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JONQUIÈRES MAPS AND $\text{SL}(2; \mathbb{C})$-COCYCLES

by

Julie DÉSERTI

Abstract. — We start the study of the family of birational maps $(f_{\alpha,\beta})$ of $\mathbb{P}^2_\mathbb{C}$ in [12]. For “$(\alpha,\beta)$ well chosen” of modulus 1 the centraliser of $f_{\alpha,\beta}$ is trivial, the topological entropy of $f_{\alpha,\beta}$ is 0, there exist two domains of linearisation: in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. On $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ any $f_{\alpha,\beta}$ can be viewed as a cocyle; using recent results about $\text{SL}(2; \mathbb{C})$-cocycles ([1]) we determine the LYAPUNOV exponent of the cocyle associated to $f_{\alpha,\beta}$.

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Introduction

In this article we deal with a family of birational maps $(f_{\alpha,\beta})$ given by

$$f_{\alpha,\beta} : \mathbb{P}^2_\mathbb{C} \rightarrow \mathbb{P}^2_\mathbb{C} \quad (x : y : z) \rightarrow ((\alpha x + y)z : \beta y(x + z) : z(x + z))$$

where $\alpha, \beta$ denote two complex numbers with modulus 1, case where we know almost nothing about the dynamics. Let us consider the set $\Omega$ of pairs of complex numbers of modulus 1 that satisfy diophantine condition. The family $(f_{\alpha,\beta})$ satisfies the following properties ([12]):

- for $(\alpha, \beta) \in \Omega$ the centraliser of $f_{\alpha,\beta}$, that is the set of birational maps of $\mathbb{P}^2_\mathbb{C}$ that commutes with $f_{\alpha,\beta}$, is isomorphic to $\mathbb{Z}$;
- the topological entropy of $f_{\alpha,\beta}$ is 0;
- rotation domains of ranks 1 and 2 coexist: there is a domain of linearisation where the orbit of a generic point under $f_{\alpha,\beta}$ is a torus, and there is an other domain of linearisation where the orbit of a generic point under $f_{\alpha,\beta}^2$ is a circle.

We can also see $f_{\alpha,\beta}$ on $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ (since all the computations of [12] have been done in an affine chart they may all be carried on $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$); the sets $\mathbb{P}^1_\mathbb{C} \times \mathbb{S}^1_\rho$, where $\mathbb{S}^1_\rho = \{ y \in \mathbb{C} | |y| = \rho \}$, are invariant.
Let us define $A_n^{\alpha,\rho} : S^1 \to M(2; \mathbb{C})$ given in terms of $A^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y \\ 1 & 1 \end{bmatrix}$ by

$$A_n^{\alpha,\rho}() = A^{\alpha,\rho}(\beta^n \cdot)A^{\alpha,\rho}(\beta^{n-1} \cdot) \ldots A^{\alpha,\rho}(\beta \cdot)A^{\alpha,\rho}().$$

To compute $f_n^{\alpha,\beta}(x,y)$ is equivalent to compute $A_n^{\alpha,\rho}(y)$ as soon as $f_k^{\alpha,\beta}(x,y) \neq (-1, \alpha)$ for any $1 \leq k \leq n$.

Using [1] we are able to determine the Lyapunov exponent of the cocycle $(A^{\alpha,\rho}, \beta)$:

**Theorem A.** — The Lyapunov exponent of $(A^{\alpha,\rho}, \beta)$ is

- positive as soon as $\rho > 1$;
- zero as soon as $\rho \leq 1$.

More precisely $f_{\alpha,\beta}$ is semi conjugate to $(\frac{\alpha y^2}{x+1}, \beta^{1/2} y)$ and the Lyapunov exponent of the cocycle $(B^{\alpha,\rho}, \beta^{1/2})$, where

$$B^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix},$$

is equal to $\max(0, \ln \rho)$.

In the next section we introduce the family $(f_{\alpha,\beta})$ and its properties (§1). Then we deal with the recent works of Avila on SL(2; C)-cocycles. In the last section we give the proof of Theorem A (see §2). Let us explain the sketch of it. We associate to $(B^{\alpha,\rho}, \beta^{1/2})$ a cocycle $(\tilde{B}^{\alpha,\rho}, \beta^{1/2})$ that belongs to SL(2; C). We first determine

$$\lim_{\rho \to 0} L(\tilde{B}^{\alpha,\rho}, \beta^{1/2}),$$

and then

$$\lim_{\rho \to +\infty} L(\tilde{B}^{\alpha,\rho}, \beta^{1/2})$$

where $L(C, \gamma)$ denotes the Lyapunov exponent of the SL(2; C)-cocycle $(C, \gamma)$. In both cases, we get 0. Using [1, Theorem 5] we obtain that $L(\tilde{B}^{\alpha,\rho}, \beta^{1/2})$ vanishes everywhere; it allows us to determine $L(A^{\alpha,\rho}, \beta)$ since

$$L(B^{\alpha,\rho}(y), \beta^{1/2}) = L(\tilde{B}^{\alpha,\rho}(y), \beta^{1/2}) + \max(0, \ln \rho),$$

and since $(A^{\alpha,\rho}, \beta)$ and $(\beta^{1/2}, B^{\alpha,\rho})$ are conjugate.

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1. Some properties of the family \((f_\alpha, \beta)\)

A rational map \(\phi\) from \(\mathbb{P}^2_C\) into itself is a map of the form

\[(x : y : z) \mapsto (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)),\]

where the \(\phi_i\)'s are some homogeneous polynomials of the same degree without common factor; \(\phi\) is birational if it admits an inverse of the same type. We will denote by Bir\((\mathbb{P}^2_C)\) the group of birational maps of \(\mathbb{P}^2_C\), also called the Cremona group. The degree of \(\phi\), denoted \(\deg \phi\), is the degree of the \(\phi_i\)'s. The degree is not a birational invariant: \(\deg \psi \phi \psi^{-1} \neq \deg \phi\) for generic birational maps \(\phi\) and \(\psi\). The first dynamical degree of \(\phi\) given by

\[\lambda(\phi) = \lim_{n \to +\infty} \frac{\deg \phi^n}{n},\]

is a birational invariant; it is strongly related to the topological entropy \(h_{\text{top}}(\phi)\) of \(\phi\) (see [17, 20])

\[h_{\text{top}}(\phi) \leq \log \lambda(\phi) \quad (1.1)\]

Any birational map \(\phi\) admits a resolution

\[\pi_1 \quad \pi_2\]

\[\mathbb{P}^2_C \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \mathbb{P}^2_C,\]

where \(\pi_1, \pi_2 : S \to \mathbb{P}^2_C\) are sequences of blow-ups (see [3] for example). The resolution is minimal if and only if no \((-1)\)-curve of \(S\) is contracted by both \(\pi_1\) and \(\pi_2\). The base-points of \(\phi\) are the points blown-up in \(\pi_1\), which can be points of \(\mathbb{P}^2_C\) or infinitely near points. We denote by \(b(\phi)\) the number of such points, which is also equal to the difference of the ranks of Pic\((S)\) and Pic\((\mathbb{P}^2_C)\), and thus equals to \(b(\phi^{-1})\). The dynamical number of base-points of \(\phi\) introduced in [8] is by definition

\[\mu(\phi) = \lim_{n \to +\infty} \frac{b(\phi^n)}{n},\]

it is a real positive number that satisfies \(\mu(\phi^n) = |n \mu(\phi)|\) for any \(n \in \mathbb{Z}\), \(\mu(\psi \phi \psi^{-1}) = \mu(\phi)\), and allows us to give a characterization of birational maps conjugate to automorphisms:

**Theorem 1.1 ([8]).** — Let \(S\) be a smooth projective surface; the birational map \(\phi \in \text{Bir}(S)\) is conjugate to an automorphism of a smooth projective surface if and only if \(\mu(\phi) = 0\).

The behavior of \(\phi \in \text{Bir}(\mathbb{P}^2_C)\) is strongly related to the behavior of \((\deg \phi^n)_{n \in \mathbb{N}}\) (see [16, 15, 8]); up to birational conjugacy exactly one of the following holds:

1. the sequence \((\deg \phi^n)_{n \in \mathbb{N}}\) is bounded and either \(\phi\) is of finite order, or \(\phi\) is an automorphism of \(\mathbb{P}^2_C\);
2. there exists an integer \(k\) such that

\[\lim_{n \to +\infty} \frac{\deg \phi^n}{n} = k^2 \frac{\mu(\phi)}{2}\]

and \(\phi\) is not an automorphism;
3. there exists an integer \( k \geq 3 \) such that
\[
\lim_{n \to +\infty} \frac{\deg \phi^n}{n^2} = k^2 \frac{\kappa(\phi)}{9}
\]
where \( \kappa(\phi) \in \mathbb{Q} \) is a birational invariant, and \( \phi \) is an automorphism;

4. the sequence \( \left( \deg \phi^n \right)_{n \in \mathbb{N}} \) grows exponentially (see [15] for more precise dynamical properties).

In the first three cases \( \lambda(\phi) = 1 \), in the last one \( \lambda(\phi) > 1 \). In case 2. (resp. 3.) the map \( \phi \) preserves a unique fibration which is rational (resp. elliptic).

In case 1. (resp. 2., resp. 3, resp. 4) we say that \( \phi \) is elliptic (resp. a JONQUIÈRES twist, resp. an HALPHEN twist, resp. hyperbolic).

Let us give some examples. Let
\[
\phi(x, y) = \left( \frac{a(y)x + b(y)}{c(y)x + d(y)}, \frac{\alpha y + \beta}{\gamma y + \delta} \right)
\]
be an element of the JONQUIÈRES group \( \text{PGL}(2; \mathbb{C}(y)) \rtimes \text{PGL}(2; \mathbb{C}) \); either \( \phi \) is elliptic (for instance \( \phi: (x : y : z) \mapsto (yz : xz : xy) \)), or \( \phi \) is a JONQUIÈRES twist (for example \( \phi: (x : y : z) \mapsto (xz : xy : z^2) \) for which the unique invariant fibration is \( y/z = \text{constant} \). The map
\[
\phi: \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}} \quad \quad (x : y : z) \mapsto ((2y + z)(y + z) : x(2y - z) : 2z(y + z))
\]
is an HALPHEN twist ([15, Proposition 9.5]). HÉNON automorphisms give by homogeneization examples of hyperbolic maps.

Clearly elliptic birational maps have a poor dynamical behavior contrary to hyperbolic ones. The study of automorphisms of positive entropy is strongly related with birational maps of \( \mathbb{P}^2_{\mathbb{C}} \):

**Theorem 1.2 ([9]).** — Let \( S \) be a compact complex surface that carries an automorphism \( \phi \) of positive topological entropy.
- Either the KODAIRA dimension of \( S \) is zero and \( \phi \) is conjugate to an automorphism on the unique minimal model of \( S \) that necessarily is a torus, or a K3 surface or an ENRIQUES surface;
- or the surface \( S \) is a non-minimal rational one, isomorphic to \( \mathbb{P}^2_{\mathbb{C}} \) blown up at \( n \) points, \( n \geq 10 \), and \( \phi \) is conjugate to a birational map of \( \mathbb{P}^2_{\mathbb{C}} \).

This yields many examples of hyperbolic birational maps for which we can establish a lot of dynamical properties ([18, 4, 5, 6, 7, 14, 13]).

Another way to measure chaos is to look at the size of centralisers. Let us give two examples. The polynomial automorphisms of \( \mathbb{C}^2 \) having rich dynamics are HÉNON maps; furthermore a polynomial automorphism of \( \mathbb{C}^2 \) is a HÉNON one if and only if its centraliser is countable. Let us now consider rational maps on \( S^1 \); if the centraliser of such maps is not trivial\(^1\), then the JULIA set is "special". The centraliser of an elliptic birational map of infinite order is uncountable ([8]). The centralisers of HALPHEN twists are described in [16]. The centraliser of an hyperbolic map is countable ([10]). In [11] we end the story by studying centralisers of JONQUIÈRES twists. If the

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1. The centraliser of a map \( \phi \) is trivial if it coincides with the iterates of \( \phi \).
fibration is fiberwise invariant, then the centraliser is uncountable; but if it isn’t, then generically the centraliser is isomorphic to \( \mathbb{Z} \). We don’t know a lot about dynamics of these maps, in this article we will thus focus on a family of such maps. We consider the JONQUIÈRES maps

\[
        f_{a,b} : \mathbb{P}_C^2 \to \mathbb{P}_C^2 \quad (x : y : z) \mapsto ((ax + y)z : (b + 1)z(x + z))
\]

where \( a, b \) denote two complex numbers with modulus 1. The base-points of \( f_{a,b} \) are

\[
(1 : 0 : 0), \quad (0 : 1 : 0), \quad (-1 : \alpha : 1).
\]

Any \( f_{a,b} \) preserves a rational fibration (the fibration \( y = \text{constant} \) in the affine chart \( z = 1 \)). Each element of the family \( \{f_{a,b}\} \) has first dynamical degree 1 hence topological entropy zero \((1,1)\); more precisely one has ([8, Example 4.3])

\[
        \mu(f_{a,b}) = \frac{1}{2}
\]

so \( f_{a,b} \) is not conjugate to an automorphism (Theorem 1.1). The centralizer of \( f_{a,b} \) is isomorphic to \( \mathbb{Z} \) (see [12, Theorem 1.6]). The idea of the proof is the following: the point \( p = (1 : \alpha : 1) \) is blown-up onto a fiber of the fibration \( y = \text{constant} \). Let \( \psi \) be an element of

\[
        \text{Cent}(f_{a,b}) = \{ g \in \text{Bir}(\mathbb{P}_C^2) | g \circ f_{a,b} = f_{a,b} \circ g \};
\]

since \( \psi \) blows down a finite number of curves there exists a positive integer \( k \) (chosen minimal) such that \( f_{a,b}^k(p) \) is not blown down by \( \psi \). Replacing \( \psi \) by \( \tilde{\psi} = \psi \circ f_{a,b}^{-1} \) one gets that \( \tilde{\psi}(p) \) is an indeterminacy point of \( f_{a,b} \). In other words \( \tilde{\psi} \) permutes the indeterminacy points of \( f_{a,b} \). A more precise study allows us to establish that \( p \) is fixed by \( \tilde{\psi} \). The pair \( (\alpha, \beta) \) being in \( \Omega \), the closure of the negative orbit of \( p \) under the action of \( f_{a,b} \) is ZARISKI dense; since \( \tilde{\psi} \) fixes any element of the orbit of \( p \) one obtains \( \tilde{\psi} = \text{id} \).

Let us recall that if \( \psi \) is an automorphism on a compact complex manifold \( M \), the FATOU set \( \mathcal{F}(\psi) \) of \( \psi \) is the set of points that have a neighborhood \( \mathcal{V} \) such that \( \{ f_{\psi}^n | n \in \mathbb{N} \} \) is a normal family. Set

\[
        \mathcal{G}(\mathcal{U}) = \{ \phi : \mathcal{U} \to \overline{\mathcal{U}} | \phi = \lim_{n \to +\infty} \psi^n \};
\]

we say that \( \mathcal{U} \) is a rotation domain if \( \mathcal{G}(\mathcal{U}) \) is a subgroup of \( \text{Aut}(\mathcal{U}) \). An equivalent definition is the following: a component \( \mathcal{U} \) of \( \mathcal{F}(\psi) \) which is invariant by \( \psi \) is a rotation domain if \( \psi, \mathcal{U} \) is conjugate to a linear rotation. If \( \mathcal{U} \) is a rotation domain, \( \mathcal{G}(\mathcal{U}) \) is a compact Lie group, and the action of \( \mathcal{G}(\mathcal{U}) \) on \( \mathcal{U} \) is analytic real. Since \( \mathcal{G}(\mathcal{U}) \) is compact, infinite, abelian Lie group, the connected component of the identity of \( \mathcal{G}(\mathcal{U}) \) is a torus of dimension \( 0 \leq d \leq \dim_\mathbb{C} M \). The integer \( d \) is the rank of the rotation domain. The rank coincides with the dimension of the closure of a generic orbit of a point in \( \mathcal{U} \).

We can also see \( f_{a,b} \) on \( \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1 \) and that is what we will do in the sequel (since all the computations of [12] have been done in an affine chart they may all be carried on \( \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1 \); the sets \( \mathbb{P}_\mathbb{C}^1 \times S_\mathbb{C}^1 \) are invariant. In [12] we show that there are two rotation domains for \( f_{a,b} \), one of
rank 1, and the other one of rank 2\(^2\); in the first case we give here a more precise statement than in [12]:

**Theorem 1.3.** — Assume that \((\alpha, \beta)\) belongs to \(\Omega\).

There exists a strictly positive real number \(r\) such that \(f_{\alpha, \beta}\) is conjugate to \((\alpha x, \beta y)\) on \(\mathbb{P}_C^1 \times \mathbb{D}(0, r)\) where \(\mathbb{D}(0, r)\) denotes the disk centered at the origin with radius \(r\).

There exists a strictly positive real number \(\tilde{r}\) such that \(f_{\alpha, \beta}^2\) is conjugate to \(\left(\frac{x}{\beta}, \frac{z}{\beta^2}\right)\) on \(\mathbb{P}_C^1 \times \mathbb{D}(0, \tilde{r})\).

**Remark 1.4.** — The point \((\alpha - 1, 0)\) is also a fixed point of \(f_{\alpha, \beta}\) where the behavior of \(f_{\alpha, \beta}\) is the same as near \((0, 0)\).

**Proof.** — The first assertion is proved in [12].

Let us consider the map \(\psi(x, z) = \left(\frac{a(z)x + b(z)}{c(z)x + 1}, z\right)\). The equation

\[
\psi^{-1} f_{\alpha, \beta}^2 \psi = \left(\frac{x}{\beta}, \frac{z}{\beta^2}\right)
\]

yields

\[
\beta a(\beta^{-2}z)c(z) + \beta a(\beta^{-2}z)a(z) - c(\beta^{-2}z)a(z) + \alpha a(\beta^{-2}z)a(z) + az(\alpha^2 a(\beta^{-2}z)c(z) - \alpha c(\beta^{-2}z)c(z) - c(\beta^{-2}z)c(z) - c(\beta^{-2}z)a(z)) = 0,
\]

\[
(\alpha + 1)z + b(z) - \beta b(\beta^{-2}z) - \alpha^2 zb(\beta^{-2}z) + zb(z) - (\alpha + \beta) b(\beta^{-2}z) b(z) = 0
\]

and

\[
(\alpha + 1)z + b(z) - \beta b(\beta^{-2}z) - \alpha^2 zb(\beta^{-2}z) + zb(z) - (\alpha + \beta) b(\beta^{-2}z) b(z) = 0
\]

Let us set

\[
a(z) = \sum_{i \geq 0} a_i z^i, \quad b(z) = \sum_{i \geq 0} b_i z^i, \quad c(z) = \sum_{i \geq 0} c_i z^i.
\]

We easily get \(a_0 = 1 - \beta, \quad b_0 = 0\) and \(c_0 = \alpha + \beta\).

Relation (1.4) implies that

\[
b_1 = \frac{\beta(1 + \alpha)}{1 - \beta} \quad \& \quad \beta b_\nu \left(1 - \beta^{1-2\nu}\right) + F_i(b_i | i < \nu) = 0 \quad \forall \nu > 1,
\]

(1.3) yields

\[
a_\nu \left(\beta^{1-2\nu} - \beta\right) + b_\nu \left((\alpha + \beta)a_0 \left(1 + \beta^{1-2\nu}\right) + c_0 \left(\beta^{2-2\nu} - 1\right)\right) + G_i(a_i, b_i, c_i | i < \nu) = 0
\]

and (1.2) to

\[
c_\nu a_0 \left(\beta - \beta^{-2\nu}\right) + c_\nu \left((\alpha + \beta)a_0 \left(1 + \beta^{-2\nu}\right) + c_0 \left(\beta^{1-2\nu} - 1\right)\right) + H_i(a_i, b_i, c_i | i < \nu) = 0
\]

2. There already exists an example of automorphism of positive entropy with rotation domains of rank 1 and 2 (see [5]), but \(f_{\alpha, \beta}\) is not conjugate to an automorphism on a rational surface.
where the $F_i$’s, $G_i$’s and $H_i$’s denote universal polynomials; this allows to compute $b_\nu$, $a_\nu$ and $c_\nu$. Thus we get a formal conjugacy of $f_{\alpha,\beta}^2$ to its linear part. Since this linear part satisfies a Rüssmann condition (see [19, Theorem 2.1] condition (2)), according to [19, Theorem 2.1] any formal linearizing map conjugating $f_{\alpha,\beta}^2$ to its linear part is convergent on a polydisc.

**2. About SL(2; $\mathbb{C}$)-cocycles**

A (one-frequency, analytic) quasiperiodic SL(2; $\mathbb{C}$)-cocycle is a pair $(A, \beta)$, where $\beta \in \mathbb{R}$ and $A : S^1 \rightarrow \text{SL}(2; \mathbb{C})$ is analytic, and defines a linear skew product acting on $\mathbb{C}^2 \times S^1$ by

$$(x, y) \mapsto (A(y) \cdot x, \beta y).$$

The iterates of the cocyle are given by $(A_n, n\beta)$ where $A_n$ is given by

$$A_n(y) = A(\beta^{n-1}y) \cdots A(y) \quad n \geq 1, \quad A_0(y) = \text{id}, \quad A_{-n}(y) = A_n(\beta^{-n}y)^{-1}.$$

The Lyapunov exponent $L(A, \beta)$ of a quasiperiodic SL(2; $\mathbb{C}$)-cocycle $(A, \beta)$ is given by

$$\lim_{n \to +\infty} \frac{1}{n} \int_{S^1} \ln ||A_n(y)|| \, dy.$$

A quasiperiodic SL(2; $\mathbb{C}$)-cocycle $(A, \beta)$ is uniformly hyperbolic if there exist analytic functions $u, s : S^1 \rightarrow \mathbb{P}^2_C$, called the unstable and stable directions, and $n \geq 1$ such that for any $y \in S^1$,

$$A(y) \cdot u(y) = u(\beta y) \quad A(y) \cdot s(y) = s(\beta y),$$

and for any unit vector $x \in s(y)$ (resp. $x \in u(y)$) we have $||A_n(y) \cdot x|| < 1$ (resp. $||A_n(y) \cdot x|| > 1$). The unstable and stable directions are uniquely characterized by those properties, and clearly $u(y) \neq s(y)$ for any $y \in S^1$. If $(A, \beta)$ is uniformly hyperbolic, then $L(A, \beta) > 0$. Let us denote by

$$\mathcal{UH} \subset C^0(\text{SL}(2; \mathbb{C}), S^1)$$

the set of $A$ such that $(A, \beta)$ is uniformly hyperbolic. Uniform hyperbolicity is a stable property: $\mathcal{UH}$ is open, and $A \mapsto L(A, \beta)$ is analytic over $\mathcal{UH}$ (regularity properties of the Lyapunov exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables).

**Definition.** — Let $(A, \beta)$ be a quasiperiodic SL(2; $\mathbb{C}$)-cocycle. If $L(A, \beta) > 0$ but $(A, \beta) \notin \mathcal{UH}$, then $(A, \beta)$ is nonuniformly hyperbolic.

If $A \in C^0(\text{SL}(2; \mathbb{C}), S^1)$ admits a holomorphic extension to $|\text{Im} \, y| < \delta$ then for $|\epsilon| < \delta$ we can define $A_\epsilon \in C^0(\text{SL}(2; \mathbb{C}), S^1)$ by

$$A_\epsilon(y) = A(y + \epsilon).$$

The Lyapunov exponent $L(A_\epsilon, \beta)$ is a convex function of $\epsilon$. We can thus introduce the following notion. The acceleration of a quasiperiodic SL(2; $\mathbb{C}$)-cocycle $(A, \beta)$ is given by

$$\omega(A, \beta) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi \epsilon} (L(A_\epsilon, \beta) - L(A, \beta)).$$
Remark 2.1. — The convexity of the LYAPUNOV exponent in function of $\varepsilon$ implies that the acceleration is decreasing.

Since the LYAPUNOV exponent is a convex and continuous function the acceleration is an upper semi-continuous function in $\mathbb{R} \setminus \mathbb{Q} \times C^0(SL(2; \mathbb{C}), S^1_0)$. The acceleration is quantized:

Theorem 2.2 ([1]). — If $(A, \beta)$ is a $SL(2; \mathbb{C})$-cocycle with $\beta \in \mathbb{R} \setminus \mathbb{Q}$, then $\omega(A, \beta)$ is always an integer.

A direct consequence is the following:

Corollary 2.3. — The function $\varepsilon \mapsto L(A_\varepsilon, \beta)$ is a piecewise affine function of $\varepsilon$.

It is thus natural to introduce the notion of regularity. A cocycle

$$(A, \beta) \in C^0(SL(2; \mathbb{C}), S^1_0) \times \mathbb{R} \setminus \mathbb{Q}$$

is regular if $L(A_\varepsilon, \beta)$ is affine for $\varepsilon$ in a neighborhood of 0. In other words $(A, \beta)$ is regular if the equality

$$L(A_\varepsilon, \beta) - L(A, \beta) = 2\pi \omega(A, \beta)$$

holds for all $\varepsilon$ small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near $(A, \beta)$. It is an open condition in $C^0(SL(2; \mathbb{C}), S^1_0) \times \mathbb{R} \setminus \mathbb{Q}$. The following statement gives a characterization of the dynamics of regular cocycles with positive LYAPUNOV exponent:

Theorem 2.4 ([1]). — Let $(A, \beta)$ be a $SL(2; \mathbb{C})$-cocycle with $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $L(A, \beta) > 0$; then $(A, \beta)$ is regular if and only if $(A, \beta)$ is $UH$.

One striking consequence is the following:

Corollary 2.5 ([1]). — For any $(A, \beta)$ in $C^0(SL(2; \mathbb{C}), S^1_0) \times \mathbb{R} \setminus \mathbb{Q}$ there exists $\varepsilon_0$ such that

- $L(A_\varepsilon, \beta) = 0$ (and $\omega(A, \beta) = 0$) for every $0 < \varepsilon < \varepsilon_0$,
- or $(A_\varepsilon, \beta) \in UH$ for every $0 < \varepsilon < \varepsilon_0$.

Remark 2.6. — Let us mention that there is a link between $SL(2; \mathbb{C})$-cocycles and SCHRÖDINGER operators (see [1] for more details).

3. Proof of Theorem A

Suppose that $\rho \neq 1$, and let us consider the cocycle $(B^{\alpha, \rho}, \beta^{1/2})$ where

$$B^{\alpha, \rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix}.$$ 

Since

$$\left(\frac{\alpha x + y}{x + 1}, \beta y\right)(x, y^2) = (x, y^2) \left(\frac{\alpha x + y^2}{x + 1}, \beta^{1/2} \right),$$
the cocycles \((A^{\alpha,\rho},\beta)\) and \((B^{\alpha,\rho},\beta^{1/2})\) have the same behavior. Using two different arguments of monodromy (one for \(\rho < 1\), and the other one for \(\rho > 1\)) we see that there is a continuous determination for the square root of \(\det B^{\alpha,\rho}(y) = \alpha - y^2\). Let us set

\[
\tilde{B}^{\alpha,\rho}(y) = \frac{1}{\sqrt{\alpha - y^2}} B^{\alpha,\rho}(y) \in \text{SL}(2; \mathbb{C})
\]

that is thus defined on two different domains of analyticity. According to Theorem 1.3 one has \(L(\tilde{B}^{\alpha,\rho},\beta^{1/2}) = 0\) when \(\rho\) is close to both 0 and \(\infty\).

Assume that \(L(\tilde{B}^{\alpha,\rho},\beta^{1/2})\) is non constant. When \(\tilde{B}^{\alpha,\rho}\) is holomorphic, so in particular when \(\rho < 1\) and \(\rho > 1\), the acceleration is decreasing (Remark 2.1); furthermore the acceleration is positive for \(\rho < 1\) and negative for \(\rho > 1\) (because \(L\) is continuous). Theorem 2.2 thus implies

\[
\omega(\tilde{B}^{\alpha,1+},\beta^{1/2}) - \omega(\tilde{B}^{\alpha,1-},\beta^{1/2}) \leq -2.
\]

By definition of \(\tilde{B}^{\alpha,\rho}\) we have

\[
L(\tilde{B}^{\alpha,\rho}(y),\beta^{1/2}) = L(B^{\alpha,\rho}(y),\beta^{1/2}) - \int_{\mathbb{P}^1} \ln \sqrt{\alpha - y^2} \, dy
\]

\[
= L(B^{\alpha,\rho}(y),\beta^{1/2}) - \max(0, \ln \rho).
\]

Even though \((B^{\alpha,\rho}(y),\beta^{1/2})\) is not a \(\text{SL}(2; \mathbb{C})\)-cocycle, the LYAPUNOV exponent is still a convex function of \(\log \rho\) (see for example [2]). The jump of \(\omega(B^{\alpha,\rho}(y),\beta^{1/2})\) is thus \(\geq 0\), and the jump for the second term of the right member is \(-1\). Therefore the jump of \(L(\tilde{B}^{\alpha,\rho}(y),\beta^{1/2})\) is \(\geq -1\): contradiction.

References


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