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Sandro Bettin, Sary Drappeau

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# EFFECTIVE ESTIMATION OF SOME OSCILLATORY INTEGRALS RELATED TO INFINITELY DIVISIBLE DISTRIBUTIONS

S. BETTIN AND S. DRAPPEAU

ABSTRACT. We present a practical framework to prove, in a simple way, two-terms asymptotic expansions for Fourier integrals

$$\mathcal{I}(t) = \int_{\mathbb{R}} (e^{it\phi(x)} - 1) d\mu(x)$$

where  $\mu$  is a probability measure on  $\mathbb{R}$  and  $\phi$  is measurable. This applies to many basic cases, in link with Levy's continuity theorem. We present applications to limit laws related to rational continued fractions coefficients.

## 1. INTRODUCTION

Let  $\mu$  be a probability measure on  $\mathbb{R}$ , and  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be  $\mu$ -measurable. The present paper is concerned with asymptotic formulæ for the Fourier integrals associated with  $\phi$  near the origin,

$$(1.1) \quad \mathcal{I}[\phi](t) := \int (e^{it\phi(x)} - 1) d\mu(x), \quad (t \rightarrow 0).$$

Such estimates are connected with the question of whether the push-forward measure  $\phi_*(\mu)$  belongs to the bassin of attraction of a stable law, see Chapter 2 of [IL71]. Our interest in this question originates from this point of view, and more specifically from the work [BD] where we study the convergence towards stable laws of the value distribution of invariants related to modular forms. In the setting of [BD], the measure  $\mu$  is the Gauss-Kuzmin distribution

$$d\mu(x) = \frac{dx}{(1+x)\log 2} \quad (x \in [0, 1]),$$

and this measure is invariant under the Gauss map  $T(x) = \{1/x\}$ , where  $\{x\} = x - [x]$  is the fractional part of  $x$ . More precisely, in [BD], we are interested in Birkhoff sums

$$(1.2) \quad \sum_{j=1}^r \phi(T^j(x)), \quad (T^r = T \circ \dots \circ T),$$

where  $x$  varies among rationals and  $r \geq 0$  is the length of the continued fractions expansion of  $x$ . In the set of rationals we consider, these sums are found to typically behave as sums of the shape

$$\sum_{j=1}^r \phi(X_j)$$

where  $(X_j)_{1 \leq j \leq r}$  are i.i.d. random variables distributed according to the Gauss-Kuzmin measure  $\mu$ . Then effective estimates for the integral (1.1), in conjunction with [BD, Theorem 3.1]

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and the Berry-Esseen inequality [Fel71, equation (XVI.3.13)] are used to obtain uniform limit theorems for the rational Birkhoff sums (1.2).

We return to the setting where  $\mu$  is an arbitrary probability measure on  $\mathbb{R}$ . Integrals (1.1) are related to the methods of asymptotic analysis mentioned *e.g.* in Chapter 9 of the monograph [Olv97]. When expressed as convolution integrals  $\int_x h(tx)f(x)dx$ , they are referred to as  $h$ -transforms in [BH86], and are also the topic of interest of the recent work [Lóp08]. The variety in assumptions and methods seems to prevent us from having a uniform framework for estimating (1.1).

The goal of the present paper is to present and prove several basic estimates through which one can give a streamlined and simple proof of an effective asymptotic expansion of the integral (1.1), including the terms of interest in central limit theorems.

**Definition 1.1.** *Given  $\alpha \in (0, 3]$  and two positive functions  $L, R$  defined in a neighborhood of 0 in  $\mathbb{R}_+^*$ , we denote by  $\mathcal{G}(\alpha, L, R)$  the set of functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that for some numbers  $c_1, c_2 \in \mathbb{R}$  and  $c_* \in \mathbb{C}$ , and all small enough  $t > 0$ , there holds*

$$(1.3) \quad \mathcal{I}[\phi](t) = ic_1t + c_2t^2 + c_*t^\alpha L(t) + O(t^3 + t^\alpha R(t)).$$

*Remark.* – If  $R = O(t^\varepsilon)$  for any  $\varepsilon > 0$  and  $\alpha < 1$ , the term  $c_1t$  in (1.3) is part of the error term, and likewise for  $c_2t^2$  if  $\alpha < 2$ .

- We will be interested in the largest one or two terms in the expansion (1.3). The case  $\alpha = 3$ ,  $L = R \equiv 1$  corresponds to an order 2 Taylor expansion.
- Whenever the expansion (1.3) holds for  $\phi$ , we will denote the coefficients by  $c_1(\phi)$ ,  $c_2(\phi)$ ,  $c_*(\phi)$  respectively.

**Theorem 1.2.** (1) *If  $\int |\phi(x)|^\alpha d\mu(x) < \infty$  for some  $\alpha \in (0, 3]$ , then  $\phi \in \mathcal{G}(\alpha, 1, 1)$ .*

(2) *Suppose that  $d\mu = f d\nu$  where  $\nu$  is the Lebesgue measure and  $f \in \mathcal{C}^1([0, 1])$ . Then for all  $a \in \mathbb{R}^*$ ,  $\beta > 3$  and  $\lambda \geq 0$ , the function*

$$\phi : (0, 1] \rightarrow \mathbb{R}, \quad \phi(x) = ax^{-\beta} |\log x|^\lambda,$$

*belongs to  $\mathcal{G}(\frac{1}{\beta}, |\log|^{\lambda/\beta+v}, |\log|^{\lambda/\beta+v-1+\varepsilon})$  for any  $\varepsilon \in (0, 1]$ , where  $v = 1$  for  $\beta \in \{1/2, 1\}$  and  $v = 0$  otherwise.*

(3) *Given two measurable functions  $\phi_1, \phi_2$ , such that  $\phi_j \in \mathcal{G}(\alpha_j, L_j, R_j)$  with  $t^{\alpha_2} L_2(t) = O(t^{\alpha_1} L_1(t))$  as  $t \rightarrow 0$ , then  $\phi_1 + \phi_2 \in \mathcal{G}(\alpha_1, L_1, R_+)$  for some positive function  $R_+$  explicit in terms of  $L_1, L_2$  and  $R_1$ .*

The three items here are special cases of Proposition 2.1, Corollary 2.3 and Proposition 2.5 below, respectively. The coefficients  $c_1, c_2$  and  $c_*$  and the function  $R_+$  are explicitly described in the precise versions below.

The proofs of all three result are rather short, but together they allow for a simple proof of the expansion (1.1) in several concrete cases:

- In Corollary 3.1, we study a function  $\phi : (0, 1] \rightarrow \mathbb{R}^2$  having an asymptotic behaviour around 0 of the shape  $x^{-1/2} |\log x|$ . The ensuing estimate we obtain is used in [BD, Theorem 2.1] to deduce a central limit theorem for central values  $\{D(1/2, x), x \in \mathbb{Q} \cap (0, 1]\}$  of the analytic continuation of the Estermann function

$$(1.4) \quad D(s, x) = \sum_{n \geq 1} \frac{\tau(n)}{n^s} e^{2\pi i n x}, \quad (\operatorname{Re}(s) > 1),$$

where  $\tau$  is the divisor function.

- In Corollaries 3.3 and 3.2, we study the functions of the shape  $\phi(x) = [1/x]^\lambda$  where  $\lambda \geq 1/2$ . These functions occur when studying the values  $\{\Sigma_\lambda(x), x \in \mathbb{Q} \cap (0, 1]\}$  of the moments of the continued fractions coefficients,

$$\Sigma_\lambda(x) = \sum_{j=1}^r a_j^\lambda, \quad (x = [0; a_1, \dots, a_r] = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}, a_r > 1),$$

see [BD, Theorems 2.5 and 9.4]. This, in turn, is applied to obtain a law of large numbers for the values of the Kashaev invariants of the  $4_1$  knot [BD, Corollary 2.6].

- In Corollary 3.4, we study the function  $\phi$  on  $(0, 1]$  given by  $\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor$ , where  $T : (0, 1] \rightarrow (0, 1]$ ,  $T(x) = \{1/x\}$  is the Gauss map. The estimate we obtain is used in [BD, Theorem 2.7] to obtain an independent proof, using dynamical systems, of a theorem of Vardi [Var93] on the convergence to a Cauchy law of the values of Dedekind sums.

## 2. ESTIMATION OF (1.1) IN GENERAL

### 2.1. Basic estimates.

2.1.1. *Taylor estimate.* The first and simplest method to obtain an estimate for (1.1) is to insert and integrate a Taylor expansion for the exponential.

**Proposition 2.1.** *Assume that for some  $\alpha \in (0, 3]$ , we have*

$$K := \int |\phi(x)|^\alpha d\mu(x) < \infty.$$

Then  $\phi \in \mathcal{G}(\alpha, 1, 1)$ , and more precisely

$$(2.1) \quad \mathcal{I}[\phi](t) = ic_1 t + c_2 t^2 + O(Kt^\alpha)$$

with  $c_1 = \int \phi d\mu$  if  $\alpha \geq 1$ , and  $c_2 = -\frac{1}{2} \int |\phi|^2 d\mu$  if  $\alpha \geq 2$ . The implied constant is absolute.

*Proof.* We use the bound  $\left| e^{iu} - \sum_{0 \leq k < \alpha} \frac{(iu)^k}{k!} \right| \ll |u|^\alpha$  with  $u = t\phi(x)$ , and integrate over  $x$ .  $\square$

Although it will not be useful for us here, we note that in the precise bound (2.1), the value of  $\alpha$  could be taken as a function of  $t$ . For example, if  $\mu$  is the Lebesgue measure on  $(0, 1)$  and  $\phi(x) = 1/x$ , we can take  $\alpha = 1 - 1/|\log t|$  and obtain  $\mathcal{I}[\phi](t) = O(t|\log t|)$ .

2.1.2. *Using properties of the Mellin transform.* When the moment  $\int |\phi|^\alpha d\mu$  diverges at some particular  $\alpha$ , we can often extract a useful expansion from the Cauchy formula and the polar behaviour of the Mellin transform. For  $x \in \mathbb{R}$ ,  $s \in \mathbb{C}$  and  $\eta \in [0, 1]$ , let

$$\phi_{s,\eta}(x) := \mathbf{1}_{\phi(x) \neq 0} |\phi(x)|^s \exp(-s \frac{\pi i}{2} (1 - \eta) \operatorname{sgn} \phi(x)), \quad \phi_s(x) := \phi_{s,0}(x).$$

Note that for  $k \in \mathbb{N}_{>0}$ ,  $\phi_k(x) = (-i\phi(x))^k$ . Define further

$$G_\eta(s) := \int \phi_{s,\eta}(x) d\mu(x).$$

**Proposition 2.2.** *Let  $\alpha \in (0, 3)$ ,  $\rho \in (0, 1)$ ,  $\delta, \eta_0 > 0$  and  $\xi \in \mathbb{R}$ . Assume that for some  $c > 0$ , we have*

$$(2.2) \quad \int_{\phi(x) \neq 0} (|\phi(x)|^c + |\phi(x)|^{-c}) d\mu(x) < \infty$$

and that the functions  $G_\eta(s)$  for  $\eta \in [0, \eta_0]$ , initially defined for  $\operatorname{Re}(s) \in (-c, c)$ , can be analytically continued to the set

$$\{s \in \mathbb{C}, 0 < \operatorname{Re}(s) \leq \alpha + \delta, s \notin [\alpha, \alpha + \delta]\}.$$

Assume further that

$$\sup_{0 \leq \eta \leq \eta_0} \int_{s=\alpha+\delta+i\tau}^{\tau \in \mathbb{R}} |\Gamma(-s)G_\eta(s)| d\tau < \infty,$$

and that there is an open neighborhood  $\mathcal{V}$  of  $[\alpha, \alpha + \delta]$  for which

$$(2.3) \quad (\alpha - s)^\xi G_0(s) = \varrho + O(|s - \alpha|^\rho), \quad s \in \mathcal{V} \setminus [\alpha, \alpha + \delta], \operatorname{Re}(s) \leq \alpha + \delta.$$

Then,  $\phi \in \mathcal{G}(\alpha, |\log|^{\xi-1+v_\alpha}, |\log|^{\xi-1+v_\alpha-\rho})$ , where  $v_\alpha = 1$  if  $\alpha = 1, 2$  and  $v_\alpha = 0$  otherwise, and with coefficients given by

$$(2.4) \quad c_1 = iG_0(1) \text{ if } \alpha > 1, \quad c_2 = \frac{1}{2}G_0(2) \text{ if } \alpha > 2, \quad c_* = \begin{cases} -\varrho/\Gamma(\xi+1), & \alpha = 1, \\ \frac{1}{2}\varrho/\Gamma(\xi+1), & \alpha = 2, \\ \varrho \frac{\Gamma(-\alpha)}{\Gamma(\xi)}, & \alpha \notin \{1, 2\}. \end{cases}$$

*Proof.* We write

$$\mathcal{I}[\phi](t) + 1 = \int e^{it\phi(x)} d\mu(x) = J_+ + J_- + J_0,$$

where  $J_\pm$  corresponds to the part of the integral restricted to  $\pm\phi > 0$ .

For all  $\varepsilon \in (0, \frac{\pi}{2}\eta_0)$ , define

$$J_+(\varepsilon) := \int_{\phi(x)>0} e^{(-\varepsilon+i)t\phi(x)} d\mu(x), \quad J_-(\varepsilon) := \int_{\phi(x)<0} e^{(\varepsilon+i)t\phi(x)} d\mu(x).$$

By dominated convergence, we have  $J_+ := \lim_{\varepsilon \rightarrow 0^+} J_+(\varepsilon)$ , and similarly for  $J_-$ . We use the Mellin transform formula for the exponential

$$e^{-y} = \frac{1}{2\pi i} \int_{-c/2-i\infty}^{-c/2+i\infty} \Gamma(-s)|y|^s e^{s \arg(y)} ds$$

valid for  $\operatorname{Re}(y) > 0$ , see [GR07, eq. 17.43.1] (the extension to non-real  $y$  is straightforward by the Stirling formula [GR07, eq. 8.327.1]). Inserting this in  $J_\pm(\varepsilon)$ , we obtain

$$J_+(\varepsilon) + J_-(\varepsilon) = \frac{1}{2\pi i} \int_{-c/2-i\infty}^{-c/2+i\infty} \Gamma(-s)G_\eta(s)|1+i\varepsilon|^s t^s ds,$$

where  $\eta = \frac{2}{\pi} \arctan \varepsilon \leq \frac{2\varepsilon}{\pi} \leq \eta_0$ . We move the contour forward to  $\operatorname{Re}(s) = \alpha + \delta$ . The simple pole at  $s = 0$  contributes  $\int_{\phi(x) \neq 0} d\mu(x)$ , and therefore by adding the contribution from  $J_0$  we get

$$\begin{aligned} J_0 + J_+(\varepsilon) + J_-(\varepsilon) &= 1 + R + \frac{1}{2\pi i} \int_{H(\alpha, \alpha+\delta)} \Gamma(-s)G_\eta(s)t^s |1+i\varepsilon|^s ds \\ &\quad + \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\alpha+\delta} \Gamma(-s)G_\eta(s)t^s |1+i\varepsilon|^s ds, \end{aligned}$$

where  $R$  consists of the contribution of the residues at 1 (if  $\alpha > 1$ ) and 2 (if  $\alpha > 2$ ). Here  $H(\alpha, \alpha + \delta)$  is a Hankel contour, going from  $\alpha + \delta - i0$  to  $\alpha + \delta + i0$  passing around  $\alpha$  from the left. The last integral is bounded by the triangle inequality, using our first hypothesis on  $G_\eta$ , which gives

$$\frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\alpha+\delta} \Gamma(-s)G_\eta(s)t^s |1+i\varepsilon|^s ds \ll t^{\alpha+\delta},$$

uniformly in  $\varepsilon$ . Passing to the limit  $\varepsilon \rightarrow 0$ , there remains to prove

$$\frac{1}{2\pi i} \int_{H(\alpha, \alpha+\delta)} \Gamma(-s)G_0(s)t^s ds = c_* t^\alpha |\log t|^{\xi-1+v_\alpha} + O(t^\alpha |\log t|^{\xi-1+v_\alpha-\rho}).$$

This is done by using our second hypothesis along with a standard Hankel contour integration argument; we refer to *e.g.* Corollary II.0.18 of [Ten15] for the details.  $\square$

An important special case is the following.

**Corollary 2.3.** *Let  $\mu$  be defined on  $[0, 1]$  by  $d\mu(x) = f(x) dx$  where  $f \in \mathcal{C}^1([0, 1])$ . Let  $a \in \mathbb{R} \setminus \{0\}$ . For all  $\beta > \frac{1}{3}$ ,  $\lambda \geq 0$ , and  $\phi$  given by*

$$\phi(x) = ax^{-\beta} |\log x|^\lambda$$

one has  $\phi \in \mathcal{G}(1/\beta, |\log|^\lambda|^\beta + v_{1/\beta}, |\log|^\lambda|^\beta + v_{1/\beta} - 1 + \varepsilon)$  for any  $\varepsilon \in (0, 1)$  and with

$$c_* = f(0) \frac{|a|^{1/\beta} e^{-\frac{\pi i \operatorname{sgn} a}{2\beta}}}{\beta^{\lambda/\beta+1}} \times \begin{cases} -(\lambda+1)^{-1}, & \beta = 1, \\ (4\lambda+2)^{-1}, & \beta = 1/2, \\ \Gamma(-1/\beta), & \beta \notin \{1, 1/2\}. \end{cases}$$

and  $c_1 = \int \phi \, d\mu$  if  $\beta < 1$  and  $c_2 = -\frac{1}{2} \int |\phi|^2 \, d\mu$  if  $\beta < \frac{1}{2}$ .

*Proof.* First, we write  $d\mu(x) = f(0)\chi(x) \, dx + xg(x) \, dx$ , where  $\chi$  is the characteristic function of the interval  $[0, 1]$  and  $g \in \mathcal{C}([0, 1])$ . For the contribution of  $\chi \, dx$  we apply Proposition 2.2 with any fixed  $c < 1/\beta$ ,  $\alpha = 1/\beta$ ,  $\xi = \lambda/\beta + 1$ , any fixed  $\rho \in (0, 1)$  and  $\delta > 0$ . By [GR07, 4.272.6], for  $\operatorname{Re}(s) < 1/\beta$  and  $\eta \in [0, 1]$  we have

$$G_\eta(s) = e^{-s\frac{\pi i}{2}(1-\eta)\operatorname{sgn}(a)} |a|^s \int_0^1 x^{-\beta s} |\log x|^{\lambda s} \, dx = e^{-s\frac{\pi i}{2}(1-\eta)\operatorname{sgn}(a)} |a|^s \frac{\Gamma(\lambda s + 1)}{(1-\beta s)^{\lambda s + 1}}.$$

Notice also that by Stirling's formula  $G_\eta(s) \ll e^{\pi(\frac{1-\eta}{2})|\tau|} |\tau|^{-1/2}$  as  $|\tau| = |\operatorname{Im} s| \rightarrow \infty$ , so that in any case  $\Gamma(-s)G_\eta(s) \ll |\tau|^{-1-\operatorname{Re}(s)}$ . Therefore the hypotheses of Proposition 2.2 are easily verified with

$$\varrho = |a|^{1/\beta} e^{-\frac{\pi i \operatorname{sgn} a}{2\beta}} \frac{\Gamma(\lambda/\beta + 1)}{\beta^{\lambda/\beta+1}}.$$

Thus,

$$\int_0^1 (e^{it\phi(x)} - 1) \, dx = itc'_1 + c'_2 t^2 + c_* t^{1/\beta} |\log t|^{\lambda/\beta + v_{1/\beta}} + O(t^{1/\beta} |\log t|^{\lambda/\beta + v_{1/\beta} - \rho})$$

with coefficients as given in (2.4) with  $G_0(1) = -i \int \phi \chi \, dx$  and  $G_0(2) = -\int \phi^2 \chi \, dx$ . Finally, as in Proposition 2.1 we deduce

$$\int (e^{it\phi(x)} - 1) x g(x) \, dx = ic''_1 t + c''_2 t^2 + O(Kt^{\alpha'})$$

for any  $0 < \alpha' < \min(3, \frac{2}{\beta})$  and with  $c''_1 = \int \phi(x) x g(x) \, dx$  if  $\alpha' > 1$  and  $c''_2 = -\frac{1}{2} \int \phi(x)^2 x g(x) \, dx$  if  $\alpha' > 2$ . The result then follows.  $\square$

## 2.2. Addition.

**Lemma 2.4.** For  $j \in \{1, 2\}$ , let  $\delta_j(x) = e^{it\phi_j(x)} - 1$ . Then

$$(2.5) \quad \begin{aligned} \mathcal{I}[\phi_1 + \phi_2](t) &= \mathcal{I}[\phi_1](t) + \mathcal{I}[\phi_2](t) + \int \delta_1(x) \delta_2(x) \, d\mu(x) \\ &= \mathcal{I}[\phi_1](t) + \mathcal{I}[\phi_2](t) + O\left(\prod_{j \in \{1, 2\}} |\operatorname{Re} \mathcal{I}[\phi_j](t)|^{1/2}\right) \end{aligned}$$

*Proof.* The first equation is simply the relation  $e^{it(\phi_1(x) + \phi_2(x))} - 1 = \delta_1(x) + \delta_2(x) + \delta_1(x)\delta_2(x)$  integrated over  $x$ . The last term is bounded using the Cauchy-Schwarz inequality

$$\left( \int |\delta_1(x) \delta_2(x)| \, d\mu(x) \right)^2 \leq \prod_{j \in \{1, 2\}} \int |\delta_j(x)|^2 \, d\mu(x)$$

and expanding the square on the right-hand side.  $\square$

**Proposition 2.5.** For  $j \in \{1, 2\}$ , let  $\alpha_j \in (0, 2]$ , let  $L_j, R_j$  be positive functions defined on a neighborhood of 0 in  $\mathbb{R}_+^*$ , and  $\phi_j \in \mathcal{G}(\alpha_j, L_j, R_j)$ . If  $\alpha_1 \leq \alpha_2$ , and under the following assumptions:

- $R_j(t), L_j(t) = t^{o(1)}$  as  $t \rightarrow 0$ ,
- $R_j(t) = O(L_j(t))$ ,
- $t^2 = O(t^{\alpha_1} L_1(t))$ ,

we have

$$\phi_1 + \phi_2 \in \mathcal{G}(\alpha_1, L_1, R_+), \quad R_+ = \begin{cases} R_1 & \text{if } \alpha_1 < \alpha_2, \\ R_1 + L_2 + \sqrt{L_1 L_2} & \text{if } \alpha_1 = \alpha_2 < 2, \\ R_1 + L_2 + \sqrt{L_1}(\sqrt{L_2} + 1) & \text{if } \alpha_1 = \alpha_2 = 2. \end{cases}$$

Moreover,

$$\begin{aligned} c_1(\phi_1 + \phi_2) &= c_1(\phi_1) + c_1(\phi_2), \\ c_*(\phi_1 + \phi_2) &= c_*(\phi_1). \end{aligned}$$

*Proof.* We use Lemma 2.4; when computing the real part in (2.5), the term  $ic_1 t$  vanishes.  $\square$

*Remark.* Note that using this result might induce a slight quantitative loss in the two cases when  $\alpha_1 = \alpha_2$ . What is gained at this price is that we are only required to study each  $\phi_j$  separately, which simplifies the analysis.

We also remark that this estimate is useful only when the term  $c_2 t^2$  is not relevant in (1.3). In the complementary case, Proposition 2.1 can be used, although the ensuing error term will typically be worse than optimal by a factor of  $|\log t|$ .

It is straightforward to generalize Proposition 2.5, affecting to each  $\phi_j$  a different value of the frequency: under the same hypotheses and notations, and additionally that  $L_j, R_j$  tend monotonically to  $+\infty$  at 0,

$$\int e^{it_1 \phi_1(x) + it_2 \phi_2(x)} d\mu(x) = 1 + ic_1(\phi_1)t_1 + ic_1(\phi_2)t_2 + c_* t_1^{\alpha_1} L_1(t_1) + O(t_+^2 + t_+^{\alpha_1} R_+(t_+)),$$

where  $c_1, c_*$  are as in the conclusion of Proposition 2.5, and  $t_+ = \max\{t_1, t_2\}$ .

### 3. APPLICATIONS

We now describe the applications we will be interested in. The measure is the Gauss-Kuzmin distribution

$$d\mu(x) = \frac{dx}{(1+x)\log 2} \quad (x \in [0, 1]).$$

The measure  $\mu$  is invariant under the Gauss map  $T(x) = \{1/x\}$  on  $(0, 1)$ , in particular,

$$(3.1) \quad \mathcal{I}[\phi \circ T](t) = \mathcal{I}[\phi](t).$$

**3.1. Central values of the Estermann function.** The first application we discuss is the ‘‘period function’’  $\phi : \mathbb{R} \rightarrow \mathbb{C}$  associated with the Estermann function (1.4), namely

$$\phi(x) = D\left(\frac{1}{2}, 1/x\right) - D\left(\frac{1}{2}, x\right),$$

initially defined in  $\mathbb{Q} \cap (0, 1]$ . By [Bet16], this function can be extended to a continuous function on  $(0, 1]$ , more precisely given by an expression of the shape (3.2) below. Interpreting  $\phi$  to be  $\mathbb{R}^2$ -valued, the analogue of the integral (1.1) is estimated using the following.

**Corollary 3.1.** *Let  $\varepsilon > 0$ ,  $\mathcal{E} : [0, 1] \rightarrow \mathbb{C}$  be a bounded, continuous function, and*

$$(3.2) \quad \phi_j(x) := \left( \frac{1}{2} x^{-1/2} (\log(1/x) + \gamma_0 - \log(8\pi) - \frac{\pi}{2}) + \zeta\left(\frac{1}{2}\right)^2 + \operatorname{Re} \mathcal{E}((-1)^j x) \right) \\ \left( \frac{(-1)^{j-1}}{2} x^{-1/2} (\log(1/x) + \gamma_0 - \log(8\pi) + \frac{\pi}{2}) + \operatorname{Im} \mathcal{E}((-1)^j x) \right).$$

*Let also  $\mathbf{u}_j := \left( \frac{1}{(-1)^{j-1}} \right)$ . Then for some vector  $\boldsymbol{\mu} \in \mathbb{R}^2$ , and all  $\mathbf{t} \in \mathbb{R}^2$ , we have*

$$\begin{aligned} & \int_0^1 e^{i\langle \mathbf{t}, \phi_1(x) + \phi_2(T(x)) \rangle} d\mu(x) \\ &= 1 + i\langle \mathbf{t}, \boldsymbol{\mu} \rangle - \frac{1}{3 \log 2} \sum_{j \in \{1, 2\}} \langle \mathbf{t}, \mathbf{u}_j \rangle^2 |\log |\langle \mathbf{t}, \mathbf{u}_j \rangle||^3 + O_\varepsilon(\|\mathbf{t}\|^2 |\log \|\mathbf{t}\||^{2+\varepsilon}). \end{aligned}$$

*Proof.* Let  $\varepsilon \in (0, 1)$ . Using Corollary 2.3 with  $\beta = 1/2$  and  $\lambda \in \{0, 1\}$ , and Proposition 2.1, we obtain

$$\begin{aligned} (x \mapsto \pm \frac{1}{2}x^{-1/2}|\log x|) &\in \mathcal{G}(2, |\log|^3, |\log|^{2+\varepsilon}), \\ (x \mapsto (\gamma_0 - \log(8\pi) + \frac{\pi}{2})x^{-1/2}) &\in \mathcal{G}(2, |\log|, |\log|^\varepsilon), \\ (x \mapsto \operatorname{Im} \mathcal{E}(\pm x)) &\in \mathcal{G}(3, 1, 1), \end{aligned}$$

as well as  $c_*(x \mapsto \pm \frac{1}{2}x^{-1/2}|\log x|) = -\frac{1}{3\log 2}$ . From Proposition 2.5 and the ensuing remark, and using the property (3.1), we obtain for  $j \in \{1, 2\}$

$$\int_0^1 (e^{i\langle \mathbf{t}, \phi_j(x) \rangle} - 1) d\mu(x) = i\langle \mathbf{t}, \boldsymbol{\mu}_j \rangle + c_* \langle \mathbf{t}, \mathbf{u}_j \rangle^2 |\log |\langle \mathbf{t}, \mathbf{u}_j \rangle||^3 + O_\varepsilon(\|\mathbf{t}\|^2 |\log \|\mathbf{t}\||^{2+\varepsilon}),$$

where  $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{R}^2$ . On the other hand, we have

$$\Delta(\mathbf{t}) := \int_0^1 (e^{i\langle \mathbf{t}, \phi_1(x) \rangle} - 1)(e^{i\langle \mathbf{t}, \phi_2(T(x)) \rangle} - 1) d\mu(x) = \int_0^1 (e^{i\langle \mathbf{t}, \phi_2(x) \rangle} - 1) F_x(\mathbf{t}) dx,$$

where

$$F_x(\mathbf{t}) = \frac{1}{\log 2} \sum_{n \geq 1} \frac{e^{i\langle \mathbf{t}, \phi_1(1/(n+x)) \rangle} - 1}{(n+x)(n+x+1)}.$$

By a Taylor expansion at order 1, we have  $|F_x(\mathbf{t})| \ll \|\mathbf{t}\|$  uniformly in  $x$ , and therefore

$$|\Delta(\mathbf{t})| \ll \|\mathbf{t}\|^2 \int_0^1 \|\phi_2(x)\| dx \ll \|\mathbf{t}\|^2.$$

By (2.5), we deduce

$$\int_0^1 e^{i\langle \mathbf{t}, \phi_1(x) + \phi_2(T(x)) \rangle} d\mu(x) = 1 + \int_0^1 (e^{i\langle \mathbf{t}, \phi_1(x) \rangle} + e^{i\langle \mathbf{t}, \phi_2(T(x)) \rangle} - 2) d\mu(x) + O(\|\mathbf{t}\|^2),$$

whence the claimed estimate.  $\square$

**3.2. Moments of continued fractions coefficients.** The next application we consider pertains to the moments functions  $\Sigma_\lambda$  of continued fractions coefficients, where  $\lambda \geq 0$  is the order of the moment. The function of interest to us here is

$$\phi_\lambda(x) = \lfloor 1/x \rfloor^\lambda.$$

The case  $\lambda < 1/2$  can be easily dealt with using Proposition 2.1, so we do not focus on it here.

A first approach is to use Proposition 2.5 to approximate  $\lfloor 1/x \rfloor$  by  $1/x$ , and then use Corollary 2.3. This leads to the following.

**Corollary 3.2.** *Let  $\lambda \geq 1/2$ . The function  $\phi_\lambda$  given by  $\phi_\lambda(x) = \lfloor 1/x \rfloor^\lambda$  satisfies the following.*

– If  $\lambda = 1/2$ , then with  $c_* = -1/(\log 2)$ , we have

$$(3.3) \quad \mathcal{I}[\phi_{1/2}](t) = ic_1 t + c_* t^2 |\log t| + O_\varepsilon(t^2 |\log t|^\varepsilon).$$

– If  $\lambda > 1/2$  and  $\lambda \neq 1$ , then with  $c_* = -\exp(-\pi i/(2\lambda))\Gamma(1 - 1/\lambda)/\log 2$ , we have

$$\mathcal{I}[\phi_\lambda](t) = (\mathbf{1}_{\lambda < 1})ic_1 t + c_* t^{1/\lambda} + O_\varepsilon(t^{1/\lambda} |\log t|^{-1+\varepsilon})$$

When  $1/2 \leq \lambda < 1$ , we have  $c_1 = \int_0^1 \phi_\lambda(x) d\mu(x)$ .

*Proof.* We write  $\phi_\lambda(x) = p_\lambda(x) + r_\lambda(x)$ , where  $p_\lambda(x) = x^{-\lambda}$  and  $r_\lambda(x) \ll_\lambda \lfloor 1/x \rfloor^{\lambda-1}$ . By Proposition 2.1, we have  $r_\lambda \in \mathcal{G}(\min(3, \frac{1}{\lambda-1/3}), 1, 1)$ .

We consider first the case  $\lambda > 1/2$ ,  $\lambda \neq 1$ . By Corollary 2.3, we have  $p_\lambda \in \mathcal{G}(\frac{1}{\lambda}, 1, |\log|^{-1+\varepsilon})$ . We deduce, by Proposition 2.5, that  $\phi_\lambda \in \mathcal{G}(\frac{1}{\lambda}, 1, |\log|^{-1+\varepsilon})$ , and this yields the second and third cases.

If  $\lambda = 1/2$ , then Corollary 2.3 implies  $p_{1/2} \in \mathcal{G}(2, |\log|, |\log|^\varepsilon)$ , and by Proposition 2.1, for some  $c \in \mathbb{R}$ , we have

$$\mathcal{I}[r_{1/2}](t) = ict + O(t^2)$$



On the other hand, since  $\left| (e^{itp_{1/2}(x)} - 1)(e^{itr_{1/2}(x)} - 1) \right| \ll t^2 \left| p_{1/2}(x)r_{1/2}(x) \right| \ll t^2$ , we get

$$\int_0^1 (e^{itp_{1/2}(x)} - 1)(e^{itr_{1/2}(x)} - 1) d\mu(x) = O(t^2).$$

By (2.5), we conclude (3.3) as claimed.  $\square$

The case  $\lambda = 1$  could be analyzed by the same method, but we chose to study it separately to obtain a more precise error term by another approach, using Proposition 2.2 directly. The associated Mellin transform  $G_0(s)$  is related to the Riemann  $\zeta$ -function.

**Corollary 3.3.** *The function  $\phi$  given by  $\phi(x) = \lfloor 1/x \rfloor$  satisfies*

$$\mathcal{I}[\phi](t) = -\frac{it}{\log 2}(\log t + \gamma_0 - \frac{\pi i}{2}) + O_\varepsilon(t^{2-\varepsilon}).$$

*Proof.* The integral (2.2) converges for all  $c < 1$ . A quick computation shows that an analytic continuation of  $G_\eta(s)$  is given by

$$G_\eta(s) = \frac{\exp(-s\frac{\pi i}{2}(1-\eta))}{\log 2} \{\zeta(2-s) + H(s)\},$$

where  $H(s) = \sum_{n \geq 1} n^s (\log(1 + \frac{1}{n(n+2)}) - \frac{1}{n^2})$  is analytic and uniformly bounded in  $\text{Re}(s) \leq 2-\varepsilon$ . We have

$$\int_{\text{Re}(s)=2-\varepsilon} |\Gamma(-s)G_\eta(s)| |ds| \ll_\varepsilon 1 + \int_0^\infty |\zeta(\varepsilon + i\tau)| \frac{d\tau}{1+\tau^2} \ll_\varepsilon 1$$

by the Stirling formula. The polar behaviour (2.3) is given by

$$G_0(s) = \frac{\exp(-s\frac{\pi i}{2})}{\log 2} \{\zeta(2-s) + H(s)\} = \frac{\exp(-s\frac{\pi i}{2})}{\log 2} \left\{ \frac{1}{1-s} + A + O(s-1) \right\}$$

for  $s$  in a neighborhood of 1, where

$$\begin{aligned} A &= \sum_{n \geq 1} \left( n \log \left( 1 + \frac{1}{n(n+2)} \right) - \log \left( 1 + \frac{1}{n} \right) \right) \\ &= - \lim_{N \rightarrow \infty} \sum_{n=1}^N \left( n \log \left( 1 + \frac{1}{n+1} \right) - (n-1) \log \left( 1 + \frac{1}{n} \right) \right) \\ &= -1. \end{aligned}$$

Applying Proposition 2.2 with  $\delta = 1/2$  and  $\alpha = 1$  yields the claimed result up to  $O(t)$ . Our more precise statement follows from noting that there is no branch cut along  $s \geq 1$  in this case, so that the residue theorem may be used. We obtain

$$\text{Res}_{s=1} \Gamma(-s)G_0(s)t^s = \frac{it}{\log 2}(\gamma_0 - \frac{\pi i}{2} + \log t),$$

whence the claimed estimate. One could go further, isolating a pole of order 2 at  $s = 2$ , and this would give an error term  $O(t^2|\log t|)$ .  $\square$

**3.3. Dedekind sums.** The final example we discuss is related to Dedekind sums, for the definition of which we refer to [BD, Section 2.4]. The ‘‘period function’’  $\phi$  relevant to us here is

$$\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor.$$

Compared with the case of  $x \mapsto \lfloor 1/x \rfloor$  studied in Corollary 3.3, the relevant exponent  $\alpha$  is again 1, but the leading term turns out to be  $t$  (the terms  $t \log t$  vanish).

**Corollary 3.4.** *The map  $\phi$  on  $(0, 1)$  given by  $\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor$  satisfies*

$$\mathcal{I}[\phi](t) = -\frac{\pi}{\log 2}t + O(t^2|\log t|^2).$$

*Proof.* We consider

$$\begin{aligned} \Delta(t) &:= \int_0^1 (e^{-it\lfloor 1/T(x) \rfloor} - 1)(e^{it\lfloor 1/x \rfloor} - 1) d\mu(x) \\ &= \int_0^1 (e^{-it\lfloor 1/x \rfloor} - 1)F_x(t) dx, \end{aligned}$$

with  $F_x(t) = \frac{1}{\log 2} \sum_{n \geq 1} \frac{e^{itn} - 1}{(n+x)(n+1+x)}$ . Since  $|e^{iu} - 1| \ll |u|^{1-1/|\log t|}$  for all  $u \in \mathbb{R}$ , we find

$$F_x(t) \ll t \sum_{n \geq 1} \frac{1}{n^{1+1/|\log t|}} \ll t|\log t|.$$

Similarly,

$$\int_0^1 |e^{-it\lfloor 1/x \rfloor} - 1| dx \ll t \int_0^1 x^{-1+1/|\log t|} dx \ll t|\log t|.$$

We thus obtain  $\Delta(t) = O((t \log t)^2)$ . Using Corollary 3.3 with the improved error term  $O(t^2|\log t|)$ , (3.1) and (2.5), we deduce

$$\int_0^1 e^{it(\lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor)} d\mu(x) = 1 + 2 \operatorname{Re} I(t) + O((t \log t)^2),$$

where  $I(t) = \int_0^1 (e^{it\lfloor 1/x \rfloor} - 1) d\mu(x)$ . Corollary 3.3 allows us to conclude.  $\square$

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SB: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, 16146 GENOVA, ITALY  
*Email address:* `bettin@dima.unige.it`

SD: AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M UMR 7373, 13453 MARSEILLE, FRANCE  
*Email address:* `sary-aurelien.drappeau@univ-amu.fr`