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JONQUIÈRES MAPS AND $\mathrm{SL}(2;\mathbb{C})$-COCYCLES

by

Julie DÉSERTI

Abstract. — We start the study of the family of birational maps $(f_{\alpha,\beta})$ of $\mathbb{P}^2_{\mathbb{C}}$ in [12]. For "(α,β) well chosen" of modulus 1 the centraliser of $f_{\alpha,\beta}$ is trivial, the topological entropy of $f_{\alpha,\beta}$ is 0, there exist two domains of linearisation : in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. On $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ any $f_{\alpha,\beta}$ can be viewed as a cocyle ; using recent results about $\mathrm{SL}(2;\mathbb{C})$-cocycles ([1]) we determine the LYAPUNOV exponent of the cocyle associated to $f_{\alpha,\beta}$.

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Introduction

In this article we deal with a family of birational maps $(f_{\alpha,\beta})$ given by

$$f_{\alpha,\beta}\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}} \qquad (x:y:z) \dashrightarrow \big((\alpha x+y)z : \beta y(x+z) : z(x+z)\big)$$

where α, β denote two complex numbers with modulus 1, case where we know almost nothing about the dynamics. Let us consider the set Ω of pairs of complex numbers of modulus 1 that satisfy diophantine condition. The family $(f_{\alpha,\beta})$ satisfies the following properties ([12]):

- for $(\alpha,\beta)\in\Omega$ the centraliser of $f_{\alpha,\beta}$, that is the set of birational maps of $\mathbb{P}^2_{\mathbb{C}}$ that commutes with $f_{\alpha,\beta}$, is isomorphic to $\mathbb{Z}$;
- the topological entropy of $f_{\alpha,\beta}$ is 0;
- rotation domains of ranks 1 and 2 coexist: there is a domain of linearisation where the orbit of a generic point under $f_{\alpha,\beta}$ is a torus, and there is an other domain of linearisation where the orbit of a generic point under $f^2_{\alpha,\beta}$ is a circle.

We can also see $f_{\alpha,\beta}$ on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ (since all the computations of [12] have been done in an affine chart they may all be carried on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$); the sets $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{S}^1_{\rho}$, where $\mathbb{S}^1_{\rho}=\{y\in\mathbb{C}\,|\,|y|=\rho\}$, are invariant.

Let us define $A_n^{\alpha,\rho}\colon \mathbb{S}^1_\rho \to \mathrm{M}(2;\mathbb{C})$ given in terms of $A^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y \\ 1 & 1 \end{bmatrix}$ by

$$A_n^{\alpha,\rho}(\cdot) = A^{\alpha,\rho}(\beta^n\,\cdot)A^{\alpha,\rho}(\beta^{n-1}\,\cdot)\dots A^{\alpha,\rho}(\beta\,\cdot)A^{\alpha,\rho}(\cdot).$$

To compute $f^n_{\alpha,\beta}(x,y)$ is equivalent to compute $A_n^{\alpha,\rho}(y)$ as soon as $f^k_{\alpha,\beta}(x,y) \neq (-1,\alpha)$ for any $1 \leq k \leq n$.

Using [1] we are able to determine the LYAPUNOV exponent of the cocycle $(A^{\alpha,\rho},\beta)$:

Theorem A. — *The* LYAPUNOV *exponent of* $(A^{\alpha,\rho},\beta)$ *is*
- *positive as soon as* $\rho > 1$*;*
- *zero as soon as* $\rho \leq 1$*.*

More precisely $f_{\alpha,\beta}$ *is semi conjugate to* $\left(\frac{\alpha x+y^2}{x+1}, \beta^{1/2}y\right)$ *and the* LYAPUNOV *exponent of the cocycle* $\left(B^{\alpha,\rho},\beta^{1/2}\right)$*, where*

$$B^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix},$$

is equal to $\max(0,\ln\rho)$*.*

In the next section we introduce the family $(f_{\alpha,\beta})$ and its properties (§1). Then we deal with the recent works of AVILA on $\mathrm{SL}(2;\mathbb{C})$-cocyles. In the last section we give the proof of Theorem A (*see* §2). Let us explain the sketch of it. We associate to $\left(B^{\alpha,\rho},\beta^{1/2}\right)$ a cocycle $\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right)$ that belongs to $\mathrm{SL}(2;\mathbb{C})$. We first determine

$$\lim_{\rho\to 0} L\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right),$$

and then

$$\lim_{\rho\to +\infty} L\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right)$$

where $L(C,\gamma)$ denotes the LYAPUNOV exponent of the $\mathrm{SL}(2;\mathbb{C})$-cocyle (C,γ). In both cases, we get 0. Using [1, Theorem 5] we obtain that $L\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right)$ vanishes everywhere; it allows us to determine $L\left(A^{\alpha,\rho},\beta\right)$ since

$$L\left(B^{\alpha,\rho}(y),\beta^{1/2}\right) = L\left(\widetilde{B}^{\alpha,\rho}(y),\beta^{1/2}\right) + \max(0,\ln\rho),$$

and since $\left(A^{\alpha,\rho},\beta\right)$ and $\left(\beta^{1/2},B^{\alpha,\rho}\right)$ are conjugate.

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1. Some properties of the family $(f_{\alpha,\beta})$

A ***rational map*** ϕ from $\mathbb{P}^2_{\mathbb{C}}$ into itself is a map of the form

$$(x:y:z) \dashrightarrow \big(\phi_0(x,y,z):\phi_1(x,y,z):\phi_2(x,y,z)\big),$$

where the ϕ_i's are some homogeneous polynomials of the same degree without common factor; ϕ is ***birational*** if it admits an inverse of the same type. We will denote by $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ the group of birational maps of $\mathbb{P}^2_{\mathbb{C}}$, also called the **CREMONA *group***. The ***degree*** of ϕ, denoted $\deg\phi$, is the degree of the ϕ_i's. The degree is not a birational invariant: $\deg\psi\phi\psi^{-1} \neq \deg\phi$ for generic birational maps ϕ and ψ. The ***first dynamical degree*** of ϕ given by

$$\lambda(\phi) = \lim_{n\to+\infty} \big(\deg\phi^n\big)^{1/n},$$

is a birational invariant; it is strongly related to the topological entropy $h_{\mathrm{top}}(\phi)$ of ϕ (*see* [17, 20])

$$h_{\mathrm{top}}(\phi) \leq \log\lambda(\phi) \tag{1.1}$$

Any birational map ϕ admits a resolution

$$\begin{tikzcd} & \mathrm{S} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2_{\mathbb{C}} \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

where $\pi_1, \pi_2 : \mathrm{S} \to \mathbb{P}^2_{\mathbb{C}}$ are sequences of blow-ups (*see* [3] for example). The resolution is ***minimal*** if and only if no (-1)-curve of S is contracted by both π_1 and π_2. The ***base-points*** of ϕ are the points blown-up in π_1, which can be points of $\mathbb{P}^2_{\mathbb{C}}$ or infinitely near points. We denote by $\mathfrak{b}(\phi)$ the number of such points, which is also equal to the difference of the ranks of $\mathrm{Pic}(\mathrm{S})$ and $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}})$, and thus equals to $\mathfrak{b}(\phi^{-1})$. The ***dynamical number of base-points of*** ϕ introduced in [8] is by definition

$$\mu(\phi) = \lim_{n\to+\infty} \frac{\mathfrak{b}(\phi^n)}{n};$$

it is a real positive number that satisfies $\mu(\phi^n) = |n\,\mu(\phi)|$ for any $n \in \mathbb{Z}$, $\mu(\psi\phi\psi^{-1}) = \mu(\phi)$, and allows us to give a characterization of birational maps conjugate to automorphisms:

Theorem 1.1 **([8])**. — *Let* S *be a smooth projective surface; the birational map* $\phi \in \mathrm{Bir}(\mathrm{S})$ *is conjugate to an automorphism of a smooth projective surface if and only if* $\mu(\phi) = 0$.

The behavior of $\phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is strongly related to the behavior of $\big(\deg\phi^n\big)_{n\in\mathbb{N}}$ (*see* [16, 15, 8]); up to birational conjugacy exactly one of the following holds:

1. the sequence $\big(\deg\phi^n\big)_{n\in\mathbb{N}}$ is bounded and either ϕ is of finite order, or ϕ is an automorphism of $\mathbb{P}^2_{\mathbb{C}}$;
2. there exists an integer k such that
$$\lim_{n\to+\infty} \frac{\deg\phi^n}{n} = k^2\,\frac{\mu(\phi)}{2}$$
and ϕ is not an automorphism;

3. there exists an integer $k \geq 3$ such that
$$\lim_{n\to+\infty} \frac{\deg \phi^n}{n^2} = k^2 \frac{\kappa(\phi)}{9}$$
where $\kappa(\phi) \in \mathbb{Q}$ is a birational invariant, and ϕ is an automorphism;
4. the sequence $\left(\deg \phi^n\right)_{n\in\mathbb{N}}$ grows exponentially (*see* [15] for more precise dynamical properties).

In the first three cases $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$. In case 2. (resp. 3.) the map ϕ preserves a unique fibration which is rational (resp. elliptic).

In case 1. (resp. 2., resp. 3, resp. 4) we say that ϕ is ***elliptic*** (resp. a **JONQUIÈRES *twist***, resp. an **HALPHEN *twist***, resp. ***hyperbolic***).

Let us give some examples. Let
$$\phi(x,y) = \left(\frac{a(y)x+b(y)}{c(y)x+d(y)}, \frac{\alpha y+\beta}{\gamma y+\delta}\right)$$
be an element of the **JONQUIÈRES *group*** $\mathrm{PGL}(2;\mathbb{C}(y)) \rtimes \mathrm{PGL}(2;\mathbb{C})$; either ϕ is elliptic (for instance $\phi\colon (x:y:z) \dashrightarrow (yz:xz:xy)$), or ϕ is a JONQUIÈRES twist (for example $\phi\colon (x:y:z) \dashrightarrow (xz:xy:z^2)$ for which the unique invariant fibration is $y/z =$ constant). The map
$$\phi\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}} \qquad (x:y:z) \dashrightarrow \big((2y+z)(y+z) : x(2y-z) : 2z(y+z)\big)$$
is an HALPHEN twist ([15, Proposition 9.5]). HÉNON automorphisms give by homogeneization examples of hyperbolic maps.

Clearly elliptic birational maps have a poor dynamical behavior contrary to hyperbolic ones. The study of automorphisms of positive entropy is strongly related with birational maps of $\mathbb{P}^2_{\mathbb{C}}$:

Theorem 1.2 **([9])**. — *Let* S *be a compact complex surface that carries an automorphism* ϕ *of positive topological entropy.*

- *Either the* KODAIRA *dimension of* S *is zero and* ϕ *is conjugate to an automorphism on the unique minimal model of* S *that necessarily is a torus, or a* $K3$ *surface or an* ENRIQUES *surface;*
- *or the surface* S *is a non-minimal rational one, isomorphic to* $\mathbb{P}^2_{\mathbb{C}}$ *blown up at* n *points,* $n \geq 10$*, and* ϕ *is conjugate to a birational map of* $\mathbb{P}^2_{\mathbb{C}}$*.*

This yields many examples of hyperbolic birational maps for which we can establish a lot of dynamical properties ([18, 4, 5, 6, 7, 14, 13]).

Another way to measure chaos is to look at the size of centralisers. Let us give two examples. The polynomial automorphisms of $\mathbb{C}^2$ having rich dynamics are HÉNON maps; furthermore a polynomial automorphism of $\mathbb{C}^2$ is a HÉNON one if and only if its centraliser is countable. Let us now consider rational maps on $\mathbb{S}^1$; if the centraliser of such maps is not trivial [(1)], then the JULIA set is "special". The centraliser of an elliptic birational map of infinite order is uncountable ([8]). The centralisers of HALPHEN twists are described in [16]. The centraliser of an hyperbolic map is countable ([10]). In [11] we end the story by studying centralisers of JONQUIÈRES twists. If the

1. The centraliser of a map ϕ is trivial if it coincides with the iterates of ϕ.

fibration is fiberwise invariant, then the centraliser is uncountable ; but if it isn't, then generically the centraliser is isomorphic to $\mathbb{Z}$. We don't know a lot about dynamics of these maps, in this article we will thus focus on a family of such maps. We consider the JONQUIÈRES maps

$$f_{\alpha,\beta}\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}} \qquad (x:y:z) \dashrightarrow \big((\alpha x+y)z : \beta y(x+z) : z(x+z)\big)$$

where α, β denote two complex numbers with modulus 1. The base-points of $f_{\alpha,\beta}$ are

$$(1:0:0), \quad (0:1:0), \quad (-1:\alpha:1).$$

Any $f_{\alpha,\beta}$ preserves a rational fibration (the fibration $y=$ constant in the affine chart $z=1$). Each element of the family $(f_{\alpha,\beta})$ has first dynamical degree 1 hence topological entropy zero (1.1); more precisely one has ([**8**, Example 4.3])

$$\mu(f_{\alpha,\beta}) = \frac{1}{2}$$

so $f_{\alpha,\beta}$ is not conjugate to an automorphism (Theorem 1.1). The centralizer of $f_{\alpha,\beta}$ is isomorphic to $\mathbb{Z}$ (*see* [**12**, Theorem 1.6]). The idea of the proof is the following: the point $p=(1:\alpha:1)$ is blown-up onto a fiber of the fibration $y=$ constant. Let ψ be an element of

$$\mathrm{Cent}(f_{\alpha,\beta}) = \big\{g \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}}) \,|\, g\circ f_{\alpha,\beta} = f_{\alpha,\beta}\circ g\big\};$$

since ψ blows down a finite number of curves there exists a positive integer k (chosen minimal) such that $f^k_{\alpha,\beta}(p)$ is not blown down by ψ. Replacing ψ by $\widetilde{\psi}=\psi f^{k-1}_{\alpha,\beta}$ one gets that $\widetilde{\psi}(p)$ is an indeterminacy point of $f_{\alpha,\beta}$. In other words $\widetilde{\psi}$ permutes the indeterminacy points of $f_{\alpha,\beta}$. A more precise study allows us to establish that p is fixed by $\widetilde{\psi}$. The pair (α,β) being in Ω, the closure of the negative orbit of p under the action of $f_{\alpha,\beta}$ is ZARISKI dense; since $\widetilde{\psi}$ fixes any element of the orbit of p one obtains $\widetilde{\psi}=\mathrm{id}$.

Let us recall that if ψ is an automorphism on a compact complex manifold M, the **FATOU *set*** $\mathcal{F}(\psi)$ of ψ is the set of points that have a neighborhood $\mathcal{V}$ such that $\big\{f^n_{|\mathcal{V}} \,|\, n\in\mathbb{N}\big\}$ is a normal family. Set

$$\mathcal{G}(\mathcal{U}) = \big\{\phi\colon \mathcal{U}\to\overline{\mathcal{U}} \,|\, \phi = \lim_{n_j\to+\infty}\psi^{n_j}\big\};$$

we say that $\mathcal{U}$ is a ***rotation domain*** if $\mathcal{G}(\mathcal{U})$ is a subgroup of $\mathrm{Aut}(\mathcal{U})$. An equivalent definition is the following: a component $\mathcal{U}$ of $\mathcal{F}(\psi)$ which is invariant by ψ is a rotation domain if $\psi_{|\mathcal{U}}$ is conjugate to a linear rotation. If $\mathcal{U}$ is a rotation domain, $\mathcal{G}(\mathcal{U})$ is a compact LIE group, and the action of $\mathcal{G}(\mathcal{U})$ on $\mathcal{U}$ is analytic real. Since $\mathcal{G}(\mathcal{U})$ is a compact, infinite, abelian LIE group, the connected component of the identity of $\mathcal{G}(\mathcal{U})$ is a torus of dimension $0\le d\le \dim_{\mathbb{C}}\mathrm{M}$. The integer d is the ***rank of the rotation domain***. The rank coincides with the dimension of the closure of a generic orbit of a point in $\mathcal{U}$.

We can also see $f_{\alpha,\beta}$ on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ and that is what we will do in the sequel $\big($since all the computations of [**12**] have been done in an affine chart they may all be carried on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\big)$; the sets $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{S}^1_{\rho}$ are invariant. In [**12**] we show that there are two rotation domains for $f^2_{\alpha,\beta}$, one of

rank 1, and the other one of rank 2 [(2)]; in the first case we give here a more precise statement than in [12]:

Theorem 1.3. — *Assume that* (α,β) *belongs to* Ω.

There exists a strictly positive real number r *such that* $f_{\alpha,\beta}$ *is conjugate to* $(\alpha x,\beta y)$ *on* $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{D}(0,r)$ *where* $\mathbb{D}(0,r)$ *denotes the disk centered at the origin with radius* r.

There exists a strictly positive real number $\widetilde{r}$ *such that* $f^2_{\alpha,\beta}$ *is conjugate to* $\left(\frac{x}{\beta},\frac{z}{\beta^2}\right)$ *on* $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{D}(0,\widetilde{r})$.

Remark 1.4. — The point $(\alpha-1,0)$ is also a fixed point of $f_{\alpha,\beta}$ where the behavior of $f_{\alpha,\beta}$ is the same as near $(0,0)$.

Proof. — The first assertion is proved in [12].

Let us consider the map $\psi(x,z)=\left(\frac{a(z)x+b(z)}{c(z)x+1},z\right)$. The equation

$$\psi^{-1}f^2_{\alpha,\beta}\psi=\left(\frac{x}{\beta},\frac{z}{\beta^2}\right)$$

yields

$$\begin{aligned}&\beta a\big(\beta^{-2}z\big)c(z)+\beta a\big(\beta^{-2}z\big)a(z)-c\big(\beta^{-2}z\big)a(z)+\alpha a\big(\beta^{-2}z\big)a(z)\\&\quad+z\big(\alpha^2 a\big(\beta^{-2}z\big)c(z)-\alpha c\big(\beta^{-2}z\big)c(z)-c\big(\beta^{-2}z\big)c(z)-c\big(\beta^{-2}z\big)a(z)\big)=0,\end{aligned}\tag{1.2}$$

$$\begin{aligned}&\beta a\big(\beta^{-2}z\big)-\beta a(z)+z\big(\alpha^2 a\big(\beta^{-2}z\big)-\alpha\beta c(z)-\beta c(z)-\beta a(z)-\alpha c\big(\beta^{-2}z\big)-c\big(\beta^{-2}z\big)\big)\\&\quad+\beta(\alpha+\beta)a(z)b\big(\beta^{-2}z\big)+(\alpha+\beta)b(z)a\big(\beta^{-2}z\big)+\beta^2 b\big(\beta^{-2}z\big)c(z)-b(z)c\big(\beta^{-2}z\big)\\&\quad+z\big(\alpha^2\beta b\big(\beta^{-2}z\big)c(z)-b(z)c\big(\beta^{-2}z\big)\big)=0\end{aligned}\tag{1.3}$$

and

$$(\alpha+1)z+b(z)-\beta b\big(\beta^{-2}z\big)-\alpha^2 zb\big(\beta^{-2}z\big)+zb(z)-(\alpha+\beta)b\big(\beta^{-2}z\big)b(z)=0\tag{1.4}$$

Let us set

$$a(z)=\sum_{i\geq 0}a_iz^i,\qquad b(z)=\sum_{i\geq 0}b_iz^i,\qquad c(z)=\sum_{i\geq 0}c_iz^i.$$

We easily get $a_0=1-\beta$, $b_0=0$ and $c_0=\alpha+\beta$.

Relation (1.4) implies that

$$b_1=\frac{\beta(1+\alpha)}{1-\beta}\qquad\&\qquad\beta b_\nu\left(1-\beta^{1-2\nu}\right)+F_i(b_i\,|\,i<\nu)=0\quad\forall\nu>1,$$

(1.3) yields

$$a_\nu\left(\beta^{1-2\nu}-\beta\right)+b_\nu\left((\alpha+\beta)a_0\left(1+\beta^{1-2\nu}\right)+c_0\left(\beta^{2-2\nu}-1\right)\right)+G_i(a_i,b_i,c_i\,|\,i<\nu)=0$$

and (1.2) to

$$c_\nu a_0\left(\beta-\beta^{-2\nu}\right)+a_\nu\left((\alpha+\beta)a_0\left(1+\beta^{-2\nu}\right)+c_0\left(\beta^{1-2\nu}-1\right)\right)+H_i(a_i,b_i,c_i\,|\,i<\nu)=0$$

2. There already exists an example of automorphism of positive entropy with rotation domains of rank 1 and 2 (*see* [5]), but $f_{\alpha,\beta}$ is not conjugate to an automorphism on a rational surface.

where the F_i's, G_i's and H_i's denote universal polynomials; this allows to compute b_ν, a_ν and c_ν. Thus we get a formal conjugacy of $f^2_{\alpha,\beta}$ to its linear part. Since this linear part satisfies a Rüssmann condition (*see* [19, Theorem 2.1] condition (2)), according to [19, Theorem 2.1] any formal linearizing map conjugating $f^2_{\alpha,\beta}$ to its linear part is convergent on a polydisc. □

2. About $\mathrm{SL}(2;\mathbb{C})$-cocycles

A (one-frequency, analytic) ***quasiperiodic*** $\mathrm{SL}(2;\mathbb{C})$***-cocycle*** is a pair (A,β), where $\beta\in\mathbb{R}$ and

$$A\colon \mathbb{S}^1_1\to \mathrm{SL}(2;\mathbb{C})$$

is analytic, and defines a linear skew product acting on $\mathbb{C}^2\times\mathbb{S}^1_1$ by

$$(x,y)\mapsto (A(y)\cdot x,\beta y).$$

The iterates of the cocyle are given by $(A_n,n\beta)$ where A_n is given by

$$A_n(y)=A\big(\beta^{n-1}y\big)\dots A(y)\quad n\geq 1,\quad A_0(y)=\mathrm{id},\quad A_{-n}(y)=A_n(\beta^{-n}y)^{-1}.$$

The **LYAPUNOV** ***exponent*** $L(A,\beta)$ of a quasiperiodic $\mathrm{SL}(2;\mathbb{C})$-cocycle (A,β) is given by

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\mathbb{S}^1_1}\ln||A_n(y)||\,\mathrm{d}y.$$

A quasiperiodic $\mathrm{SL}(2;\mathbb{C})$-cocycle (A,β) is ***uniformly hyperbolic*** if there exist analytic functions

$$u,\,s\colon \mathbb{S}^1_1\to\mathbb{P}^2_{\mathbb{C}},$$

called the ***unstable and stable directions***, and $n\geq 1$ such that for any $y\in\mathbb{S}^1_1$,

$$A(y)\cdot u(y)=u(\beta y)\qquad A(y)\cdot s(y)=s(\beta y),$$

and for any unit vector $x\in s(y)$ (resp. $x\in u(y)$) we have $||A_n(y)\cdot x||<1$ (resp. $||A_n(y)\cdot x||>1$). The unstable and stable directions are uniquely characterized by those properties, and clearly $u(y)\neq s(y)$ for any $y\in\mathbb{S}^1_1$. If (A,β) is uniformly hyperbolic, then $L(A,\beta)>0$. Let us denote by

$$\mathcal{UH}\subset C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$$

the set of A such that (A,β) is uniformly hyperbolic. Uniform hyperbolicity is a stable property: $\mathcal{UH}$ is open, and $A\mapsto L(A,\beta)$ is analytic over $\mathcal{UH}$ (regularity properties of the LYAPUNOV exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables).

Definition. — Let (A,β) be a quasiperiodic $\mathrm{SL}(2;\mathbb{C})$-cocycle. If $L(A,\beta)>0$ but $(A,\beta)\notin\mathcal{UH}$, then (A,β) is ***nonuniformly hyperbolic***.

If $A\in C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$ admits a holomorphic extension to $|\mathrm{Im}\,y|<\delta$ then for $|\varepsilon|<\delta$ we can define $A_\varepsilon\in C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$ by

$$A_\varepsilon(y)=A(y+\mathbf{i}\varepsilon).$$

The LYAPUNOV exponent $L(A_\varepsilon,\beta)$ is a convex function of ε. We can thus introduce the following notion. The ***acceleration*** of a quasiperiodic $\mathrm{SL}(2;\mathbb{C})$-cocyle (A,β) is given by

$$\omega(A,\beta)=\lim_{\varepsilon\to 0^+}\frac{1}{2\pi\varepsilon}\big(L(A_\varepsilon,\beta)-L(A,\beta)\big).$$

Remark 2.1. — The convexity of the LYAPUNOV exponent in function of ε implies that the acceleration is decreasing.

Since the LYAPUNOV exponent is a convex and continuous function the acceleration is an upper semi-continuous function in $\mathbb{R}\setminus\mathbb{Q}\times C^{\omega}\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$. The acceleration is quantized:

Theorem 2.2 **([1])**. — *If (A,β) is a $\mathrm{SL}(2;\mathbb{C})$-cocycle with $\beta\in\mathbb{R}\smallsetminus\mathbb{Q}$, then $\omega(A,\beta)$ is always an integer.*

A direct consequence is the following:

Corollary 2.3. — *The function $\varepsilon\mapsto L(A_\varepsilon,\beta)$ is a piecewise affine function of ε.*

It is thus natural to introduce the notion of regularity. A cocycle

$$(A,\beta)\in C^{\omega}\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)\times\mathbb{R}\setminus\mathbb{Q}$$

is ***regular*** if $L(A_\varepsilon,\beta)$ is affine for ε in a neighborhood of 0. In other words (A,β) is regular if the equality

$$L(A_\varepsilon,\beta)-L(A,\beta)=2\pi\varepsilon\omega(A,\beta)$$

holds for all ε small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near (A,β). It is an open condition in $C^{\omega}\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)\times\mathbb{R}\setminus\mathbb{Q}$. The following statement gives a characterization of the dynamics of regular cocycles with positive LYAPUNOV exponent:

Theorem 2.4 **([1])**. — *Let (A,β) be a $\mathrm{SL}(2;\mathbb{C})$-cocycle with $\beta\in\mathbb{R}\smallsetminus\mathbb{Q}$. Assume that $L(A,\beta)>0$; then (A,β) is regular if and only if (A,β) is $\mathcal{UH}$.*

One striking consequence is the following:

Corollary 2.5 **([1])**. — *For any (A,β) in $C^{\omega}\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)\times\mathbb{R}\setminus\mathbb{Q}$ there exists ε_0 such that*

- *$L(A_\varepsilon,\beta)=0$ (and $\omega(A,\beta)=0$) for every $0<\varepsilon<\varepsilon_0$,*
- *or $(A_\varepsilon,\beta)\in\mathcal{UH}$ for every $0<\varepsilon<\varepsilon_0$.*

Remark 2.6. — Let us mention that there is a link between $\mathrm{SL}(2;\mathbb{C})$-cocycles and SCHRÖDINGER operators (*see* [1] for more details).

3. Proof of Theorem A

Suppose that $\rho\neq 1$, and let us consider the cocycle $(B^{\alpha,\rho},\beta^{1/2})$ where

$$B^{\alpha,\rho}(y)=\begin{bmatrix}\alpha & y^2\\ 1 & 1\end{bmatrix}.$$

Since

$$\left(\frac{\alpha x+y}{x+1},\beta y\right)(x,y^2)=(x,y^2)\left(\frac{\alpha x+y^2}{x+1},\beta^{1/2}y\right)$$

the cocycles $(A^{\alpha,\rho},\beta)$ and $(B^{\alpha,\rho},\beta^{1/2})$ have the same behavior. Using two different arguments of monodromy (one for $\rho<1$, and the other one for $\rho>1$) we see that there is a continuous determination for the square root of $\det B^{\alpha,\rho}(y)=\alpha-y^2$. Let us set

$$\widetilde{B}^{\alpha,\rho}(y)=\frac{1}{\sqrt{\alpha-y^2}}B^{\alpha,\rho}(y)\in \mathrm{SL}(2;\mathbb{C})$$

that is thus defined on two different domains of analyticity.

According to Theorem 1.3 one has $L\big(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\big)=0$ when ρ is close to both 0 and ∞.

Assume that $L\big(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\big)$ is non constant. When $\widetilde{B}^{\alpha,\rho}$ is holomorphic, so in particular when $\rho<1$ and $\rho>1$, the acceleration is decreasing (Remark 2.1); furthermore the acceleration is positive for $\rho<1$ and negative for $\rho>1$ (because L is continuous). Theorem 2.2 thus implies

$$\omega\big(\widetilde{B}^{\alpha,1^+},\beta^{1/2}\big)-\omega\big(\widetilde{B}^{\alpha,1^-},\beta^{1/2}\big)\leq -2.$$

By definition of $\widetilde{B}^{\alpha,\rho}$ we have

$$\begin{aligned} L\big(\widetilde{B}^{\alpha,\rho}(y),\beta^{1/2}\big) &= L\big(B^{\alpha,\rho}(y),\beta^{1/2}\big)-\int_{\mathbb{S}^1_\rho}\ln\sqrt{\alpha-y^2}\,\mathrm{d}y \\ &= L\big(B^{\alpha,\rho}(y),\beta^{1/2}\big)-\max(0,\ln\rho). \end{aligned}$$

Even though $\big(B^{\alpha,\rho}(y),\beta^{1/2}\big)$ is not a $\mathrm{SL}(2;\mathbb{C})$-cocycle, the LYAPUNOV exponent is still a convex function of $\log\rho$ (*see for example* [**2**]). The jump of $\omega(B^{\alpha,\rho}(y),\beta^{1/2})$ is thus ≥ 0, and the jump for the second term of the right member is -1. Therefore the jump of $L\big(\widetilde{B}^{\alpha,\rho}(y),\beta^{1/2}\big)$ is ≥ -1: contradiction.

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