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Relaxation in a Fuzzy Dark Matter Halo. II. Self-consistent kinetic equations

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ABSTRACT

Fuzzy dark matter (FDM) is composed of ultra-light bosons having a de Broglie wavelength that is comparable to the size of the stellar component of galaxies at typical galactic velocities. FDM behaves like cold dark matter on large scales. However, on the scale of the de Broglie wavelength, an FDM halo exhibits density fluctuations that lead to relaxation, a process similar to the two-body relaxation that occurs in classical gravitational N-body systems and is described by the Fokker–Planck equation. We derive the FDM analog of that kinetic equation, and solve it to find the evolution of the velocity distribution in a spatially homogeneous FDM halo. We also determine the dielectric function and the dispersion relation for linear waves in an FDM halo.

1. INTRODUCTION

Cosmological models based on cold dark matter (CDM) explain most features of the cosmic microwave background, large-scale structure, and other cosmological phenomena. However, CDM has been less successful in predicting the properties of small-scale structure, such as the abundance of dwarf galaxies and the dark-matter density profiles near the centers of galaxies (e.g., Weinberg et al. 2015; Bullock & Boylan-Kolchin 2017). This shortcoming may reflect either our limited understanding of baryonic physics on these scales or deviations of the behavior of the dark matter from the predictions of CDM.

Fuzzy dark matter (FDM) is dark matter composed of bosons with mass $m_b \simeq 10^{-21}$ – 10^{-22} eV, so small that the de Broglie wavelength

$$\lambda = \frac{h}{m_b v} = 1.20 \text{ kpc} \frac{10^{-22} \text{ eV}}{m_b} \frac{100 \text{ km s}^{-1}}{v} \quad (1)$$

is comparable to galaxy scales at a typical galaxy velocity v (see, e.g., Hu et al. 2000; Marsh 2016; Hui et al. 2017). FDM behaves like CDM on scales much larger than the de Broglie wavelength and thus preserves the success of CDM in explaining the properties of large-scale structure and the cosmic microwave background. However, on small scales FDM behaves quite differently from CDM. In particular, FDM exhibits density fluctuations on the scale of the de Broglie wavelength that arise from interference patterns. These fluctuations never damp, in contrast to the fluctuations in the density of CDM that arise from incomplete phase mixing. The gravitational field from these fluctuations scatters both condensed baryonic objects—stars, globular clusters, black holes, etc.—and the FDM waves themselves.

Hui et al. (2017) argued, and Bar-Or et al. (2019) (hereafter Paper I) showed explicitly, that scattering of condensed objects by the FDM fluctuations can be analyzed by treating the FDM fluctuations as quasiparticles with effective mass of order the mass contained within the typical angular de Broglie wavelength, $\lambda_\sigma = \hbar/(m_b \sigma)$ with σ the one-dimensional velocity dispersion in the galaxy (see also El-Zant et al. 2020). In particular, Paper I computed the diffusion coefficients originating from a homogeneous FDM background that can be used in the classical Fokker–Planck (FP) equation to describe the evolution of the distribution function (DF) of a population of stars or other condensed objects.

The goal of this paper is to describe the evolution of the FDM distribution function itself due to scattering by these same fluctuations, by deriving the appropriate wave version of the FP equation. We shall focus on systems that are homogeneous on large scales after averaging over the fluctuations, although our results can be applied to inhomogeneous systems so long as the system size is large compared to the de Broglie wavelength. Relaxation of the FDM distribution may lead to the formation of a central soliton or Bose–Einstein condensate, but we do not study the formation of the condensate here.

The relaxation time of a test particle orbiting in a stellar system of density ρ_0 and velocity dispersion σ , composed of classical particles of mass m , is (Binney & Tremaine 2008, eq. 7.106)

$$t_{\text{relax}} \simeq 0.34 \frac{\sigma^3}{G^2 m \rho_0 \ln \Lambda}, \quad (2)$$

where $\ln \Lambda$ is the Coulomb logarithm, with $\Lambda \simeq R/b$ where R is the size of the system and b is the larger of Gm/σ^2 and the size of the test particle. In an FDM halo the effective mass is $m_{\text{eff}} \simeq \rho_0 \lambda_\sigma^3$ so the relaxation of the FDM halo takes place on a timescale

$$t_{\text{relax}} \simeq \frac{m_b^3 \sigma^6}{G^2 \rho_0^2 \hbar^3 \ln(R/\lambda_\sigma)}. \quad (3)$$

Our aim is to place this approximate result on a solid quantitative foundation.

Many of our results have appeared already in the literature in several contexts: weak turbulence, the nonlinear Schrödinger equation, quantum plasmas, etc. (see, e.g., [Levkov et al. 2018](#)). Nevertheless we have found it simpler and more transparent to provide self-contained derivations.

The present paper is organized as follows. In Section 2, we discuss the relaxation of particles and waves. In particular, we present a closed kinetic equation describing the self-consistent relaxation of a homogeneous FDM halo under the effects of its self-generated fluctuations. In Section 3, we present a first heuristic derivation of that kinetic equation relying on the Boltzmann–Nordheim–Uehling–Uhlenbeck (BNUU) equation. In Section 4, we revisit that same derivation, this time starting from a quasi-linear expansion of the coupled Schrödinger–Poisson (SP) system. In Section 5, we discuss the linear stability of FDM halo, and in Section 6 we present some applications of this generalized kinetic equation. Finally, we conclude in Section 7.

2. RELAXATION AND KINETIC EQUATIONS

We shall work with an infinite halo that is homogeneous in a time-averaged sense, and characterized by a mean density ρ_0 . We also invoke the Jeans swindle, that is, we ignore any acceleration due to ρ_0 ([Binney & Tremaine 2008](#), §5.2.2), and focus only on the effects due to the density perturbations. We define the halo DF $F_b(\mathbf{v})$ such that $F_b(\mathbf{v}) d\mathbf{r} d\mathbf{v}$ is the time-averaged mass of halo particles in the phase-space volume element $d\mathbf{r} d\mathbf{v}$. Thus the mean density and one-dimensional velocity dispersion are given by

$$\rho_0 = \int d\mathbf{v} F_b(\mathbf{v}); \quad 3\rho_0 \sigma^2 = \int d\mathbf{v} v^2 F_b(\mathbf{v}). \quad (4)$$

In order to highlight the connections between the classical and quantum cases, we will successively consider the case of the relaxation of classical particles induced by a classical halo (Section 2.1), then the relaxation of classical particles induced by a fuzzy halo (Section 2.2), and finally the relaxation of a fuzzy halo induced by itself (Section 2.3).

2.1. Relaxation of particles by particles

First we review relaxation in a halo composed of classical particles of mass m . Let $F(\mathbf{v})$ be the DF of a population of point-like classical test objects of mass m_t . The evolution of $F(\mathbf{v})$ due to interactions between the test objects and the background halo particles is described by the classical Landau equation ([Landau 1936](#); [Lifshitz & Pitaevskii 1981](#); [Chavanis 2013](#)), which reads

$$\frac{\partial F(\mathbf{v})}{\partial t} = 2G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \left[m F_b(\mathbf{v}') \frac{\partial F(\mathbf{v})}{\partial v_j} - m_t \frac{\partial F_b(\mathbf{v}')}{\partial v'_j} F(\mathbf{v}) \right], \quad (5)$$

where

$$u_{ij}(\mathbf{v}) = \int d\hat{\mathbf{k}} \hat{k}_i \hat{k}_j \delta_D(\hat{\mathbf{k}} \cdot \mathbf{v}) = \pi \frac{\partial^2 v}{\partial v_i \partial v_j} = \pi \frac{v^2 \delta_{ij} - v_i v_j}{v^3}, \quad (6)$$

is the collision kernel. Here $\hat{\mathbf{k}} = \mathbf{k}/k$ is a unit vector along \mathbf{k} and summation over repeated Cartesian indices is assumed. The Coulomb logarithm reads $\ln \Lambda = \ln(k_{\text{max}}/k_{\text{min}})$, with k_{max} and k_{min} the maximum and minimum wavenumbers that contribute to the relaxation. Typically k_{min}^{-1} is approximately the size of the halo or the radius of the orbit of the test object, and $k_{\text{max}} \simeq \sigma^2/[G(m_t + m)]$ corresponds to the scale associated with strong deflections in two-body encounters. The first term in Eq. (5) represents diffusion and is independent of the mass of the test objects, while the second term represents dynamical friction (also called friction due to polarization), and is independent of the mass of the halo particles at fixed halo density. This equation is equivalent to the standard FP equation of [Chandrasekhar \(1942\)](#).

2.2. Relaxation of particles by waves

Let us now assume that the diffusion of the classical test particles is being sourced by a background fuzzy halo composed of ultra-light particles of individual mass m_b , which must therefore be treated as waves.

As shown in Paper I, the Landau equation (5) for the evolution of classical test particles becomes

$$\frac{\partial F(\mathbf{v})}{\partial t} = 2G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \left\{ \left[m_b + \frac{h^3}{m_b^3} F_b(\mathbf{v}') \right] F_b(\mathbf{v}') \frac{\partial F(\mathbf{v})}{\partial v_j} - m_t \frac{\partial F_b(\mathbf{v}')}{\partial v'_j} F(\mathbf{v}) \right\}; \quad (7)$$

here $h = 2\pi\hbar$ and the factor k_{\max} in the Coulomb logarithm is modified to $k_{\max} \simeq \min\{m_b\sigma/\hbar, \sigma^2/[G(m_t + m_b)]\}$. The similarity with Eq. (5) is striking. The main difference arises in the diffusion term, where the mass of the background classical particles m is now replaced with $[m_b + (h/m_b)^3 F_b(\mathbf{v}')]^3$.

2.3. Relaxation of waves by waves

The Landau equation can also be generalized to describe the self-consistent evolution of a halo DF composed of ultra-light fuzzy particles. In this case, Eq. (7) becomes

$$\frac{\partial F_b(\mathbf{v})}{\partial t} = 2G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \left\{ \left[m_b + \frac{h^3}{m_b^3} F_b(\mathbf{v}') \right] F_b(\mathbf{v}') \frac{\partial F_b(\mathbf{v})}{\partial v_j} - \left[m_b + \frac{h^3}{m_b^3} F_b(\mathbf{v}) \right] \frac{\partial F_b(\mathbf{v}')}{\partial v'_j} F_b(\mathbf{v}) \right\}. \quad (8)$$

Once again, the similarity with Eq. (7) is striking. The main difference arises in the friction component, where m_t , the mass of the test particle, is now replaced with $[m_b + (h/m_b)^3 F_b(\mathbf{v})]$ (see also Lancaster et al. 2020 for an extensive discussion of dynamical friction in FDM halos). Deriving Eq. (8) is one of the main goals of the present paper. We give a physically motivated derivation of this result relying on the BNUU equation in Section 3, and a more rigorous derivation starting from the quasi-linear expansion of the SP system in Section 4. Before that, let us first discuss some of the main properties of Eq. (8).

2.4. Some properties of the Landau equation

First, we note that Eq. (8), like Eqs. (5) and (7), is a flux-conservative equation, that is, it has the form $\partial F_b/\partial t = -\partial F_i/\partial v_i$, with F_i the mass flux in direction v_i . Moreover, Eq. (8) can be written as a FP equation

$$\frac{\partial F_b(\mathbf{v})}{\partial t} = -\frac{\partial}{\partial v_i} [D_i(\mathbf{v}) F_b(\mathbf{v})] + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} [D_{ij}(\mathbf{v}) F_b(\mathbf{v})], \quad (9)$$

where the flux is

$$F_i = -D_i(\mathbf{v}) F_b(\mathbf{v}) + \frac{1}{2} \frac{\partial}{\partial v_j} [D_{ij}(\mathbf{v}) F_b(\mathbf{v})]. \quad (10)$$

The first- and second-order diffusion coefficients are given by

$$D_i(\mathbf{v}) = D_i^c(\mathbf{v}) + D_i^b(\mathbf{v}); \quad D_{ij}(\mathbf{v}) = D_{ij}^c(\mathbf{v}) + D_{ij}^b(\mathbf{v}). \quad (11)$$

Here, $D_i^c(\mathbf{v})$ and $D_{ij}^c(\mathbf{v})$ are the classical diffusion coefficients, while D_i^b and D_{ij}^b (“b” for boson) capture the contributions associated with wave interference. The first-order or “drift” coefficients read

$$\begin{aligned} D_i^c(\mathbf{v}) &= 4G^2 m_b \ln \Lambda \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \frac{\partial F_b(\mathbf{v}')}{\partial v'_j}, \\ D_i^b(\mathbf{v}) &= 2G^2 \frac{h^3}{m_b^3} \ln \Lambda \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \left[F_b(\mathbf{v}) \frac{\partial F_b(\mathbf{v}')}{\partial v'_j} + \frac{\partial F_b^2(\mathbf{v}')}{\partial v'_j} \right]. \end{aligned} \quad (12)$$

while the second-order or “diffusion” coefficients are

$$\begin{aligned} D_{ij}^c(\mathbf{v}) &= 4G^2 m_b \ln \Lambda \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') F_b(\mathbf{v}'), \\ D_{ij}^b(\mathbf{v}) &= 4G^2 \frac{h^3}{m_b^3} \ln \Lambda \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') F_b^2(\mathbf{v}'). \end{aligned} \quad (13)$$

We can rewrite these coefficients in terms of Rosenbluth potentials (Rosenbluth et al. 1957). We use the relations

$$u_{ij}(\mathbf{v} - \mathbf{v}') = \pi \frac{\partial^2}{\partial v_i \partial v_j} |\mathbf{v} - \mathbf{v}'|; \quad \frac{\partial}{\partial v_j} u_{ij}(\mathbf{v} - \mathbf{v}') = 2\pi \frac{\partial}{\partial v_i} \frac{1}{|\mathbf{v} - \mathbf{v}'|}. \quad (14)$$

Equation (12) then becomes

$$\begin{aligned}
D_i^c(\mathbf{v}) &= 8\pi G^2 m_b \ln \Lambda \frac{\partial}{\partial v_i} \int d\mathbf{v}' \frac{F_b(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|}, \\
D_i^b(\mathbf{v}) &= 4\pi G^2 \frac{h^3}{m_b^3} \ln \Lambda \left[\frac{\partial}{\partial v_i} \int d\mathbf{v}' \frac{F_b^2(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} + F_b(\mathbf{v}) \frac{\partial}{\partial v_i} \int d\mathbf{v}' \frac{F_b(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} \right] \\
&= 4\pi G^2 m_{\text{eff}} \ln \Lambda \left[\frac{\partial}{\partial v_i} \left[\int d\mathbf{v}' \frac{F_{\text{eff}}(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} \right] + \frac{F_{\text{eff}}(\mathbf{v})}{F_b(\mathbf{v})} \frac{\partial}{\partial v_i} \left[\int d\mathbf{v}' \frac{F_b(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} \right] \right]. \tag{15}
\end{aligned}$$

while Eq. (13) becomes

$$\begin{aligned}
D_{ij}^c(\mathbf{v}) &= 4\pi G^2 m_b \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int d\mathbf{v}' |\mathbf{v} - \mathbf{v}'| F_b(\mathbf{v}'), \\
D_{ij}^b(\mathbf{v}) &= 4\pi G^2 \frac{h^3}{m_b^3} \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int d\mathbf{v}' |\mathbf{v} - \mathbf{v}'| F_b^2(\mathbf{v}') \\
&= 4\pi G^2 m_{\text{eff}} \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int d\mathbf{v}' |\mathbf{v} - \mathbf{v}'| F_{\text{eff}}(\mathbf{v}'). \tag{16}
\end{aligned}$$

In these expressions, we introduced in particular an effective DF, $F_{\text{eff}}(\mathbf{v})$, and an effective mass, m_{eff} , through

$$F_{\text{eff}}(\mathbf{v}) = \frac{\int d\mathbf{v}' F_b(\mathbf{v}')}{\int d\mathbf{v}' F_b^2(\mathbf{v}')} F_b^2(\mathbf{v}); \quad m_{\text{eff}} = \frac{h^3 \int d\mathbf{v} F_b^2(\mathbf{v})}{m_b^3 \int d\mathbf{v} F_b(\mathbf{v})}. \tag{17}$$

As already discussed in Paper I and recovered in Eq. (16), the fuzzy diffusion term, $D_{ij}^b(\mathbf{v})$, is identical to the classical one, $D_{ij}^c(\mathbf{v})$, except that the mass of the particle and the DF are replaced by their effective counterparts m_{eff} and F_{eff} . In particular, when the underlying DF is Maxwellian, these diffusion coefficients are the same as those in a halo of classical particles with mass $m_{\text{eff}} = \rho_0 (\lambda_\sigma / \sqrt{4\pi})^3$ and effective velocity dispersion $\sigma_{\text{eff}} = \sigma / \sqrt{2}$. As a result, in the fuzzy case, the relaxation time from Eq. (2) becomes

$$t_{\text{relax}} \simeq 0.34 \frac{\sigma_{\text{eff}}^3}{G^2 m_{\text{eff}} \rho_0 \ln \Lambda} = 0.34 \frac{m_b^3 \sigma^6}{G^2 (2\pi)^{3/2} h^3 \rho_0^2 \ln \Lambda}. \tag{18}$$

When the DF is isotropic, i.e., when $F_b(\mathbf{v})$ depends only on $v = |\mathbf{v}|$, Eq. (8) can be rewritten in an even simpler form. To do so, we rely once again on the properties of the Rosenbluth potentials, see Eqs. (L22) and (L23) of Binney & Tremaine (2008). In particular, we can write

$$\int d\tilde{\mathbf{v}}' u_{ij}(\mathbf{v} - \mathbf{v}') = \begin{cases} 4\pi^2 \left(\frac{\delta_{ij}}{v} - \frac{v_i v_j}{v^3} - \frac{v'^2 \delta_{ij}}{3v^3} + \frac{v'^2 v_i v_j}{v^5} \right) & \text{for } v' < v, \\ \frac{8\pi^2}{3} \frac{\delta_{ij}}{v'} & \text{for } v < v', \end{cases} \tag{19}$$

as well as

$$\int d\tilde{\mathbf{v}}' u_{ij}(\mathbf{v} - \mathbf{v}') v'_j = \begin{cases} \frac{8\pi^2}{3} \frac{v'^2 v_i}{v^3} & \text{for } v' < v, \\ \frac{8\pi^2}{3} \frac{v_i}{v'} & \text{for } v < v', \end{cases} \tag{20}$$

where we recall that the sum over j is implied in the second equation. In order to get that last relation, we used the fact that $u_{ij}(\mathbf{v})v_j = 0$ in conjunction with Eq. (19).

Then, after some lengthy manipulations, Eq. (8) becomes

$$\begin{aligned}
\frac{\partial F_b(v)}{\partial t} &= \frac{16\pi^2 G^2 \ln \Lambda}{3} \frac{1}{v^2} \frac{d}{dv} \left\{ \frac{1}{v} \frac{dF_b(v)}{dv} \int_0^v dv' v'^4 F_b(v') \left[m_b + \frac{h^3}{m_b^3} F_b(v') \right] \right. \\
&\quad \left. + v^2 \frac{dF_b(v)}{dv} \int_v^\infty dv' v' F_b(v') \left[m_b + \frac{h^3}{m_b^3} F_b(v') \right] + 3F_b(v) \left[m_b + \frac{h^3}{m_b^3} F_b(v) \right] \int_0^v dv' v'^2 F_b(v') \right\}. \tag{21}
\end{aligned}$$

In the limit of high phase-space density, i.e., many particles per unit phase-space cell of volume h^3 , or

$$\frac{h^3}{m_b^4} F_b \gg 1, \quad \text{or} \quad m_{\text{eff}} \simeq \rho_0 \left(\frac{h}{m_b \sigma} \right)^3 \gg m_b, \quad (22)$$

we can reduce Eqs. (8) and (21) to

$$\begin{aligned} \frac{\partial F_b(\mathbf{v})}{\partial t} &= 2G^2 \ln \Lambda \frac{h^3}{m_b^3} \frac{\partial}{\partial v_i} \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \left[F_b^2(\mathbf{v}') \frac{\partial F_b(\mathbf{v})}{\partial v_j} - F_b^2(\mathbf{v}) \frac{\partial F_b(\mathbf{v}')}{\partial v'_j} \right] \\ &= \frac{16\pi^2 G^2 \ln \Lambda}{3} \frac{h^3}{m_b^3} \frac{1}{v^2} \frac{d}{dv} \left[\frac{1}{v} \frac{dF_b(v)}{dv} \int_0^v dv' v'^4 F_b^2(v') + v^2 \frac{dF_b(v)}{dv} \int_v^{+\infty} dv' v' F_b^2(v') + 3F_b^2(v) \int_0^v dv' v'^2 F_b(v') \right]. \end{aligned} \quad (23)$$

The second of these is equivalent to Eq. (S23) in [Levkov et al. \(2018\)](#).

The generic steady state of Eq. (21) is the Bose-Einstein DF,

$$F_b(v) = \frac{m_b^4}{h^3} \frac{1}{e^{\frac{1}{2}\beta v^2} / z - 1}. \quad (24)$$

In that expression, the inverse temperature β and the fugacity z are determined from the conservation of mass and kinetic energy per unit volume, following Eq. (4). In practice, one has

$$\rho_0 = \frac{m_b^4 (2\pi)^{3/2}}{h^3 \beta^{3/2}} \text{Li}_{3/2}(z); \quad \rho_0 \sigma^2 = \frac{m_b^4 (2\pi)^{3/2}}{h^3 \beta^{5/2}} \text{Li}_{5/2}(z), \quad (25)$$

with $\text{Li}_n(z) = \sum_{k=1}^{+\infty} z^k / k^n$ the polylogarithm. Combining these equations to eliminate β , we can write

$$\sigma^3 = \frac{h^3 \rho_0}{m_b^4 (2\pi)^{3/2}} \frac{\text{Li}_{5/2}^{3/2}(z)}{\text{Li}_{3/2}^{5/2}(z)}, \quad (26)$$

which has a minimum value, σ_c , reached at $z = 1$,

$$\begin{aligned} \sigma_c^3 &= \frac{\zeta^{3/2}(5/2)}{\zeta^{5/2}(3/2)} \frac{h^3 \rho_0}{m_b^4 (2\pi)^{3/2}} \\ &\simeq 1.09 (100 \text{ km s}^{-1})^3 \frac{\rho_0}{0.01 M_\odot \text{ pc}^{-3}} \left(\frac{m_b}{20 \text{ eV}} \right)^{-4}, \end{aligned} \quad (27)$$

with $\zeta(x) = \text{Li}_x(1)$ the Riemann zeta function. The corresponding value of β is then given by

$$\beta_c = \frac{\zeta(5/2)}{\zeta(3/2)} \frac{1}{\sigma_c^2} = \left[\frac{m_b \zeta(3/2)}{\rho_0} \right]^{2/3} \frac{m_b^2}{2\pi \hbar^2}. \quad (28)$$

Systems with an initial velocity dispersion below σ_c cannot occupy the steady state given by Eq. (24). In this case the steady state is given by

$$F_b(v) = \frac{m_b^4}{h^3} \frac{1}{e^{\frac{1}{2}\beta v^2} - 1} + \rho_s \delta_D(\mathbf{v}), \quad (29)$$

where the last term is the Bose-Einstein condensate or soliton¹. In that expression, the inverse temperature follows from the second of Eqs. (25) and reads

$$\beta^{5/2} = \frac{m_b^4 (2\pi)^{3/2}}{h^3 \rho_0 \sigma^2} \zeta(5/2). \quad (30)$$

Finally, the mass density in the soliton is determined from the conservation of mass and energy. As such, it is given by

$$\rho_s = \left(1 - \frac{\beta_c^{3/2}}{\beta^{3/2}} \right) \rho_0 = \left(1 - \frac{\sigma^{6/5}}{\sigma_c^{6/5}} \right) \rho_0. \quad (31)$$

As stated in Section 1, we do not consider the formation of the condensate or soliton in this paper.

¹ For the typical boson masses $m_b = 10^{-21}$ – 10^{-22} eV suggested for FDM, σ_c is much larger than galaxy velocity dispersions and the thermal equilibrium state is always a condensate.

3. DERIVING THE KINETIC EQUATION FROM THE BOLTZMANN–NORDHEIM–UEHLING–UHLENBECK (BNUU) EQUATION

In this section we give a (relatively) simple derivation of Eq. (8). The BNUU equation (Nordheim 1928; Uehling & Uhlenbeck 1933; Erdős et al. 2004) is a heuristic generalization of the Boltzmann equation to quantum systems, see also. For a homogeneous system the BNUU equation reads

$$\begin{aligned} \frac{\partial f(\mathbf{p}_1)}{\partial t} = & \int d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 S^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) \{ f(\mathbf{p}_3) f(\mathbf{p}_4) [1 + \epsilon h^3 f(\mathbf{p}_1)] [1 + \epsilon h^3 f(\mathbf{p}_2)] \\ & - f(\mathbf{p}_1) f(\mathbf{p}_2) [1 + \epsilon h^3 f(\mathbf{p}_3)] [1 + \epsilon h^3 f(\mathbf{p}_4)] \}, \end{aligned} \quad (32)$$

with $\epsilon = 0$ for a classical system, $+1$ for bosons, and -1 for fermions. Here \mathbf{p} is the momentum and $f(\mathbf{p}) d\mathbf{r} d\mathbf{p}$ is the number of particles in a phase-space volume element $d\mathbf{r} d\mathbf{p}$ (the dependence of the DF $f(\mathbf{p})$ on time is not shown explicitly). The function $S^{(4)}$ describes the rate at which particles with momenta \mathbf{p}_1 and \mathbf{p}_2 are scattered to momenta \mathbf{p}_3 and \mathbf{p}_4 . More precisely, $S^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) f(\mathbf{p}_1) f(\mathbf{p}_2) d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4$ is the rate per unit volume at which particles are scattered from the momentum-space volumes $d\mathbf{p}_1$ and $d\mathbf{p}_2$ into the volumes $d\mathbf{p}_3$ and $d\mathbf{p}_4$. For $\epsilon = 0$, Eq. (32) reduces to the classical Boltzmann equation, and for $\epsilon = -1$ factors such as $1 - h^3 f(\mathbf{p}_3)$ ensure that no particles are scattered into states that are fully occupied according to the Pauli principle.

Since momentum is conserved in collisions, the function $S^{(4)}$ must contain a factor $\delta_D(\mathbf{p}_3 + \mathbf{p}_4 - \mathbf{p}_1 - \mathbf{p}_2)$ and we use this to carry out the integral over \mathbf{p}_4 . Then we replace \mathbf{p}_3 by the momentum transfer $\mathbf{q} = \mathbf{p}_3 - \mathbf{p}_1$ to obtain

$$\begin{aligned} \frac{\partial f(\mathbf{p}_1)}{\partial t} = & \int d\mathbf{p}_2 d\mathbf{q} S(\mathbf{p}_1 + \frac{1}{2}\mathbf{q}, \mathbf{p}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q}) \{ f(\mathbf{p}_1 + \mathbf{q}) f(\mathbf{p}_2 - \mathbf{q}) [1 + \epsilon h^3 f(\mathbf{p}_1)] [1 + \epsilon h^3 f(\mathbf{p}_2)] \\ & - f(\mathbf{p}_1) f(\mathbf{p}_2) [1 + \epsilon h^3 f(\mathbf{p}_1 + \mathbf{q})] [1 + \epsilon h^3 f(\mathbf{p}_2 - \mathbf{q})] \}. \end{aligned} \quad (33)$$

Here we have rewritten $S^{(4)}$ as a function of three variables, $S(\mathbf{p}_1 + \frac{1}{2}\mathbf{q}, \mathbf{p}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q}) \equiv S^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 + \mathbf{q}, \mathbf{p}_2 - \mathbf{q}) \delta_D(\mathbf{p}_1 + \mathbf{q} + \mathbf{p}_2 - \mathbf{q} - \mathbf{p}_1 - \mathbf{p}_2)$. For gravitational scattering between particles of mass m_b , S is given by (e.g., Goodman 1983)

$$S(\mathbf{a}, \mathbf{b}; \mathbf{q}) = \frac{4G^2 m_b^5}{q^4} \delta_D[(\mathbf{b} - \mathbf{a}) \cdot \mathbf{q}]. \quad (34)$$

In that expression, the delta function ensures that the relative momentum is conserved in the collision, $|\mathbf{p}_4 - \mathbf{p}_3| = |\mathbf{p}_2 - \mathbf{p}_1|$, and the factor q^{-4} reflects the angular dependence of the Coulomb differential scattering cross-section, $|\sin^2(\theta/2)|^{-4}$, where θ is the scattering angle.

We now assume that \mathbf{q} is small (equivalent to the FP approximation of weak deflections) and that S varies slowly with \mathbf{q} in its first two arguments. This allows us then to expand Eq. (33) to second order in \mathbf{q} . We abbreviate the notation by writing $f_1 = f(\mathbf{p}_1)$, $\partial_t f = \partial f / \partial t$, $\partial_{1,i} f = \partial f / \partial p_{1i}$, $\partial_{1,ij} f = \partial^2 f / \partial p_{1i} \partial p_{1j}$, etc. We also assume summation over repeated indices. Then

$$\begin{aligned} \partial_t f(\mathbf{p}_1) = & \int d\mathbf{p}_2 d\mathbf{q} \left(S + \frac{1}{2} q_i \partial_{1,i} S - \frac{1}{2} q_i \partial_{2,i} S \right) \\ & \times \left[(f_1 + q_i \partial_{1,i} f_1 + \frac{1}{2} q_i q_j \partial_{1,ij} f_1) (f_2 - q_k \partial_{2,k} f_2 + \frac{1}{2} q_k q_m \partial_{2,km} f_2) (1 + \epsilon h^3 f_1) (1 + \epsilon h^3 f_2) \right. \\ & \left. - f_1 f_2 (1 + \epsilon h^3 f_1 + \epsilon h^3 q_i \partial_{1,i} f_1 + \frac{1}{2} \epsilon h^3 q_i q_j \partial_{1,ij} f_1) (1 + \epsilon h^3 f_2 - \epsilon h^3 q_k \partial_{2,k} f_2 + \frac{1}{2} \epsilon h^3 q_k q_m \partial_{2,km} f_2) \right]; \end{aligned} \quad (35)$$

here the function $S(\mathbf{p}_1 + \frac{1}{2}\mathbf{q}, \mathbf{p}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q})$ was only expanded to first order in \mathbf{q} , as the terms within brackets vanish at zero order in \mathbf{q} . In that expression, S stands for $S(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q})$ which is even in \mathbf{q} by Eq. (34) (or more generally by detailed balance). As a result, the terms that are first order in \mathbf{q} vanish when integrated. Thus, Eq. (35) simplifies to

$$\begin{aligned} \partial_t f(\mathbf{p}_1) = & \int d\mathbf{p}_2 d\mathbf{q} q_i q_j \left\{ \frac{1}{2} S [\partial_{1,ij} f_1 f_2 (1 + \epsilon h^3 f_2) + \partial_{2,ij} f_2 f_1 (1 + \epsilon h^3 f_1) - 2 \partial_{1,i} f_1 \partial_{2,j} f_2 (1 + \epsilon h^3 f_1 + \epsilon h^3 f_2)] \right. \\ & \left. + \frac{1}{2} (\partial_{1,i} S - \partial_{2,i} S) [\partial_{1,j} f_1 f_2 (1 + \epsilon h^3 f_2) - \partial_{2,j} f_2 f_1 (1 + \epsilon h^3 f_1)] \right\}. \end{aligned} \quad (36)$$

Integrating the term involving $\partial_{2,i}S$ by parts gives

$$\begin{aligned}\partial_t f(\mathbf{p}_1) &= \int d\mathbf{p}_2 d\mathbf{q} q_i q_j \left\{ \frac{1}{2} S [\partial_{1,ij} f_1 f_2 (1 + \epsilon h^3 f_2) - \partial_{1,i} f_1 \partial_{2,j} f_2 (1 + 2\epsilon h^3 f_1)] \right. \\ &\quad \left. + \frac{1}{2} \partial_{1,i} S [\partial_{1,j} f_1 f_2 (1 + \epsilon h^3 f_2) - \partial_{2,j} f_2 f_1 (1 + \epsilon h^3 f_1)] \right\} \\ &= \frac{1}{2} \partial_{1,i} \int d\mathbf{p}_2 d\mathbf{q} q_i q_j S [\partial_{1,j} f_1 f_2 (1 + \epsilon h^3 f_2) - \partial_{2,j} f_2 f_1 (1 + \epsilon h^3 f_1)]\end{aligned}\quad (37)$$

Following Eq. (34), the integral over \mathbf{q} yields

$$\int d\mathbf{q} q_i q_j S(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) = 4G^2 m_b^5 \ln \Lambda u_{ij}(\mathbf{p}_1 - \mathbf{p}_2), \quad (38)$$

where u_{ij} is the collision kernel from Eq. (6), and $\Lambda = q_{\max}/q_{\min}$ with q_{\max} and q_{\min} the maximum and minimum impact parameters included in the integral.

Now change variables from momentum \mathbf{p} to velocity $\mathbf{v} = \mathbf{p}/m_b$, from \mathbf{v}_1 and \mathbf{v}_2 to \mathbf{v} and \mathbf{v}' , and from the number density in position-momentum phase space $f(\mathbf{p})$ to the mass density in position-velocity space $F_b(\mathbf{v}) = m_b^4 f(m_b \mathbf{v})$. We find

$$\frac{\partial F_b(\mathbf{v})}{\partial t} = 2G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \left\{ \left[m_b + \epsilon \frac{h^3}{m_b^3} F_b(\mathbf{v}') \right] F_b(\mathbf{v}') \frac{\partial F_b(\mathbf{v})}{\partial v_j} - \left[1 + \epsilon \frac{h^3}{m_b^3} F_b(\mathbf{v}) \right] \frac{\partial F_b(\mathbf{v}')}{\partial v'_j} F_b(\mathbf{v}) \right\}. \quad (39)$$

which proves that Eq. (8) follows directly from the BNUU equation. Of course, the present derivation remains heuristic because it stems from the heuristic BNUU equation (32). In the following section, we will present a more careful derivation of Eq. (8), through a detailed study of the self-consistent dynamics of the wavefunction of an FDM halo.

4. DERIVING THE KINETIC EQUATION FROM THE SCHRÖDINGER–POISSON EQUATION

We now revisit the derivation of the kinetic equation (8) by studying the gravitational interaction of waves $\psi(\mathbf{r}, t)$ that evolve according to the SP equations (Ruffini & Bonazzola 1969)

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m_b} \nabla^2 \psi(\mathbf{r}, t) + m_b \Phi(\mathbf{r}, t) \psi(\mathbf{r}, t), \quad (40)$$

$$\nabla^2 \Phi(\mathbf{r}, t) = 4\pi G (|\psi(\mathbf{r}, t)|^2 - \rho_0). \quad (41)$$

In the Poisson equation, the term $-\rho_0$ appears because of the Jeans swindle, already described in Section 2. We also note that the wavefunction $\psi(\mathbf{r}, t)$ is normalized so that $|\psi(\mathbf{r}, t)|^2$ is a mass density, i.e., one has $\langle |\psi(\mathbf{r}, t)|^2 \rangle = \rho_0$.

The Wigner function (Wigner 1932)

$$W(\mathbf{r}, \mathbf{v}, t) = \int \frac{d\mathbf{s}}{(2\pi)^3} e^{-i\mathbf{v}\cdot\mathbf{s}} \psi\left(\mathbf{r} + \frac{1}{2}\hbar\mathbf{s}/m_b, t\right) \psi^*\left(\mathbf{r} - \frac{1}{2}\hbar\mathbf{s}/m_b, t\right) \quad (42)$$

is the quantum analog to the discrete classical phase-space DF, $F_d(\mathbf{r}, \mathbf{v}, t) = m_b \sum_{i=1}^N \delta_D[\mathbf{r} - \mathbf{r}_i(t)] \delta_D[\mathbf{v} - \mathbf{v}_i(t)]$. As such, $W(\mathbf{r}, \mathbf{v}, t)$ can be interpreted classically as the density of bosons in the infinitesimal volume $d\mathbf{r}d\mathbf{v}$. When averaged over realisations, $\overline{W} = \langle W \rangle$ is the quantum analog to the mean phase-space DF, $F_b = \langle F_d \rangle$, i.e., the DF whose kinetic equation we want to derive. With our conventions, we also note that W and \overline{W} satisfy the normalisations

$$\int d\mathbf{v} W(\mathbf{r}, \mathbf{v}, t) = |\psi(\mathbf{r}, t)|^2; \quad \int d\mathbf{v} \overline{W}(\mathbf{v}) = \rho_0, \quad (43)$$

where we used the relation $\int d\mathbf{v}/(2\pi)^3 e^{-i\mathbf{v}\cdot\mathbf{s}} = \delta_D(\mathbf{s})$.

The time evolution of $W(\mathbf{r}, \mathbf{v}, t)$ follows from the Schrödinger equation (40). More precisely, using the relation

$$\nabla^2 \psi(\mathbf{r} \pm \frac{1}{2}\hbar\mathbf{s}/m_b, t) = \pm \frac{2m_b}{\hbar} \frac{\partial}{\partial \mathbf{s}} \cdot \left[\nabla \psi(\mathbf{r} \pm \frac{1}{2}\hbar\mathbf{s}/m_b, t) \right] \quad (44)$$

to perform an integration by parts with respect to $d\mathbf{s}$ in Eq. (42), we obtain

$$\begin{aligned}\frac{\partial W(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla W(\mathbf{r}, \mathbf{v}, t) &= i \frac{m_b}{\hbar} \int \frac{d\mathbf{s}}{(2\pi)^3} e^{-i\mathbf{v}\cdot\mathbf{s}} \left[\Phi\left(\mathbf{r} - \frac{1}{2}\hbar\mathbf{s}/m_b, t\right) - \Phi\left(\mathbf{r} + \frac{1}{2}\hbar\mathbf{s}/m_b, t\right) \right] \\ &\quad \times \psi\left(\mathbf{r} + \frac{1}{2}\hbar\mathbf{s}/m_b, t\right) \psi^*\left(\mathbf{r} - \frac{1}{2}\hbar\mathbf{s}/m_b, t\right).\end{aligned}\quad (45)$$

To better emphasize the analogy with the classical case, we can rewrite Eq. (45) in the shorter form

$$\frac{\partial W(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla W(\mathbf{r}, \mathbf{v}, t) = \mathfrak{D}[\Phi(\mathbf{r}, t), W(\mathbf{r}, \mathbf{v}, t)], \quad (46)$$

where the non-linear term from the r.h.s. reads

$$\mathfrak{D}[\Phi(\mathbf{r}, t), W(\mathbf{r}, \mathbf{v}, t)] \equiv i \int d\mathbf{k} \int_{\mathcal{B}} \frac{d\omega}{2\pi} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \widehat{\Phi}(\mathbf{k}, \omega) \widehat{\mathfrak{D}}_{\mathbf{k}}[W(\mathbf{r}, \mathbf{v}, t)]. \quad (47)$$

In this expression, the Laplace-Fourier transformed potential, $\widehat{\Phi}(\mathbf{k}, \omega)$, is introduced with the convention

$$\widehat{\Phi}(\mathbf{k}, \omega) = \int \frac{d\mathbf{r}}{(2\pi)^3} \int_0^\infty dt e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \Phi(\mathbf{r}, t); \quad \Phi(\mathbf{r}, t) = \int d\mathbf{k} \int_{\mathcal{B}} \frac{d\omega}{2\pi} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \widehat{\Phi}(\mathbf{k}, \omega), \quad (48)$$

where the Bromwich contour, \mathcal{B} , has to pass above all the poles of the integrand, i.e., $\text{Im}(\omega)$ has to be large enough. In Eq. (47), we also defined the finite-difference operator

$$\widehat{\mathfrak{D}}_{\mathbf{k}}[W(\mathbf{v})] = \frac{m_b}{\hbar} \left[W\left(\mathbf{v} + \frac{\hbar}{2m_b} \mathbf{k}\right) - W\left(\mathbf{v} - \frac{\hbar}{2m_b} \mathbf{k}\right) \right]. \quad (49)$$

We already note that in the classical limit, $\hbar \rightarrow 0$, this finite-difference operator satisfies

$$\widehat{\mathfrak{D}}_{\mathbf{k}}[W(\mathbf{r}, \mathbf{v}, t)] \rightarrow \mathbf{k} \cdot \frac{\partial W(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}}; \quad \mathfrak{D}[\Phi(\mathbf{r}, t), W(\mathbf{r}, \mathbf{v}, t)] \rightarrow \nabla \Phi(\mathbf{r}, t) \cdot \frac{\partial W(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}}; \quad (50)$$

so that Eq. (46) reduces to the classical collisionless Boltzmann equation (Binney & Tremaine 2008)

$$\frac{\partial W}{\partial t} + \mathbf{v} \cdot \nabla W - \nabla \Phi \cdot \frac{\partial W}{\partial \mathbf{v}} = 0. \quad (51)$$

Our goal now is to describe how the unavoidable quantum fluctuations in the system lead to the relaxation of the halo's underlying mean DF. To pursue this, we perform a quasi-linear expansion of Eq. (46). This is a standard procedure in kinetic theory and we generally follow the method presented in Chavanis (2012b) in the context of the relaxation of classical discrete self-gravitating systems.

We write the Wigner distribution as a perturbation around its ensemble average,

$$W = \overline{W} + f. \quad (52)$$

Since we used the Jeans swindle in Eq. (41), we note that $\Phi(\mathbf{r}, t)$ is already the fluctuating potential. Separating the average and fluctuating components of Eq. (46), we obtain two evolution equations for the instantaneous fluctuations, $\partial f / \partial t$, and the system's mean DF, $\partial \overline{W} / \partial t$. They read

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla f(\mathbf{r}, \mathbf{v}, t) = \mathfrak{D}[\Phi(\mathbf{r}, t), \overline{W}(\mathbf{r}, \mathbf{v}, t)], \quad (53)$$

$$\frac{\partial \overline{W}(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla \overline{W}(\mathbf{r}, \mathbf{v}, t) = \langle \mathfrak{D}[\Phi(\mathbf{r}, t), f(\mathbf{r}, \mathbf{v}, t)] \rangle, \quad (54)$$

where Eq. (53) has been linearized in the perturbations by neglecting the quadratic term $\mathfrak{D}[\Phi(\mathbf{r}, t), f(\mathbf{r}, \mathbf{v}, t)]$ therein. These equations are valid in the weak-coupling limit where $t_d \ll t_{\text{relax}}$. In the present case, the dynamical time t_d and the relaxation time t_{relax} are approximately

$$t_d \simeq \max \left[\frac{\hbar}{m_b \sigma^2}, (G\rho_0)^{-1/2} \right]; \quad t_{\text{relax}} \simeq \min \left[\frac{m_b^3 \sigma^6}{G^2 \rho_0^2 \hbar^3}, \frac{\sigma^3}{G^2 m_b \rho_0} \right]. \quad (55)$$

Here the classical dynamical time is $(G\rho_0)^{-1/2}$ and the classical relaxation time is given by Eq. (2); if the wavelike nature of the particles is important the dynamical time is $\sim \lambda_\sigma / \sigma$, the time taken for a particle traveling at the typical speed to cross a de Broglie wavelength, and the relaxation time is given by Eq. (3).

In the weak-coupling limit, we can assume that \overline{W} is a (quasi)-stationary solution of the l.h.s. of Eq. (54), i.e., we can assume that $\overline{W} = \overline{W}(\mathbf{v}, t)$ is homogeneous and therefore only slowly changes in time. The l.h.s. of Eq. (54) is then the ensemble-averaged collisionless Boltzmann equation that describes the free streaming of particles in the absence of any potential perturbations, while its r.h.s. is the collision term. In particular, we note that since both the potential $\Phi(\mathbf{r}, t)$ and the perturbed DF $f(\mathbf{r}, \mathbf{v}, t)$ are quadratic in the wavefunction, the collision term from Eq. (54) involves a product of four factors $\psi(\mathbf{r}, t)$ evaluated at different locations.

Following these assumptions, we can take the Laplace-Fourier transform of Eq. (53), which gives

$$\widehat{f}(\mathbf{k}, \mathbf{v}, \omega) = -\frac{\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})]}{\omega - \mathbf{k} \cdot \mathbf{v}} \widehat{\Phi}(\mathbf{k}, \omega) + i \frac{\widehat{f}_0(\mathbf{k}, \mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v}}, \quad (56)$$

where the Fourier transforms with respect to space and the Laplace transform with respect to time are defined with the conventions of Eq. (48). We also assumed that $\overline{W}(\mathbf{v})$ can be taken as constant on the timescales over which the fluctuations evolve. In Eq. (56), we also introduced

$$\widehat{f}_0(\mathbf{k}, \mathbf{v}) = \int \frac{d\mathbf{r}}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{r}} f_0(\mathbf{r}, \mathbf{v}), \quad (57)$$

as the Fourier transform of the fluctuations of the DF at the initial time, $f_0(\mathbf{r}, \mathbf{v}) = f(\mathbf{r}, \mathbf{v}, t=0)$.

Using the normalisations from Eq. (43), we can rewrite the Poisson equation (41) in the simple form

$$\nabla^2 \Phi(\mathbf{r}, t) = 4\pi G \int d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t), \quad (58)$$

which becomes in Fourier-Laplace space

$$\widehat{\Phi}(\mathbf{k}, \omega) = -\frac{4\pi G}{k^2} \int d\mathbf{v} \widehat{f}(\mathbf{k}, \mathbf{v}, \omega). \quad (59)$$

We now have at our disposal Eqs. (56) and (59), which jointly couple the DF and potential fluctuations, $\widehat{f}(\mathbf{k}, \mathbf{v}, \omega)$ and $\widehat{\Phi}(\mathbf{k}, \mathbf{v}, \omega)$. Solving these self-consistently amounts then to accounting for collective effects, i.e., accounting for the ability of the system to amplify its own self-generated perturbations. To make progress, the traditional solution is to act on both sides of Eq. (56) with the same operator as in the r.h.s. of Eq. (59). One immediately obtains

$$\widehat{\Phi}(\mathbf{k}, \omega) = \frac{1}{\epsilon(\mathbf{k}, \omega)} \frac{4\pi G}{ik^2} \int d\mathbf{v} \frac{\widehat{f}_0(\mathbf{k}, \mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v}}, \quad (60)$$

where the dielectric function is

$$\epsilon(\mathbf{k}, \omega) = 1 - \frac{4\pi G}{k^2} \int d\mathbf{v} \frac{\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})]}{\omega - \mathbf{k} \cdot \mathbf{v}}. \quad (61)$$

As in Eq. (50), one can straightforwardly obtain the classical dielectric function through the substitution $\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}] \rightarrow \mathbf{k} \cdot \partial \overline{W} / \partial \mathbf{v}$. As usual, the dielectric function can be rewritten using Landau's prescription. Following our convention from Eq. (48), this amounts to making the replacement $\omega_{\mathbf{R}} \rightarrow \omega_{\mathbf{R}} + i0^+$ and using the Plemelj formula

$$\frac{1}{\omega_{\mathbf{R}} + i0^+} = \mathcal{P} \left(\frac{1}{\omega_{\mathbf{R}}} \right) - i\pi \delta_{\mathbf{D}}(\omega_{\mathbf{R}}), \quad (62)$$

for $\omega_{\mathbf{R}} \in \mathbb{R}$ and with \mathcal{P} being Cauchy's principal value. We will further discuss the properties of that dielectric function in Section 5, when investigating the linear stability of the present system.

We can now turn to Eq. (54) in order to relate the long-term evolution of the mean DF, $\partial \overline{W}(\mathbf{v}, t) / \partial t$, to the correlations of the initial DF fluctuations, $\widehat{f}_0(\mathbf{k}, \mathbf{v})$. Starting from Eq. (54), we can use the definition from Eq. (47) to write

$$\frac{\partial \overline{W}(\mathbf{v}, t)}{\partial t} = i \int d\mathbf{k} d\mathbf{k}' e^{i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}} \widehat{\mathcal{D}}_{\mathbf{k}'} \left[\int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} e^{-i(\omega + \omega')t} \langle \widehat{f}(\mathbf{k}, \mathbf{v}, \omega) \widehat{\Phi}(\mathbf{k}', \omega') \rangle \right], \quad (63)$$

In principle the r.h.s. depends on position \mathbf{r} but we shall argue below that this dependence vanishes. Since Eq. (56) has two terms, we get two contributions,

$$\frac{\partial \overline{W}(\mathbf{v}, t)}{\partial t} = F_1(\mathbf{v}) + F_2(\mathbf{v}), \quad (64)$$

where we introduced

$$\begin{aligned} F_1(\mathbf{v}) &= -\int d\mathbf{k}d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \widehat{\mathcal{D}}_{\mathbf{k}'} \left[\int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{e^{-i(\omega+\omega')t}}{\omega - \mathbf{k}\cdot\mathbf{v}} \langle \widehat{f}_0(\mathbf{k}, \mathbf{v}) \widehat{\Phi}(\mathbf{k}', \omega') \rangle \right] \\ F_2(\mathbf{v}) &= -i \int d\mathbf{k}d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \widehat{\mathcal{D}}_{\mathbf{k}'} \left[\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})] \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{e^{-i(\omega+\omega')t}}{\omega - \mathbf{k}\cdot\mathbf{v}} \langle \widehat{\Phi}(\mathbf{k}, \omega) \widehat{\Phi}(\mathbf{k}', \omega') \rangle \right], \end{aligned} \quad (65)$$

which, as we will show, respectively capture the contributions from the drift and diffusion components. To proceed further, we rewrite these two expressions using Eq. (60). After some manipulations, these two components become

$$\begin{aligned} F_1(\mathbf{v}) &= i \int d\mathbf{k}d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \widehat{\mathcal{D}}_{\mathbf{k}'} \left[\frac{4\pi G}{k'^2} \int d\mathbf{v}' \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(\mathbf{k}', \omega')} \frac{e^{-i(\omega+\omega')t}}{(\omega - \mathbf{k}\cdot\mathbf{v})(\omega' - \mathbf{k}'\cdot\mathbf{v}')} \langle \widehat{f}_0(\mathbf{k}, \mathbf{v}) \widehat{f}_0(\mathbf{k}', \mathbf{v}') \rangle \right], \\ F_2(\mathbf{v}) &= i \int d\mathbf{k}d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \widehat{\mathcal{D}}_{\mathbf{k}'} \left[\frac{(4\pi G)^2}{k^2 k'^2} \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})] \int d\mathbf{v}' d\mathbf{v}'' \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(\mathbf{k}, \omega) \epsilon(\mathbf{k}', \omega')} \frac{e^{-i(\omega+\omega')t}}{(\omega - \mathbf{k}\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v}')(\omega' - \mathbf{k}'\cdot\mathbf{v}'')} \right. \\ &\quad \left. \times \langle \widehat{f}_0(\mathbf{k}, \mathbf{v}') \widehat{f}_0(\mathbf{k}', \mathbf{v}'') \rangle \right]. \end{aligned} \quad (66)$$

To pursue the calculation further, we must now characterize the properties of the correlations of the initial fluctuations in the system. In a homogeneous system, such correlations can only depend on the positions \mathbf{r} and \mathbf{r}' of the two points through their difference $\mathbf{r} - \mathbf{r}'$. Moreover we assume that the particles' velocities are chosen independently from the time-averaged DF $F_b(\mathbf{v})$, so the correlation function must vanish if $\mathbf{v} \neq \mathbf{v}'$. Therefore we can write

$$\begin{aligned} \langle f_0(\mathbf{r}, \mathbf{v}) f_0(\mathbf{r}', \mathbf{v}') \rangle &= \delta_D(\mathbf{v} - \mathbf{v}') C(\mathbf{r} - \mathbf{r}', \mathbf{v}), \\ \langle \widehat{f}_0(\mathbf{k}, \mathbf{v}) \widehat{f}_0(\mathbf{k}', \mathbf{v}') \rangle &= \delta_D(\mathbf{k} + \mathbf{k}') \delta_D(\mathbf{v}' - \mathbf{v}') \widehat{C}(\mathbf{k}, \mathbf{v}), \end{aligned} \quad (67)$$

with $\widehat{C}(\mathbf{k}, \mathbf{v})$ the Fourier transform of the correlation function $C(\mathbf{r} - \mathbf{r}', \mathbf{v})$. Since the DF $W(\mathbf{r}, \mathbf{v}, t)$ is real, $f_0(\mathbf{r}, \mathbf{v})$ is also real. Therefore $C(\mathbf{r}, \mathbf{v})$ is real and from its definition it is also an even function of \mathbf{r} . Thus, $\widehat{C}(\mathbf{k}, \mathbf{v})$ is real and an even function of \mathbf{k} .

We postpone to Appendix B the explicit calculation of the function $\widehat{C}(\mathbf{k}, \mathbf{v})$. Inserting Eq. (67) in Eq. (66), we get

$$\begin{aligned} F_1(\mathbf{v}) &= -i \int d\mathbf{k} \widehat{\mathcal{D}}_{\mathbf{k}} \left[\frac{4\pi G}{k^2} \widehat{C}(\mathbf{k}, \mathbf{v}) \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(-\mathbf{k}, \omega')} \frac{e^{-i(\omega+\omega')t}}{(\omega - \mathbf{k}\cdot\mathbf{v})(\omega' + \mathbf{k}\cdot\mathbf{v})} \right], \\ F_2(\mathbf{v}) &= -i \int d\mathbf{k} \widehat{\mathcal{D}}_{\mathbf{k}} \left[\frac{(4\pi G)^2}{k^4} \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})] \int d\mathbf{v}' \widehat{C}(\mathbf{k}, \mathbf{v}') \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(\mathbf{k}, \omega) \epsilon(-\mathbf{k}, \omega')} \frac{e^{-i(\omega+\omega')t}}{(\omega - \mathbf{k}\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v}')(\omega' + \mathbf{k}\cdot\mathbf{v}')} \right], \end{aligned} \quad (68)$$

where we have used the relation $\widehat{\mathcal{D}}_{-\mathbf{k}}[\overline{W}(\mathbf{v})] = -\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})]$.

If we assume that the system is initially stable, then all the inverse Laplace transforms present in Eq. (68) can be explicitly computed, as detailed in Appendix A. We get

$$\begin{aligned} F_1(\mathbf{v}) &= i \int d\mathbf{k} \widehat{\mathcal{D}}_{\mathbf{k}} \left[\frac{4\pi G}{k^2} \frac{\widehat{C}(\mathbf{k}, \mathbf{v})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}) \right], \\ F_2(\mathbf{v}) &= -i \int d\mathbf{k} \widehat{\mathcal{D}}_{\mathbf{k}} \left(\frac{(4\pi G)^2}{k^4} \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})] \int d\mathbf{v}' \frac{\widehat{C}(\mathbf{k}, \mathbf{v}')}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}')|^2} \left\{ \mathcal{P} \left[\frac{1}{\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')} \right] + i\pi \delta_D[\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')] \right\} \right). \end{aligned} \quad (69)$$

The dielectric function $\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})$ is given by equation (61) and the Plemelj formula (62),

$$\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}) = 1 - \frac{4\pi G}{k^2} \mathcal{P} \int d\mathbf{v}' \frac{\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v}')] }{\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')} + \frac{4\pi^2 G}{k^2} i \int d\mathbf{v}' \delta_D[\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')] \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v}')]. \quad (70)$$

Since $\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})]$ is an odd function of \mathbf{k} , the contribution of the real part of the dielectric function integrates to zero in the expression for $F_1(\mathbf{v})$ (this result also follows from the physical argument that $F_1(\mathbf{v})$ must be real). Similar arguments can be used to simplify the expression for $F_2(\mathbf{v})$. All in all, we get

$$\begin{aligned} F_1(\mathbf{v}) &= -\pi \int d\mathbf{k} \widehat{\mathcal{D}}_{\mathbf{k}} \left[\frac{(4\pi G)^2}{k^4} \int d\mathbf{v}' \delta_D[\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')] \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v}')] \frac{\widehat{C}(\mathbf{k}, \mathbf{v})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \right], \\ F_2(\mathbf{v}) &= \pi \int d\mathbf{k} \widehat{\mathcal{D}}_{\mathbf{k}} \left[\frac{(4\pi G)^2}{k^4} \int d\mathbf{v}' \delta_D[\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')] \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v}')] \frac{\widehat{C}(\mathbf{k}, \mathbf{v}')}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}')|^2} \right]. \end{aligned} \quad (71)$$

Glancing back at Eq. (64), we can finally rewrite the kinetic equation as

$$\frac{\partial \overline{W}}{\partial t} = \pi \int d\mathbf{k} \widehat{\mathcal{D}}_{\mathbf{k}} \left[\frac{(4\pi G)^2}{k^4} \int d\mathbf{v}' \frac{\delta_{\text{D}}[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \left(\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})] \widehat{C}(\mathbf{k}, \mathbf{v}') - \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v}')] \widehat{C}(\mathbf{k}, \mathbf{v}) \right) \right]. \quad (72)$$

Equation (72) is equivalent to Eq. (9) in [Kadomtsev & Pogutse \(1970\)](#) except that there the gravitational interactions are replaced with Coulomb interactions, paying careful attention to the change of sign of the attraction, while the finite-difference operators are replaced with their classical limits, as in Eq. (50). In order to finalize the calculation, it now only remains to compute explicitly the autocorrelation function, $\widehat{C}(\mathbf{k}, \mathbf{v})$, as defined in Eq. (67).

We compute this correlation function in Appendix B,

$$\widehat{C}(\mathbf{k}, \mathbf{v}) = \frac{1}{(2\pi)^3} \left[m_{\text{b}} + \frac{h^3}{m_{\text{b}}^3} \overline{W}(\mathbf{v}) \right] \overline{W}(\mathbf{v}). \quad (73)$$

As a consequence, Eq. (72) becomes

$$\begin{aligned} \frac{\partial \overline{W}(\mathbf{v})}{\partial t} = 2G^2 \int d\mathbf{k} \int d\mathbf{v}' \widehat{\mathcal{D}}_{\mathbf{k}} \left\{ \frac{\delta_{\text{D}}[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]}{k^4 |\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})] \overline{W}(\mathbf{v}') \left[m_{\text{b}} + \frac{h^3}{m_{\text{b}}^3} \overline{W}(\mathbf{v}') \right] \right. \\ \left. - \frac{\delta_{\text{D}}[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]}{k^4 |\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v}')] \overline{W}(\mathbf{v}) \left[m_{\text{b}} + \frac{h^3}{m_{\text{b}}^3} \overline{W}(\mathbf{v}) \right] \right\}. \quad (74) \end{aligned}$$

The FP equation is based on the approximation that deflections are weak or that the momentum change due to gravitational scattering is small (see discussion preceding Eq. 35). In the present context, this corresponds to the approximation that \mathbf{k} is small compared to the characteristic scale of changes in $\overline{W}(\mathbf{v})$. In this case we can replace the discrete derivatives in $\widehat{\mathcal{D}}_{\mathbf{k}}[\overline{W}(\mathbf{v})]$ with their continuous analogs, as in the first of Eqs. (50). Equation (74) then becomes

$$\begin{aligned} \frac{\partial \overline{W}(\mathbf{v})}{\partial t} = 2G^2 \int d\mathbf{k} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \int d\mathbf{v}' \frac{\delta_{\text{D}}[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]}{k^4 |\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \\ \times \left\{ \mathbf{k} \cdot \frac{\partial \overline{W}(\mathbf{v}')}{\partial \mathbf{v}} \overline{W}(\mathbf{v}') \left[m_{\text{b}} + \frac{h^3}{m_{\text{b}}^3} \overline{W}(\mathbf{v}') \right] - \mathbf{k} \cdot \frac{\partial \overline{W}(\mathbf{v})}{\partial \mathbf{v}'} \overline{W}(\mathbf{v}) \left[m_{\text{b}} + \frac{h^3}{m_{\text{b}}^3} \overline{W}(\mathbf{v}) \right] \right\}. \quad (75) \end{aligned}$$

Equation (75) is the main result of this section, a Balescu–Lenard (BL) type kinetic equation describing the self-consistent relaxation of a homogeneous FDM halo. In particular, we note that this kinetic equation involves the dielectric function, $1/|\epsilon|^2$, which describes how the fluctuations are dressed by collective effects.

In Section 5 we show that this system is unstable for perturbations with wavenumber smaller than the effective Jeans scale $\sim \min\{k_{\text{J}}, \tilde{k}_{\text{J}}\}$, where k_{J} is the classical Jeans wavenumber and \tilde{k}_{J} its quantum analog. Assuming that the system is much smaller than the effective Jeans allows us to neglect collective effects and set $1/|\epsilon|^2 \rightarrow 1$. Equation (75) finally becomes

$$\begin{aligned} \frac{\partial \overline{W}(\mathbf{v})}{\partial t} = 2G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\mathbf{v}' u_{ij}(\mathbf{v} - \mathbf{v}') \\ \times \left\{ \frac{\partial \overline{W}(\mathbf{v}')}{\partial v_j} \overline{W}(\mathbf{v}') \left[m_{\text{b}} + \frac{h^3}{m_{\text{b}}^3} \overline{W}(\mathbf{v}') \right] - \frac{\partial \overline{W}(\mathbf{v})}{\partial v'_j} \overline{W}(\mathbf{v}) \left[m_{\text{b}} + \frac{h^3}{m_{\text{b}}^3} \overline{W}(\mathbf{v}) \right] \right\} \quad (76) \end{aligned}$$

where $u_{ij}(\mathbf{v})$ is defined in Eq. (6). We have therefore reached our final result, as we have recovered the kinetic equation (8) describing the self-consistent relaxation of a homogeneous FDM halo.

5. LINEAR STABILITY

As for many physical systems, the dielectric function (61) is central to understanding the dynamical behavior of an FDM halo. To illustrate the utility of this function we explore the stability properties of the halo. Throughout this section, we will assume for simplicity that the unperturbed DF is isotropic, $F_{\text{b}}(\mathbf{v}) = F_{\text{b}}(v)$.

In the classical or particle limit, an infinite homogeneous system is susceptible to an instability characterized by the classical Jeans wavenumber k_{J} ([Jeans 1902](#); [Binney & Tremaine 2008](#)). If the DF is Maxwellian,

$$F_{\text{b}}(\mathbf{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-|\mathbf{v}|^2/(2\sigma^2)}, \quad (77)$$

then

$$k_J = \frac{(4\pi G \rho_0)^{1/2}}{\sigma}, \quad (78)$$

and perturbations with $k \leq k_J$ are unstable, while ones with $k > k_J$ are stable.

In contrast, a halo composed of waves rather than particles that has zero velocity dispersion (i.e., the unperturbed wavefunction $\psi(\mathbf{r}, t) = \text{cst.}$) is unstable to perturbations with wavenumber $k < \tilde{k}_J$ where the quantum Jeans wavenumber is (see, e.g., Khlopov et al. 1985; Bianchi et al. 1990; Hu et al. 2000; Chavanis 2011)

$$\tilde{k}_J = 2 \left(\frac{\pi G \rho_0 m_b^2}{\hbar^2} \right)^{1/4}. \quad (79)$$

In an FDM halo with non-zero velocity dispersion, the effective Jeans wavenumber can be determined from the dielectric function (61). For an isotropic DF,

$$\epsilon(k, \omega) = 1 - \frac{4\pi G m_b}{k^2 \hbar} \int \frac{du}{\omega - ku} \left[F_{\parallel} \left(u + \frac{\hbar k}{2m_b} \right) - F_{\parallel} \left(u - \frac{\hbar k}{2m_b} \right) \right], \quad (80)$$

with $F_{\parallel}(u) = \int dv_1 dv_2 F_b[(v_1^2 + v_2^2 + u^2)^{1/2}]$ and v_1 and v_2 are the velocities along the two axes perpendicular to \mathbf{k} . The system is linearly unstable if there exists a frequency ω in the upper half of the complex plane and a real wavenumber k such that $\epsilon(k, \omega) = 0$. In order to investigate the system's stability, we place ourselves at the limit of marginal stability, i.e., we assume that $\text{Im}(\omega) \rightarrow 0$. In that case, we can use the Plemelj formula from Eq. (62) to rewrite Eq. (80) as

$$\begin{aligned} \epsilon(k, \omega) = & 1 - \frac{4\pi G m_b}{k^2 \hbar} \mathcal{P} \int \frac{du}{\omega - ku} \left[F_{\parallel} \left(u + \frac{\hbar k}{2m_b} \right) - F_{\parallel} \left(u - \frac{\hbar k}{2m_b} \right) \right] \\ & + i \frac{4\pi^2 G m_b}{k^3 \hbar} \left[F_{\parallel} \left(\frac{\omega}{k} + \frac{\hbar k}{2m_b} \right) - F_{\parallel} \left(\frac{\omega}{k} - \frac{\hbar k}{2m_b} \right) \right]. \end{aligned} \quad (81)$$

In order to have $\epsilon(k, \omega) = 0$, both the real part and the imaginary part of Eq. (81) must vanish. Let us assume for simplicity that the DF $F_b(v)$ is a monotonic decreasing function of $v = |\mathbf{v}|$ (i.e., we ignore two-stream instabilities). Then it is straightforward to show that $F_{\parallel}(u)$ is an even function of u , monotonic decreasing for $u > 0$ and monotonic increasing for $u < 0$. As a consequence, the second term from Eq. (81) vanishes if and only if $k = 0$ or $\omega = 0$. Because that first possibility is not of physical interest, we may then assume that $\omega = 0$ when investigating the system's marginal stability.

For simplicity, let us now assume that the system's unperturbed DF is Maxwellian (see Eq. (77)). In that case, we can rewrite the dielectric function from Eq. (80) as

$$\begin{aligned} \epsilon(k, \omega) = & 1 - \frac{4\pi G m_b \rho_0}{\sqrt{2\pi} k^2 \sigma \hbar} \int \frac{dv}{\omega - kv} \left[e^{-[v + \hbar k / (2m_b)]^2 / (2\sigma^2)} - e^{-[v - \hbar k / (2m_b)]^2 / (2\sigma^2)} \right] \\ = & 1 - \left(\frac{k_J}{k} \right)^3 \frac{1}{2\sqrt{\pi}\eta} \int \frac{dx}{\varpi - x} \left[e^{-[x + k\eta / (2k_J)]^2} - e^{-[x - k\eta / (2k_J)]^2} \right] \\ = & 1 - \left(\frac{k_J}{k} \right)^3 \frac{1}{2\eta} \left[Z \left(\varpi - \frac{k\eta}{2k_J} \right) - Z \left(\varpi + \frac{k\eta}{2k_J} \right) \right] \end{aligned} \quad (82)$$

In that expression, we introduced the rescaled frequency $\varpi = \omega / (\sqrt{2} k \sigma)$, as well as the dimensionless ratio

$$\begin{aligned} \eta = & \sqrt{2} \left(\frac{k_J}{\tilde{k}_J} \right)^2 = \frac{\sqrt{2\pi G \rho_0} \hbar}{\sigma^2 m_b} \\ \simeq & 0.0315 \left(\frac{m_b}{10^{-22} \text{ eV}} \right)^{-1} \left(\frac{\rho_0}{0.01 M_{\odot} \text{ pc}^{-3}} \right)^{1/2} \left(\frac{\sigma}{100 \text{ km s}^{-1}} \right)^{-2}. \end{aligned} \quad (83)$$

We note that $\eta \ll 1$ corresponds to the classical limit, while $\eta \gg 1$ is associated with the quantum limit. In the last line of Eq. (82), we introduced the plasma dispersion function (see, e.g., [Fried & Conte 1961](#)), defined as

$$Z(\varpi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s - \varpi} = \begin{cases} \int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s - \varpi} & \text{if } \text{Im}(\varpi) > 0, \\ \mathcal{P} \int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s - \varpi} + \pi i e^{-\varpi^2} & \text{if } \text{Im}(\varpi) = 0, \\ \int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s - \varpi} + 2\pi i e^{-\varpi^2} & \text{if } \text{Im}(\varpi) < 0. \end{cases} \quad (84)$$

Following Eq. (82), the requirement of marginal stability at $\varpi = 0$ now leads to the implicit relation

$$k_c^3 = \frac{k_J^3}{2\eta} \left[Z\left(-\frac{k_c\eta}{2k_J}\right) - Z\left(\frac{k_c\eta}{2k_J}\right) \right], \quad (85)$$

with k_c the critical wavenumber at marginal stability. For $x \in \mathbb{R}$, we can use the two expansions ([Fried & Conte 1961](#))

$$Z(x) \simeq \begin{cases} i\sqrt{\pi}e^{-x^2} - 2x & \text{for } |x| \ll 1, \\ i\sqrt{\pi}e^{-x^2} - \frac{1}{x} & \text{for } |x| \gg 1. \end{cases} \quad (86)$$

Then we can approximate the r.h.s. of Eq. (85) as

$$k_c \simeq \begin{cases} k_J & \text{for } \eta \ll 1, \\ \tilde{k}_J & \text{for } \eta \gg 1. \end{cases} \quad (87)$$

We therefore recover the known result that in the limit $\eta \ll 1$ ($k_J \ll \tilde{k}_J$), the system's stability is determined by classical physics and the critical wavenumber is k_J , while in the limit $\eta \gg 1$ ($k_J \gg \tilde{k}_J$), stability is dominated by quantum effects and the critical wavenumber is \tilde{k}_J . This behaviour is illustrated in Fig. 1, where we show the critical wavenumber for a Maxwellian DF as determined from Eq. (85), as a function of the ratio k_J/\tilde{k}_J .

[Chavanis \(2011\)](#) derived a simple dispersion relation by assuming that the fuzzy halo is a fluid with a sound speed c_s . Assuming that this parameter is a proxy for the halo velocity dispersion, i.e., $c_s = \sigma$, Eq. (138) of [Chavanis \(2011\)](#) gives the simple dispersion relation

$$k_{c_s}^2 = \frac{k_J^2}{\eta^2} \left[\sqrt{1 + 2\eta^2} - 1 \right], \quad (88)$$

so that, in this model, perturbations with $k > k_{c_s}$ are stable, while ones with $k < k_{c_s}$ are unstable. The prediction of Eq. (88) is also illustrated in Fig. 1; it correctly recovers the transition between the classical and quantum regimes as one varies the ratio k_J/\tilde{k}_J .

In Fig. 2, we illustrate the importance of the dressing of relaxation by collective effects, through the factor $1/|\epsilon(k, \omega)|^2$ as appearing in Eq. (75). Here the dielectric function $\epsilon(k, \omega)$ is given by Eq. (82). We note that this dressing becomes negligible (i.e., $\epsilon(k, \omega) \rightarrow 1$) as the system becomes more stable (i.e., $k \gg k_c$). Moreover, for a given value of k/k_c , as quantum effects become more important (i.e., $\eta \gg 1$) collective effects become negligible, $\epsilon(k, \omega) \sim \eta^{-3}$.

6. NUMERICAL APPLICATION

In this final section we present some numerical simulations of FDM halos, and compare them with the predictions from the kinetic theory derived above. In order to investigate the long-term relaxation of the system, we also present direct time integrations of the diffusion equation itself, in particular highlighting the unavoidable formation of the central soliton for cold enough initial conditions.

We use numerical methods similar to those of [Levkov et al. \(2018\)](#). Importantly, in order to evade the Jeans instability, we assume $G < 0$, which ensures that the system is stable and does not affect kinetic equations such as (8) as they only contain G^2 . We consider a three-dimensional box of length L that is discretized in K^3 cells. Each location on the grid is characterized by a position $\mathbf{r} = \Delta_r \mathbf{n}$, with $\Delta_r = L/K$ and $\mathbf{n} \in \{0, \dots, K-1\}^3$. At each grid location, we track the local value of the wavefunction $\psi_{\mathbf{r}}(t)$ as well as the gravitational potential $\Phi_{\mathbf{r}}(t)$.

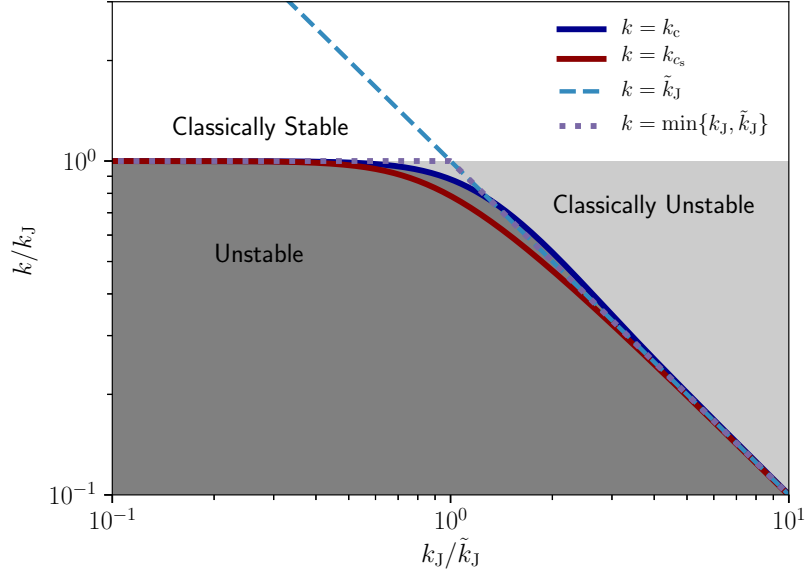


Figure 1. Linear stability of a Maxwellian DF. The critical wavenumber k_c that separates stable and unstable perturbations is shown as a function of the ratio between the classical Jeans wavenumber k_J and the quantum Jeans wavenumber \tilde{k}_J (blue line). Perturbations with wavenumber $k > k_c$ are stable and perturbations with wavenumber $k < k_c$ are unstable. This critical wavenumber k_c can be approximated by $\min\{k_J, \tilde{k}_J\}$ (dotted line). The critical wavenumber k_{c_s} for a fluid system (see Eq. 88 and Chavanis 2011) is also shown.

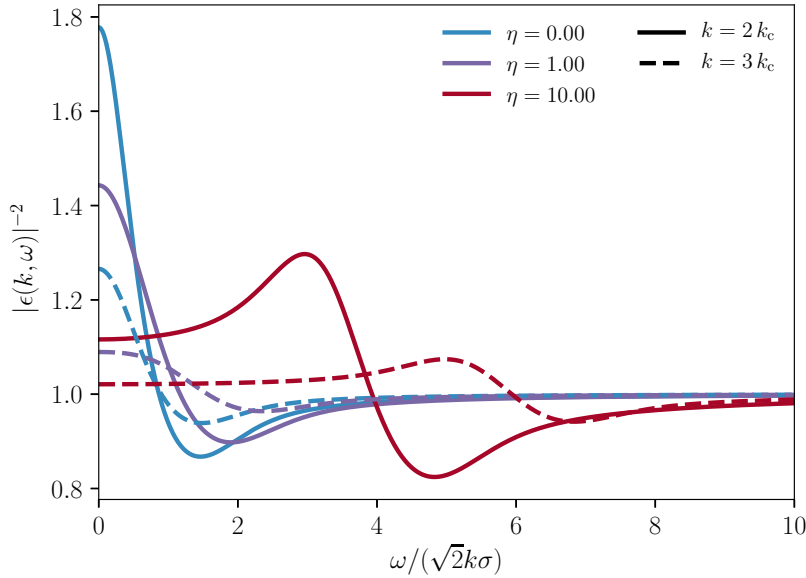


Figure 2. Illustration of the self-gravitating dressing of perturbations by collective effects, as measured by the dielectric function $1/|\epsilon(k, \omega)|^2$, for different values of the stability parameter k/k_c , and the quantum parameter η .

The initial conditions are set so that the wavefunction approximates a uniform density Maxwellian DF, as defined in Eq. (77). To do so, we naturally perform the discrete Fourier expansion

$$\hat{\psi}_{\mathbf{k}}(t) = \frac{\Delta_r^3}{(2\pi)^3} \sum_{\mathbf{r}} \psi_{\mathbf{r}}(t) e^{-i\mathbf{k}\cdot\mathbf{r}}; \quad \psi_{\mathbf{r}}(t) = \Delta_k^3 \sum_{\mathbf{k}} \hat{\psi}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (89)$$

where we introduced $\mathbf{k} = \Delta_k \mathbf{n}$, with $\Delta_k = 2\pi/L$ and $\mathbf{n} \in \{-K/2, \dots, -1, 1, \dots, K/2\}^3$. Each of the Fourier wavenumbers is then initialized with

$$\widehat{\psi}_{\mathbf{k}}(0) = \sqrt{f_k(k)} e^{i\phi_{\mathbf{k}}}; \quad f_k(k) = \frac{\hbar^3}{\Delta_k^3 m_b^3} F_b(v = \hbar k/m_b), \quad (90)$$

where $\phi_{\mathbf{k}}$ is a random phase uniformly distributed in $[0, 2\pi]$ and uncorrelated on the \mathbf{k} -grid, that is $\langle \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \rangle \propto \delta_{\mathbf{k}\mathbf{k}'}$. Once the wavefunction is known, we can compute its associated density $|\psi_{\mathbf{r}}|^2$. This is subsequently used in the Poisson equation (40) to estimate the potential $\Phi_{\mathbf{r}}$. The calculation of the potential is performed in Fourier space, using a FFT (i.e., assuming periodic boundary conditions), and further accelerated by GPU.

Once $\psi_{\mathbf{r}}(t)$ and $\Phi_{\mathbf{r}}(t)$ are known, we may proceed with the forward integration in time of the SP equations. This is performed through appropriate sequences of kick and drift operators, given by

$$\begin{aligned} \text{Drift: } \widehat{\psi}_{\mathbf{k}} &\rightarrow \widehat{\psi}_{\mathbf{k}} e^{-i\hbar\Delta t|\mathbf{k}|^2/(2m_b)}, \\ \text{Kick: } \psi_{\mathbf{r}} &\rightarrow \psi_{\mathbf{r}} e^{-i\Delta t m_b \Phi_{\mathbf{r}}/\hbar}. \end{aligned} \quad (91)$$

The timestep Δt and order of the integrator have to be picked carefully. In practice, we used a sixth-order explicit symplectic integrator (Yoshida 1990). Once we are able to perform numerical simulations of the system, we measure the value of $\partial F_b(v = \hbar k/m_b)/\partial t$ by fitting the function $t \rightarrow \langle |\widehat{\psi}_{\mathbf{k}}(t)|^2 \rangle \Delta_k^3 m_b^3 / \hbar^3$ with a linear function of t , and performing an ensemble average over 10^3 realisations with different initial conditions.

Having integrated the FDM dynamics, we may now compare the results with the prediction from kinetic theory. In the limit where collective effects can be neglected, the system's relaxation is described by Eq. (8). In particular, for an isotropic system with a Maxwellian DF, as in Eq. (77), We note that the flux generated by the classical diffusion coefficients D_i^c and D_{ij}^c (see Eq. 11) vanishes exactly, so the only surviving contributions are from the wave diffusion coefficients D_i^b and D_{ij}^b . With this simplification we can obtain from Eq. (21) the diffusion flux at $t = 0$ through

$$\left. \frac{\partial F_b(v)}{\partial t} \right|_{t=0} = \frac{(2\pi)^{1/2} G^2 \hbar^3 \rho_0^3 \ln \Lambda}{2m_b^3 \sigma^9} \left[18 e^{-3v^2/(2\sigma^2)} + \pi^{1/2} \frac{\sigma}{v} e^{-v^2/(2\sigma^2)} \text{erf}(v/\sigma) - 2^{7/2} \pi^{1/2} \frac{\sigma}{v} e^{-v^2/\sigma^2} \text{erf}[v/(2^{1/2}\sigma)] \right], \quad (92)$$

an exact quantitative expression of the approximate relaxation time (3).

In Fig. 3, we compare the numerical simulations with the prediction from Eq. (92), using an estimate $\ln \Lambda \simeq 2.1$. As illustrated

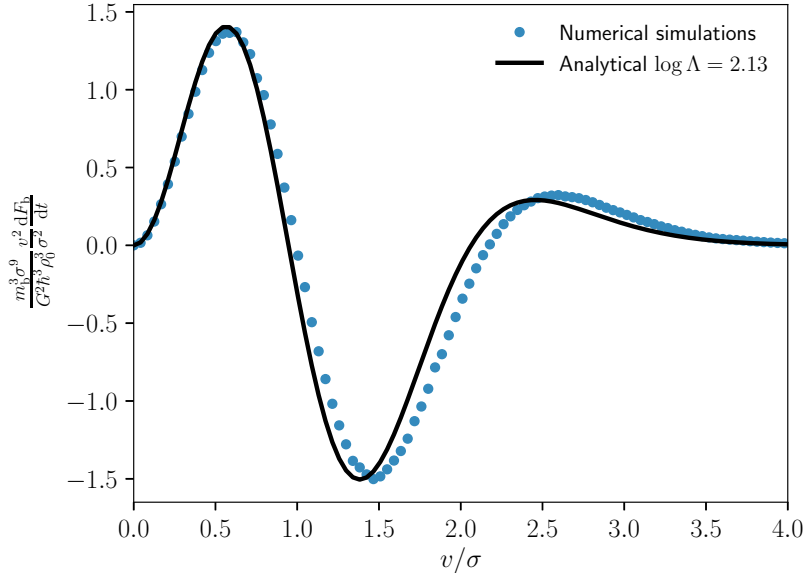


Figure 3. The initial rate of change of a Maxwellian DF, as measured in the numerical simulations is in a good agreement with the analytical prediction from Eq. (92) with $\ln \Lambda \simeq 2.1$. For our numerical experiments, we used a box of size $L = 150 \times \lambda_\sigma$ with $\lambda_\sigma = \hbar/(m_b \sigma)$, discretized in 256^3 cells. We picked the initial density to be $\rho_0 = 50 \times m_b \lambda_\sigma$ and fixed the (negative) gravitational constant to $G = -0.06 \times \sigma \hbar/m_b$.

in this figure, we recover a good agreement between the numerical simulations and the kinetic prediction.

In addition to integrating the SP equations directly, we investigated the evolution of the DF itself by directly integrating forward in time the isotropic Landau equation (21). To do so, we divide the interval $0 \leq v \leq v_{\max}$ onto a regular grid. At each of the grid locations, the isotropic integrals from Eq. (21) are computed using explicit second-order integration rules. Once the evolution rate, $\partial F_b(v)/\partial t$, has been determined on the velocity grid, we integrate it forward in time using a first-order explicit Euler method with a timestep given by $\Delta t = 10^{-3} \times \min_v [F_b(v)/(\partial F_b(v)/\partial t)]$.

Using that method, we show in Fig. 4 that a Maxwellian DF with $\sigma \simeq 1.3\sigma_c$ (see Eq. (27)) relaxes to a Bose-Einstein steady state in a few relaxation times (as defined in Eq. (18)). In that same figure, we also recover that when $\sigma = \sigma_c$, the DF develops a

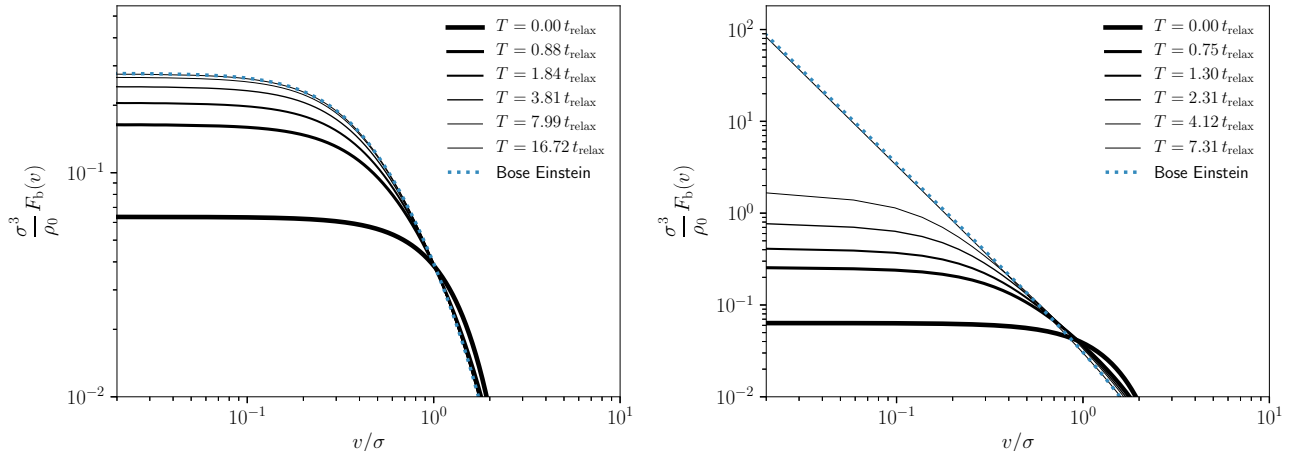


Figure 4. Relaxation of a Maxwellian DF with $\sigma = 1.3\sigma_c$ (left panel) and $\sigma = \sigma_c$ (right panel). Here, the integration of Eq. (21) was performed on a linear grid in v with 2^{11} cells and $v_{\max} = 10 \times \sigma$. Time has been rescaled as a function of t_{relax} , as defined in Eq. (18).

$F_b \propto 1/v^2$ cusp at small v on finite time, so that the system is almost on the verge of forming the central soliton.

7. CONCLUSIONS

In this paper, we investigated the self-consistent relaxation of an FDM halo driven by the unavoidable and undamped quantum fluctuations that it must sustain. The main result was presented in Eq. (8), which is the appropriate generalisation of the classical Fokker–Planck (FP) equation to the quantum case. We showed how this kinetic equation can be derived either from the heuristic Boltzmann–Nordheim–Uehling–Uhlenbeck (BNUU) equation (Section 3), or from the quasi-linear perturbation of the Schrödinger–Poisson (SP) system (Section 4). We showed in particular how the diffusion can be accelerated through collective effects that can dress the perturbations. The strength of the collective effects is encapsulated in the dielectric function (Eq. 61), as illustrated in Eq. (75) with a Balescu–Lenard (BL) type kinetic equation. We subsequently described in Section 5 the linear stability of a homogeneous FDM halo, making clear the connection between the classical and quantum limits. Finally, we illustrated some of these results in Section 6 using tailored numerical simulations.

Of course, the present paper is only a first step towards a description of the evolution of an FDM halo. First, it is important to extend the present derivation to inhomogeneous FDM halos. So long as the typical de Broglie wavelength λ_σ is small compared to the size of the halo, a good first approximation would be to treat the FDM diffusion coefficients (15) and (16) as local diffusion coefficients in an inhomogeneous FP equation analogous to (9). For more accurate analyses, one would benefit in particular from the recent progress in the context of classical self-gravitating systems that recently led to the derivation of the inhomogeneous BL equation (Heyvaerts 2010; Chavanis 2012a). As highlighted in Figure 4, FDM halos with a sufficiently small velocity dispersion—which includes all galaxy velocity dispersions if the particle mass $m_b \ll 1 \text{ eV}$ —unavoidably relax to a Bose-Einstein condensate. Describing the formation and growth of that condensate is essential for understanding the long-term fate of FDM halos. Finally, should FDM prove to be a viable alternative to classical CDM, the present kinetic theory needs to be implemented in a cosmological context (see, e.g., Amin & Mocz 2019).

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APPENDIX

A. COMPUTING THE INVERSE LAPLACE TRANSFORMS

In this section, we compute explicitly the Laplace transforms from Eq. (68). The approach we follow is very similar to the one from Schekochihin (2017).

Let us first start with the inverse Laplace transforms appearing in the expression for $F_1(\mathbf{v})$ in Eq. (68). We want to evaluate

$$I(\mathbf{k}, \mathbf{v}) = \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(-\mathbf{k}, \omega')} \frac{e^{-i(\omega+\omega')t}}{(\omega - \mathbf{k} \cdot \mathbf{v})(\omega' + \mathbf{k} \cdot \mathbf{v})}. \quad (\text{A1})$$

Recall that each integral is along a Bromwich contour \mathcal{B} , a horizontal contour that passes above all the poles of the integrands. We assume that the system is linearly stable, so the function $\omega \rightarrow 1/\epsilon(\mathbf{k}, \omega)$ has no poles in the upper half of the complex plane. We carry out the integrals by lowering the integration contours to very negative imaginary values, so that $e^{-i(\omega+\omega')t}$ vanishes. Assuming that t is large enough for the transients associated with the system's damped modes (i.e., the contributions from the poles of the dielectric functions) to be negligible, only the contributions from the poles on the real axis $\omega = \mathbf{k} \cdot \mathbf{v}$ and $\omega' = -\mathbf{k} \cdot \mathbf{v}$ remain. Paying careful attention to the direction of integration, we note that these poles each contribute $-2\pi i$ times the associated residue. Using these arguments, we obtain

$$I(\mathbf{k}, \mathbf{v}) = -\frac{\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2}. \quad (\text{A2})$$

Here we have used the symmetry relation

$$\epsilon(-\mathbf{k}, -\omega_R) = \epsilon^*(\mathbf{k}, \omega_R), \quad (\text{A3})$$

for $\omega_R \in \mathbb{R}$, which directly follows from Eq. (61).

The second integral to compute appears in the expression for $F_2(\mathbf{v})$ in Eq. (68). We must evaluate

$$J(\mathbf{k}, \mathbf{v}, \mathbf{v}') = \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(\mathbf{k}, \omega) \epsilon(-\mathbf{k}, \omega')} \frac{e^{-i(\omega+\omega')t}}{(\omega - \mathbf{k} \cdot \mathbf{v})(\omega - \mathbf{k} \cdot \mathbf{v}')(\omega' + \mathbf{k} \cdot \mathbf{v}')}. \quad (\text{A4})$$

Using the same approach as in Eq. (A1), we first perform the integral over ω' , noting that it involves a single pole along the real axis at $\omega' = -\mathbf{k} \cdot \mathbf{v}'$, the other poles being damped. Equation (A4) then becomes

$$J(\mathbf{k}, \mathbf{v}, \mathbf{v}') = -i \frac{e^{i\mathbf{k} \cdot \mathbf{v}' t}}{\epsilon^*(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')} \int \frac{d\omega}{2\pi} \frac{1}{\epsilon(\mathbf{k}, \omega)} \frac{e^{-i\omega t}}{(\omega - \mathbf{k} \cdot \mathbf{v})(\omega - \mathbf{k} \cdot \mathbf{v}')} \quad (\text{A5})$$

For the remaining integral there are two poles on the real axis, which each contribute $-2\pi i$ times the associated residue. We obtain

$$\begin{aligned} J(\mathbf{k}, \mathbf{v}, \mathbf{v}') &= -\frac{e^{i\mathbf{k} \cdot \mathbf{v}' t}}{\epsilon^*(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')} \left[\frac{1}{\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})} \frac{e^{-i\mathbf{k} \cdot \mathbf{v} t}}{\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')} + \frac{1}{\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')} \frac{e^{-i\mathbf{k} \cdot \mathbf{v}' t}}{\mathbf{k} \cdot (\mathbf{v}' - \mathbf{v})} \right] \\ &= \frac{1}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')|^2} \frac{1}{\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')} \left[1 - \frac{\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')}{\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})} e^{-i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}') t} \right]. \end{aligned} \quad (\text{A6})$$

Relying on our assumption of timescale separation, i.e., the assumption that the fluctuations evolve on timescales much faster than the mean system, we can take the limit $t \rightarrow \infty$ of Eq. (A6). We then use the identity

$$\lim_{t \rightarrow \infty} \frac{e^{-ixt}}{x} = -i\pi \delta_D(x) \quad (\text{A7})$$

to simplify Eq. (A6) to

$$J(\mathbf{k}, \mathbf{v}, \mathbf{v}') = \frac{1}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')|^2} \left\{ \mathcal{P} \left[\frac{1}{\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')} \right] + i\pi \delta_D[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')] \right\}. \quad (\text{A8})$$

B. COMPUTING THE CORRELATIONS OF THE POTENTIAL FLUCTUATIONS

In Section 4, we showed that the evolution of the FDM halo DF is sourced by the correlations of the initial fluctuations in the system. These correlations are described by the function $\widehat{C}(\mathbf{k}, \mathbf{v})$, the Fourier transform of the correlation function, as defined in Eq. (67). Let us now explicitly compute this correlation.

In order to introduce that calculation, let us start by considering the classical case. In that regime, the system's discrete DF is given by

$$F_d(\mathbf{r}, \mathbf{v}, t) = m_b \sum_{i=1}^N \delta_D[\mathbf{r} - \mathbf{r}_i(t)] \delta_D[\mathbf{v} - \mathbf{v}_i(t)], \quad (\text{B9})$$

where at the initial time, the phase-space positions and velocities of the particles are drawn independently from another, uniformly in space, and according to the DF $F_b(\mathbf{v})$ for their velocities. Similarly to Eq. (52), the instantaneous fluctuations in the system's DF are given by $f = F_d - F_b$. At the initial time, we can then write

$$\langle f_0(\mathbf{r}, \mathbf{v}) f_0(\mathbf{r}', \mathbf{v}') \rangle = m_b^2 \sum_{i,j} \langle \delta_D(\mathbf{r} - \mathbf{r}_i) \delta_D(\mathbf{v} - \mathbf{v}_i) \delta_D(\mathbf{r}' - \mathbf{r}_j) \delta_D(\mathbf{v}' - \mathbf{v}_j) \rangle - F_b(\mathbf{v}) F_b(\mathbf{v}'), \quad (\text{B10})$$

where we dropped the time dependence ($t = 0$) to shorten the notation. As the particles are chosen independently, there are two types of terms in the double sum, depending on whether $i = j$ or $i \neq j$. We then get

$$\begin{aligned} \langle f_0(\mathbf{r}, \mathbf{v}) f_0(\mathbf{r}', \mathbf{v}') \rangle &= m_b^2 \delta_D(\mathbf{r} - \mathbf{r}') \delta_D(\mathbf{v} - \mathbf{v}') \sum_i \langle \delta_D(\mathbf{r} - \mathbf{r}_i) \delta_D(\mathbf{v} - \mathbf{v}_i) \rangle \\ &\quad + m_b^2 \sum_{i \neq j} \langle \delta_D(\mathbf{r} - \mathbf{r}_i) \delta_D(\mathbf{v} - \mathbf{v}_i) \rangle \langle \delta_D(\mathbf{r}' - \mathbf{r}_j) \delta_D(\mathbf{v}' - \mathbf{v}_j) \rangle \\ &\quad - F_b(\mathbf{v}) F_b(\mathbf{v}'). \end{aligned} \quad (\text{B11})$$

Since F_b obeys the normalisation convention $\int d\mathbf{v} F_b(\mathbf{v}) = \rho_0$, we have

$$\langle \delta_D(\mathbf{r} - \mathbf{r}_i) \delta_D(\mathbf{v} - \mathbf{v}_i) \rangle = \frac{1}{Nm_b} F_b(\mathbf{v}). \quad (\text{B12})$$

As a consequence, in the limit $N \gg 1$, the last two terms in Eq. (B11) cancel and we have

$$\langle f_0(\mathbf{r}, \mathbf{v}) f_0(\mathbf{r}', \mathbf{v}') \rangle = m_b F_b(\mathbf{v}) \delta_D(\mathbf{r} - \mathbf{r}') \delta_D(\mathbf{v} - \mathbf{v}'). \quad (\text{B13})$$

Following the convention from Eq. (48), this can be rewritten in Fourier space as

$$\langle \widehat{f}_0(\mathbf{k}, \mathbf{v}) \widehat{f}_0(\mathbf{k}', \mathbf{v}') \rangle = \frac{1}{(2\pi)^3} m_b F_b(\mathbf{v}) \delta_D(\mathbf{k} + \mathbf{k}') \delta_D(\mathbf{v} - \mathbf{v}'), \quad (\text{B14})$$

and the needed correlation function from Eq. (67) is then

$$\widehat{C}(\mathbf{k}, \mathbf{v}) = \frac{1}{(2\pi)^3} m_b F_b(\mathbf{v}). \quad (\text{B15})$$

We note that this correlation function is independent of \mathbf{k} , a consequence of our assumption that the initial positions of the particles are chosen independently from a homogeneous distribution.

Let us now adapt this calculation to the FDM case and compute the statistics of the persistent fluctuations present in the Wigner distribution function of the halo. We consider the following wavefunction

$$\psi(\mathbf{r}, t) = \int d\mathbf{k} \varphi(\mathbf{k}) e^{i[\mathbf{k} \cdot \mathbf{r} - \omega(k)t]}. \quad (\text{B16})$$

In the limit where the potential fluctuations $\Phi(\mathbf{r}, t)$ vanish, this wavefunction is a solution of the free Schrödinger equation provided that it satisfies the dispersion relation

$$\omega(k) = \frac{\hbar k^2}{2m_b}. \quad (\text{B17})$$

Let us now assume that the wavefunction in \mathbf{k} -space, $\varphi(\mathbf{k})$, is the sum of Gaussian wavepackets of the form

$$\varphi(\mathbf{k}) = A \sum_{i=1}^N e^{i\phi_i} e^{-i\mathbf{k}\cdot\mathbf{r}_i} e^{-|\mathbf{k}-m_b\mathbf{v}_i/\hbar|^2\varepsilon^2}. \quad (\text{B18})$$

In that expression, $\{\mathbf{r}_i, \mathbf{v}_i\}$ are random positions and velocities drawn independently from the DF $F_b(\mathbf{v})$, and $\{\phi_i\}$ are independent random phases. In addition, ε is an ad hoc parameter, so that ε and $\hbar/(2m_b\varepsilon)$ are respectively the initial uncertainties in the positions and velocities. This parameter will prove to be useful in managing our asymptotic developments. In Eq. (B18), we also introduced the prefactor A that is tuned to satisfy the normalisation condition stemming from Eq. (43), namely that $\langle |\psi(\mathbf{r}, t)|^2 \rangle = \rho_0$.

Let us now determine the value of A . Starting from Eq. (B16), we write

$$\langle |\psi(\mathbf{r}, t)|^2 \rangle = \int d\mathbf{k} d\mathbf{k}' \langle \varphi(\mathbf{k}) \varphi^*(\mathbf{k}') \rangle e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} e^{-it[\omega(\mathbf{k})-\omega(\mathbf{k}')]}. \quad (\text{B19})$$

The two-point correlation function of $\varphi(\mathbf{k})$ is

$$\begin{aligned} \langle \varphi(\mathbf{k}) \varphi^*(\mathbf{k}') \rangle &= A^2 \sum_{i,j} \langle e^{i(\phi_i-\phi_j)} e^{-i\mathbf{k}\cdot\mathbf{r}_i} e^{i\mathbf{k}'\cdot\mathbf{r}_j} e^{-|\mathbf{k}-m_b\mathbf{v}_i/\hbar|^2\varepsilon^2} e^{-|\mathbf{k}'-m_b\mathbf{v}_j/\hbar|^2\varepsilon^2} \rangle \\ &= A^2 \sum_i \langle e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i} e^{-|\mathbf{k}-m_b\mathbf{v}_i/\hbar|^2\varepsilon^2} e^{-|\mathbf{k}'-m_b\mathbf{v}_i/\hbar|^2\varepsilon^2} \rangle \\ &= A^2 \sum_i \int d\mathbf{r} d\mathbf{v} \langle \delta_D(\mathbf{r}-\mathbf{r}_i) \delta_D(\mathbf{v}-\mathbf{v}_i) \rangle e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} e^{-|\mathbf{k}-m_b\mathbf{v}/\hbar|^2\varepsilon^2} e^{-|\mathbf{k}'-m_b\mathbf{v}/\hbar|^2\varepsilon^2}. \end{aligned} \quad (\text{B20})$$

To get the second line we noted that $\langle e^{i(\phi_i-\phi_j)} \rangle$ is non-zero only for $i = j$. Using Eq. (B12), one can write

$$\begin{aligned} \langle \varphi(\mathbf{k}) \varphi^*(\mathbf{k}') \rangle &= \frac{A^2}{m_b} \int d\mathbf{r} d\mathbf{v} F_b(\mathbf{v}) e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} e^{-|\mathbf{k}-m_b\mathbf{v}/\hbar|^2\varepsilon^2} e^{-|\mathbf{k}'-m_b\mathbf{v}/\hbar|^2\varepsilon^2} \\ &= \frac{(2\pi)^3 A^2}{m_b} \delta_D(\mathbf{k}-\mathbf{k}') \int d\mathbf{v} F_b(\mathbf{v}) e^{-2|\mathbf{k}-m_b\mathbf{v}/\hbar|^2\varepsilon^2}. \end{aligned} \quad (\text{B21})$$

We can now use this result to pursue the simplification of Eq. (B19). We have

$$\langle |\psi(\mathbf{r}, t)|^2 \rangle = \frac{(2\pi)^3 A^2}{m_b} \int d\mathbf{k} d\mathbf{v} F_b(\mathbf{v}) e^{-2|\mathbf{k}-m_b\mathbf{v}/\hbar|^2\varepsilon^2} = \frac{2^{3/2}\pi^{9/2} A^2}{\varepsilon^3 m_b} \int d\mathbf{v} F_b(\mathbf{v}). \quad (\text{B22})$$

Recalling from Eq. (43) that $\int d\mathbf{v} F_b(\mathbf{v}) = \rho_0$, we obtain the value of A as

$$A = \frac{\varepsilon^{3/2} m_b^{1/2}}{2^{3/4}\pi^{9/4}}. \quad (\text{B23})$$

Relying on the definition from Eq. (42), we can now compute the Wigner function $W(\mathbf{r}, \mathbf{v}, t)$ associated with Eq. (B16). Following some cumbersome calculations, it takes the form

$$\begin{aligned} W(\mathbf{r}, \mathbf{v}, t) &= \int \frac{d\mathbf{s}}{(2\pi)^3} e^{-i\mathbf{v}\cdot\mathbf{s}} \psi\left(\mathbf{r} + \frac{1}{2}\hbar\mathbf{s}/m_b, t\right) \psi^*\left(\mathbf{r} - \frac{1}{2}\hbar\mathbf{s}/m_b, t\right) \\ &= \left(\frac{2m_b}{\hbar}\right)^3 \int d\mathbf{k}_1 d\mathbf{k}_2 \varphi(\mathbf{k}_1) \varphi^*(\mathbf{k}_2) \delta_D\left(\mathbf{k}_1 + \mathbf{k}_2 - \frac{2m_b}{\hbar}\mathbf{v}\right) e^{i(\mathbf{k}_1 - \mathbf{k}_2)\cdot(\mathbf{r} - \mathbf{v}t)} \\ &= m_b \left(\frac{m_b}{\pi\hbar}\right)^3 \sum_{i,j} e^{i(\phi_i - \phi_j)} \exp\left[-\frac{2\varepsilon^2 m_b^2}{\hbar^2} \left|\frac{1}{2}(\mathbf{v}_i + \mathbf{v}_j) - \mathbf{v}\right|^2\right] \exp\left[-\frac{1}{2\varepsilon^2} \left|\frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j) - \mathbf{r} + \mathbf{v}t\right|^2\right] \\ &\quad \times \exp\left\{-i\frac{m_b}{\hbar} [(\mathbf{r}_i - \mathbf{r}_j)\cdot\mathbf{v} + \left[\frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j) - \mathbf{r} + \mathbf{v}t\right]\cdot(\mathbf{v}_i - \mathbf{v}_j)]\right\}. \end{aligned} \quad (\text{B24})$$

In the second line we have used the relation $\omega(k_2) - \omega(k_1) = \frac{1}{2}(\hbar/m_b)(k_2^2 - k_1^2) = \frac{1}{2}(\hbar/m_b)(\mathbf{k}_2 + \mathbf{k}_1)\cdot(\mathbf{k}_2 - \mathbf{k}_1)$; when multiplied by the delta function in that equation this reduces to $\mathbf{v}\cdot(\mathbf{k}_2 - \mathbf{k}_1)$.

In order to check these calculations, let us now follow the same method as in Eq. (B20) to compute the ensemble-averaged Wigner function. Owing to the presence of the factor $e^{i(\phi_i - \phi_j)}$ in Eq. (B24), only the contributions with $i = j$ remain. All in all, we obtain

$$\begin{aligned} \overline{W}(\mathbf{r}, \mathbf{v}, t) &= \left(\frac{m_b}{\pi\hbar}\right)^3 \int d\mathbf{r}' d\mathbf{v}' F_b(\mathbf{v}') \exp\left[-\frac{2\varepsilon^2 m_b^2}{\hbar^2} |\mathbf{v}' - \mathbf{v}|^2\right] \exp\left[-\frac{1}{2\varepsilon^2} |\mathbf{r}' - \mathbf{r} + \mathbf{v}t|^2\right] \\ &\simeq F_b(\mathbf{v}), \end{aligned} \quad (\text{B25})$$

To get the last line, we assumed that $\varepsilon \gg \lambda_\sigma = \hbar/(m_b\sigma)$, with σ the system's typical velocity; physically, this means that the size of the wavepacket is much larger than the typical de Broglie wavelength. In that limit, we can then use the replacement

$$e^{-\alpha|\mathbf{v}|^2/\sigma^2} \xrightarrow{\alpha \gg 1} \frac{\pi^{3/2}}{\alpha^{3/2}} \delta_D(\mathbf{v}/\sigma), \quad (\text{B26})$$

with $\alpha^{-1} = \hbar^2/(2\varepsilon^2 m_b^2 \sigma^2)$ our small parameter. Thus we have recovered in Eq. (B25) the known result that the ensemble-averaged Wigner function is the same as the DF.

Having computed the Wigner distribution in Eq. (B24), we can now find the correlation of its fluctuations at the initial time, as required by Eq. (67). Following Eq. (52), we write

$$\langle f_0(\mathbf{r}, \mathbf{v}) f_0(\mathbf{r}', \mathbf{v}') \rangle = \langle W(\mathbf{r}, \mathbf{v}) W(\mathbf{r}', \mathbf{v}') \rangle - \overline{W}(\mathbf{r}, \mathbf{v}) \overline{W}(\mathbf{r}', \mathbf{v}'), \quad (\text{B27})$$

where all the functions are evaluated at the initial time. Glancing back at Eq. (B24), we note that the Wigner function can be rewritten in the shorter form

$$W(\mathbf{r}, \mathbf{v}) = \sum_{i,j} e^{i(\phi_i - \phi_j)} g_{ij}(\mathbf{r}, \mathbf{v}), \quad (\text{B28})$$

where the expression for the function $g_{ij}(\mathbf{r}, \mathbf{v})$ naturally follows from Eq. (B24). As a result, the ensemble average in the r.h.s. of Eq. (B27) takes the form

$$\langle W(\mathbf{r}, \mathbf{v}) W(\mathbf{r}', \mathbf{v}') \rangle = \sum_{\substack{i,j \\ k,l}} \langle e^{i(\phi_i - \phi_j + \phi_k - \phi_l)} g_{ij}(\mathbf{r}, \mathbf{v}) g_{kl}(\mathbf{r}', \mathbf{v}') \rangle \quad (\text{B29})$$

Because the wavepackets are drawn independently, the phase term $\langle e^{i(\phi_i - \phi_j + \phi_k - \phi_l)} \rangle$ is non-zero only in three cases, namely (i) $i = j = k = l$; (ii) $i = j$ and $k = l$ with $i \neq k$; (iii) $i = l$ and $j = k$ with $i \neq k$. In the limit $N \gg 1$, we can then rewrite Eq. (B29) as

$$\langle W(\mathbf{r}, \mathbf{v}) W(\mathbf{r}', \mathbf{v}') \rangle = N \langle g_{ii}(\mathbf{r}, \mathbf{v}) g_{ii}(\mathbf{r}', \mathbf{v}') \rangle + N^2 \langle g_{ii}(\mathbf{r}, \mathbf{v}) g_{kk}(\mathbf{r}', \mathbf{v}') \rangle + N^2 \langle g_{ik}(\mathbf{r}, \mathbf{v}) g_{ki}(\mathbf{r}', \mathbf{v}') \rangle, \quad (\text{B30})$$

where, in the last two terms, it is understood that $\{\mathbf{r}_i, \mathbf{v}_i\}$ and $\{\mathbf{r}_k, \mathbf{v}_k\}$ are two independent sets of random variables. Let us now compute in turn each of the terms appearing in Eq. (B30). We can first write

$$\begin{aligned} N \langle g_{ii}(\mathbf{r}, \mathbf{v}) g_{ii}(\mathbf{r}', \mathbf{v}') \rangle &= m_b \left(\frac{m_b}{\pi\hbar}\right)^6 \int d\mathbf{v}_i F_b(\mathbf{v}_i) \exp\left[-\frac{2\varepsilon^2 m_b^2}{\hbar^2} (|\mathbf{v}_i - \mathbf{v}|^2 + |\mathbf{v}_i - \mathbf{v}'|^2)\right] \\ &\quad \times \int d\mathbf{r}_i \exp\left[-\frac{1}{2\varepsilon^2} (|\mathbf{r}_i - \mathbf{r}|^2 + |\mathbf{r}_i - \mathbf{r}'|^2)\right] \\ &= m_b \left(\frac{m_b}{\pi\hbar}\right)^6 \exp\left[-\frac{\varepsilon^2 m_b^2}{\hbar^2} |\mathbf{v} - \mathbf{v}'|^2\right] \int d\mathbf{v}_i F_b(\mathbf{v}_i) \exp\left[-\frac{4\varepsilon^2 m_b^2}{\hbar^2} \left|\mathbf{v}_i - \frac{1}{2}(\mathbf{v} + \mathbf{v}')\right|^2\right] \\ &\quad \times \exp\left[-\frac{1}{4\varepsilon^2} |\mathbf{r} - \mathbf{r}'|^2\right] \int d\mathbf{r}_i \exp\left[-\frac{1}{\varepsilon^2} \left|\mathbf{r}_i - \frac{1}{2}(\mathbf{r} + \mathbf{r}')\right|^2\right] \\ &\simeq m_b \delta_D(\mathbf{r} - \mathbf{r}') \delta_D(\mathbf{v} - \mathbf{v}') \overline{W}(\mathbf{v}), \end{aligned} \quad (\text{B31})$$

where to get the last line, we assumed once again that $\varepsilon \gg \lambda_\sigma$, and used the asymptotic replacement from Eq. (B26). The second term from Eq. (B30) reads

$$\begin{aligned} N^2 \langle g_{ii}(\mathbf{r}, \mathbf{v}) g_{kk}(\mathbf{r}', \mathbf{v}') \rangle &= \left(\frac{m_b}{\pi\hbar}\right)^3 \int d\mathbf{r}_i d\mathbf{v}_i F_b(\mathbf{v}_i) \exp\left[-\frac{2\varepsilon^2 m_b^2}{\hbar^2} |\mathbf{v}_i - \mathbf{v}|^2\right] \exp\left[-\frac{1}{2\varepsilon^2} |\mathbf{r}_i - \mathbf{r}|^2\right] \\ &\quad \times \left(\frac{m_b}{\pi\hbar}\right)^3 \int d\mathbf{r}_k d\mathbf{v}_k F_b(\mathbf{v}_k) \exp\left[-\frac{2\varepsilon^2 m_b^2}{\hbar^2} |\mathbf{v}_k - \mathbf{v}'|^2\right] \exp\left[-\frac{1}{2\varepsilon^2} |\mathbf{r}_k - \mathbf{r}'|^2\right] \\ &= \overline{W}(\mathbf{r}, \mathbf{v}) \overline{W}(\mathbf{r}', \mathbf{v}'), \end{aligned} \quad (\text{B32})$$

where we used the result from Eq. (B25). The last term from Eq. (B30) then reads

$$\begin{aligned}
N^2 \langle g_{ik}(\mathbf{r}, \mathbf{v}) g_{ki}(\mathbf{r}', \mathbf{v}') \rangle &= \left(\frac{m_b}{\pi \hbar} \right)^6 \int d\mathbf{r}_i d\mathbf{v}_i d\mathbf{r}_k d\mathbf{v}_k F_b(\mathbf{v}_i) F_b(\mathbf{v}_k) \exp\left[-\frac{2\varepsilon^2 m_b^2}{\hbar^2} \left(\left| \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_k) - \mathbf{v} \right|^2 + \left| \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_k) - \mathbf{v}' \right|^2 \right)\right] \\
&\quad \times \exp\left[-\frac{1}{2\varepsilon^2} \left(\left| \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_k) - \mathbf{r} \right|^2 + \left| \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_k) - \mathbf{r}' \right|^2 \right)\right] \\
&\quad \times \exp\left\{-i \frac{m_b}{\hbar} [(\mathbf{r}_i - \mathbf{r}_k) \cdot (\mathbf{v} - \mathbf{v}') - (\mathbf{r} - \mathbf{r}') \cdot (\mathbf{v}_i - \mathbf{v}_k)]\right\} \\
&= \left(\frac{m_b}{\pi \hbar} \right)^6 \exp\left[-\frac{\varepsilon^2 m_b^2}{\hbar^2} |\mathbf{v} - \mathbf{v}'|^2\right] \exp\left[-\frac{1}{4\varepsilon^2} |\mathbf{r} - \mathbf{r}'|^2\right] \\
&\quad \times \int d\mathbf{r}_i d\mathbf{v}_i d\mathbf{r}_k d\mathbf{v}_k F_b(\mathbf{v}_i) F_b(\mathbf{v}_k) \exp\left[-\frac{4\varepsilon^2 m_b^2}{\hbar^2} \left| \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_k) - \frac{1}{2}(\mathbf{v} + \mathbf{v}') \right|^2\right] \\
&\quad \times \exp\left[-\frac{1}{\varepsilon^2} \left| \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_k) - \frac{1}{2}(\mathbf{r} + \mathbf{r}') \right|^2\right] \exp\left\{-i \frac{m_b}{\hbar} [(\mathbf{r}_i - \mathbf{r}_k) \cdot (\mathbf{v} - \mathbf{v}') - (\mathbf{r} - \mathbf{r}') \cdot (\mathbf{v}_i - \mathbf{v}_k)]\right\}.
\end{aligned} \tag{B33}$$

At this stage, the asymptotic formula from Eq. (B26) has to be used carefully, because the complex exponential from Eq. (B33) is rapidly fluctuating. To clarify this calculation, let us perform the change of variables

$$\begin{cases} \boldsymbol{\sigma}_r = \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_k), \\ \boldsymbol{\delta}_r = \mathbf{r}_i - \mathbf{r}_k, \end{cases} \quad ; \quad \begin{cases} \boldsymbol{\sigma}_v = \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_k), \\ \boldsymbol{\delta}_v = \mathbf{v}_i - \mathbf{v}_k. \end{cases} \tag{B34}$$

Equation (B33) becomes

$$\begin{aligned}
N^2 \langle g_{ik}(\mathbf{r}, \mathbf{v}) g_{ki}(\mathbf{r}', \mathbf{v}') \rangle &= \left(\frac{m_b}{\pi \hbar} \right)^6 \exp\left[-\frac{\varepsilon^2 m_b^2}{\hbar^2} |\mathbf{v} - \mathbf{v}'|^2\right] \exp\left[-\frac{1}{4\varepsilon^2} |\mathbf{r} - \mathbf{r}'|^2\right] \\
&\quad \times \int d\boldsymbol{\sigma}_v d\boldsymbol{\delta}_v F_b(\boldsymbol{\sigma}_v + \frac{1}{2}\boldsymbol{\delta}_v) F_b(\boldsymbol{\sigma}_v - \frac{1}{2}\boldsymbol{\delta}_v) \exp\left[-\frac{4\varepsilon^2 m_b^2}{\hbar^2} \left| \boldsymbol{\sigma}_v - \frac{1}{2}(\mathbf{v} + \mathbf{v}') \right|^2\right] \exp\left[-i \frac{m_b}{\hbar} (\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{\delta}_v\right] \\
&\quad \times \int d\boldsymbol{\sigma}_r d\boldsymbol{\delta}_r \exp\left[-\frac{1}{\varepsilon^2} \left| \boldsymbol{\sigma}_r - \frac{1}{2}(\mathbf{r} + \mathbf{r}') \right|^2\right] \exp\left[-i \frac{m_b}{\hbar} \boldsymbol{\delta}_r \cdot (\mathbf{v} - \mathbf{v}')\right].
\end{aligned} \tag{B35}$$

In this expression, we note that the exponential factor $\exp[-\frac{1}{4\varepsilon^2} |\mathbf{r} - \mathbf{r}'|^2]$ is nearly zero unless $|\mathbf{r} - \mathbf{r}'| \lesssim \varepsilon$. In that regime, the complex exponential $\exp[-i \frac{m_b}{\hbar} (\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{\delta}_v]$ will average to nearly zero unless $|\boldsymbol{\delta}_v| \lesssim \hbar/(\varepsilon m_b) = \sigma \lambda_\sigma / \varepsilon$. Given our assumption that $\varepsilon \gg \lambda_\sigma$, we conclude that the dominant contribution to the integral comes from $|\boldsymbol{\delta}_v| \ll \sigma$. We recall that the typical variance of $F_b(\mathbf{v})$ is σ , so in Eq. (B35) we may perform the replacement $F_b(\boldsymbol{\sigma}_v + \frac{1}{2}\boldsymbol{\delta}_v) F_b(\boldsymbol{\sigma}_v - \frac{1}{2}\boldsymbol{\delta}_v) \rightarrow F_b^2(\boldsymbol{\sigma}_v)$. We get

$$\begin{aligned}
N^2 \langle g_{ik}(\mathbf{r}, \mathbf{v}) g_{ki}(\mathbf{r}', \mathbf{v}') \rangle &\simeq 2^6 \delta_D(\mathbf{r} - \mathbf{r}') \delta_D(\mathbf{v} - \mathbf{v}') \int d\boldsymbol{\sigma}_r d\boldsymbol{\sigma}_v \exp\left[-\frac{1}{\varepsilon^2} |\boldsymbol{\sigma}_r - \mathbf{r}'|^2\right] F_b^2(\boldsymbol{\sigma}_v) \exp\left[-\frac{4\varepsilon^2 m_b^2}{\hbar^2} |\boldsymbol{\sigma}_v - \mathbf{v}'|^2\right] \\
&\simeq \frac{\hbar^3}{m_b^3} \delta_D(\mathbf{r} - \mathbf{r}') \delta_D(\mathbf{v} - \mathbf{v}') \overline{W}^2(\mathbf{v}),
\end{aligned} \tag{B36}$$

Gathering together Eqs. (B31), (B32), and (B35), we can now rewrite the correlation from Eq. (B27) to obtain

$$\langle f_0(\mathbf{r}, \mathbf{v}) f_0(\mathbf{r}', \mathbf{v}') \rangle = \left[m_b + \frac{\hbar^3}{m_b^3} \overline{W}(\mathbf{v}) \right] \overline{W}(\mathbf{v}) \delta_D(\mathbf{r} - \mathbf{r}') \delta_D(\mathbf{v} - \mathbf{v}'). \tag{B37}$$

Following Eqs. (48) and (67), we finally obtain the needed correlation function as

$$\widehat{C}(\mathbf{k}, \mathbf{v}) = \frac{1}{(2\pi)^3} \left[m_b + \frac{\hbar^3}{m_b^3} \overline{W}(\mathbf{v}) \right] \overline{W}(\mathbf{v}), \tag{B38}$$

which reduces to the classical correlation function (B15) as $\hbar \rightarrow 0$.

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