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Compressibility Correction to Kutta-Joukowski and Maskell's formulae using vortex-force theory

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Nomenclature

c_{ref}	=	reference chord
C_D	=	drag coefficient
C_{D_i}	=	lift-induced drag coefficient
$C_{D_{i,VdV}}$	=	lift-induced drag coefficient computed by Van der Vooren's formulation
C_{D_P}	=	profile drag coefficient
$C_{D_{P,VdV}}$	=	profile drag coefficient computed by Van der Vooren's formulation
C_L	=	lift coefficient
d	=	distance between S_e and the body surface
$(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$	=	wind-fixed frame
\mathbf{F}	=	aerodynamic force
H	=	total enthalpy
\mathbf{l}	=	Lamb vector $\mathbf{l} = \boldsymbol{\omega} \times \mathbf{u}$
M_∞	=	freestream Mach number
n	=	dimension of the physical space $n = 2, 3$
\mathbf{n}	=	unit normal
p	=	pressure
R	=	perfect gas constant $R = 287.058 \text{ J.kg}^{-1}.\text{K}^{-1}$
Re	=	Reynolds number
\mathbf{r}	=	position vector
S_e	=	external boundary
s	=	entropy

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T	=	temperature
\mathbf{u}	=	velocity vector $\mathbf{u} = (U_\infty + \Delta u_x) \mathbf{e}_x + \Delta u_y \mathbf{e}_y + \Delta u_z \mathbf{e}_z$
u^2	=	velocity squared $u^2 = (U_\infty + \Delta u_x)^2 + \Delta u_y^2 + \Delta u_z^2$
U_∞	=	freestream velocity $U_\infty = U_\infty \mathbf{e}_x$
$\Delta \mathbf{u}$	=	perturbation velocity vector $\Delta \mathbf{u} = \Delta u_x \mathbf{e}_x + \Delta u_y \mathbf{e}_y + \Delta u_z \mathbf{e}_z$
(x, y, z)	=	wind-fixed coordinate system
α	=	angle of attack
γ	=	ratio of specific heats $\gamma = 1.4$
μ	=	dynamic viscosity
Ω	=	fluid domain
$\boldsymbol{\omega}$	=	vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$
ρ	=	density
ρ_∞	=	freestream density
Σ	=	trefftz plane
$\boldsymbol{\tau}$	=	viscous-stress tensor
∇	=	gradient operator $\nabla = \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y + \frac{\partial}{\partial z} \mathbf{e}_z$

I. Introduction

THE current understanding of aerodynamic lift and lift-induced drag still rests upon the works of the pioneering aerodynamicists Kutta, Joukowski and Prandtl at the beginning of the twentieth century. Indeed, assuming an inviscid and irrotational incompressible flow, Kutta [1] and later Joukowski [2] both established the same circulation-based lift formula now referred to as the Kutta-Joukowski theorem. However, this formula was initially left on the sidelines as it provides no drag: this is *d'Alembert's paradox*. Besides, it was also argued that no circulation can be generated in an inviscid and barotropic flow according to Kelvin's theorem. Prandtl overcame this issue by considering effectively inviscid flows for which the viscosity $\mu \rightarrow 0$ (i.e. $Re \rightarrow \infty$) but $\mu \neq 0$. In such conditions, the circulation is set by the Kutta condition and confined to the infinitely thin boundary layers. Prandtl replaced the lifting surface by a distribution of horseshoe vortices and modeled the wake as a vortex sheet [3]. For an effectively inviscid incompressible steady flow, he developed the lifting-line theory and introduced for the first time the concept of lift-induced drag. Nevertheless, the aforementioned developments were not applicable to finite-Reynolds-number flows.

The study of viscous flows was held by Bryant and Williams [4] in wind-tunnel experiments which emphasized that the Kutta-Joukowski theorem still holds in such conditions. Besides, the ever more pressing need for formulations able to deal with viscous and high-lifting flows encouraged aerodynamicists to wage further research on more accurate

lift-induced drag predictions. Maskell [5] proposed a formula that computes the lift-induced drag in incompressible flows from wake surveys for which the measurements can be confined to the vortical wake. This method is still widely used today [6–9] and Giles and Cummings [10] even managed to theoretically link Maskell’s formula to Prandtl’s lifting-line theory.

As a consequence, our physical understanding of aerodynamic lift and lift-induced drag is still strongly inherited from theories developed within the framework of incompressible flows (see Kroo’s review [11] for instance). It has now become a challenge to investigate the effects of compressibility on the lift and lift-induced drag. For instance, with a linear far-field analysis, Liu *et al.* [12, 13] recently showed that the Kutta-Joukowski theorem still holds in compressible steady viscous flows. More recently, Schmitz proposed an extension of the Kutta-Joukowski theorem in viscous incompressible flows [14]. Using a partial-pressure-field decomposition, he later extended this study to compressible flows [15].

The present paper addresses the link between the vortex-force theory, Kutta-Joukowski theorem and a lift-induced drag formula (first derived by Maskell [5] in incompressible flows) in the case of compressible steady viscous flows. The vortex-force theory defines the lift and the lift-induced drag as the sum of the vortex force (volume integral of the Lamb vector $\boldsymbol{l} = \boldsymbol{\omega} \times \boldsymbol{u}$ where \boldsymbol{u} is the velocity and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity) and a volume integral involving the density gradient [16]: our work investigates the contribution of the density gradient. It is proven both theoretically and numerically that the sum of these two volume integrals yields counterparts of Kutta-Joukowski theorem and Maskell’s lift-induced drag formula in compressible flows plus a correction which vanishes as the integration domain is increased.

II. Presentation of the vortex-force theory

The vortex-force theory relies on a *far-field* momentum balance in a control volume of fluid Ω surrounding the aircraft and bounded by an external closed surface S_e (see Fig.1). This momentum balance is based on unconventional formulae referred to as *derivative moment transformations* (DMT) presented in Appendix A. Wu *et al.* [17] derived an exact expression of the aerodynamic force but, as noticed by Liu [18], their definition could not provide any breakdown of the lift and the drag in compressible flows. This issue was circumvented by Mele and Tognaccini [16] who derived the following force formula:

$$\boldsymbol{F} = \boldsymbol{F}_{\rho l} + \boldsymbol{F}_{m_\rho} + \boldsymbol{F}_{S_e} + \boldsymbol{F}_\tau \quad (1)$$

with

$$\mathbf{F}_{\rho l} = - \int_{\Omega} \rho \mathbf{l} dv \quad (2)$$

$$\mathbf{F}_{m_{\rho}} = - \frac{1}{n-1} \int_{\Omega} \mathbf{r} \times \left(\nabla \rho \times \nabla \left(\frac{u^2}{2} \right) \right) dv = - \int_{\Omega} \mathbf{m}_{\rho} dv \quad (3)$$

$$\mathbf{F}_{S_e} = - \frac{1}{n-1} \oint_{S_e} \mathbf{r} \times (\mathbf{n} \times \rho \mathbf{l}) dS \quad (4)$$

$$\mathbf{F}_{\tau} = \oint_{S_e} \boldsymbol{\tau} \cdot \mathbf{n} dS + \frac{1}{n-1} \oint_{S_e} \mathbf{r} \times (\mathbf{n} \times \nabla \cdot \boldsymbol{\tau}) dS \quad (5)$$

where $n = 2, 3$ is the space dimension, ρ is the density, $u^2 = \|\mathbf{u}\|^2$, $\boldsymbol{\tau}$ is the viscous-stress tensor and \mathbf{n} is the unit normal pointing outwards. Eq.(1) involves several terms containing the position vector \mathbf{r} which raises the question whether the decomposition depends on the reference point chosen for moment computation. Fournis *et al.* [19] showed that the breakdown sensitivity to \mathbf{r} vanishes as S_e retreats to infinity and proposed an invariant formulation based on *far-field* flow symmetries. Former studies numerically highlighted that $\mathbf{F}_{\rho l} + \mathbf{F}_{m_{\rho}}$ provides the total lift and the lift-induced drag in compressible flows while the surface integral \mathbf{F}_{S_e} yields the profile drag (friction and viscous pressure drag) and no lift [16, 20–22]. \mathbf{F}_{τ} is neglected in practice [23], $\mathbf{F}_{\rho l}$ is the vortex force and $\mathbf{F}_{m_{\rho}}$ is an integral containing the density gradient in compressible flows.

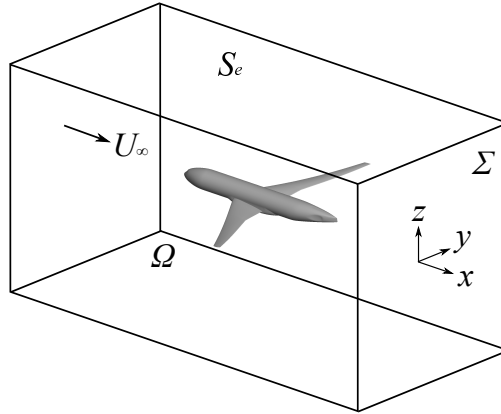


Fig. 1 Control volume

Mele and Tognaccini [16] numerically emphasized that the incompressible Kutta-Joukowski theorem still holds in transonic and supersonic regimes suggesting that $\mathbf{F}_{m_{\rho}}$ is nothing but a compressibility correction to $\mathbf{F}_{\rho l}$ [20]. Besides, Kang *et al.* [22] established a link with Maskell's formula by subtracting \mathbf{F}_{S_e} to the total *far-field* drag.

The aim of the next section is to theoretically understand why should $\mathbf{F}_{m_{\rho}}$ be added to $\mathbf{F}_{\rho l}$ in the definition of the lift and lift-induced drag in compressible flows. It also proposes an analytical link with Kutta-Joukowski theorem and Maskell's formula.

III. Kutta-Joukowski theorem and Maskell's formula in compressible flows

In steady incompressible flows, only the vortex force $\mathbf{F}_{\rho l}$ contributes to the lift and the lift-induced drag and Marongiu *et al.* [24] highlighted the link between $\mathbf{F}_{\rho l}$, Kutta-Joukowski theorem and Maskell's lift-induced drag formula. The goal is to understand the role played by $\mathbf{F}_{m\rho}$ in compressible flows. One may write:

$$\mathbf{F}_{\rho l} + \mathbf{F}_{m\rho} = - \int_{\Omega} \left(\rho \mathbf{l} - \frac{u^2}{2} \nabla \rho \right) dv + \frac{1}{n-1} \oint_{S_e} \mathbf{r} \times \left(\mathbf{n} \times \frac{u^2}{2} \nabla \rho \right) dS \quad (6)$$

where the first DMT Eq.(A.1) has been applied on $\mathbf{F}_{m\rho}$. By using the relation

$$\rho \mathbf{l} - \frac{u^2}{2} \nabla \rho = \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} - \rho \frac{u^2}{2} \mathbf{1} \right) \quad (7)$$

along with the decomposition of the velocity vector $\mathbf{u} = \mathbf{U}_{\infty} + \Delta \mathbf{u}$ and the application of the divergence theorem, Eq.(6) may be written as follows:

$$\begin{aligned} \mathbf{F}_{\rho l} + \mathbf{F}_{m\rho} &= \oint_{S_e} \rho (\mathbf{U}_{\infty} \cdot \Delta \mathbf{u}) \mathbf{n} dS - \oint_{S_e} \rho \Delta \mathbf{u} (\mathbf{U}_{\infty} \cdot \mathbf{n}) dS \\ &\quad - \oint_{S_e} \rho \mathbf{U}_{\infty} (\mathbf{U}_{\infty} \cdot \mathbf{n}) dS - \oint_{S_e} \rho \mathbf{U}_{\infty} (\Delta \mathbf{u} \cdot \mathbf{n}) dS \\ &\quad + \oint_{S_e} \rho \frac{\Delta u^2}{2} \mathbf{n} dS - \oint_{S_e} \rho \Delta \mathbf{u} (\Delta \mathbf{u} \cdot \mathbf{n}) dS \\ &\quad + \frac{U_{\infty}^2}{2} \oint_{S_e} \Delta \rho \mathbf{n} dS + \frac{1}{n-1} \oint_{S_e} \mathbf{r} \times \left(\mathbf{n} \times \frac{u^2}{2} \nabla \rho \right) dS \end{aligned} \quad (8)$$

where ρ_{∞} is the freestream density and $\Delta \rho = \rho - \rho_{\infty}$. The first line of Eq.(8) is the surface integral of the double-cross product $\mathbf{U}_{\infty} \times (\mathbf{n} \times \rho \Delta \mathbf{u})$. The second line vanishes due to the steady continuity equation $\oint_{S_e} \rho \mathbf{u} \cdot \mathbf{n} dS = 0$. The velocity perturbation $\Delta \mathbf{u}$ is nonzero only in the wake provided that S_e does not lie too close to the body skin. Therefore, integrals involving $\Delta \mathbf{u}$ can be limited to a wake plane Σ on which $\mathbf{n} = \mathbf{e}_x$. Finally, Eq.(8) reduces to:

$$\begin{aligned} \mathbf{F}_{\rho l} + \mathbf{F}_{m\rho} &= \mathbf{U}_{\infty} \times \oint_{S_e} \mathbf{n} \times \rho \Delta \mathbf{u} dS \\ &\quad + \int_{\Sigma} \rho \frac{\Delta u^2}{2} \mathbf{n} dS - \int_{\Sigma} \rho \Delta \mathbf{u} (\Delta \mathbf{u} \cdot \mathbf{n}) dS \\ &\quad + \frac{U_{\infty}^2}{2} \oint_{S_e} \Delta \rho \mathbf{n} dS + \frac{1}{n-1} \oint_{S_e} \mathbf{r} \times \left(\mathbf{n} \times \frac{u^2}{2} \nabla \rho \right) dS \end{aligned} \quad (9)$$

The first line of Eq.(9) is a compressible Kutta-Joukowski formula \mathbf{F}_{KJ} containing the local density ρ instead of ρ_{∞} . The surface integral $\oint_{S_e} \mathbf{n} \times \rho \Delta \mathbf{u} dS$ is equal to $\rho_{\infty} \mathbf{\Gamma}^*$ where $\mathbf{\Gamma}^*$ is a circulation vector which accounts for compressibility effects. Using the *far-field* symmetries of the wake introduced by Fournis *et al.* [19], it is possible to show that $\mathbf{\Gamma}^* \cdot \mathbf{e}_x = \mathbf{\Gamma}^* \cdot \mathbf{e}_z = 0$. Hence it contributes solely to the lift while the second line contributes to the lift and the drag:

its contribution to the lift, denoted $-\rho_\infty \delta\Gamma^*$, also accounts for compressibility effects and is to be included in the Kutta-Joukowski formula [14]. The third line is present only in compressible flows as it involves density gradients and will be referred to as $F_{\nabla\rho}$. Using Eq.(A.2) to recast the last line, $F_{\rho l} + F_{m_\rho}$ can be expressed as follows:

$$\begin{aligned}
F_{\rho l} + F_{m_\rho} &= \underbrace{-\rho_\infty U_\infty (\Gamma^* + \delta\Gamma^*) \mathbf{e}_z}_{\text{Compressible K-J theorem } F_{\text{KJ}}} \\
&\quad + \underbrace{\left(\frac{1}{2} \int_\Sigma \rho (\Delta u_y^2 + \Delta u_z^2 - \Delta u_x^2) dS \right) \mathbf{e}_x}_{\text{Compressible Maskell's formula } F_{\text{MSK}}} \\
&\quad + \underbrace{\frac{1}{n-1} \oint_{S_e} \mathbf{r} \times \left(\mathbf{n} \times \frac{u^2 - U_\infty^2}{2} \nabla \rho \right) dS}_{\text{Compressibility correction } F_{\nabla\rho}}
\end{aligned} \tag{10}$$

with

$$\Gamma^* = \oint_{S_e} \mathbf{n} \times \frac{\rho}{\rho_\infty} \Delta \mathbf{u} dS = -\Gamma^* \mathbf{e}_y \tag{11}$$

$$\delta\Gamma^* = \frac{1}{U_\infty} \int_\Sigma \frac{\rho}{\rho_\infty} \Delta u_x \Delta u_z dS \tag{12}$$

In the end, it is possible to derive a Kutta-Joukowski lift theorem F_{KJ} and a counterpart of Maskell's formula F_{MSK} in compressible flows plus a compressibility correction $F_{\nabla\rho}$. In incompressible flows $\rho = \rho_\infty$ such that $F_{\nabla\rho} = \mathbf{0}$ and Eq.(10) reduces to the generalized Kutta-Joukowski theorem derived by Schmitz [14] plus the original Maskell's formula. In compressible flows, the circulation $\Gamma^* + \delta\Gamma^*$ defined in Eqs.(11) and (12) accounts for the density variations in the flow where $\delta\Gamma^*$ is a viscous correction to the total circulation due to the presence of Δu_x . Similarly, Maskell's formula F_{MSK} of Eq.(10) contains ρ instead of ρ_∞ as it is the case in incompressible flows. The theoretical asymptotic study in Appendix B shows that the compressibility correction $F_{\nabla\rho}$ effectively vanishes in the far wake. Thus, $F_{\rho l} + F_{m_\rho}$ relates the vortex-force theory to extensions of both the Kutta-Joukowski theorem and Maskell's formula to compressible flows. This analytical result is confirmed by a numerical application in the next section.

IV. Analysis of the results

The theoretical study presented above has been applied to Reynolds-Averaged Navier-Stokes simulations in the ONERA-SAFRAN-AIRBUS elsA CFD solver. A subsonic computation and a transonic computation have been performed on a fine mesh of the ONERA M6 wing with 10,200,000 points. The flight conditions are summarized in Table 1. This mesh was used by Mayeur *et al.* [25] in CFD computations using the elsA solver that were compared with wind-tunnel experiments. The y^+ is always lower than one and the far field lies 185 reference chord lengths away from the wing surface. Spalart-Allmaras turbulence model and Roe's numerical scheme were applied and a good

Flight conditions	M_∞	Re	α
Subsonic case	0.6	10^7	3°
Transonic case	0.84	1.2×10^7	3.06°

Table 1 Flight conditions applied on the M6 wing

convergence was reached since the residuals decreased by 10^6 . The same computations were performed on a coarser grid of 3,800,000 points [25]. The y_+ always revealed to be close to one. The spurious drag associated with this mesh is higher since the *near-field* drag (which is the standard integration of the pressure and shear stresses on the wing surface) differs by 3 counts for both cases between the two grids. The total drag coefficients computed by Van der Vooren’s formula and vortex-force theory respectively differ by 0.7 and 0.5 counts in the subsonic case and by 0.6 and 1.4 counts in the transonic case between the two grids. The corresponding lift-induced drag coefficients differ by 0.5 and 1.3 counts in the subsonic case and by 1.3 and 3.5 counts in the transonic case which suggests that the coarsening does not impact the drag breakdown very much. An isometric view of the ONERA M6 wing is shown in Fig.2 along with a Q-criterion isosurface and an ω_x modulus contour in the wake. The dissipation of the vortices is weak in this case, which is an important point regarding the computation of the lift-induced drag on the wake plane.

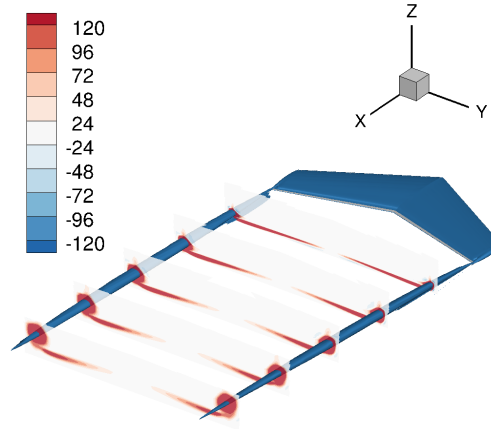


Fig. 2 Computed Q-criterion for the visualization of the tip vortices along with an ω_x modulus contour in the wake of the ONERA M6 wing with $M_\infty = 0.6$, $Re = 10^7$, $\alpha = 3^\circ$

The decomposition of the drag in profile and lift-induced contributions was sought, the profile drag being defined by the term $F_{S_e} + F_\tau$. F_τ is fairly small in high-Reynolds-number flows [26] and is usually neglected, but the following expression allows one to account for it even if it is small, making the overall force prediction more accurate when S_e lies close to the boundary layers. Let s , h and $H = h + \frac{u^2}{2}$ be respectively the specific entropy, enthalpy and total enthalpy. Let p be the pressure, R the perfect gas constant and T the temperature. By using the steady Crocco-Vazsonyi equation

$$\nabla \cdot \boldsymbol{\tau} - \rho \boldsymbol{l} = \rho \nabla H - \rho T \nabla s \quad (13)$$

along with the perfect gas law $p = \rho RT$, it is possible to express $\mathbf{F}_{S_e} + \mathbf{F}_\tau$ as follows:

$$\mathbf{F}_{S_e} + \mathbf{F}_\tau = \mathbf{F}_{sH} = \oint_{S_e} \boldsymbol{\tau} \cdot \mathbf{n} dS - \frac{1}{n-1} \oint_{S_e} \mathbf{r} \times \left(\mathbf{n} \times \left(\frac{p}{R} \nabla s - \rho \nabla H \right) \right) dS \quad (14)$$

The latter definition is better suited for numerical applications since it contains only first-order derivatives which allows for a more accurate profile drag estimation on the external surface S_e . Moreover, the profile drag recovers its irreversible character since it involves the entropy s and the total enthalpy H . It should be noted that the drag integrals derived by Giles and Cummings [10] and tested by Hunt *et al.* [9] are first-order approximations of \mathbf{F}_{MSK} of Eq.(10) and \mathbf{F}_{sH} of Eq.(14).

The sensitivity of the different contributors to the aerodynamic force with respect to the size of the control volume Ω has been investigated. The integration domain is extended by increasing d/c_{ref} , where d is the distance between S_e and the body surface and $c_{ref} = 0.64607 \text{ m}$ is the reference chord. This distance is computed by the turbulence model in the elsA solver. The force decomposition provided by the vortex-force theory stabilized after one reference-chord length and a half downstream of the trailing edge, therefore confining the integration domain to a better refined part of the grid, a crucial aspect for accurately predicting the aerodynamic force. In the following tables, the drag breakdown computed by Eqs.(10) and (14) will be compared to the thermodynamic method from Destarac and Van der Vooren [27] (subscript \bullet_{VDV}) for $d/c_{ref} = 2$. The profile drag is only of viscous origin in the subsonic case while in the transonic case, it also accounts for the wave drag. The very same Kutta-Joukowski-Maskell formula Eq.(10) can be derived in transonic flows provided that one starts from the alternative expression of \mathbf{F}_{m_ρ} derived by Mele *et al.* [21]. Indeed, it was shown that the original expression given by Eq.(3) is unable to predict the correct contribution due to inaccuracies in the shock wave region where the numerical scheme is only first order accurate. Therefore, in Tables 2 and 4, \mathbf{F}_{m_ρ} is computed with Eq.(3) in the subsonic case and with Eq.(25) of Ref.[21] in the transonic case. The results for the total drag will also be compared to the *near-field* drag.

The results are summarized in Tables 2 to 4. The correction vanished fast, after half a reference-chord length downstream of the trailing edge for both lift and drag. In Table 2 it is shown that the lift-induced drag computed by $\mathbf{F}_{\rho l} + \mathbf{F}_{m_\rho}$ is only 0.62 drag counts lower than that computed by its equivalent expression $\mathbf{F}_{MSK} + \mathbf{F}_{KJ} + \mathbf{F}_{\nabla \rho}$ and 6.63 drag counts lower than that computed by Van der Vooren's formula in the subsonic case. In the transonic case, the lift-induced drag computed by $\mathbf{F}_{\rho l} + \mathbf{F}_{m_\rho}$ is 1.5 counts lower than that provided by $\mathbf{F}_{MSK} + \mathbf{F}_{KJ} + \mathbf{F}_{\nabla \rho}$ and 10.23 counts lower than that predicted by Van der Vooren's formula. Moreover \mathbf{F}_{MSK} always yields higher values for the lift-induced drag and is closer (5.2 and 6.68 drag counts lower) to the values given by Van der Vooren's formula. The compressibility correction is in absolute value higher in the transonic case but vanishes as d/c_{ref} increases. The lift predictions are also in perfect agreement. It confirms that $\mathbf{F}_{\rho l} + \mathbf{F}_{m_\rho}$ constitutes a robust definition of the lift and the lift-induced drag in compressible flows as it is consistent with Kutta-Joukowski theorem and Maskell's formula. As expected, \mathbf{F}_{sH}

Term	Subsonic case		Transonic case	
	$C_{D_i} \times 10^4$	C_L	$C_{D_i} \times 10^4$	C_L
$F_{\rho l}$ (2)	53.35	0.14582	89.05	0.07522
F_{m_ρ}	-21.80	0.06731	-37.46	0.19799
$F_{\rho l} + F_{m_\rho}$	31.56	0.21313	51.59	0.27321
F_{MSK}	32.99	0	55.14	0
F_{KJ}	0	0.21303	0	0.27274
$F_{\nabla\rho}$	-0.80	0.00005	-2.05	0.00041
$F_{MSK} + F_{KJ} + F_{\nabla\rho}$	32.18	0.21308	53.09	0.27315
$C_{D_{i,vdV}}$	38.19	-	61.82	-
<i>Near-field</i> lift	-	0.21244	-	0.27209

Table 2 Lift and lift-induced drag results for the ONERA M6 wing with $d/c_{ref} = 2$.

Term	Subsonic case		Transonic case	
	$C_{D_P} \times 10^4$	C_L	$C_{D_P} \times 10^4$	C_L
F_{sH} (14)	80.72	-0.00058	117.74	-0.00098
$C_{D_{P,vdV}}$	74.18	-	109.17	-

Table 3 Profile drag results for the ONERA M6 wing with $d/c_{ref} = 2$.

Term	Subsonic case		Transonic case	
	$C_D \times 10^4$	C_L	$C_D \times 10^4$	C_L
$F_{\rho l} + F_{m_\rho} + F_{sH}$	112.28	0.21254	169.41	0.27223
$F_{MSK} + F_{KJ} + F_{\nabla\rho} + F_{sH}$	112.90	0.21250	170.83	0.27217
<i>Near-field</i> force	113.56	0.21244	172.23	0.27209
Total drag VdV	112.37	-	170.99	-

Table 4 Total force results for the ONERA M6 wing with $d/c_{ref} = 2$.

only accounts for the profile drag and no lift (see Table 3). Its prediction is always higher than that provided by Van der Vooren's formula (6.54 drag counts in the subsonic case and 8.57 in the transonic case) such that the total drag computed by the vortex-force theory and Van der Vooren's formula never differ from each other by more than two counts (see Table 4): in the end the drag decomposition computed by the vortex-force theory differs from that provided by the thermodynamic method but they both predict the same total drag. Van der Vooren's formula was developed with a model in which there is no vortex in the wake such that the drag is only of profile nature. The lift-induced drag is then computed by subtracting the profile drag to the total drag. On the contrary, no such assumption is made in the vortex-force theory in which the Lamb vector plays a key role as it is involved in the lift, the lift-induced drag and the profile drag, suggesting that the phenomenological contributions to the total force are all coupled. Therefore, the

difference in drag decomposition given by those two formulae is not surprising but is still a key issue under investigation.

V. Conclusion

A theoretical link between the vortex-force theory, Kutta-Joukowski theorem and Maskell's formula was emphasized in compressible viscous flows. Consequently and for the first time, the present study analytically evidences the consistency of the lift and lift-induced drag definition given by the vortex-force theory. Along with the connection to Kutta-Joukowski theorem and Maskell's formula, a compressibility correction involving density gradients was shown to vanish as the size of the integration domain increases. It was confirmed by RANS simulations over the ONERA M6 wing. It suggests that, as soon as the domain boundaries lie sufficiently far, the lift and the lift-induced drag are fully given by compressible counterparts of the circulation theorem and Maskell's formula.

Furthermore, the drag breakdown provided by the vortex-force theory was compared to a thermodynamic formulation already used for industrial purposes. The results show that the two formulae yield two different decompositions although they remain close to each other. Hence, the vortex-force theory may well constitute a robust alternative to former methods for predicting and decomposing the drag.

Appendix A: Derivative moment transformations

Let \mathbf{f} be a differentiable vector field, Φ be a differentiable scalar field, Ω be a domain bounded by $\partial\Omega$ and S be a surface bounded by ∂S . The *derivative moment transformations* state that in an n -dimensional space with $n = 2, 3$:

$$\int_{\Omega} \mathbf{f} \, dv = \frac{1}{n-1} \int_{\Omega} \mathbf{r} \times (\nabla \times \mathbf{f}) \, dv - \frac{1}{n-1} \oint_{\partial\Omega} \mathbf{r} \times (\mathbf{n} \times \mathbf{f}) \, dS \quad (\text{A.1})$$

$$\int_S \Phi \mathbf{n} \, dS = -\frac{1}{n-1} \int_S \mathbf{r} \times (\mathbf{n} \times \nabla \Phi) \, dS + \frac{1}{n-1} \oint_{\partial S} \Phi \mathbf{r} \times d\mathbf{r} \quad (\text{A.2})$$

Appendix B: Asymptotic behavior of $F_{\nabla\rho}$ in compressible flow

The asymptotic behavior of the velocity perturbation in the wake was addressed by Schlichting [28] (p.732) and White [29] (p.481) for planar (two-dimensional) and circular (three-dimensional) wakes in incompressible flows using the two-dimensional form of Navier-Stokes equations in the (x, z) -plane. In such conditions, and assuming that the same study can be held in the (x, y) -plane, the lateral extent of the planar wake behaves as $y, z = O(x^{1/2})$ while the velocity perturbation behaves as $\Delta u_x = O(x^{-1/2})$. Hence the incompressible continuity equation yields $\Delta u_y, \Delta u_z = O(x^{-1})$. For circular wakes, $y, z = O(x^{1/3})$ while $\Delta u_x = O(x^{-2/3})$ which yields $\Delta u_y, \Delta u_z = O(x^{-4/3})$. Experimental studies conducted on three-dimensional wakes highlighted that the asymptotic behavior of the longitudinal velocity defect becomes equivalent to that observed for circular wakes in the far field [30]. Furthermore, Kuo and Baldwin [31] noticed that the lateral and vertical width of the three-dimensional wake progressively increase as $x^{1/3}$ as in the case of a circular

wake.

The aforementioned results were established in incompressible flows whereas the flows considered in this study are compressible. Since the compressibility correction $F_{\nabla\rho}$ derived above involves the velocity perturbation, its asymptotic behavior will be sought only in the wake far enough downstream, where the flow is assumed incompressible following Schlichting [28] and Bradshaw [32]. In the far wake the local density is equal to its freestream value plus a small perturbation $\rho = \rho_\infty + \delta\rho$ with $\delta\rho = O(x^{-\epsilon})$ with $\epsilon > 0$. Besides, the correction can be written as follows:

$$\mathbf{F}_{\nabla\rho} = \frac{1}{n-1} \int_{\Sigma} \left(U_\infty \Delta u_x + \frac{\Delta u_x^2 + \Delta u_y^2 + \Delta u_z^2}{2} \right) \begin{pmatrix} y (\partial\rho/\partial y) + z (\partial\rho/\partial z) \\ -x (\partial\rho/\partial z) \\ -x (\partial\rho/\partial z) \end{pmatrix} dS \quad (\text{B.1})$$

It is then possible to show that $F_{\nabla\rho}$ asymptotically behaves as:

$$\mathbf{F}_{\nabla\rho} = \begin{pmatrix} O(x^{-(\epsilon+\eta)}) \\ O(x^{-\epsilon}) \\ O(x^{-\epsilon}) \end{pmatrix} \quad \text{with} \quad \eta = \begin{cases} \frac{1}{2} & \text{planar wake} \\ \frac{2}{3} & \text{circular wake} \end{cases}$$

Therefore, the compressibility correction is theoretically expected to vanish when Σ retreats to infinity, in the case of a planar or a circular wake sufficiently far downstream of the obstacle.

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