



**HAL**  
open science

# REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE $S^2$

Julie Déserti

► **To cite this version:**

Julie Déserti. REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE  $S^2$ . Journal of Singularities, 2014. hal-03000453

**HAL Id: hal-03000453**

**<https://hal.science/hal-03000453>**

Submitted on 11 Nov 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE $\mathbb{S}^2$

by

Julie Déserti

---

November 19, 2018

**Abstract.** — In [10] it is proved that any morphism from a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$  to the group of analytic diffeomorphisms of  $\mathbb{S}^2$  has a finite image as soon as  $n \geq 5$ . The case  $n = 4$  is also claimed to follow along the same arguments; in fact this is not straightforward and this case indeed needs a modification of the argument. In this paper we recall the strategy for  $n \geq 5$  and then focus on the case  $n = 4$ .

*2010 Mathematics Subject Classification.* — 58D05, 58B25

## 1. Introduction

After the works of Margulis ([14, 19]) on the linear representations of lattices of simple, real Lie groups with  $\mathbb{R}$ -rank larger than 1, some authors, like Zimmer, suggest to study the actions of lattices on compact manifolds ([20, 21, 22, 23]). One of the main conjectures of this program is the following: let us consider a connected, simple, real Lie group  $G$  and let  $\Gamma$  be a lattice of  $G$  of  $\mathbb{R}$ -rank larger than 1. If there exists a morphism of infinite image from  $\Gamma$  to the group of diffeomorphisms of a compact manifold  $M$ , then the  $\mathbb{R}$ -rank of  $G$  is bounded by the dimension of  $M$ . There are a lot of contributions in that direction ([3, 4, 5, 7, 8, 9, 10, 11, 16, 17]). In this article we will focus on the embeddings of subgroups of finite index of  $\mathrm{SL}(n, \mathbb{Z})$  into the group  $\mathrm{Diff}^0(\mathbb{S}^2)$  of real analytic diffeomorphisms of  $\mathbb{S}^2$  (see [10]).

The article is organized as follows. First of all we will recall the strategy of [10]: the study of the nilpotent subgroups of  $\mathrm{Diff}^0(\mathbb{S}^2)$  implies that such subgroups are metabelian. But subgroups of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ , for  $n \geq 5$ , contain nilpotent subgroups of length  $n - 1$  of finite index which are not metabelian; as a consequence Ghys gets the following statement.

**Theorem A ([10]).** — *Let  $\Gamma$  be a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ . As soon as  $n \geq 5$  there is no embedding of  $\Gamma$  into  $\mathrm{Diff}^0(\mathbb{S}^2)$ .*

To study nilpotent subgroups of  $\mathrm{Diff}^0(\mathbb{S}^2)$  one has to study nilpotent subgroups of  $\mathrm{Diff}_+^0(\mathbb{S}^1)$  (see §2) and then nilpotent subgroups of the group of formal diffeomorphisms of  $\mathbb{C}^2$  (see §3). The last section is devoted to establish the following result.

**Theorem B.** — *Let  $\Gamma$  be a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ . As soon as  $n \geq 4$  there is no embedding of  $\Gamma$  into  $\mathrm{Diff}^{\omega}(\mathbb{S}^2)$ .*

The proof relies on the characterization, up to isomorphism, of nilpotent subalgebras of length 3 of the algebra of formal vector fields of  $\mathbb{C}^2$  which vanish at the origin.

**Acknowledgements.** — The author would like to thank Dominique Cerveau and Étienne Ghys for interesting discussions and advices.

## 2. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^1$

Let  $G$  be a group; let us set

$$G^{(0)} = G \quad \& \quad G^{(i)} = [G, G^{(i-1)}] \quad \forall i \geq 1.$$

The group  $G$  is *nilpotent* if there exists an integer  $n$  such that  $G^{(n)} = \{\mathrm{id}\}$ ; the *length of nilpotence* of  $G$  is the smallest integer  $k$  such that  $G^{(k)} = \{\mathrm{id}\}$ .

Set

$$G_{(0)} = G \quad \& \quad G_{(i)} = [G_{(i-1)}, G_{(i-1)}] \quad \forall i \geq 1.$$

The group  $G$  is *solvable* if  $G_{(n)} = \{\mathrm{id}\}$  for a certain  $n$ ; the *length of solvability* of  $G$  is the smallest integer  $k$  such that  $G_{(k)} = \{\mathrm{id}\}$ .

We say that the group  $G$  (resp. algebra  $\mathfrak{g}$ ) is *metabelian* if  $[G, G]$  (resp.  $[\mathfrak{g}, \mathfrak{g}]$ ) is abelian.

**Proposition 2.1 ([10]).** — *Any nilpotent subgroup of  $\mathrm{Diff}_+^{\omega}(\mathbb{S}^1)$  is abelian.*

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\mathrm{Diff}_+^{\omega}(\mathbb{S}^1)$ . Assume that  $G$  is not abelian; it thus contains a Heisenberg group

$$\langle f, g, h \mid [f, g] = h, [f, h] = [g, h] = \mathrm{id} \rangle.$$

The application “rotation number“

$$\mathrm{Diff}_+^{\omega}(\mathbb{S}^1) \rightarrow \mathbb{R}/\mathbb{Z}, \quad \psi \mapsto \lim_{n \rightarrow +\infty} \frac{\psi^n(x) - x}{n}$$

is not a morphism but its restriction to a solvable subgroup is ([1]). Thus the rotation number of  $h$  is zero and the set  $\mathrm{Fix}(h)$  of fixed points of  $h$  is non-empty and finite. Considering some iterates of  $f$  and  $g$  instead of  $f$  and  $g$  one can assume that  $f$  and  $g$  fix any point of  $\mathrm{Fix}(h)$ . The set of fixed points of a non trivial element of  $\langle f, g \rangle$  is finite and invariant by  $h$  so the action of  $\langle f, g \rangle$  is free <sup>(1)</sup> on each component of  $\mathbb{S}^1 \setminus \mathrm{Fix}(h)$ . But the action of a free group on  $\mathbb{R}$  is abelian: contradiction.  $\square$

---

1. The stabilizer of every point is trivial, *i.e.* the action of a non trivial element of  $\langle f, g \rangle$  has no fixed point.

### 3. Nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$

Let us denote  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  the group of formal diffeomorphisms of  $\mathbb{C}^2$ , *i.e.* the formal completion of the group of germs of holomorphic diffeomorphisms at 0. For any  $i$  let  $\text{Diff}_i$  be the quotient of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  by the normal subgroups of formal diffeomorphisms tangent to the identity with multiplicity  $i$ ; it can be viewed as the set of jets of diffeomorphisms at order  $i$  with the law of composition with truncation at order  $i$ . Note that  $\text{Diff}_i$  is a complex linear algebraic group. One can see  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  as the projective limit of the  $\text{Diff}_i$ 's:  $\widehat{\text{Diff}}(\mathbb{C}^2, 0) = \varprojlim \text{Diff}_i$ . Let us denote by  $\widehat{\chi}(\mathbb{C}^2, 0)$  the algebra of formal vector fields in  $\mathbb{C}^2$  vanishing at 0. One can define the set  $\chi_i$  of the  $i$ -th jets of vector fields; one has  $\varprojlim \chi_i = \widehat{\chi}(\mathbb{C}^2, 0)$ .

Let  $\widehat{O}(\mathbb{C}^2)$  be the ring of formal series in two variables and let  $\widehat{K}(\mathbb{C}^2)$  be its fraction field;  $O_i$  is the set of elements of  $\widehat{O}(\mathbb{C}^2)$  truncated at order  $i$ .

The family  $(\exp_i: \chi_i \rightarrow \text{Diff}_i)_i$  is filtered, *i.e.* compatible with the truncation. We then define the exponential application as follows:  $\exp = \varprojlim \exp_i: \widehat{\chi}(\mathbb{C}^2, 0) \rightarrow \widehat{\text{Diff}}(\mathbb{C}^2, 0)$ .

As in the classical case, if  $X$  belongs to  $\widehat{\chi}(\mathbb{C}^2, 0)$ , then  $\exp(X)$  can be seen as the “flow at time 1” of  $X$ . Indeed an element  $X_i$  of  $\chi_i$  can be seen as a derivation of  $O_i$ ; so it can be written  $S_i + N_i$  where  $S_i$  and  $N_i$  are two semi-simple, resp. nilpotent derivations which commute. Taking the limit, one gets  $X = S + N$  where  $S$  is a semi-simple vector field and  $N$  a nilpotent one and  $[S, N] = \text{id}$  (*see* [15]). A semi-simple vector field is a formal vector field conjugate to a diagonal linear vector field which is complete. A vector field is nilpotent if and only if its linear part is; let us remark that the usual flow  $\varphi_t$  of a nilpotent vector field is polynomial in  $t$

$$\varphi_t(x) = \sum_I P_I(t)x^I, \quad P_I \in (\mathbb{C}[t])^2$$

so  $\varphi_1(x)$  is well defined. As a consequence  $\exp(tX) = \exp(tS)\exp(tN)$  is well defined for  $t = 1$ . Note that the Jordan decomposition is purely formal: if  $X$  is holomorphic,  $S$  and  $N$  are not necessary holomorphic.

**Proposition 3.1** ([10]). — *Any nilpotent subalgebra of  $\widehat{\chi}(\mathbb{C}^2, 0)$  is metabelian.*

*Proof.* — Let  $\mathfrak{l}$  be a nilpotent subalgebra of  $\widehat{\chi}(\mathbb{C}^2, 0)$  and let  $Z(\mathfrak{l})$  be its center. Since  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is a vector space of dimension 2 over  $\widehat{K}(\mathbb{C}^2)$  one has the following alternative:

- the dimension of the subspace generated by  $Z(\mathfrak{l})$  in  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is 1;
- the dimension of the subspace generated by  $Z(\mathfrak{l})$  in  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is 2.

Let us study these different cases.

Under the first assumption there exists an element  $X$  of  $Z(\mathfrak{l})$  having the following property: any vector field of  $Z(\mathfrak{l})$  can be written  $uX$  with  $u$  in  $\widehat{K}(\mathbb{C}^2)$ . Let us consider the subalgebra  $\mathfrak{g}$  of  $\mathfrak{l}$  given by

$$\mathfrak{g} = \{\tilde{X} \in \mathfrak{l} \mid \exists u \in \widehat{K}(\mathbb{C}^2), \tilde{X} = uX\}.$$

Since  $X$  belongs to  $Z(\mathfrak{l})$ , the algebra  $\mathfrak{g}$  is abelian; it is also an ideal of  $\mathfrak{l}$ . Let us assume that  $\mathfrak{l}$  is not abelian: let  $Y$  be an element of  $\mathfrak{l}$  whose projection on  $\mathfrak{l}/\mathfrak{g}$  is non trivial and central. Any vector field of  $\mathfrak{l}$  can be written as  $uX + vY$  with  $u, v$  in  $\widehat{K}(\mathbb{C}^2)$ . As  $X$  belongs to  $Z(\mathfrak{l})$  and  $Y$  is central modulo  $\mathfrak{g}$  one has

$$X(u) = X(v) = Y(v) = 0.$$

The vector fields  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  being some linear combinations of  $X$  and  $Y$  with coefficients in  $\widehat{K}(\mathbb{C}^2, 0)$ , the partial derivatives of  $v$  are zero so  $v$  is a constant. Therefore  $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{g}$ ; but  $\mathfrak{g}$  is abelian so  $\mathfrak{l}$  is metabelian.

In the second case  $Z(\mathfrak{l})$  has two elements  $X$  and  $Y$  which are linearly independent on  $\widehat{K}(\mathbb{C}^2)$ . Any vector field of  $\mathfrak{l}$  can be written as  $uX + vY$  with  $u$  and  $v$  in  $\widehat{K}(\mathbb{C}^2)$ . Since  $X$  and  $Y$  belong to  $Z(\mathfrak{l})$  one has

$$X(u) = X(v) = Y(u) = Y(v) = 0.$$

As a consequence  $u$  and  $v$  are constant, *i.e.*  $\mathfrak{l} \subset \{uX + vY \mid u, v \in \mathbb{C}\}$ ; in particular  $\mathfrak{l}$  is abelian.  $\square$

**Proposition 3.2 ([10]).** — *Any nilpotent subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  is metabelian.*

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  of length  $k$ . Let us denote  $G_i$  the projection of  $G$  on  $\text{Diff}_i$ . The Zariski closure  $\overline{G}_i$  of  $G_i$  in  $\text{Diff}_i$  is an algebraic nilpotent subgroup of length  $k$ . It is sufficient to prove that  $\overline{G}_i$  is metabelian.

Since  $\overline{G}_i$  is a complex algebraic subgroup it is the direct product of the subgroup  $\overline{G}_{i,u}$  of its unipotent elements and the subgroup  $\overline{G}_{i,s}$  of its semi-simple elements (*see for example* [2]).

An element of  $\text{Diff}_i$  is unipotent if and only if its linear part, which is in  $\text{GL}(2, \mathbb{C})$ , is; so  $\overline{G}_{i,s}$  projects injectively onto a nilpotent subgroup of  $\text{GL}(2, \mathbb{C})$ . Therefore  $\overline{G}_{i,s}$  is abelian.

Let us now consider  $\overline{G}_{i,u}$ ; this group is the exponential of a nilpotent Lie algebra  $\mathfrak{l}_i$  of  $\chi_i$  of length  $k$ . Taking the limit one thus obtains the existence of a nilpotent subalgebra  $\mathfrak{l}$  of  $\widehat{\chi}(\mathbb{C}^2, 0)$  of length  $k$  such that  $\exp(\mathfrak{l})$  projects onto  $\overline{G}_{i,u}$  for any  $i$ . According to Proposition 3.1 the subalgebra  $\mathfrak{l}$  and thus  $\overline{G}_{i,u}$  are metabelian.  $\square$

#### 4. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^2$

**Proposition 4.1 ([10]).** — *Any nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$  has a finite orbit.*

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ ; up to finite index one can assume that the elements of  $G$  preserve the orientation. Let  $\phi$  be a non trivial element of  $G$  which commutes with  $G$ . Let  $\text{Fix}(\phi)$  be the set of fixed points of  $\phi$ ; it is a non empty analytic subspace of  $\mathbb{S}^2$  invariant by  $G$ . If  $p$  is an isolated fixed point of  $\phi$ , then the orbit of  $p$  under the action of  $G$  is finite. So it is sufficient to study the case where  $\text{Fix}(\phi)$  only contains curves; there are thus two possibilities:

- $\text{Fix}(\phi)$  is a singular analytic curve whose set of singular points is a finite orbit for  $G$ ;
- $\text{Fix}(\phi)$  is a smooth analytic curve, not necessary connected. One of the connected component of  $\mathbb{S}^2 \setminus \text{Fix}(\phi)$  is a disk denoted  $\mathbb{D}$ . Any subgroup  $\Gamma$  of finite index of  $G$  which contains  $\phi$  fixes  $\mathbb{D}$ . Let us consider an element  $\gamma$  of  $\Gamma$  and a fixed point  $m$  of  $\gamma$  which is in  $\overline{\mathbb{D}}$ . By construction  $\phi$  has no fixed point in  $\mathbb{D}$  so according to the Brouwer Theorem  $(\phi^k(m))_k$  has a limit point on the boundary  $\partial\mathbb{D}$  of  $\overline{\mathbb{D}}$ . Therefore  $\gamma$  has at least one fixed point on  $\partial\mathbb{D}$ . The group  $\Gamma$  thus acts on the circle  $\partial\mathbb{D}$  and any of its elements has a fixed point on  $\mathbb{D}$ . Then  $\Gamma$  has a fixed point on  $\partial\mathbb{D}$  (Proposition 2.1).  $\square$

**Theorem 4.2 ([10]).** — *Any nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$  is metabelian.*

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$  and let  $\Gamma$  be a subgroup of finite index of  $G$  having a fixed point  $m$  (such a subgroup exists according to Proposition 4.1). One can embed  $\Gamma$  into  $\widehat{\text{Diff}}(\mathbb{R}^2, 0)$ , and so into  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ , by considering the jets of infinite order of elements of  $\Gamma$  in  $m$ . According to Proposition 3.2 the group  $\Gamma$  is metabelian.

One can assume that  $G$  is a finitely generated group.

Let us first assume that  $G$  has no element of finite order. Then  $G$  is a cocompact lattice of the nilpotent, simply connected Lie group  $G \otimes \mathbb{R}$  (*see* [18]). The group  $G$  is metabelian if and only if  $G \otimes \mathbb{R}$  is; but  $\Gamma$  is metabelian so  $G \otimes \mathbb{R}$  also.

Finally let us consider the case where  $G$  has at least one element of finite order. The set of such elements is a normal subgroup of  $G$  which thus intersects non trivially the center  $Z(G)$  of  $G$ . Let us consider a non trivial element  $\phi$  of  $Z(G)$  which has finite order. Let us recall that a finite group of diffeomorphisms of the sphere is conjugate to a group of isometries. Denote by  $G^+$  the subgroup of elements of  $G$  which preserve the orientation. It is thus sufficient to prove that  $G^+$  is metabelian; indeed if  $\phi$  does not preserve the orientation,  $\phi$  has order 2 and  $G = \mathbb{Z}/2\mathbb{Z} \times G^+$ . So let us assume that  $\phi$  preserves the orientation;  $\phi$  is conjugate to a direct isometry of  $\mathbb{S}^2$  and has exactly two fixed points on the sphere. The group  $G$  has thus an invariant set of two elements. By considering germs in the neighborhood of these two points, one gets that  $G$  can be embedded into  $2 \cdot \text{Diff}(\mathbb{R}^2, 0)$  <sup>(2)</sup> and thus into  $2 \cdot \text{Diff}(\mathbb{C}^2, 0)$ :

$$1 \longrightarrow \text{Diff}(\mathbb{C}^2, 0) \longrightarrow 2 \cdot \text{Diff}(\mathbb{C}^2, 0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Let us remark that  $2 \cdot \text{Diff}(\mathbb{C}^2, 0)$  is the projective limit of the algebraic groups  $2 \cdot \text{Diff}_i$ . The end of the proof is thus the same as the proof of Proposition 3.2 except that the subgroup of the semi-simple elements of  $2 \cdot \text{Diff}_i$  embeds now in  $2 \cdot \text{GL}(2, \mathbb{C})$ ; it is metabelian because it contains an abelian subgroup of index 2.  $\square$

Let  $\Gamma$  be a subgroup of finite index of  $\text{SL}(n, \mathbb{Z})$  for  $n \geq 5$ . Since  $\Gamma$  contains nilpotent subgroups of finite index of length  $n - 1$  (for example the group of upper triangular unipotent matrices) which are not metabelian one gets the following statement.

**Corollary 4.3** ([10]). — *Let  $\Gamma$  be a subgroup of finite index of  $\text{SL}(n, \mathbb{Z})$ ; as soon as  $n \geq 5$  there is no embedding of  $\Gamma$  into  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ .*

### 5. Nilpotent subgroups of length 3 of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Let us precise Proposition 3.1 for nilpotent subalgebras of length 3 of  $\widehat{\chi}(\mathbb{C}^2, 0)$ . Let  $\mathfrak{l}$  be such an algebra. The center  $Z(\mathfrak{l})$  of  $\mathfrak{l}$  generates a subspace of dimension at most 1 of  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ , for else  $\mathfrak{l}$  would be abelian (Proposition 3.1) and this is impossible under our assumptions. So let us assume that the dimension of the subspace generated by  $Z(\mathfrak{l})$  in  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is 1. There exists an element  $X$  in  $Z(\mathfrak{l})$  with the following property: any element of  $Z(\mathfrak{l})$  can be written  $uX$  with  $u$  in  $\widehat{K}(\mathbb{C}^2)$ . Let  $\mathfrak{g}$  denote the abelian ideal of  $\mathfrak{l}$  defined by

$$\mathfrak{g} = \{ \widetilde{X} \in \mathfrak{l} \mid \exists u \in \widehat{K}(\mathbb{C}^2), \widetilde{X} = uX \}.$$

By hypothesis  $\mathfrak{l}$  is not abelian. Let  $Y$  be in  $\mathfrak{l}$ ; assume that its projection onto  $\mathfrak{l}/\mathfrak{g}$  is a non trivial element of  $Z(\mathfrak{l}/\mathfrak{g})$ . Any vector field of  $\mathfrak{l}$  can be written

$$uX + vY, \quad u, v \in \widehat{K}(\mathbb{C}^2).$$

Since  $X$ , resp.  $Y$  belongs to  $Z(\mathfrak{l})$ , resp.  $Z(\mathfrak{l}/\mathfrak{g})$  and since the length of  $\mathfrak{l}$  is 3, one has

$$X(u) = Y^3(u) = X(v) = Y(v) = 0. \quad (5.1)$$

If  $X$  and  $Y$  are non singular, one can choose formal coordinates  $x$  and  $y$  such that  $X = \frac{\partial}{\partial x}$  and  $Y = \frac{\partial}{\partial y}$ . The previous conditions can be thus translated as follows:  $v$  is a constant and  $u$  is a polynomial in  $y$  of degree 2. We will see that we have a similar property without assumption on  $X$  and  $Y$ .

---

2. Let  $G$  be a group and let  $q$  be a positive integer;  $q \cdot G$  denotes the semi-direct product of  $G^q$  by  $\mathbb{Z}/q\mathbb{Z}$  under the action of the cyclic permutation of the factors.

**Lemma 5.1.** — *Let  $X$  and  $Y$  be two vector fields of  $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$  that commute and are not colinear. One can assume that  $(X, Y) = \left(\frac{\partial}{\partial \tilde{x}}, \frac{\partial}{\partial \tilde{y}}\right)$  where  $\tilde{x}$  and  $\tilde{y}$  are two independent variables in a Liouvillian extension of  $\widehat{K}(\mathbb{C}^2, 0)$ .*

*Proof.* — Since  $X$  and  $Y$  are non colinear, there exist two 1-forms  $\alpha, \beta$  with coefficients in  $\widehat{K}(\mathbb{C}^2)$  such that

$$\alpha(X) = 1, \quad \alpha(Y) = 0, \quad \beta(X) = 0, \quad \beta(Y) = 1.$$

The vector fields  $X$  and  $Y$  commute if and only if  $\alpha$  and  $\beta$  are closed (this statement of linear algebra is true for convergent meromorphic vector fields and is also true in the completion). The 1-form  $\alpha$  is closed so according to [6] one has

$$\alpha = \sum_{i=1}^r \lambda_i \frac{d\widehat{\phi}_i}{\widehat{\phi}_i} + d\left(\frac{\widehat{\Psi}_1}{\widehat{\Psi}_2}\right) = d\left(\sum_{i=1}^r \lambda_i \log \widehat{\phi}_i + \frac{\widehat{\Psi}_1}{\widehat{\Psi}_2}\right)$$

where  $\widehat{\Psi}_1, \widehat{\Psi}_2$  and the  $\widehat{\phi}_i$  denote some formal series and the  $\lambda_i$  some complex numbers. One has a similar expression for  $\beta$ . So there exists a Liouvillian extension  $\kappa$  of  $\widehat{K}(\mathbb{C}^2)$  having two elements  $\tilde{x}$  and  $\tilde{y}$  with  $\alpha = d\tilde{x}$  and  $\beta = d\tilde{y}$ . One thus has

$$X(\tilde{x}) = 1, \quad X(\tilde{y}) = 0, \quad Y(\tilde{x}) = 0, \quad Y(\tilde{y}) = 1.$$

□

From (5.1) one gets:  $v$  is a constant and  $u$  is a polynomial in  $\tilde{y}$  of degree 2; so one proves the following statement.

**Proposition 5.2.** — *Let  $\mathfrak{l}$  be a nilpotent subalgebra of  $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$  of length 3. Then  $\mathfrak{l}$  is isomorphic to a subalgebra of*

$$\mathfrak{n} = \left\{ P(\tilde{y}) \frac{\partial}{\partial \tilde{x}} + \alpha \frac{\partial}{\partial \tilde{y}} \mid \alpha \in \mathbb{C}, P \in \mathbb{C}[\tilde{y}], \deg P = 2 \right\}.$$

**Remark 5.3.** — We use a real version of this statement whose proof is an adaptation of the previous one: a nilpotent subalgebra  $\mathfrak{l}$  of length 3 of  $\widehat{\mathcal{X}}(\mathbb{R}^2, 0)$  is isomorphic to a subalgebra of

$$\mathfrak{n} = \left\{ P(\tilde{y}) \frac{\partial}{\partial \tilde{x}} + \alpha \frac{\partial}{\partial \tilde{y}} \mid \alpha \in \mathbb{R}, P \in \mathbb{R}[\tilde{y}], \deg P = 2 \right\}.$$

**Theorem 5.4.** — *Let  $\Gamma$  be a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ ; as soon as  $n \geq 4$  there is no embedding of  $\Gamma$  into  $\mathrm{Diff}^0(\mathbb{S}^2)$ .*

*Proof.* — Let  $U(4, \mathbb{Z})$  (resp.  $U(4, \mathbb{R})$ ) be the subgroup of unipotent upper triangular matrices of  $\mathrm{SL}(4, \mathbb{Z})$  (resp.  $\mathrm{SL}(4, \mathbb{R})$ ); it is a nilpotent subgroup of length 3. Assume that there exists an embedding from a subgroup  $\Gamma$  of finite index of  $\mathrm{SL}(4, \mathbb{Z})$  into  $\mathrm{Diff}^0(\mathbb{S}^2)$ . Up to finite index  $\Gamma$  contains  $U(4, \mathbb{Z})$ . Let us set  $H = \rho(U(4, \mathbb{Z}))$ . Up to finite index  $H$  has a fixed point (Proposition 4.1). One can thus see  $H$  as a subgroup of  $\mathrm{Diff}(\mathbb{R}^2, 0) \subset \widehat{\mathrm{Diff}}(\mathbb{R}^2, 0)$  up to finite index.

Let us denote  $j^1$  the morphism from  $\widehat{\mathrm{Diff}}(\mathbb{R}^2, 0)$  to  $\mathrm{Diff}_j$ . Up to conjugation  $j^1(\rho(U(4, \mathbb{Z})))$  is a subgroup of

$$\left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix} \mid \lambda \in \mathbb{R}^*, t \in \mathbb{R} \right\}.$$

Up to index 2 one can thus assume that  $j^1 \circ \rho$  takes values in the connected, simply connected group  $T$  defined by

$$T = \left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix} \mid \lambda, t \in \mathbb{R}, \lambda > 0 \right\}.$$

Let us set

$$\text{Diff}_i(T) = \{f \in \text{Diff}_i \mid j^1(f) \in T\};$$

the group  $\text{Diff}_i(T)$  is a connected, simply connected, nilpotent and algebraic group. The morphism

$$\rho_i: U(4, \mathbb{Z}) \rightarrow \text{Diff}_i$$

can be extended to a unique continuous morphism  $\tilde{\rho}_i: U(4, \mathbb{R}) \rightarrow \text{Diff}_i(T)$  (see [13, 12]) so to an algebraic morphism<sup>(3)</sup>. Let us note that  $\tilde{\rho}_i(U(4, \mathbb{Z}))$  is an algebraic subgroup of  $\text{Diff}_i(T)$  which contains  $\rho_i(U(4, \mathbb{Z}))$ ; in particular  $\overline{H_i} = \overline{\rho_i(U(4, \mathbb{Z}))} \subset \tilde{\rho}_i(U(4, \mathbb{R}))$ . By construction the family  $(H_i)_i$  is filtered; since the extension is unique, the family  $(\tilde{\rho}_i)_i$  is also filtered. Therefore  $K = \varprojlim \overline{H_i}$  is well defined. Since  $\rho$  is injective,  $H$  is a nilpotent subgroup of length 3; as  $H \subset K$  and as any  $\overline{H_i}$  is nilpotent of length at most 3 the group  $K$  is nilpotent of length at most 3. For  $i$  sufficiently large  $\tilde{\rho}_i(U(4, \mathbb{R}))$  is nilpotent of length 3; this group is connected so its Lie algebra is also nilpotent of length 3. Therefore the image of

$$D\tilde{\rho} := \varprojlim D\tilde{\rho}_i: \mathfrak{u}(4, \mathbb{R}) \rightarrow \hat{\chi}(\mathbb{R}^2, 0)$$

is isomorphic to  $\mathfrak{n}$  (Proposition 5.2). So there exists a surjective map  $\psi$  from  $\mathfrak{u}(4, \mathbb{R})$  onto  $\mathfrak{n}$ . The kernel of  $\psi$  is an ideal of  $\mathfrak{u}(4, \mathbb{R})$  of dimension 2; hence  $\ker \psi = \langle \delta_{14}, a\delta_{13} + b\delta_{24} \rangle$  where the  $\delta_{ij}$  denote the Kronecker matrices. One concludes by remarking that  $\dim Z(\mathfrak{u}(4, \mathbb{R})/\ker \psi) = 2$  whereas  $\dim Z(\mathfrak{n}) = 1$ .  $\square$

**Corollary 5.5.** — *The image of a morphism from a subgroup of  $\text{SL}(n, \mathbb{Z})$  of finite index to  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$  is finite as soon as  $n \geq 4$ .*

## References

- [1] C. Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991.
- [2] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [3] M. Burger and N. Monod. Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc. (JEMS)*, 1(2):199–235, 1999.
- [4] S. Cantat. Version kählérienne d’une conjecture de Robert J. Zimmer. *Ann. Sci. École Norm. Sup. (4)*, 37(5):759–768, 2004.
- [5] S. Cantat. Sur les groupes de transformations birationnelles des surfaces. *Ann. of Math. (2)*, 174(1):299–340, 2011.
- [6] D. Cerveau and J.-F. Mattei. *Formes intégrables holomorphes singulières*, volume 97 of *Astérisque*. Société Mathématique de France, Paris, 1982. With an English summary.
- [7] J. Déserti. Groupe de Cremona et dynamique complexe: une approche de la conjecture de Zimmer. *Int. Math. Res. Not.*, pages Art. ID 71701, 27, 2006.

3. Let  $N_1$  and  $N_2$  be two connected, simply connected, nilpotent and algebraic subgroups on  $\mathbb{R}$ ; any continuous morphism between  $N_1$  and  $N_2$  is algebraic.



- [8] B. Farb and P. Shalen. Real-analytic actions of lattices. *Invent. Math.*, 135(2):273–296, 1999.
- [9] J. Franks and M. Handel. Area preserving group actions on surfaces. *Geom. Topol.*, 7:757–771 (electronic), 2003.
- [10] É. Ghys. Sur les groupes engendrés par des difféomorphismes proches de l’identité. *Bol. Soc. Brasil. Mat. (N.S.)*, 24(2):137–178, 1993.
- [11] É. Ghys. Actions de réseaux sur le cercle. *Invent. Math.*, 137(1):199–231, 1999.
- [12] A. I. Malcev. On a class of homogeneous spaces. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.*, 13:9–32, 1949.
- [13] A. I. Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Translation*, 1951(39):33, 1951.
- [14] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1991.
- [15] J. Martinet. Normalisation des champs de vecteurs holomorphes (d’après A.-D Brjuno). In *Séminaire Bourbaki (1980/1981)*, volume 901 of *Lecture Notes in Math.*, pages Exp. No. 765, pp. 103–119. Springer, Berlin, 1981.
- [16] A. Navas. Actions de groupes de Kazhdan sur le cercle. *Ann. Sci. École Norm. Sup. (4)*, 35(5):749–758, 2002.
- [17] L. Polterovich. Growth of maps, distortion in groups and symplectic geometry. *Invent. Math.*, 150(3):655–686, 2002.
- [18] M. S. Raghunathan. *Discrete subgroups of Lie groups*. Springer-Verlag, New York, 1972. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68*.
- [19] E. B. Vinberg, V. V. Gorbatsevich, and O. V. Shvartsman. Discrete subgroups of Lie groups. In *Lie groups and Lie algebras, II*, volume 21 of *Encyclopaedia Math. Sci.*, pages 1–123, 217–223. Springer, Berlin, 2000.
- [20] R. J. Zimmer. Kazhdan groups acting on compact manifolds. *Invent. Math.*, 75(3):425–436, 1984.
- [21] R. J. Zimmer. On connection-preserving actions of discrete linear groups. *Ergodic Theory Dynam. Systems*, 6(4):639–644, 1986.
- [22] R. J. Zimmer. Actions of semisimple groups and discrete subgroups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 1247–1258, Providence, RI, 1987. Amer. Math. Soc.
- [23] R. J. Zimmer. Lattices in semisimple groups and invariant geometric structures on compact manifolds. In *Discrete groups in geometry and analysis (New Haven, Conn., 1984)*, volume 67 of *Progr. Math.*, pages 152–210. Birkhäuser Boston, Boston, MA, 1987.