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► **To cite this version:**

Julie Déserti. REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE S^2 . Journal of Singularities, 2014. <hal-03000453>

HAL Id: hal-03000453

<https://hal.science/hal-03000453v1>

Submitted on 11 Nov 2020

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REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE \mathbb{S}^2

by

Julie Déserti

November 19, 2018

Abstract. — In [10] it is proved that any morphism from a subgroup of finite index of $\mathrm{SL}(n, \mathbb{Z})$ to the group of analytic diffeomorphisms of \mathbb{S}^2 has a finite image as soon as $n \geq 5$. The case $n = 4$ is also claimed to follow along the same arguments; in fact this is not straightforward and this case indeed needs a modification of the argument. In this paper we recall the strategy for $n \geq 5$ and then focus on the case $n = 4$.

2010 Mathematics Subject Classification. — 58D05, 58B25

1. Introduction

After the works of Margulis ([14, 19]) on the linear representations of lattices of simple, real Lie groups with \mathbb{R} -rank larger than 1, some authors, like Zimmer, suggest to study the actions of lattices on compact manifolds ([20, 21, 22, 23]). One of the main conjectures of this program is the following: let us consider a connected, simple, real Lie group G and let Γ be a lattice of G of \mathbb{R} -rank larger than 1. If there exists a morphism of infinite image from Γ to the group of diffeomorphisms of a compact manifold M , then the \mathbb{R} -rank of G is bounded by the dimension of M . There are a lot of contributions in that direction ([3, 4, 5, 7, 8, 9, 10, 11, 16, 17]). In this article we will focus on the embeddings of subgroups of finite index of $\mathrm{SL}(n, \mathbb{Z})$ into the group $\mathrm{Diff}^0(\mathbb{S}^2)$ of real analytic diffeomorphisms of \mathbb{S}^2 (see [10]).

The article is organized as follows. First of all we will recall the strategy of [10]: the study of the nilpotent subgroups of $\mathrm{Diff}^0(\mathbb{S}^2)$ implies that such subgroups are metabelian. But subgroups of finite index of $\mathrm{SL}(n, \mathbb{Z})$, for $n \geq 5$, contain nilpotent subgroups of length $n - 1$ of finite index which are not metabelian; as a consequence Ghys gets the following statement.

Theorem A ([10]). — *Let Γ be a subgroup of finite index of $\mathrm{SL}(n, \mathbb{Z})$. As soon as $n \geq 5$ there is no embedding of Γ into $\mathrm{Diff}^0(\mathbb{S}^2)$.*

To study nilpotent subgroups of $\mathrm{Diff}^0(\mathbb{S}^2)$ one has to study nilpotent subgroups of $\mathrm{Diff}_+^0(\mathbb{S}^1)$ (see §2) and then nilpotent subgroups of the group of formal diffeomorphisms of \mathbb{C}^2 (see §3). The last section is devoted to establish the following result.

Theorem B. — *Let Γ be a subgroup of finite index of $\mathrm{SL}(n, \mathbb{Z})$. As soon as $n \geq 4$ there is no embedding of Γ into $\mathrm{Diff}^{\omega}(\mathbb{S}^2)$.*

The proof relies on the characterization, up to isomorphism, of nilpotent subalgebras of length 3 of the algebra of formal vector fields of \mathbb{C}^2 which vanish at the origin.

Acknowledgements. — The author would like to thank Dominique Cerveau and Étienne Ghys for interesting discussions and advices.

2. Nilpotent subgroups of the group of analytic diffeomorphisms of \mathbb{S}^1

Let G be a group; let us set

$$G^{(0)} = G \quad \& \quad G^{(i)} = [G, G^{(i-1)}] \quad \forall i \geq 1.$$

The group G is *nilpotent* if there exists an integer n such that $G^{(n)} = \{\mathrm{id}\}$; the *length of nilpotence* of G is the smallest integer k such that $G^{(k)} = \{\mathrm{id}\}$.

Set

$$G_{(0)} = G \quad \& \quad G_{(i)} = [G_{(i-1)}, G_{(i-1)}] \quad \forall i \geq 1.$$

The group G is *solvable* if $G_{(n)} = \{\mathrm{id}\}$ for a certain n ; the *length of solvability* of G is the smallest integer k such that $G_{(k)} = \{\mathrm{id}\}$.

We say that the group G (resp. algebra \mathfrak{g}) is *metabelian* if $[G, G]$ (resp. $[\mathfrak{g}, \mathfrak{g}]$) is abelian.

Proposition 2.1 ([10]). — *Any nilpotent subgroup of $\mathrm{Diff}_+^{\omega}(\mathbb{S}^1)$ is abelian.*

Proof. — Let G be a nilpotent subgroup of $\mathrm{Diff}_+^{\omega}(\mathbb{S}^1)$. Assume that G is not abelian; it thus contains a Heisenberg group

$$\langle f, g, h \mid [f, g] = h, [f, h] = [g, h] = \mathrm{id} \rangle.$$

The application “rotation number“

$$\mathrm{Diff}_+^{\omega}(\mathbb{S}^1) \rightarrow \mathbb{R}/\mathbb{Z}, \quad \psi \mapsto \lim_{n \rightarrow +\infty} \frac{\Psi^n(x) - x}{n}$$

is not a morphism but its restriction to a solvable subgroup is ([1]). Thus the rotation number of h is zero and the set $\mathrm{Fix}(h)$ of fixed points of h is non-empty and finite. Considering some iterates of f and g instead of f and g one can assume that f and g fix any point of $\mathrm{Fix}(h)$. The set of fixed points of a non trivial element of $\langle f, g \rangle$ is finite and invariant by h so the action of $\langle f, g \rangle$ is free ⁽¹⁾ on each component of $\mathbb{S}^1 \setminus \mathrm{Fix}(h)$. But the action of a free group on \mathbb{R} is abelian: contradiction. \square

1. The stabilizer of every point is trivial, *i.e.* the action of a non trivial element of $\langle f, g \rangle$ has no fixed point.

3. Nilpotent subgroups of the group of formal diffeomorphisms of \mathbb{C}^2

Let us denote $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ the group of formal diffeomorphisms of \mathbb{C}^2 , *i.e.* the formal completion of the group of germs of holomorphic diffeomorphisms at 0. For any i let Diff_i be the quotient of $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ by the normal subgroups of formal diffeomorphisms tangent to the identity with multiplicity i ; it can be viewed as the set of jets of diffeomorphisms at order i with the law of composition with truncation at order i . Note that Diff_i is a complex linear algebraic group. One can see $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ as the projective limit of the Diff_i 's: $\widehat{\text{Diff}}(\mathbb{C}^2, 0) = \varprojlim \text{Diff}_i$. Let us denote by $\widehat{\chi}(\mathbb{C}^2, 0)$ the algebra of formal vector fields in \mathbb{C}^2 vanishing at 0. One can define the set χ_i of the i -th jets of vector fields; one has $\varprojlim \chi_i = \widehat{\chi}(\mathbb{C}^2, 0)$.

Let $\widehat{O}(\mathbb{C}^2)$ be the ring of formal series in two variables and let $\widehat{K}(\mathbb{C}^2)$ be its fraction field; O_i is the set of elements of $\widehat{O}(\mathbb{C}^2)$ truncated at order i .

The family $(\exp_i: \chi_i \rightarrow \text{Diff}_i)_i$ is filtered, *i.e.* compatible with the truncation. We then define the exponential application as follows: $\exp = \varprojlim \exp_i: \widehat{\chi}(\mathbb{C}^2, 0) \rightarrow \widehat{\text{Diff}}(\mathbb{C}^2, 0)$.

As in the classical case, if X belongs to $\widehat{\chi}(\mathbb{C}^2, 0)$, then $\exp(X)$ can be seen as the “flow at time 1” of X . Indeed an element X_i of χ_i can be seen as a derivation of O_i ; so it can be written $S_i + N_i$ where S_i and N_i are two semi-simple, resp. nilpotent derivations which commute. Taking the limit, one gets $X = S + N$ where S is a semi-simple vector field and N a nilpotent one and $[S, N] = \text{id}$ (*see* [15]). A semi-simple vector field is a formal vector field conjugate to a diagonal linear vector field which is complete. A vector field is nilpotent if and only if its linear part is; let us remark that the usual flow φ_t of a nilpotent vector field is polynomial in t

$$\varphi_t(x) = \sum_I P_I(t)x^I, \quad P_I \in (\mathbb{C}[t])^2$$

so $\varphi_1(x)$ is well defined. As a consequence $\exp(tX) = \exp(tS)\exp(tN)$ is well defined for $t = 1$. Note that the Jordan decomposition is purely formal: if X is holomorphic, S and N are not necessary holomorphic.

Proposition 3.1 ([10]). — *Any nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2, 0)$ is metabelian.*

Proof. — Let \mathfrak{l} be a nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2, 0)$ and let $Z(\mathfrak{l})$ be its center. Since $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is a vector space of dimension 2 over $\widehat{K}(\mathbb{C}^2)$ one has the following alternative:

- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is 1;
- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is 2.

Let us study these different cases.

Under the first assumption there exists an element X of $Z(\mathfrak{l})$ having the following property: any vector field of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let us consider the subalgebra \mathfrak{g} of \mathfrak{l} given by

$$\mathfrak{g} = \{\tilde{X} \in \mathfrak{l} \mid \exists u \in \widehat{K}(\mathbb{C}^2), \tilde{X} = uX\}.$$

Since X belongs to $Z(\mathfrak{l})$, the algebra \mathfrak{g} is abelian; it is also an ideal of \mathfrak{l} . Let us assume that \mathfrak{l} is not abelian: let Y be an element of \mathfrak{l} whose projection on $\mathfrak{l}/\mathfrak{g}$ is non trivial and central. Any vector field of \mathfrak{l} can be written as $uX + vY$ with u, v in $\widehat{K}(\mathbb{C}^2)$. As X belongs to $Z(\mathfrak{l})$ and Y is central modulo \mathfrak{g} one has

$$X(u) = X(v) = Y(v) = 0.$$

The vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ being some linear combinations of X and Y with coefficients in $\widehat{K}(\mathbb{C}^2, 0)$, the partial derivatives of v are zero so v is a constant. Therefore $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{g}$; but \mathfrak{g} is abelian so \mathfrak{l} is metabelian.

In the second case $Z(\mathfrak{l})$ has two elements X and Y which are linearly independent on $\widehat{K}(\mathbb{C}^2)$. Any vector field of \mathfrak{l} can be written as $uX + vY$ with u and v in $\widehat{K}(\mathbb{C}^2)$. Since X and Y belong to $Z(\mathfrak{l})$ one has

$$X(u) = X(v) = Y(u) = Y(v) = 0.$$

As a consequence u and v are constant, *i.e.* $\mathfrak{l} \subset \{uX + vY \mid u, v \in \mathbb{C}\}$; in particular \mathfrak{l} is abelian. \square

Proposition 3.2 ([10]). — *Any nilpotent subgroup of $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ is metabelian.*

Proof. — Let G be a nilpotent subgroup of $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ of length k . Let us denote G_i the projection of G on Diff_i . The Zariski closure \overline{G}_i of G_i in Diff_i is an algebraic nilpotent subgroup of length k . It is sufficient to prove that \overline{G}_i is metabelian.

Since \overline{G}_i is a complex algebraic subgroup it is the direct product of the subgroup $\overline{G}_{i,u}$ of its unipotent elements and the subgroup $\overline{G}_{i,s}$ of its semi-simple elements (*see for example* [2]).

An element of Diff_i is unipotent if and only if its linear part, which is in $\text{GL}(2, \mathbb{C})$, is; so $\overline{G}_{i,s}$ projects injectively onto a nilpotent subgroup of $\text{GL}(2, \mathbb{C})$. Therefore $\overline{G}_{i,s}$ is abelian.

Let us now consider $\overline{G}_{i,u}$; this group is the exponential of a nilpotent Lie algebra \mathfrak{l}_i of χ_i of length k . Taking the limit one thus obtains the existence of a nilpotent subalgebra \mathfrak{l} of $\widehat{\chi}(\mathbb{C}^2, 0)$ of length k such that $\exp(\mathfrak{l})$ projects onto $\overline{G}_{i,u}$ for any i . According to Proposition 3.1 the subalgebra \mathfrak{l} and thus $\overline{G}_{i,u}$ are metabelian. \square

4. Nilpotent subgroups of the group of analytic diffeomorphisms of \mathbb{S}^2

Proposition 4.1 ([10]). — *Any nilpotent subgroup of $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ has a finite orbit.*

Proof. — Let G be a nilpotent subgroup of $\text{Diff}^{\text{an}}(\mathbb{S}^2)$; up to finite index one can assume that the elements of G preserve the orientation. Let ϕ be a non trivial element of G which commutes with G . Let $\text{Fix}(\phi)$ be the set of fixed points of ϕ ; it is a non empty analytic subspace of \mathbb{S}^2 invariant by G . If p is an isolated fixed point of ϕ , then the orbit of p under the action of G is finite. So it is sufficient to study the case where $\text{Fix}(\phi)$ only contains curves; there are thus two possibilities:

- $\text{Fix}(\phi)$ is a singular analytic curve whose set of singular points is a finite orbit for G ;
- $\text{Fix}(\phi)$ is a smooth analytic curve, not necessary connected. One of the connected component of $\mathbb{S}^2 \setminus \text{Fix}(\phi)$ is a disk denoted \mathbb{D} . Any subgroup Γ of finite index of G which contains ϕ fixes \mathbb{D} . Let us consider an element γ of Γ and a fixed point m of γ which is in $\overline{\mathbb{D}}$. By construction ϕ has no fixed point in \mathbb{D} so according to the Brouwer Theorem $(\phi^k(m))_k$ has a limit point on the boundary $\partial\mathbb{D}$ of $\overline{\mathbb{D}}$. Therefore γ has at least one fixed point on $\partial\mathbb{D}$. The group Γ thus acts on the circle $\partial\mathbb{D}$ and any of its elements has a fixed point on \mathbb{D} . Then Γ has a fixed point on $\partial\mathbb{D}$ (Proposition 2.1). \square

Theorem 4.2 ([10]). — *Any nilpotent subgroup of $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ is metabelian.*

Proof. — Let G be a nilpotent subgroup of $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ and let Γ be a subgroup of finite index of G having a fixed point m (such a subgroup exists according to Proposition 4.1). One can embed Γ into $\widehat{\text{Diff}}(\mathbb{R}^2, 0)$, and so into $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$, by considering the jets of infinite order of elements of Γ in m . According to Proposition 3.2 the group Γ is metabelian.

One can assume that G is a finitely generated group.

Let us first assume that G has no element of finite order. Then G is a cocompact lattice of the nilpotent, simply connected Lie group $G \otimes \mathbb{R}$ (*see* [18]). The group G is metabelian if and only if $G \otimes \mathbb{R}$ is; but Γ is metabelian so $G \otimes \mathbb{R}$ also.

Finally let us consider the case where G has at least one element of finite order. The set of such elements is a normal subgroup of G which thus intersects non trivially the center $Z(G)$ of G . Let us consider a non trivial element ϕ of $Z(G)$ which has finite order. Let us recall that a finite group of diffeomorphisms of the sphere is conjugate to a group of isometries. Denote by G^+ the subgroup of elements of G which preserve the orientation. It is thus sufficient to prove that G^+ is metabelian; indeed if ϕ does not preserve the orientation, ϕ has order 2 and $G = \mathbb{Z}/2\mathbb{Z} \times G^+$. So let us assume that ϕ preserves the orientation; ϕ is conjugate to a direct isometry of \mathbb{S}^2 and has exactly two fixed points on the sphere. The group G has thus an invariant set of two elements. By considering germs in the neighborhood of these two points, one gets that G can be embedded into $2 \cdot \text{Diff}(\mathbb{R}^2, 0)$ ⁽²⁾ and thus into $2 \cdot \text{Diff}(\mathbb{C}^2, 0)$:

$$1 \longrightarrow \text{Diff}(\mathbb{C}^2, 0) \longrightarrow 2 \cdot \text{Diff}(\mathbb{C}^2, 0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Let us remark that $2 \cdot \text{Diff}(\mathbb{C}^2, 0)$ is the projective limit of the algebraic groups $2 \cdot \text{Diff}_i$. The end of the proof is thus the same as the proof of Proposition 3.2 except that the subgroup of the semi-simple elements of $2 \cdot \text{Diff}_i$ embeds now in $2 \cdot \text{GL}(2, \mathbb{C})$; it is metabelian because it contains an abelian subgroup of index 2. \square

Let Γ be a subgroup of finite index of $\text{SL}(n, \mathbb{Z})$ for $n \geq 5$. Since Γ contains nilpotent subgroups of finite index of length $n - 1$ (for example the group of upper triangular unipotent matrices) which are not metabelian one gets the following statement.

Corollary 4.3 ([10]). — *Let Γ be a subgroup of finite index of $\text{SL}(n, \mathbb{Z})$; as soon as $n \geq 5$ there is no embedding of Γ into $\text{Diff}^{\text{an}}(\mathbb{S}^2)$.*

5. Nilpotent subgroups of length 3 of the group of analytic diffeomorphisms of \mathbb{S}^2

Let us precise Proposition 3.1 for nilpotent subalgebras of length 3 of $\widehat{\chi}(\mathbb{C}^2, 0)$. Let \mathfrak{l} be such an algebra. The center $Z(\mathfrak{l})$ of \mathfrak{l} generates a subspace of dimension at most 1 of $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$, for else \mathfrak{l} would be abelian (Proposition 3.1) and this is impossible under our assumptions. So let us assume that the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is 1. There exists an element X in $Z(\mathfrak{l})$ with the following property: any element of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let \mathfrak{g} denote the abelian ideal of \mathfrak{l} defined by

$$\mathfrak{g} = \{ \widetilde{X} \in \mathfrak{l} \mid \exists u \in \widehat{K}(\mathbb{C}^2), \widetilde{X} = uX \}.$$

By hypothesis \mathfrak{l} is not abelian. Let Y be in \mathfrak{l} ; assume that its projection onto $\mathfrak{l}/\mathfrak{g}$ is a non trivial element of $Z(\mathfrak{l}/\mathfrak{g})$. Any vector field of \mathfrak{l} can be written

$$uX + vY, \quad u, v \in \widehat{K}(\mathbb{C}^2).$$

Since X , resp. Y belongs to $Z(\mathfrak{l})$, resp. $Z(\mathfrak{l}/\mathfrak{g})$ and since the length of \mathfrak{l} is 3, one has

$$X(u) = Y^3(u) = X(v) = Y(v) = 0. \quad (5.1)$$

If X and Y are non singular, one can choose formal coordinates x and y such that $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$. The previous conditions can be thus translated as follows: v is a constant and u is a polynomial in y of degree 2. We will see that we have a similar property without assumption on X and Y .

2. Let G be a group and let q be a positive integer; $q \cdot G$ denotes the semi-direct product of G^q by $\mathbb{Z}/q\mathbb{Z}$ under the action of the cyclic permutation of the factors.

Lemma 5.1. — *Let X and Y be two vector fields of $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$ that commute and are not colinear. One can assume that $(X, Y) = \left(\frac{\partial}{\partial \tilde{x}}, \frac{\partial}{\partial \tilde{y}}\right)$ where \tilde{x} and \tilde{y} are two independent variables in a Liouvillian extension of $\widehat{K}(\mathbb{C}^2, 0)$.*

Proof. — Since X and Y are non colinear, there exist two 1-forms α, β with coefficients in $\widehat{K}(\mathbb{C}^2)$ such that

$$\alpha(X) = 1, \quad \alpha(Y) = 0, \quad \beta(X) = 0, \quad \beta(Y) = 1.$$

The vector fields X and Y commute if and only if α and β are closed (this statement of linear algebra is true for convergent meromorphic vector fields and is also true in the completion). The 1-form α is closed so according to [6] one has

$$\alpha = \sum_{i=1}^r \lambda_i \frac{d\widehat{\phi}_i}{\widehat{\phi}_i} + d\left(\frac{\widehat{\Psi}_1}{\widehat{\Psi}_2}\right) = d\left(\sum_{i=1}^r \lambda_i \log \widehat{\phi}_i + \frac{\widehat{\Psi}_1}{\widehat{\Psi}_2}\right)$$

where $\widehat{\Psi}_1, \widehat{\Psi}_2$ and the $\widehat{\phi}_i$ denote some formal series and the λ_i some complex numbers. One has a similar expression for β . So there exists a Liouvillian extension κ of $\widehat{K}(\mathbb{C}^2)$ having two elements \tilde{x} and \tilde{y} with $\alpha = d\tilde{x}$ and $\beta = d\tilde{y}$. One thus has

$$X(\tilde{x}) = 1, \quad X(\tilde{y}) = 0, \quad Y(\tilde{x}) = 0, \quad Y(\tilde{y}) = 1.$$

□

From (5.1) one gets: v is a constant and u is a polynomial in \tilde{y} of degree 2; so one proves the following statement.

Proposition 5.2. — *Let \mathfrak{l} be a nilpotent subalgebra of $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$ of length 3. Then \mathfrak{l} is isomorphic to a subalgebra of*

$$\mathfrak{n} = \left\{ P(\tilde{y}) \frac{\partial}{\partial \tilde{x}} + \alpha \frac{\partial}{\partial \tilde{y}} \mid \alpha \in \mathbb{C}, P \in \mathbb{C}[\tilde{y}], \deg P = 2 \right\}.$$

Remark 5.3. — We use a real version of this statement whose proof is an adaptation of the previous one: a nilpotent subalgebra \mathfrak{l} of length 3 of $\widehat{\mathcal{X}}(\mathbb{R}^2, 0)$ is isomorphic to a subalgebra of

$$\mathfrak{n} = \left\{ P(\tilde{y}) \frac{\partial}{\partial \tilde{x}} + \alpha \frac{\partial}{\partial \tilde{y}} \mid \alpha \in \mathbb{R}, P \in \mathbb{R}[\tilde{y}], \deg P = 2 \right\}.$$

Theorem 5.4. — *Let Γ be a subgroup of finite index of $\mathrm{SL}(n, \mathbb{Z})$; as soon as $n \geq 4$ there is no embedding of Γ into $\mathrm{Diff}^0(\mathbb{S}^2)$.*

Proof. — Let $U(4, \mathbb{Z})$ (resp. $U(4, \mathbb{R})$) be the subgroup of unipotent upper triangular matrices of $\mathrm{SL}(4, \mathbb{Z})$ (resp. $\mathrm{SL}(4, \mathbb{R})$); it is a nilpotent subgroup of length 3. Assume that there exists an embedding from a subgroup Γ of finite index of $\mathrm{SL}(4, \mathbb{Z})$ into $\mathrm{Diff}^0(\mathbb{S}^2)$. Up to finite index Γ contains $U(4, \mathbb{Z})$. Let us set $H = \rho(U(4, \mathbb{Z}))$. Up to finite index H has a fixed point (Proposition 4.1). One can thus see H as a subgroup of $\mathrm{Diff}(\mathbb{R}^2, 0) \subset \widehat{\mathrm{Diff}}(\mathbb{R}^2, 0)$ up to finite index.

Let us denote j^1 the morphism from $\widehat{\mathrm{Diff}}(\mathbb{R}^2, 0)$ to Diff_j . Up to conjugation $j^1(\rho(U(4, \mathbb{Z})))$ is a subgroup of

$$\left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix} \mid \lambda \in \mathbb{R}^*, t \in \mathbb{R} \right\}.$$

Up to index 2 one can thus assume that $j^1 \circ \rho$ takes values in the connected, simply connected group T defined by

$$T = \left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix} \mid \lambda, t \in \mathbb{R}, \lambda > 0 \right\}.$$

Let us set

$$\text{Diff}_i(T) = \{f \in \text{Diff}_i \mid j^1(f) \in T\};$$

the group $\text{Diff}_i(T)$ is a connected, simply connected, nilpotent and algebraic group. The morphism

$$\rho_i: U(4, \mathbb{Z}) \rightarrow \text{Diff}_i$$

can be extended to a unique continuous morphism $\tilde{\rho}_i: U(4, \mathbb{R}) \rightarrow \text{Diff}_i(T)$ (see [13, 12]) so to an algebraic morphism⁽³⁾. Let us note that $\tilde{\rho}_i(U(4, \mathbb{Z}))$ is an algebraic subgroup of $\text{Diff}_i(T)$ which contains $\rho_i(U(4, \mathbb{Z}))$; in particular $\overline{H_i} = \overline{\rho_i(U(4, \mathbb{Z}))} \subset \tilde{\rho}_i(U(4, \mathbb{R}))$. By construction the family $(H_i)_i$ is filtered; since the extension is unique, the family $(\tilde{\rho}_i)_i$ is also filtered. Therefore $K = \varprojlim \overline{H_i}$ is well defined. Since ρ is injective, H is a nilpotent subgroup of length 3; as $H \subset K$ and as any $\overline{H_i}$ is nilpotent of length at most 3 the group K is nilpotent of length at most 3. For i sufficiently large $\tilde{\rho}_i(U(4, \mathbb{R}))$ is nilpotent of length 3; this group is connected so its Lie algebra is also nilpotent of length 3. Therefore the image of

$$D\tilde{\rho} := \varprojlim D\tilde{\rho}_i: \mathfrak{u}(4, \mathbb{R}) \rightarrow \hat{\chi}(\mathbb{R}^2, 0)$$

is isomorphic to \mathfrak{n} (Proposition 5.2). So there exists a surjective map ψ from $\mathfrak{u}(4, \mathbb{R})$ onto \mathfrak{n} . The kernel of ψ is an ideal of $\mathfrak{u}(4, \mathbb{R})$ of dimension 2; hence $\ker \psi = \langle \delta_{14}, a\delta_{13} + b\delta_{24} \rangle$ where the δ_{ij} denote the Kronecker matrices. One concludes by remarking that $\dim Z(\mathfrak{u}(4, \mathbb{R})/\ker \psi) = 2$ whereas $\dim Z(\mathfrak{n}) = 1$. \square

Corollary 5.5. — *The image of a morphism from a subgroup of $\text{SL}(n, \mathbb{Z})$ of finite index to $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ is finite as soon as $n \geq 4$.*

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