpha, \rho}(y), \beta^{1/2}) &= L(B^{\alpha, \rho}(y), \beta^{1/2}) - \int_{\mathbb{S}_\beta^1} \ln \sqrt{\alpha - y^2} dy \\ &= L(B^{\alpha, \rho}(y), \beta^{1/2}) - \max(0, \ln \rho). \end{aligned}$$

Even though $(B^{\alpha, \rho}(y), \beta^{1/2})$ is not a $SL(2; \mathbb{C})$ -cocycle, the LYAPUNOV exponent is still a convex function of $\log \rho$ (see for example [2]). The jump of $\omega(B^{\alpha, \rho}(y), \beta^{1/2})$ is thus ≥ 0 , and the jump for the second term of the right member is -1 . Therefore the jump of $L(\tilde{B}^{\alpha, \rho}(y), \beta^{1/2})$ is ≥ -1 : contradiction.

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JULIE DÉSERTI, Institut de Mathématiques de Jussieu-
Paris Rive Gauche, UMR 7586, Université Paris Diderot, Bâtiment Sophie Germain, Case 7012, 75205 Paris
Cedex 13, France. • *E-mail* : deserti@math.univ-paris-diderot.fr