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A Dynamic Viscoelastic Problem with Friction and Rate-depending Contact Interactions

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Dedicated to Professor Meir Shillor on the occasion of his 70th birthday

Keywords

Dynamic problems, contact interactions, Coulomb friction, viscoelasticity, set-valued mapping.

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Abstract

The aim of this work is to study a dynamic problem that constitutes a unified approach to describe some rate-depending interactions between the boundaries of two viscoelastic bodies, including relaxed unilateral contact, pointwise friction or adhesion conditions. The classical formulation of the problem is presented and two variational formulations are given as three and four-field evolution implicit equations. Based on some approximation results and an equivalent fixed point problem for a multivalued function, we prove the existence of solutions to these variational evolution problems.

1 Introduction

This paper is concerned with the extension of some recent existence results proved for a class of nonsmooth dynamic contact problems which describe various surface interactions between the boundaries of two Kelvin-Voigt viscoelastic bodies. These interactions can include some relaxed unilateral contact, Coulomb friction or adhesion conditions.

Existence and approximation of solutions to the quasistatic elastic problems have been studied for different contact conditions. The quasistatic unilateral contact problems with local Coulomb friction have been studied in

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[1, 28, 29], adhesion laws were analyzed in [27, 9] and the normal compliance models have been investigated by several authors, see e.g. [16, 14, 30] and references therein.

Dynamic frictional contact problems with normal compliance laws have been studied in [21, 16, 17, 3, 23] and local friction laws were considered in [15, 18, 19, 12, 5, 10], for viscoelastic bodies. Dynamic frictionless problems with adhesion have been studied in [4, 20, 32] and dynamic viscoelastic problems coupling unilateral contact, recoverable adhesion and nonlocal friction have been analyzed in [11, 6].

Using the Clarke subdifferential, the variational formulations of various nonsmooth contact problems were given as hemivariational inequalities, see [22, 23, 24, 25] and references therein.

Based on Ky Fan's fixed point theorem, an elastic contact problem with relaxed unilateral conditions and pointwise Coulomb friction in the static case was studied in [26], the extension to an elastic quasistatic contact problem was investigated in [8] and the corresponding viscoelastic dynamic case was analyzed in [7].

This work extends the results in [7] to the case of a coefficient of friction depending on the sliding velocity. Using new three and four-field variational formulations, expressed as an evolution variational equation coupled with pointwise constraints, existence and improved regularity results are established.

The paper is organized as follows. In Section 2 the classical formulation of the dynamic contact problem is presented and two variational formulations are given. Section 3 is devoted to establish some auxiliary approximation results. In Section 4 the existence of a solution is proved for an equivalent fixed point problem by using the Ky Fan's theorem.

2 Classical and variational formulations

We consider two viscoelastic bodies, characterized by a Kelvin-Voigt constitutive law, which occupy the reference domains Ω^α of \mathbb{R}^3 with Lipschitz boundaries $\Gamma^\alpha = \partial\Omega^\alpha$, $\alpha = 1, 2$. Let Γ_U^α , Γ_F^α and Γ_C^α be three open disjoint sufficiently smooth parts of Γ^α such that $\Gamma^\alpha = \bar{\Gamma}_U^\alpha \cup \bar{\Gamma}_F^\alpha \cup \bar{\Gamma}_C^\alpha$ and, to simplify the estimates, $\text{meas}(\Gamma_U^\alpha) > 0$, $\alpha = 1, 2$. We assume the small deformation hypothesis and we use Cartesian coordinates representations.

Let $\mathbf{y}^\alpha(\mathbf{x}^\alpha, t)$ denote the position at time $t \in [0, T]$, where $0 < T < +\infty$, of the material point represented by the Cartesian coordinates $\mathbf{x}^\alpha = (x_1^\alpha, x_2^\alpha, x_3^\alpha)$ in the reference configuration Ω^α , and $\mathbf{u}^\alpha(\mathbf{x}^\alpha, t) = \mathbf{y}^\alpha(\mathbf{x}^\alpha, t) - \mathbf{x}^\alpha$ denote the displacement vector of \mathbf{x}^α at time t , with the Cartesian coordi-

nates $\mathbf{u}^\alpha = (u_1^\alpha, u_2^\alpha, u_3^\alpha)$.

Let $\boldsymbol{\varepsilon}^\alpha = (\varepsilon_{ij}(u^\alpha))$, and $\boldsymbol{\sigma}^\alpha = (\sigma_{ij}^\alpha)$, be the infinitesimal strain tensor and the stress tensor, respectively, corresponding to Ω^α , $\alpha = 1, 2$.

Assume that the displacements $\mathbf{u}^\alpha = \mathbf{0}$ on $\Gamma_U^\alpha \times (0, T)$, $\alpha = 1, 2$, and that the densities of both bodies are equal to 1. Let $\mathbf{f}_1 = (\mathbf{f}_1^1, \mathbf{f}_1^2)$ and $\mathbf{f}_2 = (\mathbf{f}_2^1, \mathbf{f}_2^2)$ denote the given body forces in $\Omega^1 \cup \Omega^2$ and tractions on $\Gamma_F^1 \cup \Gamma_F^2$, respectively. The initial displacements and velocities of the bodies are denoted by $\mathbf{u}_0 = (\mathbf{u}_0^1, \mathbf{u}_0^2)$, $\mathbf{u}_1 = (\mathbf{u}_1^1, \mathbf{u}_1^2)$ and the usual summation convention will be used for $i, j, k, l = 1, 2, 3$.

Suppose that the solids can be in contact between the potential contact surfaces Γ_C^1 and Γ_C^2 which can be parametrized by two C^1 functions, φ^1, φ^2 , defined on an open and bounded subset Ξ of \mathbb{R}^2 , such that $\varphi^1(\xi) - \varphi^2(\xi) \geq 0 \forall \xi \in \Xi$ and each Γ_C^α is the graph of φ^α on Ξ that is $\Gamma_C^\alpha = \{(\xi, \varphi^\alpha(\xi)) \in \mathbb{R}^3; \xi \in \Xi\}$, $\alpha = 1, 2$, see e.g. [2]. Define the initial normalized gap between the two contact surfaces by

$$g_0(\xi) = \frac{\varphi^1(\xi) - \varphi^2(\xi)}{\sqrt{1 + |\nabla \varphi^1(\xi)|^2}} \quad \forall \xi \in \Xi.$$

Let \mathbf{n}^α denote the unit outward normal vector to Γ^α , $\alpha = 1, 2$. We introduce the following notations for the normal and tangential components of a displacement field \mathbf{v}^α , of the relative displacement corresponding to $\mathbf{v} := (\mathbf{v}^1, \mathbf{v}^2)$ and of the stress vector $\boldsymbol{\sigma}^\alpha \mathbf{n}^\alpha$, respectively, on Γ_C^α , $\alpha = 1, 2$:

$$\begin{aligned} \mathbf{v}^\alpha(\xi, t) &:= \mathbf{v}^\alpha(\xi, \varphi^\alpha(\xi), t), \quad v_N^\alpha(\xi, t) := \mathbf{v}^\alpha(\xi, t) \cdot \mathbf{n}^\alpha(\xi), \\ v_N(\xi, t) &:= v_N^1(\xi, t) + v_N^2(\xi, t), \quad [v_N](\xi, t) := v_N(\xi, t) - g_0(\xi), \\ \mathbf{v}_T^\alpha(\xi, t) &:= \mathbf{v}^\alpha(\xi, t) - v_N^\alpha(\xi, t) \mathbf{n}^\alpha(\xi), \quad \mathbf{v}_T(\xi, t) := \mathbf{v}_T^1(\xi, t) - \mathbf{v}_T^2(\xi, t), \\ \sigma_N^\alpha(\xi, t) &:= (\boldsymbol{\sigma}^\alpha(\xi, t) \mathbf{n}^\alpha(\xi)) \cdot \mathbf{n}^\alpha(\xi), \quad \boldsymbol{\sigma}_T^\alpha(\xi, t) = \boldsymbol{\sigma}^\alpha(\xi, t) \mathbf{n}^\alpha(\xi) - \sigma_N^\alpha(\xi, t) \mathbf{n}^\alpha(\xi), \end{aligned}$$

for all $\xi \in \Xi$ and for all $t \in [0, T]$. Let $g := -[u_N] = g_0 - u_N^1 - u_N^2$ be the gap corresponding to the solution $\mathbf{u} := (\mathbf{u}^1, \mathbf{u}^2)$.

Let $\mathcal{A}^\alpha = (\mathcal{A}_{ijkl}^\alpha)$, $\mathcal{B}^\alpha = (\mathcal{B}_{ijkl}^\alpha)$ denote the components of the elasticity tensor and the viscosity tensor corresponding to Ω^α , respectively, satisfying the following classical symmetry and ellipticity conditions: $\mathcal{C}_{ijkl}^\alpha = \mathcal{C}_{jikl}^\alpha = \mathcal{C}_{klij}^\alpha \in L^\infty(\Omega^\alpha)$, $\forall i, j, k, l = 1, 2, 3$, $\exists \alpha_{C^\alpha} > 0$ such that $\mathcal{C}_{ijkl}^\alpha \tau_{ij} \tau_{kl} \geq \alpha_{C^\alpha} \tau_{ij} \tau_{ij} \quad \forall \tau = (\tau_{ij})$ verifying $\mathcal{C}_{ijkl}^\alpha = \mathcal{A}_{ijkl}^\alpha$, $\mathcal{C}^\alpha = \mathcal{A}^\alpha$ or $\mathcal{C}_{ijkl}^\alpha = \mathcal{B}_{ijkl}^\alpha$, $\mathcal{C}^\alpha = \mathcal{B}^\alpha \quad \forall i, j, k, l = 1, 2, 3$, $\alpha = 1, 2$.

Let $\underline{\kappa}, \bar{\kappa} : \mathbb{R}^2 \rightarrow \mathbb{R}$ be two mappings with $\underline{\kappa}$ lower semicontinuous and $\bar{\kappa}$ upper semicontinuous, satisfying the following conditions:

$$\underline{\kappa}(s) \leq \bar{\kappa}(s) \text{ and } 0 \notin (\underline{\kappa}(s), \bar{\kappa}(s)) \quad \forall s \in \mathbb{R}^2, \quad (1)$$

$$\exists r_0 \geq 0 \text{ such that } \max(|\underline{\kappa}(s)|, |\bar{\kappa}(s)|) \leq r_0 \quad \forall s \in \mathbb{R}^2. \quad (2)$$

Let $\mu : \Xi \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$ be a bounded function such that for a.e. $\xi \in \Xi$ $\mu(\xi, \cdot)$ is Lipschitz continuous with the Lipschitz constant independent of ξ , and for every $\mathbf{v} \in \mathbb{R}^3$ $\mu(\cdot, \mathbf{v})$ is measurable.

Consider the following dynamic viscoelastic contact problem with Coulomb friction.

Problem P_c : Find $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$ and, for all $t \in (0, T)$,

$$\dot{\mathbf{u}}^\alpha - \operatorname{div} \boldsymbol{\sigma}^\alpha(\mathbf{u}^\alpha, \dot{\mathbf{u}}^\alpha) = \mathbf{f}_1^\alpha \text{ in } \Omega^\alpha, \quad (3)$$

$$\boldsymbol{\sigma}^\alpha(\mathbf{u}^\alpha, \dot{\mathbf{u}}^\alpha) = \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\mathbf{u}^\alpha) + \mathcal{B}^\alpha \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha) \text{ in } \Omega^\alpha, \quad (4)$$

$$\mathbf{u}^\alpha = \mathbf{0} \text{ on } \Gamma_U^\alpha, \boldsymbol{\sigma}^\alpha \mathbf{n}^\alpha = \mathbf{f}_2^\alpha \text{ on } \Gamma_F^\alpha, \alpha = 1, 2, \quad (5)$$

$$\boldsymbol{\sigma}^1 \mathbf{n}^1 + \boldsymbol{\sigma}^2 \mathbf{n}^2 = \mathbf{0} \text{ in } \Xi, \quad (6)$$

$$\underline{\kappa}([u_N], \dot{u}_N) \leq \sigma_N \leq \bar{\kappa}([u_N], \dot{u}_N) \text{ in } \Xi, \quad (7)$$

$$|\boldsymbol{\sigma}_T| \leq \mu(\dot{\mathbf{u}}_T) |\sigma_N| \text{ in } \Xi \text{ and} \quad (8)$$

$$\dot{\mathbf{u}}_T \neq \mathbf{0} \Rightarrow \boldsymbol{\sigma}_T = -\mu(\dot{\mathbf{u}}_T) |\sigma_N| \frac{\dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|},$$

where $\boldsymbol{\sigma}^\alpha = \boldsymbol{\sigma}^\alpha(\mathbf{u}^\alpha, \dot{\mathbf{u}}^\alpha)$, $\alpha = 1, 2$, $\sigma_N := \sigma_N^1$, $\boldsymbol{\sigma}_T := \boldsymbol{\sigma}_T^1$ and μ is the sliding velocity dependent coefficient of friction. Different choices for $\underline{\kappa}$, $\bar{\kappa}$ will give various contact and friction conditions, see e.g. [7].

To give the variational formulations, we adopt the following notations:

$$\mathbf{H}^s(\Omega^\alpha) := H^s(\Omega^\alpha; \mathbb{R}^3), \alpha = 1, 2, \mathbf{H}^s := \mathbf{H}^s(\Omega^1) \times \mathbf{H}^s(\Omega^2),$$

$$\langle \mathbf{v}, \mathbf{w} \rangle_{-s, s} = \langle \mathbf{v}^1, \mathbf{w}^1 \rangle_{\mathbf{H}^{-s}(\Omega^1) \times \mathbf{H}^s(\Omega^1)} + \langle \mathbf{v}^2, \mathbf{w}^2 \rangle_{\mathbf{H}^{-s}(\Omega^2) \times \mathbf{H}^s(\Omega^2)}$$

$$\forall \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in \mathbf{H}^{-s}, \forall \mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2) \in \mathbf{H}^s, \forall s \in \mathbb{R},$$

$$\mathbf{H} := \mathbf{H}^0 = L^2(\Omega^1; \mathbb{R}^3) \times L^2(\Omega^2; \mathbb{R}^3), \mathbf{V} := \mathbf{V}^1 \times \mathbf{V}^2, \text{ where}$$

$$\mathbf{V}^\alpha = \{\mathbf{v}^\alpha \in \mathbf{H}^1(\Omega^\alpha); \mathbf{v}^\alpha = \mathbf{0} \text{ a.e. on } \Gamma_U^\alpha\}, \alpha = 1, 2.$$

$(\mathbf{H}, |\cdot|)$ and $(\mathbf{V}, \|\cdot\|)$ are Hilbert spaces with the associated inner products denoted by (\cdot, \cdot) and by $\langle \cdot, \cdot \rangle$, respectively.

Define $\Xi_T = \Xi \times (0, T)$ and the closed convex cones $L_+^2(\Xi)$, $L_+^2(\Xi_T)$ as follows:

$$L_+^2(\Xi) := \{\delta \in L^2(\Xi); \delta \geq 0 \text{ a.e. in } \Xi\},$$

$$L_+^2(\Xi_T) := \{\eta \in L^2(\Xi_T); \eta \geq 0 \text{ a.e. in } \Xi_T\}.$$

Let a, b be two bilinear, continuous and symmetric mappings defined by

$$a(\mathbf{v}, \mathbf{w}) = a^1(\mathbf{v}^1, \mathbf{w}^1) + a^2(\mathbf{v}^2, \mathbf{w}^2), \quad b(\mathbf{v}, \mathbf{w}) = b^1(\mathbf{v}^1, \mathbf{w}^1) + b^2(\mathbf{v}^2, \mathbf{w}^2)$$

$$\forall \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2), \mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2) \in \mathbf{H}^1, \text{ where, for } \alpha = 1, 2,$$

$$a^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \int_{\Omega^\alpha} \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\mathbf{v}^\alpha) \cdot \boldsymbol{\varepsilon}(\mathbf{w}^\alpha) dx, \quad b^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \int_{\Omega^\alpha} \mathcal{B}^\alpha \boldsymbol{\varepsilon}(\mathbf{v}^\alpha) \cdot \boldsymbol{\varepsilon}(\mathbf{w}^\alpha) dx.$$

As $\text{meas}(\Gamma_{\mathcal{V}}^\alpha) > 0$ and the components of $\mathcal{A}^\alpha, \mathcal{B}^\alpha, \alpha = 1, 2$, satisfy the ellipticity conditions, by Korn's inequality it follows that a and b are \mathbf{V} -elliptic in the following sense:

$$\exists m_a, m_b > 0 \quad a(\mathbf{v}, \mathbf{v}) \geq m_a \|\mathbf{v}\|^2, \quad b(\mathbf{v}, \mathbf{v}) \geq m_b \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{V}. \quad (9)$$

Assume $\mathbf{f}_1^\alpha \in W^{1,\infty}(0, T; L^2(\Omega^\alpha; \mathbb{R}^d)), \mathbf{f}_2^\alpha \in W^{1,\infty}(0, T; L^2(\Gamma_F^\alpha; \mathbb{R}^d)), \alpha = 1, 2, \mathbf{u}_0, \mathbf{u}_1 \in \mathbf{V}, g_0 \in L_+^2(\Xi)$, and define the following mapping:

$$\begin{aligned} \mathbf{f} \in W^{1,\infty}(0, T; \mathbf{H}^1), \quad \langle \mathbf{f}, \mathbf{v} \rangle &= \sum_{\alpha=1,2} \int_{\Omega^\alpha} \mathbf{f}_1^\alpha \cdot \mathbf{v}^\alpha \, dx + \sum_{\alpha=1,2} \int_{\Gamma_F^\alpha} \mathbf{f}_2^\alpha \cdot \mathbf{v}^\alpha \, ds \\ \forall \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in \mathbf{H}^1, \quad \forall t \in [0, T]. \end{aligned}$$

Assume the following compatibility conditions: $[u_{0N}] \leq 0, \bar{\kappa}([u_{0N}]) = 0$ a.e. in Ξ and $\exists \mathbf{p}_0 \in \mathbf{H}$ such that

$$(\mathbf{p}_0, \mathbf{v}) + a(\mathbf{u}_0, \mathbf{v}) + b(\mathbf{u}_1, \mathbf{v}) = \langle \mathbf{f}(0), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}. \quad (10)$$

For every $\zeta = (\zeta_1, \zeta_2) \in L^2(0, T; (L^2(\Xi))^2) = (L^2(\Xi_T))^2$, define the following nonempty, closed, and convex sets:

$$\begin{aligned} \Lambda^0(\zeta_1, \zeta_2) &= \{\eta \in L^2(\Xi_T); \underline{\kappa} \circ (\zeta_1, \zeta_2) \leq \eta \leq \bar{\kappa} \circ (\zeta_1, \zeta_2) \text{ a.e. in } \Xi_T\}, \\ \Lambda_+^0(\zeta_1, \zeta_2) &= \{\eta \in L_+^2(\Xi_T); \underline{\kappa}_+ \circ (\zeta_1, \zeta_2) \leq \eta \leq \bar{\kappa}_+ \circ (\zeta_1, \zeta_2) \text{ a.e. in } \Xi_T\}, \\ \Lambda_-^0(\zeta_1, \zeta_2) &= \{\eta \in L_+^2(\Xi_T); \bar{\kappa}_- \circ (\zeta_1, \zeta_2) \leq \eta \leq \underline{\kappa}_- \circ (\zeta_1, \zeta_2) \text{ a.e. in } \Xi_T\}, \end{aligned}$$

where, for each $r \in \mathbb{R}$, $r_+ := \max(0, r)$ and $r_- := \max(0, -r)$ denote the positive and negative parts, respectively.

Also, for every $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$, define the following nonempty and closed sets:

$$\begin{aligned} \Lambda^1(\mathbf{w}) &= \{(\eta, \boldsymbol{\varsigma}) \in L^2(\Xi_T) \times (L^2(\Xi_T))^3; \eta \in \Lambda^0([w_N], \dot{w}_N), \\ &\quad |\boldsymbol{\varsigma}| \leq \mu(\dot{\mathbf{w}}_T) |\eta|, \boldsymbol{\varsigma} \cdot \dot{\mathbf{w}}_T + \mu(\dot{\mathbf{w}}_T) |\eta| |\dot{\mathbf{w}}_T| = 0 \text{ a.e. in } \Xi_T\}, \\ \Lambda^2(\mathbf{w}) &= \{(\eta, \boldsymbol{\varsigma}) \in L^2(\Xi_T) \times (L^2(\Xi_T))^3; \eta_+ \in \Lambda_+^0([w_N], \dot{w}_N), \eta_- \in \Lambda_-^0([w_N], \dot{w}_N), \\ &\quad |\boldsymbol{\varsigma}| \leq \mu(\dot{\mathbf{w}}_T) (\eta_+ + \eta_-), \boldsymbol{\varsigma} \cdot \dot{\mathbf{w}}_T + \mu(\dot{\mathbf{w}}_T) (\eta_+ + \eta_-) |\dot{\mathbf{w}}_T| = 0 \text{ a.e. in } \Xi_T\}, \\ \Lambda^3(\mathbf{w}) &= \{(\eta_1, \eta_2, \boldsymbol{\varsigma}) \in (L^2(\Xi_T))^5; \eta_1 \in \Lambda_+^0([w_N], \dot{w}_N), \eta_2 \in \Lambda_-^0([w_N], \dot{w}_N), \\ &\quad |\boldsymbol{\varsigma}| \leq \mu(\dot{\mathbf{w}}_T) (\eta_1 + \eta_2), \boldsymbol{\varsigma} \cdot \dot{\mathbf{w}}_T + \mu(\dot{\mathbf{w}}_T) (\eta_1 + \eta_2) |\dot{\mathbf{w}}_T| = 0 \text{ a.e. in } \Xi_T\}. \end{aligned}$$

Since $\text{meas}(\Xi) < \infty$ and $\underline{\kappa}, \bar{\kappa}$ satisfy (2), it follows that for all $\zeta \in L^2(0, T; (L^2(\Xi))^2)$ the sets $\Lambda^0(\zeta), \Lambda_+^0(\zeta)$ and $\Lambda_-^0(\zeta)$ are bounded in norm in $L^2(0, T; L^2(\Xi)) = L^2(\Xi_T)$ by $R_0 = r_0(\text{meas}(\Xi))^{1/2}T$ and are bounded in norm in $L^\infty(0, T; L^\infty(\Xi))$ by r_0 .

As the coefficient of friction μ is a bounded function, it follows also that for all $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$ the sets $\Lambda^1(\mathbf{w})$, $\Lambda^2(\mathbf{w})$, and $\Lambda^3(\mathbf{w})$ are bounded in norm. Thus, there exists $R_1 > 0$ such that $\Lambda^3(\mathbf{w}) \subset D_0 \times D_1$ for all $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$, where $D_0 = \{(\eta_1, \eta_2) \in (L^2(\Xi_T))^2; \|\eta_1\|_{L^2(\Xi_T)} \leq R_0, \|\eta_2\|_{L^2(\Xi_T)} \leq R_0\}$ and $D_1 = \{\boldsymbol{\varsigma} \in (L^2(\Xi_T))^3; \|\boldsymbol{\varsigma}\|_{(L^2(\Xi_T))^3} \leq R_1\}$.

A first variational formulation of the problem P_c is the following.

Problem P_v^1 : Find $\mathbf{u} \in C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V})$, $\lambda \in L^2(\Xi_T)$, $\boldsymbol{\gamma} \in (L^2(\Xi_T))^3$, such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$, $(\lambda, \boldsymbol{\gamma}) \in \Lambda^1(\mathbf{u})$, and

$$\begin{aligned} & (\dot{\mathbf{u}}(T), \mathbf{v}(T)) - \int_0^T (\dot{\mathbf{u}}, \dot{\mathbf{v}}) dt + \int_0^T \{a(\mathbf{u}, \mathbf{v}) + b(\dot{\mathbf{u}}, \mathbf{v})\} dt \\ & - \int_0^T \{(\lambda, v_N)_{L^2(\Xi)} + (\boldsymbol{\gamma}, \mathbf{v}_T)_{(L^2(\Xi))^3}\} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle dt + (\mathbf{u}_1, \mathbf{v}(0)) \quad (11) \\ & \forall \mathbf{v} \in L^\infty(0, T; \mathbf{V}) \cap W^{1,2}(0, T; \mathbf{H}). \end{aligned}$$

where $(\cdot, \cdot)_{L^2(\Xi)}$ and $(\cdot, \cdot)_{(L^2(\Xi))^3}$ denote the inner products of the corresponding spaces.

The formal equivalence between the variational problem P_v^1 and the classical problem (3)–(8) can be easily proved by using Green's formula and an integration by parts, where the Lagrange multipliers λ , $\boldsymbol{\gamma}$ satisfy the relations $\lambda = \sigma_N$, $\boldsymbol{\gamma} = \boldsymbol{\sigma}_T$.

The sets $\Lambda^0(\zeta_1, \zeta_2)$, $\Lambda_+^0(\zeta_1, \zeta_2)$ and $\Lambda_-^0(\zeta_1, \zeta_2)$ have the following useful properties, see [7].

Lemma 2.1. *Let $(\zeta_1, \zeta_2) \in (L^2(\Xi))^2$ and $(\eta_1, \eta_2) \in \Lambda_+^0(\zeta_1, \zeta_2) \times \Lambda_-^0(\zeta_1, \zeta_2)$. Then $\eta_1 \eta_2 = 0$ a.e. in Ξ_T and there exists $\eta \in \Lambda^0(\zeta_1, \zeta_2)$ such that $\eta_+ = \eta_1$, $\eta_- = \eta_2$ a.e. in Ξ_T .*

Since $\lambda \in \Lambda^0([u_N], \dot{u}_N)$ if and only if $(\lambda_+, \lambda_-) \in (\Lambda_+^0([u_N], \dot{u}_N) \times \Lambda_-^0([u_N], \dot{u}_N))$, from the previous lemma it follows that the variational problem P_v^1 is clearly equivalent with the following problem denoted by P_v^2 , in the sense that it has the same solutions \mathbf{u} , $\boldsymbol{\gamma}$ as the problem P_v^1 and the solutions λ_1 , λ_2 satisfy the relation $\lambda = \lambda_1 - \lambda_2$, where λ is a solution of P_v^1 .

Problem P_v^2 : Find $\mathbf{u} \in C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V})$, $(\lambda_1, \lambda_2) \in (L^2(\Xi_T))^2$,

$\gamma \in (L^2(\Xi_T))^3$, such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$, $(\lambda_1, \lambda_2, \gamma) \in \Lambda^3(\mathbf{u})$, and

$$\begin{aligned} & (\dot{\mathbf{u}}(T), \mathbf{v}(T)) - \int_0^T (\dot{\mathbf{u}}, \dot{\mathbf{v}}) dt + \int_0^T \{a(\mathbf{u}, \mathbf{v}) + b(\dot{\mathbf{u}}, \mathbf{v})\} dt \\ & - \int_0^T \{(\lambda_1 - \lambda_2, v_N)_{L^2(\Xi)} + (\gamma, \mathbf{v}_T)_{(L^2(\Xi))^3}\} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle dt + (\mathbf{u}_1, \mathbf{v}(0)) \quad (12) \\ & \quad \forall \mathbf{v} \in L^\infty(0, T; \mathbf{V}) \cap W^{1,2}(0, T; \mathbf{H}). \end{aligned}$$

The existence of solutions to problem P_v^2 will be established by using an equivalent fixed point problem which will be presented in the following section.

3 A fixed point problem formulation

By an immediate application of Theorem 3.2 proved in [10] and using similar arguments to those that enabled to prove Lemma 3.2 in [7], one obtains the following existence and uniqueness result.

Lemma 3.1. *For each $(\eta_1, \eta_2) \in (W^{1,\infty}(0, T; L^2(\Xi)))^2$, $\varsigma \in (W^{1,\infty}(0, T; L^2(\Xi)))^3$ with $\eta_1(0) = \eta_2(0) = 0$, $\varsigma(0) = \mathbf{0}$, there exists a unique solution $\mathbf{u} = \mathbf{u}_{(\eta_1, \eta_2, \varsigma)}$ of the following evolution variational equation: find $\mathbf{u} \in W^{2,2}(0, T; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V})$, such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$, and for almost all $t \in (0, T)$*

$$\begin{aligned} & (\ddot{\mathbf{u}}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\dot{\mathbf{u}}, \mathbf{v}) - (\eta_1 - \eta_2, v_N)_{L^2(\Xi)} \\ & - (\varsigma, \mathbf{v}_T)_{(L^2(\Xi))^3} = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}. \quad (13) \end{aligned}$$

We shall also use the following estimate result.

Lemma 3.2. *Let $(\eta_1, \eta_2), (\delta_1, \delta_2) \in (W^{1,\infty}(0, T; L^2(\Xi)))^2$ such that $\eta_1(0) = \eta_2(0) = \delta_1(0) = \delta_2(0) = 0$, $\varsigma_1, \varsigma_2 \in (W^{1,\infty}(0, T; L^2(\Xi)))^3$ such that $\varsigma_{1,2}(0) = \mathbf{0}$, and let $\mathbf{u}_{(\eta_1, \eta_2, \varsigma_1)}$, $\mathbf{u}_{(\delta_1, \delta_2, \varsigma_2)}$ be the corresponding solutions of (13). Then there exists a constant $C_0 > 0$, independent of (η_1, η_2) , (δ_1, δ_2) , and ς_1, ς_2 , such that for all $t \in [0, T]$*

$$\begin{aligned} & |\dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)}(t) - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)}(t)|^2 + \|\mathbf{u}_{(\eta_1, \eta_2, \varsigma_1)}(t) - \mathbf{u}_{(\delta_1, \delta_2, \varsigma_2)}(t)\|^2 \\ & \quad + \int_0^t \|\dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)} - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)}\|^2 d\tau \quad (14) \\ & \leq C_0 \int_0^t \{(\eta_1 - \eta_2 - \delta_1 + \delta_2, \dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)N} - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)N})_{L^2(\Xi)} \\ & \quad + (\varsigma_1 - \varsigma_2, \dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)T} - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)T})_{(L^2(\Xi))^3}\} d\tau. \end{aligned}$$

Proof. Let $(\eta_1, \eta_2), (\delta_1, \delta_2) \in (W^{1,\infty}(0, T; L^2(\Xi)))^2$, $\boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2 \in (W^{1,\infty}(0, T; L^2(\Xi)))^3$ with $\mathbf{u}_1 := \mathbf{u}_{(\eta_1, \eta_2, \boldsymbol{\varsigma}_1)}$, $\mathbf{u}_2 := \mathbf{u}_{(\delta_1, \delta_2, \boldsymbol{\varsigma}_2)}$ the corresponding solutions of (13) which exist according to Lemma 3.1. Taking in each equation $\mathbf{v} = \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2$, for a.e. $\tau \in (0, T)$ it follows that

$$\begin{aligned} & (\ddot{\mathbf{u}}_1 - \ddot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) + a(\mathbf{u}_1 - \mathbf{u}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) + b(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) \\ &= (\eta_1 - \eta_2 - \delta_1 + \delta_2, \dot{u}_{1N} - \dot{u}_{2N})_{L^2(\Xi)} + (\boldsymbol{\varsigma}_1 - \boldsymbol{\varsigma}_2, \dot{\mathbf{u}}_{1T} - \dot{\mathbf{u}}_{2T})_{(L^2(\Xi))^3}. \end{aligned}$$

Since the solutions $\mathbf{u}_1, \mathbf{u}_2$ verify the same initial conditions and a is symmetric, by integrating over $(0, t)$ it follows that for all $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} |\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)|^2 + \frac{1}{2} a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) + \int_0^t b(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) d\tau \\ &= \int_0^t \{(\eta_1 - \eta_2 - \delta_1 + \delta_2, \dot{u}_{1N} - \dot{u}_{2N})_{L^2(\Xi)} + (\boldsymbol{\varsigma}_1 - \boldsymbol{\varsigma}_2, \dot{\mathbf{u}}_{1T} - \dot{\mathbf{u}}_{2T})_{(L^2(\Xi))^3}\} d\tau. \end{aligned}$$

Using the \mathbf{V} -ellipticity of a and b , the estimate (14) follows. \square

The following compactness theorem proved in [31] will be used several times in this paper.

Theorem 3.3. *Let X, U and Y be three Banach spaces such that $X \subset U \subset Y$ with compact embedding from X into U .*

(i) *Let \mathcal{F} be bounded in $L^p(0, T; X)$, where $1 \leq p < \infty$, and $\partial\mathcal{F}/\partial t := \{\dot{f}; f \in \mathcal{F}\}$ be bounded in $L^1(0, T; Y)$. Then \mathcal{F} is relatively compact in $L^p(0, T; U)$.*

(ii) *Let \mathcal{F} be bounded in $L^\infty(0, T; X)$ and $\partial\mathcal{F}/\partial t$ be bounded in $L^r(0, T; Y)$, where $r > 1$. Then \mathcal{F} is relatively compact in $C([0, T]; U)$.*

As $\mathcal{D}(0, T; L^2(\Xi))$ is dense in $L^2(0, T; L^2(\Xi))$, it follows that for every $(\eta_1, \eta_2) \in (L_+^2(\Xi_T))^2$ and every $\boldsymbol{\varsigma} \in (L^2(\Xi_T))^3$, there exist $(\eta_1^n, \eta_2^n)_n$ in $(L_+^2(\Xi_T))^2 \cap (W^{1,\infty}(0, T; L^2(\Xi)))^2$, $(\boldsymbol{\varsigma}^n)_n$ in $(W^{1,\infty}(0, T; L^2(\Xi)))^3$ such that $\eta_1^n(0) = \eta_2^n(0) = 0$, $\boldsymbol{\varsigma}^n(0) = \mathbf{0}$, for all $n \in \mathbb{N}$, $\eta_1^n \rightarrow \eta_1$, $\eta_2^n \rightarrow \eta_2$ in $L^2(\Xi_T)$, and $\boldsymbol{\varsigma}^n \rightarrow \boldsymbol{\varsigma}$ in $(L^2(\Xi_T))^3$.

Theorem 3.4. *Under the assumptions of Section 2, for every $(\eta_1, \eta_2) \in (L_+^2(\Xi_T))^2$ and every $\boldsymbol{\varsigma} \in (L^2(\Xi_T))^3$, let $(\eta_1^n, \eta_2^n)_n$ be a sequence in $(L_+^2(\Xi_T))^2 \cap (W^{1,\infty}(0, T; L^2(\Xi)))^2$ and $(\boldsymbol{\varsigma}^n)_n$ be a sequence in $(W^{1,\infty}(0, T; L^2(\Xi)))^3$ such that $\eta_1^n(0) = \eta_2^n(0) = 0$, $\boldsymbol{\varsigma}^n(0) = \mathbf{0}$, for all $n \in \mathbb{N}$, $\eta_1^n \rightharpoonup \eta_1$, $\eta_2^n \rightharpoonup \eta_2$ in $L^2(\Xi_T)$, and $\boldsymbol{\varsigma}^n \rightharpoonup \boldsymbol{\varsigma}$ in $(L^2(\Xi_T))^3$. Let $\mathbf{u}_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)}$ be the solution of (13) corresponding to $(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)$, for every $n \in \mathbb{N}$. Then $(\mathbf{u}_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)})_n$ is strongly convergent in $C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V})$, its limit, denoted by*

$\mathbf{u} := \mathbf{u}_{(\eta_1, \eta_2, \boldsymbol{\varsigma})}$, is independent of the chosen sequences weakly converging to $(\eta_1, \eta_2, \boldsymbol{\varsigma})$ with the same properties as $(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)$ and is a solution of the following evolution variational equation: $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$,

$$\begin{aligned} & (\dot{\mathbf{u}}(T), \mathbf{v}(T)) - \int_0^T (\dot{\mathbf{u}}, \dot{\mathbf{v}}) dt + \int_0^T \{a(\mathbf{u}, \mathbf{v}) + b(\dot{\mathbf{u}}, \mathbf{v})\} dt \\ & - \int_0^T \{(\eta_1 - \eta_2, v_N)_{L^2(\Xi)} + (\boldsymbol{\varsigma}, \mathbf{v}_T)_{(L^2(\Xi))^3}\} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle dt + (\mathbf{u}_1, \mathbf{v}(0)) \end{aligned} \quad (15)$$

$$\forall \mathbf{v} \in L^\infty(0, T; \mathbf{V}) \cap W^{1,2}(0, T; \mathbf{H}).$$

Proof. Assume $(\eta_1, \eta_2) \in (L^2_+(\Xi_T))^2$, $\boldsymbol{\varsigma} \in (L^2(\Xi_T))^3$, $(\eta_1^n, \eta_2^n) \in (L^2_+(\Xi_T))^2 \cap (W^{1,\infty}(0, T; L^2(\Xi)))^2$, $\boldsymbol{\varsigma}^n \in (W^{1,\infty}(0, T; L^2(\Xi)))^3$ such that $\eta_1^n(0) = \eta_2^n(0) = 0$, for all $n \in \mathbb{N}$, $\eta_1^n \rightharpoonup \eta_1$, $\eta_2^n \rightharpoonup \eta_2$ in $L^2(\Xi_T)$, and $\boldsymbol{\varsigma}^n \rightharpoonup \boldsymbol{\varsigma}$ in $(L^2(\Xi_T))^3$. Then, by Lemma 3.1, for every $n \in \mathbb{N}$ there exists a unique solution of the following variational equation: find $\mathbf{u}_n := \mathbf{u}_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)} \in W^{2,2}(0, T; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V})$, such that $\mathbf{u}_n(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}_n(0) = \mathbf{u}_1$, and for almost all $t \in (0, T)$

$$\begin{aligned} & (\ddot{\mathbf{u}}_n, \mathbf{v}) + a(\mathbf{u}_n, \mathbf{v}) + b(\dot{\mathbf{u}}_n, \mathbf{v}) - (\eta_1^n - \eta_2^n, v_N)_{L^2(\Xi)} \\ & - (\boldsymbol{\varsigma}^n, \mathbf{v}_T)_{(L^2(\Xi))^3} = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \quad (16)$$

For $\mathbf{v} = \dot{\mathbf{u}}_n$, and integrating over $(0, t)$ with $t \in (0, T]$, we derive

$$\begin{aligned} & \int_0^t (\ddot{\mathbf{u}}_n, \dot{\mathbf{u}}_n) d\tau + \int_0^t a(\mathbf{u}_n, \dot{\mathbf{u}}_n) d\tau + \int_0^t b(\dot{\mathbf{u}}_n, \dot{\mathbf{u}}_n) d\tau \\ & - \int_0^t (\eta_1^n - \eta_2^n, \dot{u}_{nN})_{L^2(\Xi)} d\tau - \int_0^t (\boldsymbol{\varsigma}^n, \dot{\mathbf{u}}_{nT})_{(L^2(\Xi))^3} d\tau = \int_0^t \langle \mathbf{f}, \dot{\mathbf{u}}_n \rangle d\tau \end{aligned}$$

and so for every $t \in (0, T]$ we have

$$\begin{aligned} & \frac{1}{2} |\dot{\mathbf{u}}_n(t)|^2 + \frac{1}{2} a(\mathbf{u}_n(t), \mathbf{u}_n(t)) + \int_0^t b(\dot{\mathbf{u}}_n, \dot{\mathbf{u}}_n) d\tau \\ & = \int_0^t (\eta_1^n - \eta_2^n, \dot{u}_{nN})_{L^2(\Xi)} d\tau + \int_0^t (\boldsymbol{\varsigma}^n, \dot{\mathbf{u}}_{nT})_{(L^2(\Xi))^3} d\tau \\ & \quad + \int_0^t \langle \mathbf{f}, \dot{\mathbf{u}}_n \rangle d\tau + \frac{1}{2} |\mathbf{u}_1|^2 + \frac{1}{2} a(\mathbf{u}_0, \mathbf{u}_0). \end{aligned}$$

By the relations (9), we obtain

$$\begin{aligned} & \frac{1}{2} |\dot{\mathbf{u}}_n(t)|^2 + \frac{m_a}{2} \|\mathbf{u}_n(t)\|^2 + m_b \int_0^t \|\dot{\mathbf{u}}_n\|^2 d\tau \\ & \leq k_1 \int_0^t (\|\eta_1^n\|_{L^2(\Xi)} + \|\eta_2^n\|_{L^2(\Xi)} + \|\boldsymbol{\varsigma}^n\|_{(L^2(\Xi))^3}) \|\dot{\mathbf{u}}_n\| d\tau \\ & + \int_0^t \|\mathbf{f}\| \|\dot{\mathbf{u}}_n\| d\tau + \frac{1}{2} |\mathbf{u}_1|^2 + \frac{M_a}{2} \|\mathbf{u}_0\|^2 \quad \forall n \in \mathbb{N}, \quad \forall t \in (0, T], \end{aligned}$$

where k_1 is a positive constant independent of n and M_a is a positive continuity constant of a .

Since the sequences $(\eta_1^n, \eta_2^n)_n, (\boldsymbol{\varsigma}^n)_n$ are bounded in $(L^2(\Xi_T))^2, (L^2(\Xi_T))^3$, respectively, by Young's inequality it follows that there exists a positive constant C_1 , depending only on $a, b, \mathbf{f}, \mathbf{u}_0, \mathbf{u}_1, k_1$, the bounds of $(\eta_1^n, \eta_2^n)_n$ and $(\boldsymbol{\varsigma}^n)_n$, such that the following estimates hold:

$$\forall n \in \mathbb{N}, |\dot{\mathbf{u}}_n(t)| \leq C_1, \|\mathbf{u}_n(t)\| \leq C_1 \quad \forall t \in [0, T], \|\dot{\mathbf{u}}_n\|_{L^2(0, T; \mathbf{V})} \leq C_1. \quad (17)$$

Using (16) for $\mathbf{v} = \boldsymbol{\psi}$, we see that for all $\boldsymbol{\psi} \in L^2(0, T; \mathbf{H}_0^1)$ with $\mathbf{H}_0^1 := H_0^1(\Omega^1; \mathbb{R}^3) \times H_0^1(\Omega^2; \mathbb{R}^3)$

$$\int_0^T (\ddot{\mathbf{u}}_n, \boldsymbol{\psi}) dt + \int_0^T a(\mathbf{u}_n, \boldsymbol{\psi}) dt + \int_0^T b(\dot{\mathbf{u}}_n, \boldsymbol{\psi}) dt = \int_0^T \langle \mathbf{f}, \boldsymbol{\psi} \rangle dt.$$

This relation and the estimates (17) imply that there exists a positive constant C_2 having the same properties as C_1 and satisfying the estimate

$$\forall n \in \mathbb{N}, \|\ddot{\mathbf{u}}_n\|_{L^2(0, T; \mathbf{H}_0^{-1})} \leq C_2, \quad (18)$$

where $\mathbf{H}_0^{-1} := H_0^{-1}(\Omega^1; \mathbb{R}^3) \times H_0^{-1}(\Omega^2; \mathbb{R}^3)$.

From (17), (18), it follows that there exist a subsequence $(\mathbf{u}_{n_k})_k$ and \mathbf{u} such that

$$\begin{aligned} \dot{\mathbf{u}}_{n_k} &\rightharpoonup^* \dot{\mathbf{u}} \text{ in } L^\infty(0, T; \mathbf{H}), \quad \mathbf{u}_{n_k} \rightharpoonup^* \mathbf{u} \text{ in } L^\infty(0, T; \mathbf{V}), \\ \dot{\mathbf{u}}_{n_k} &\rightharpoonup \dot{\mathbf{u}} \text{ in } L^2(0, T; \mathbf{V}), \quad \ddot{\mathbf{u}}_{n_k} \rightharpoonup \ddot{\mathbf{u}} \text{ in } L^2(0, T; \mathbf{H}_0^{-1}). \end{aligned}$$

According to Theorem 3.3 with

$$\mathcal{F} = (\dot{\mathbf{u}}_{n_k})_k, \quad X = \mathbf{V}, \quad U = \mathbf{H}^\iota, \quad \hat{Y} = \mathbf{H}_0^{-1}, \quad p = 2,$$

we obtain

$$\dot{\mathbf{u}}_{n_k} \rightarrow \dot{\mathbf{u}} \text{ in } L^2(0, T; \mathbf{H}^\iota),$$

where $1 > \iota > \frac{1}{2}$, so that, by the trace theorem, the last convergence implies

$$\dot{\mathbf{u}}_{n_k} \rightarrow \dot{\mathbf{u}} \text{ in } L^2(0, T; (L^2(\Xi))^3) = (L^2(\Xi_T))^3. \quad (19)$$

By Lemma 3.2, for all $l, m \in \mathbb{N}$ and for all $t \in [0, T]$,

$$\begin{aligned} &|\dot{\mathbf{u}}_l(t) - \dot{\mathbf{u}}_m(t)|^2 + \|\mathbf{u}_l(t) - \mathbf{u}_m(t)\|^2 + \int_0^t \|\dot{\mathbf{u}}_l - \dot{\mathbf{u}}_m\|^2 d\tau \\ &\leq C_0 \int_0^t (\eta_1^l - \eta_1^m - \eta_2^l + \eta_2^m, \dot{\mathbf{u}}_{lN} - \dot{\mathbf{u}}_{mN})_{L^2(\Xi)} d\tau \\ &\quad + C_0 \int_0^t (\boldsymbol{\varsigma}^l - \boldsymbol{\varsigma}^m, \dot{\mathbf{u}}_{lT} - \dot{\mathbf{u}}_{mT})_{(L^2(\Xi))^3} d\tau. \end{aligned} \quad (20)$$

Using the weak convergence properties of $(\eta_1^n)_n$, $(\eta_2^n)_n$, $(\boldsymbol{\varsigma}^n)_n$, and the strong convergence property (19), we can pass to limits in the previous estimates corresponding to $t = T$ for $(\mathbf{u}_{n_k})_k$ and so we obtain that $(\mathbf{u}_{n_k})_k$ is a Cauchy sequence in $W^{1,2}(0, T; \mathbf{V})$ and

$$\mathbf{u}_{n_k} \rightarrow \mathbf{u} \text{ in } W^{1,2}(0, T; \mathbf{V}).$$

Now, if $(\mathbf{u}'_{n'_k})_k$ is another subsequence of $(\mathbf{u}'_n)_n$ such that

$$\begin{aligned} \dot{\mathbf{u}}'_{n'_k} &\rightharpoonup^* \dot{\mathbf{u}}' \text{ in } L^\infty(0, T; \mathbf{H}), \quad \mathbf{u}'_{n'_k} \rightharpoonup^* \mathbf{u}' \text{ in } L^\infty(0, T; \mathbf{V}), \\ \dot{\mathbf{u}}'_{n'_k} &\rightharpoonup \dot{\mathbf{u}}' \text{ in } L^2(0, T; \mathbf{V}), \quad \ddot{\mathbf{u}}'_{n'_k} \rightharpoonup \ddot{\mathbf{u}}' \text{ in } L^2(0, T; \mathbf{H}_0^{-1}). \end{aligned}$$

then, using the same arguments as above, we have

$$\mathbf{u}'_{n'_k} \rightarrow \mathbf{u}' \text{ in } W^{1,2}(0, T; \mathbf{V})$$

and passing to limits in (20) with $l = n'_k$, $m = n_k$ we obtain that $\mathbf{u}' = \mathbf{u}$, so that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ in } W^{1,2}(0, T; \mathbf{V}). \quad (21)$$

By (20), the Cauchy-Schwarz inequality and the trace properties, there exists a positive constant C_3 such that for all $l, m \in \mathbb{N}$ and for all $t \in [0, T]$,

$$\begin{aligned} &|\dot{\mathbf{u}}_l(t) - \dot{\mathbf{u}}_m(t)|^2 + \|\mathbf{u}_l(t) - \mathbf{u}_m(t)\|^2 + \int_0^t \|\dot{\mathbf{u}}_l - \dot{\mathbf{u}}_m\|^2 d\tau \\ &\leq C_0 \int_0^t \|\eta_1^l - \eta_2^l - \eta_1^m + \eta_2^m\|_{L^2(\Xi)} \|\dot{\mathbf{u}}_{lN} - \dot{\mathbf{u}}_{mN}\|_{L^2(\Xi)} d\tau \\ &\quad + C_0 \int_0^t \|\boldsymbol{\varsigma}^l - \boldsymbol{\varsigma}^m\|_{(L^2(\Xi))^3} \|\dot{\mathbf{u}}_{lT} - \dot{\mathbf{u}}_{mT}\|_{(L^2(\Xi))^3} d\tau \\ &\leq C_0 \int_0^t (\|\eta_1^l - \eta_2^l - \eta_1^m + \eta_2^m\|_{L^2(\Xi)} + \|\boldsymbol{\varsigma}^l - \boldsymbol{\varsigma}^m\|_{(L^2(\Xi))^3}) \|\dot{\mathbf{u}}_l - \dot{\mathbf{u}}_m\|_{(L^2(\Xi))^3} d\tau \\ &\leq C_3 \int_0^T (\|\eta_1^l - \eta_2^l - \eta_1^m + \eta_2^m\|_{L^2(\Xi)} + \|\boldsymbol{\varsigma}^l - \boldsymbol{\varsigma}^m\|_{(L^2(\Xi))^3}) \|\dot{\mathbf{u}}_l - \dot{\mathbf{u}}_m\| d\tau. \end{aligned}$$

Passing to limits in the previous estimates, it follows that $(\mathbf{u}_n)_n$ is a Cauchy sequence in $C^1([0, T]; \mathbf{H}) \cap C([0, T]; \mathbf{V})$ and

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ in } C^1([0, T]; \mathbf{H}) \cap C([0, T]; \mathbf{V}). \quad (22)$$

Now, let $(\delta_1^n, \delta_2^n)_n$ be a sequence in $(L^2_+(\Xi_T))^2 \cap (W^{1,\infty}(0, T; L^2(\Xi)))^2$ and $(\boldsymbol{\varpi}^n)_n$ be a sequence in $(W^{1,\infty}(0, T; L^2(\Xi)))^3$ such that $\delta_1^n(0) = \delta_2^n(0) = 0$, $\boldsymbol{\varpi}^n(0) = \mathbf{0}$, for all $n \in \mathbb{N}$, $\delta_1^n \rightharpoonup \eta_1$, $\delta_2^n \rightharpoonup \eta_2$ in $L^2(\Xi_T)$, and $\boldsymbol{\varpi}^n \rightharpoonup \boldsymbol{\varsigma}$ in $(L^2(\Xi_T))^3$. If $\mathbf{u}_{(\delta_1^n, \delta_2^n, \boldsymbol{\varpi}^n)}$ is the solution of (13) corresponding to $(\delta_1^n, \delta_2^n, \boldsymbol{\varpi}^n)$,

for every $n \in \mathbb{N}$, then, using similar arguments as above for the union of the two sequences $(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)_n$ and $(\delta_1^n, \delta_2^n, \boldsymbol{\varpi}^n)_n$, it follows that

$$\mathbf{u}_{(\delta_1^n, \delta_2^n, \boldsymbol{\varpi}^n)} \rightarrow \mathbf{u} \text{ in } C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}).$$

It remains to prove that the unique limit \mathbf{u} of this class of approximating sequences is a solution of (15). For all $\mathbf{v} \in L^\infty(0, T; \mathbf{V}) \cap W^{1,2}(0, T; \mathbf{H})$, integrating over $(0, T)$ in (16) yields

$$\begin{aligned} \int_0^T (\dot{\mathbf{u}}_n, \mathbf{v}) + a(\mathbf{u}_n, \mathbf{v}) dt + \int_0^T b(\dot{\mathbf{u}}_n, \mathbf{v}) dt - \int_0^T (\eta_1^n - \eta_2^n, v_N)_{L^2(\Xi)} dt \\ - \int_0^T (\boldsymbol{\varsigma}^n, \mathbf{v}_T)_{(L^2(\Xi))^3} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle dt \end{aligned} \quad (23)$$

and integrating by parts the first term in (23) implies

$$\begin{aligned} (\dot{\mathbf{u}}_n(T), \mathbf{v}(T)) - (\mathbf{u}_1, \mathbf{v}(0)) - \int_0^T (\dot{\mathbf{u}}_n, \dot{\mathbf{v}}) dt + \int_0^T \{a(\mathbf{u}_n, \mathbf{v}) + b(\dot{\mathbf{u}}_n, \mathbf{v})\} dt \\ - \int_0^T \{(\eta_1^n - \eta_2^n, v_N)_{L^2(\Xi)} + (\boldsymbol{\varsigma}^n, \mathbf{v}_T)_{(L^2(\Xi))^3}\} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle dt \end{aligned} \quad (24)$$

$$\forall \mathbf{v} \in L^\infty(0, T; \mathbf{V}) \cap W^{1,2}(0, T; \mathbf{H}).$$

Passing to the limits by using (21) and (22), it follows that \mathbf{u} is a solution of (15). \square

Let $\Phi : (L^2_+(\Xi_T))^2 \times (L^2(\Xi_T))^3 \rightarrow 2^{(L^2_+(\Xi_T))^2 \times (L^2(\Xi_T))^3} \setminus \{\emptyset\}$ be the set-valued mapping defined by

$$\Phi(\eta_1, \eta_2, \boldsymbol{\varsigma}) = \Lambda^3(\mathbf{u}_{(\eta_1, \eta_2, \boldsymbol{\varsigma})}) \quad \forall (\eta_1, \eta_2, \boldsymbol{\varsigma}) \in (L^2_+(\Xi_T))^2 \times (L^2(\Xi_T))^3, \quad (25)$$

where $\mathbf{u}_{(\eta_1, \eta_2, \boldsymbol{\varsigma})}$ is the solution of the variational equation (15) which corresponds to $(\eta_1, \eta_2, \boldsymbol{\varsigma})$ by the procedure described in Theorem 3.4.

As $(\lambda_1, \lambda_2, \boldsymbol{\gamma})$ is a fixed point of Φ , i.e. $(\lambda_1, \lambda_2, \boldsymbol{\gamma}) \in \Phi(\lambda_1, \lambda_2, \boldsymbol{\gamma})$, if and only if $(\mathbf{u}_{(\lambda_1, \lambda_2, \boldsymbol{\gamma})}, \lambda_1, \lambda_2, \boldsymbol{\gamma})$ is a solution of the Problem P_v^2 , we consider a new problem, which consists in finding a fixed point of the set-valued mapping Φ , called also multifunction.

4 Existence of a solution to the contact problem

We shall prove the existence of a fixed point of the multifunction Φ by using a corollary of the Ky Fan's fixed point theorem [13], proved in [26] in the particular case of a reflexive Banach space.

Definition 4.1. Let Y be a reflexive Banach space, D a weakly closed set in Y , and $F : D \rightarrow 2^Y \setminus \{\emptyset\}$ be a multivalued function. F is called sequentially weakly upper semicontinuous if $z_n \rightharpoonup z$, $y_n \in F(z_n)$ and $y_n \rightharpoonup y$ imply $y \in F(z)$.

Proposition 4.1. ([26]) Let Y be a reflexive Banach space, D a convex, closed and bounded set in Y , and $F : D \rightarrow 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous multivalued function such that $F(z)$ is convex for every $z \in D$. Then F has a fixed point.

Theorem 4.2. Under the assumptions of Section 2, there exists $(\lambda_1, \lambda_2, \gamma) \in (L^2_+(\Xi_T))^2 \times (L^2(\Xi_T))^3$ such that $(\lambda_1, \lambda_2, \gamma) \in \Phi(\lambda_1, \lambda_2, \gamma)$. For each fixed point $(\lambda_1, \lambda_2, \gamma)$ of the multifunction Φ , $(\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}, \lambda, \gamma)$, with $\lambda = \lambda_1 - \lambda_2$, is a solution of the Problem P_v^1 and $(\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}, \lambda_1, \lambda_2, \gamma)$ is a solution of the Problem P_v^2 .

Proof. We apply Proposition 4.1 to $Y = (L^2(\Xi_T))^5$, $F = \Phi$ and $D = [(L^2_+(\Xi_T))^2 \cap D_0] \times [(L^2(\Xi_T))^3 \cap D_1]$.

The set $D \subset (L^2(\Xi_T))^5$ is clearly convex, closed, and bounded.

Since for each $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$ the set $\Lambda^3(\mathbf{w})$ is nonempty, closed, and convex, it follows that $\Phi(\eta_1, \eta_2, \varsigma)$ is a nonempty, closed, and convex subset of D for every $(\eta_1, \eta_2, \varsigma) \in D$.

In order to prove that the multifunction Φ is sequentially weakly upper semicontinuous, let $(\eta_1^n, \eta_2^n, \varsigma^n) \rightharpoonup (\eta_1, \eta_2, \varsigma)$, $(\eta_1^n, \eta_2^n, \varsigma^n) \in D$, $(\delta_1^n, \delta_2^n, \varpi^n) \in \Phi(\eta_1^n, \eta_2^n, \varsigma^n) \forall n \in \mathbb{N}$, $(\delta_1^n, \delta_2^n, \varpi^n) \rightharpoonup (\delta_1, \delta_2, \varpi)$ and let us verify that $(\delta_1, \delta_2, \varpi) \in \Phi(\eta_1, \eta_2, \varsigma)$. Using the Theorem 3.4 for each $(\eta_1^n, \eta_2^n, \varsigma^n)$, and the remark preceding this theorem, it follows that there exists a sequence $(\hat{\eta}_1^n, \hat{\eta}_2^n, \hat{\varsigma}^n)_n$ such that $(\hat{\eta}_1^n, \hat{\eta}_2^n) \in (L^2_+(\Xi_T))^2 \cap (W^{1,\infty}(0, T; L^2(\Xi)))^2$, $\hat{\varsigma}^n \in (W^{1,\infty}(0, T; L^2(\Xi)))^3$, $\hat{\eta}_1^n(0) = \hat{\eta}_2^n(0) = 0$, $\hat{\varsigma}^n(0) = \mathbf{0}$, for all $n \in \mathbb{N}$, and

$$(\hat{\eta}_1^n, \hat{\eta}_2^n, \hat{\varsigma}^n) - (\eta_1^n, \eta_2^n, \varsigma^n) \rightarrow \mathbf{0} \text{ in } (L^2(\Xi_T))^5, \quad (26)$$

$$\mathbf{u}_{(\hat{\eta}_1^n, \hat{\eta}_2^n, \hat{\varsigma}^n)} - \mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)} \rightarrow \mathbf{0} \text{ in } C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}), \quad (27)$$

where $\mathbf{u}_{(\hat{\eta}_1^n, \hat{\eta}_2^n, \hat{\varsigma}^n)}$ is the solution of (13) corresponding to $(\hat{\eta}_1^n, \hat{\eta}_2^n, \hat{\varsigma}^n)$, $\mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}$ is the solution of (15) corresponding to $(\eta_1^n, \eta_2^n, \varsigma^n)$.

As $(\eta_1^n, \eta_2^n, \varsigma^n) \rightharpoonup (\eta_1, \eta_2, \varsigma)$, by using (26), we have

$$(\hat{\eta}_1^n, \hat{\eta}_2^n, \hat{\varsigma}^n) \rightharpoonup (\eta_1, \eta_2, \varsigma) \text{ in } (L^2(\Xi_T))^5, \text{ and, by Theorem 3.4,}$$

$$\mathbf{u}_{(\hat{\eta}_1^n, \hat{\eta}_2^n, \hat{\varsigma}^n)} \rightarrow \mathbf{u}_{(\eta_1, \eta_2, \varsigma)} \text{ in } C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}), \quad (28)$$

where $\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}$ is the solution of (15) corresponding to $(\eta_1, \eta_2, \varsigma)$.

We adopt the following notations: $\mathbf{u}_n := \mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}$, $u_{nN} := u_{(\eta_1^n, \eta_2^n, \varsigma^n)N}$, $\mathbf{u}_\eta := \mathbf{u}_{(\eta_1, \eta_2, \varsigma)}$, $u_{\eta N} := u_{(\eta_1, \eta_2, \varsigma)N}$.

Thus, by (27) and the triangle inequality, we obtain

$$\mathbf{u}_n \rightarrow \mathbf{u}_\eta \text{ in } C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}), \quad (29)$$

which implies

$$\mathbf{u}_n \rightarrow \mathbf{u}_\eta, \quad \dot{\mathbf{u}}_n \rightarrow \dot{\mathbf{u}}_\eta \text{ in } (L^2(\Xi_T))^3. \quad (30)$$

Now, by Lemma 2.1, if $(\delta_1^n, \delta_2^n, \boldsymbol{\varpi}^n) \in \Phi(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n) = \boldsymbol{\Lambda}^3(\mathbf{u}_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)})$ for all $n \in \mathbb{N}$, then

$$\underline{\kappa}([u_{nN}], \dot{u}_{nN}) \leq \delta_1^n - \delta_2^n \leq \bar{\kappa}([u_{nN}], \dot{u}_{nN}) \text{ a.e. in } \Xi_T, \quad (31)$$

$$|\boldsymbol{\varpi}^n| \leq \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n) \text{ a.e. in } \Xi_T, \quad (32)$$

$$\boldsymbol{\varpi}^n \cdot \dot{\mathbf{u}}_{nT} + \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n) |\dot{\mathbf{u}}_{nT}| = 0 \text{ a.e. in } \Xi_T, \quad \forall n \in \mathbb{N}. \quad (33)$$

The relations (31) are equivalent to

$$\int_\omega \underline{\kappa}([u_{nN}], \dot{u}_{nN}) \leq \int_\omega (\delta_1^n - \delta_2^n) \leq \int_\omega \bar{\kappa}([u_{nN}], \dot{u}_{nN}),$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$.

Passing to limits according to Fatou's lemma, by using (30), the semi-continuity of $\underline{\kappa}$ and $\bar{\kappa}$, the relation (2), and the convergence property $\int_\omega (\delta_1^n - \delta_2^n) \rightarrow \int_\omega (\delta_1 - \delta_2)$, we obtain

$$\int_\omega \underline{\kappa}([u_{\eta N}], \dot{u}_{\eta N}) \leq \int_\omega (\delta_1 - \delta_2) \leq \int_\omega \bar{\kappa}([u_{\eta N}], \dot{u}_{\eta N}),$$

for every measurable subset $\omega \subset \Xi_T$, which implies

$$\underline{\kappa}([u_{\eta N}], \dot{u}_{\eta N}) \leq \delta_1 - \delta_2 \leq \bar{\kappa}([u_{\eta N}], \dot{u}_{\eta N}) \text{ a.e. in } \Xi_T. \quad (34)$$

The relation (32) is equivalent to

$$\int_\omega |\boldsymbol{\varpi}^n| \leq \int_\omega \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n),$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$.

As $\mu(\xi, \cdot)$ is Lipschitz continuous with the Lipschitz constant independent of ξ , by using (30) it is easy to see that

$$\dot{\mathbf{u}}_{nT} \rightarrow \dot{\mathbf{u}}_{\eta T} \text{ in } (L^2(\Xi_T))^3, \quad \mu(\dot{\mathbf{u}}_{nT}) \rightarrow \mu(\dot{\mathbf{u}}_{\eta T}) \text{ in } L^2(\Xi_T), \quad (35)$$

so that passing to limits we obtain

$$\int_{\omega} |\varpi| \leq \liminf \int_{\omega} |\varpi^n| \leq \lim \int_{\omega} \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n) = \int_{\omega} \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2).$$

Thus

$$\int_{\omega} |\varpi| \leq \int_{\omega} \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2),$$

for every measurable subset $\omega \subset \Xi_T$, which implies

$$|\varpi| \leq \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2) \text{ a.e. in } \Xi_T. \quad (36)$$

Now, we consider the relation (33) which is equivalent to

$$\int_{\omega} \varpi^n \cdot \dot{\mathbf{u}}_{nT} + \int_{\omega} \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n) |\dot{\mathbf{u}}_{nT}| = 0, \quad (37)$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$. By (35) we have

$$\mu(\dot{\mathbf{u}}_{nT}) |\dot{\mathbf{u}}_{nT}| \rightarrow \mu(\dot{\mathbf{u}}_{\eta T}) |\dot{\mathbf{u}}_{\eta T}| \text{ in } L^1(\Xi_T), \quad (38)$$

and, by Lemma 2.1 and the relations (2), (31),

$$\delta_1^n + \delta_2^n \rightharpoonup^* \delta_1 + \delta_2 \text{ in } L^\infty(\Xi_T). \quad (39)$$

Passing to limits in (37) by using (38) and (39), we obtain

$$\int_{\omega} \varpi \cdot \dot{\mathbf{u}}_{\eta T} + \int_{\omega} \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2) |\dot{\mathbf{u}}_{\eta T}| = 0,$$

for every measurable subset $\omega \subset \Xi_T$, which implies

$$\varpi \cdot \dot{\mathbf{u}}_{\eta T} + \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2) |\dot{\mathbf{u}}_{\eta T}| = 0. \quad (40)$$

By (29), (34), (36), (40), it follows that $(\delta_1, \delta_2, \varpi) \in \Phi(\eta_1, \eta_2, \mathfrak{s})$, so that, by applying Proposition 4.1, the theorem is proved. \square

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