



**HAL**  
open science

# Effective estimation of some oscillatory integrals related to infinitely divisible distributions

Sandro Bettin, Sary Drappeau

► **To cite this version:**

Sandro Bettin, Sary Drappeau. Effective estimation of some oscillatory integrals related to infinitely divisible distributions. 2020. hal-02998544v1

**HAL Id: hal-02998544**

**<https://hal.science/hal-02998544v1>**

Preprint submitted on 10 Nov 2020 (v1), last revised 3 Feb 2022 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# EFFECTIVE ESTIMATION OF SOME OSCILLATORY INTEGRALS RELATED TO INFINITELY DIVISIBLE DISTRIBUTIONS

S. BETTIN AND S. DRAPPEAU

ABSTRACT. We present a practical framework to prove, in a simple way, two-terms asymptotic expansions for Fourier integrals

$$\mathcal{I}(t) = \int_{\mathbb{R}} (e^{it\phi(x)} - 1) d\mu(x)$$

where  $\mu$  is a probability measure on  $\mathbb{R}$  and  $\phi$  is measurable. This applies to many basic cases, in link with Levy's continuity theorem. We present applications to limit laws related to rational continued fractions coefficients.

## 1. INTRODUCTION

Let  $\mu$  be a probability measure on  $\mathbb{R}$ , and  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be  $\mu$ -measurable. The present paper is concerned with asymptotic formulæ for the Fourier integrals associated with  $\phi$  near the origin,

$$(1.1) \quad \mathcal{I}[\phi](t) := \int (e^{it\phi(x)} - 1) d\mu(x), \quad (t \rightarrow 0).$$

Such estimates are connected with the question of whether the push-forward measure  $\phi_*(\mu)$  belongs to the bassin of attraction of a stable law, see Chapter 2 of [IL71]. Our interest in this question originates from this point of view, and more specifically from the work [BD] where we study the convergence towards stable laws of the value distribution of invariants related to modular forms. In the setting of [BD], the measure  $\mu$  is the Gauss-Kuzmin distribution

$$d\mu(x) = \frac{dx}{(1+x)\log 2} \quad (x \in [0, 1]),$$

and this measure is invariant under the Gauss map  $T(x) = \{1/x\}$ , where  $\{x\} = x - [x]$  is the fractional part of  $x$ . More precisely, in [BD], we are interested in Birkhoff sums

$$(1.2) \quad \sum_{j=1}^r \phi(T^j(x)), \quad (T^r = T \circ \dots \circ T),$$

where  $x$  varies among rationals and  $r \geq 0$  is the length of the continued fractions expansion of  $x$ . In the set of rationals we consider, these sums are found to typically behave as sums of the shape

$$\sum_{j=1}^r \phi(X_j)$$

where  $(X_j)_{1 \leq j \leq r}$  are i.i.d. random variables distributed according to the Gauss-Kuzmin measure  $\mu$ . Then effective estimates for the integral (1.1), in conjunction with [BD, Theorem 3.1]

---

*Date:* October 13, 2020.

*2010 Mathematics Subject Classification.* 41A60 (Primary); 60E07, 60E10 (Secondary).

*Key words and phrases.* Fourier integral, characteristic function, infinitely divisible distribution, asymptotic expansion, limit law.

The authors thank the referee for his or her remarks and a careful reading of the paper. This paper was partially written during a visit of S. Bettin at the Aix-Marseille University and a visit of S. Drappeau at the University of Genova. The authors thank both Institution for the hospitality and Aix-Marseille University and INdAM for the financial support for these visits. S. Bettin is member of the INdAM group GNAMPA and his work is partially supported by PRIN 2017 "Geometric, algebraic and analytic methods in arithmetic".

and the Berry-Esseen inequality [Fel71, equation (XVI.3.13)] are used to obtain uniform limit theorems for the rational Birkhoff sums (1.2).

We return to the setting where  $\mu$  is an arbitrary probability measure on  $\mathbb{R}$ . Integrals (1.1) are related to the methods of asymptotic analysis mentioned *e.g.* in Chapter 9 of the monograph [Olv97]. When expressed as convolution integrals  $\int_x h(tx)f(x)dx$ , they are referred to as  $h$ -transforms in [BH86], and are also the topic of interest of the recent work [Lóp08]. The variety in assumptions and methods seems to prevent us from having a uniform framework for estimating (1.1).

The goal of the present paper is to present and prove several basic estimates through which one can give a streamlined and simple proof of an effective asymptotic expansion of the integral (1.1), including the terms of interest in central limit theorems.

**Definition 1.1.** *Given  $\alpha \in (0, 3]$  and two positive functions  $L, R$  defined in a neighborhood of 0 in  $\mathbb{R}_+^*$ , we denote by  $\mathcal{G}(\alpha, L, R)$  the set of functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that for some numbers  $c_1, c_2 \in \mathbb{R}$  and  $c_* \in \mathbb{C}$ , and all small enough  $t > 0$ , there holds*

$$(1.3) \quad \mathcal{I}[\phi](t) = ic_1t + c_2t^2 + c_*t^\alpha L(t) + O(t^3 + t^\alpha R(t)).$$

*Remark.* – If  $R = O(t^\varepsilon)$  for any  $\varepsilon > 0$  and  $\alpha < 1$ , the term  $c_1t$  in (1.3) is part of the error term, and likewise for  $c_2t^2$  if  $\alpha < 2$ .

- We will be interested in the largest one or two terms in the expansion (1.3). The case  $\alpha = 3$ ,  $L = R \equiv 1$  corresponds to an order 2 Taylor expansion.
- Whenever the expansion (1.3) holds for  $\phi$ , we will denote the coefficients by  $c_1(\phi)$ ,  $c_2(\phi)$ ,  $c_*(\phi)$  respectively.

**Theorem 1.2.** (1) *If  $\int |\phi(x)|^\alpha d\mu(x) < \infty$  for some  $\alpha \in (0, 3]$ , then  $\phi \in \mathcal{G}(\alpha, 1, 1)$ .*

(2) *Suppose that  $d\mu = f d\nu$  where  $\nu$  is the Lebesgue measure and  $f \in \mathcal{C}^1([0, 1])$ . Then for all  $a \in \mathbb{R}^*$ ,  $\beta > 3$  and  $\lambda \geq 0$ , the function*

$$\phi : (0, 1] \rightarrow \mathbb{R}, \quad \phi(x) = ax^{-\beta} |\log x|^\lambda,$$

*belongs to  $\mathcal{G}(\frac{1}{\beta}, |\log|^{\lambda/\beta+v}, |\log|^{\lambda/\beta+v-1+\varepsilon})$  for any  $\varepsilon \in (0, 1]$ , where  $v = 1$  for  $\beta \in \{1/2, 1\}$  and  $v = 0$  otherwise.*

(3) *Given two measurable functions  $\phi_1, \phi_2$ , such that  $\phi_j \in \mathcal{G}(\alpha_j, L_j, R_j)$  with  $t^{\alpha_2} L_2(t) = O(t^{\alpha_1} L_1(t))$  as  $t \rightarrow 0$ , then  $\phi_1 + \phi_2 \in \mathcal{G}(\alpha_1, L_1, R_+)$  for some positive function  $R_+$  explicit in terms of  $L_1, L_2$  and  $R_1$ .*

The three items here are special cases of Proposition 2.1, Corollary 2.3 and Proposition 2.5 below, respectively. The coefficients  $c_1, c_2$  and  $c_*$  and the function  $R_+$  are explicitly described in the precise versions below.

The proofs of all three result are rather short, but together they allow for a simple proof of the expansion (1.1) in several concrete cases:

- In Corollary 3.1, we study a function  $\phi : (0, 1] \rightarrow \mathbb{R}^2$  having an asymptotic behaviour around 0 of the shape  $x^{-1/2} |\log x|$ . The ensuing estimate we obtain is used in [BD, Theorem 2.1] to deduce a central limit theorem for central values  $\{D(1/2, x), x \in \mathbb{Q} \cap (0, 1]\}$  of the analytic continuation of the Estermann function

$$(1.4) \quad D(s, x) = \sum_{n \geq 1} \frac{\tau(n)}{n^s} e^{2\pi i n x}, \quad (\operatorname{Re}(s) > 1),$$

where  $\tau$  is the divisor function.

- In Corollaries 3.3 and 3.2, we study the functions of the shape  $\phi(x) = [1/x]^\lambda$  where  $\lambda \geq 1/2$ . These functions occur when studying the values  $\{\Sigma_\lambda(x), x \in \mathbb{Q} \cap (0, 1]\}$  of the moments of the continued fractions coefficients,

$$\Sigma_\lambda(x) = \sum_{j=1}^r a_j^\lambda, \quad (x = [0; a_1, \dots, a_r] = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}, a_r > 1),$$

see [BD, Theorems 2.5 and 9.4]. This, in turn, is applied to obtain a law of large numbers for the values of the Kashaev invariants of the  $4_1$  knot [BD, Corollary 2.6].

- In Corollary 3.4, we study the function  $\phi$  on  $(0, 1]$  given by  $\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor$ , where  $T : (0, 1] \rightarrow (0, 1]$ ,  $T(x) = \{1/x\}$  is the Gauss map. The estimate we obtain is used in [BD, Theorem 2.7] to obtain an independent proof, using dynamical systems, of a theorem of Vardi [Var93] on the convergence to a Cauchy law of the values of Dedekind sums.

## 2. ESTIMATION OF (1.1) IN GENERAL

### 2.1. Basic estimates.

2.1.1. *Taylor estimate.* The first and simplest method to obtain an estimate for (1.1) is to insert and integrate a Taylor expansion for the exponential.

**Proposition 2.1.** *Assume that for some  $\alpha \in (0, 3]$ , we have*

$$K := \int |\phi(x)|^\alpha d\mu(x) < \infty.$$

Then  $\phi \in \mathcal{G}(\alpha, 1, 1)$ , and more precisely

$$(2.1) \quad \mathcal{I}[\phi](t) = ic_1 t + c_2 t^2 + O(Kt^\alpha)$$

with  $c_1 = \int \phi d\mu$  if  $\alpha \geq 1$ , and  $c_2 = -\frac{1}{2} \int |\phi|^2 d\mu$  if  $\alpha \geq 2$ . The implied constant is absolute.

*Proof.* We use the bound  $\left| e^{iu} - \sum_{0 \leq k < \alpha} \frac{(iu)^k}{k!} \right| \ll |u|^\alpha$  with  $u = t\phi(x)$ , and integrate over  $x$ .  $\square$

Although it will not be useful for us here, we note that in the precise bound (2.1), the value of  $\alpha$  could be taken as a function of  $t$ . For example, if  $\mu$  is the Lebesgue measure on  $(0, 1)$  and  $\phi(x) = 1/x$ , we can take  $\alpha = 1 - 1/|\log t|$  and obtain  $\mathcal{I}[\phi](t) = O(t|\log t|)$ .

2.1.2. *Using properties of the Mellin transform.* When the moment  $\int |\phi|^\alpha d\mu$  diverges at some particular  $\alpha$ , we can often extract a useful expansion from the Cauchy formula and the polar behaviour of the Mellin transform. For  $x \in \mathbb{R}$ ,  $s \in \mathbb{C}$  and  $\eta \in [0, 1]$ , let

$$\phi_{s,\eta}(x) := \mathbf{1}_{\phi(x) \neq 0} |\phi(x)|^s \exp(-s \frac{\pi i}{2} (1 - \eta) \operatorname{sgn} \phi(x)), \quad \phi_s(x) := \phi_{s,0}(x).$$

Note that for  $k \in \mathbb{N}_{>0}$ ,  $\phi_k(x) = (-i\phi(x))^k$ . Define further

$$G_\eta(s) := \int \phi_{s,\eta}(x) d\mu(x).$$

**Proposition 2.2.** *Let  $\alpha \in (0, 3)$ ,  $\rho \in (0, 1)$ ,  $\delta, \eta_0 > 0$  and  $\xi \in \mathbb{R}$ . Assume that for some  $c > 0$ , we have*

$$(2.2) \quad \int_{\phi(x) \neq 0} (|\phi(x)|^c + |\phi(x)|^{-c}) d\mu(x) < \infty$$

and that the functions  $G_\eta(s)$  for  $\eta \in [0, \eta_0]$ , initially defined for  $\operatorname{Re}(s) \in (-c, c)$ , can be analytically continued to the set

$$\{s \in \mathbb{C}, 0 < \operatorname{Re}(s) \leq \alpha + \delta, s \notin [\alpha, \alpha + \delta]\}.$$

Assume further that

$$\sup_{0 \leq \eta \leq \eta_0} \int_{s=\alpha+\delta+i\tau}^{\tau \in \mathbb{R}} |\Gamma(-s)G_\eta(s)| d\tau < \infty,$$

and that there is an open neighborhood  $\mathcal{V}$  of  $[\alpha, \alpha + \delta]$  for which

$$(2.3) \quad (\alpha - s)^\xi G_0(s) = \varrho + O(|s - \alpha|^\rho), \quad s \in \mathcal{V} \setminus [\alpha, \alpha + \delta], \operatorname{Re}(s) \leq \alpha + \delta.$$

Then,  $\phi \in \mathcal{G}(\alpha, |\log|^{\xi-1+v_\alpha}, |\log|^{\xi-1+v_\alpha-\rho})$ , where  $v_\alpha = 1$  if  $\alpha = 1, 2$  and  $v_\alpha = 0$  otherwise, and with coefficients given by

$$(2.4) \quad c_1 = iG_0(1) \text{ if } \alpha > 1, \quad c_2 = \frac{1}{2}G_0(2) \text{ if } \alpha > 2, \quad c_* = \begin{cases} -\varrho/\Gamma(\xi+1), & \alpha = 1, \\ \frac{1}{2}\varrho/\Gamma(\xi+1), & \alpha = 2, \\ \varrho \frac{\Gamma(-\alpha)}{\Gamma(\xi)}, & \alpha \notin \{1, 2\}. \end{cases}$$

*Proof.* We write

$$\mathcal{I}[\phi](t) + 1 = \int e^{it\phi(x)} d\mu(x) = J_+ + J_- + J_0,$$

where  $J_\pm$  corresponds to the part of the integral restricted to  $\pm\phi > 0$ .

For all  $\varepsilon \in (0, \frac{\pi}{2}\eta_0)$ , define

$$J_+(\varepsilon) := \int_{\phi(x)>0} e^{(-\varepsilon+i)t\phi(x)} d\mu(x), \quad J_-(\varepsilon) := \int_{\phi(x)<0} e^{(\varepsilon+i)t\phi(x)} d\mu(x).$$

By dominated convergence, we have  $J_+ := \lim_{\varepsilon \rightarrow 0^+} J_+(\varepsilon)$ , and similarly for  $J_-$ . We use the Mellin transform formula for the exponential

$$e^{-y} = \frac{1}{2\pi i} \int_{-c/2-i\infty}^{-c/2+i\infty} \Gamma(-s) |y|^s e^{s \arg(y)} ds$$

valid for  $\operatorname{Re}(y) > 0$ , see [GR07, eq. 17.43.1] (the extension to non-real  $y$  is straightforward by the Stirling formula [GR07, eq. 8.327.1]). Inserting this in  $J_\pm(\varepsilon)$ , we obtain

$$J_+(\varepsilon) + J_-(\varepsilon) = \frac{1}{2\pi i} \int_{-c/2-i\infty}^{-c/2+i\infty} \Gamma(-s) G_\eta(s) |1 + i\varepsilon|^s t^s ds,$$

where  $\eta = \frac{2}{\pi} \arctan \varepsilon \leq \frac{2\varepsilon}{\pi} \leq \eta_0$ . We move the contour forward to  $\operatorname{Re}(s) = \alpha + \delta$ . The simple pole at  $s = 0$  contributes  $\int_{\phi(x) \neq 0} d\mu(x)$ , and therefore by adding the contribution from  $J_0$  we get

$$\begin{aligned} J_0 + J_+(\varepsilon) + J_-(\varepsilon) &= 1 + R + \frac{1}{2\pi i} \int_{H(\alpha, \alpha+\delta)} \Gamma(-s) G_\eta(s) t^s |1 + i\varepsilon|^s ds \\ &\quad + \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\alpha+\delta} \Gamma(-s) G_\eta(s) t^s |1 + i\varepsilon|^s ds, \end{aligned}$$

where  $R$  consists of the contribution of the residues at 1 (if  $\alpha > 1$ ) and 2 (if  $\alpha > 2$ ). Here  $H(\alpha, \alpha + \delta)$  is a Hankel contour, going from  $\alpha + \delta - i0$  to  $\alpha + \delta + i0$  passing around  $\alpha$  from the left. The last integral is bounded by the triangle inequality, using our first hypothesis on  $G_\eta$ , which gives

$$\frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\alpha+\delta} \Gamma(-s) G_\eta(s) t^s |1 + i\varepsilon|^s ds \ll t^{\alpha+\delta},$$

**uniformly in  $\varepsilon$ .** Passing to the limit  $\varepsilon \rightarrow 0$ , there remains to prove

$$\frac{1}{2\pi i} \int_{H(\alpha, \alpha+\delta)} \Gamma(-s) G_0(s) t^s ds = c_* t^\alpha |\log t|^{\xi-1+v_\alpha} + O(t^\alpha |\log t|^{\xi-1+v_\alpha-\rho}).$$

This is done by using our second hypothesis along with a standard Hankel contour integration argument; we refer to *e.g.* Corollary II.0.18 of [Ten15] for the details.  $\square$

An important special case is the following.

**Corollary 2.3.** *Let  $\mu$  be defined on  $[0, 1]$  by  $d\mu(x) = f(x) dx$  where  $f \in C^1([0, 1])$ . Let  $a \in \mathbb{R} \setminus \{0\}$ . For all  $\beta > \frac{1}{3}$ ,  $\lambda \geq 0$ , and  $\phi$  given by*

$$\phi(x) = ax^{-\beta} |\log x|^\lambda$$

one has  $\phi \in \mathcal{G}(1/\beta, |\log|^{\lambda/\beta+v_{1/\beta}}, |\log|^{\lambda/\beta+v_{1/\beta}-1+\varepsilon})$  for any  $\varepsilon \in (0, 1)$  and with

$$c_* = f(0) \frac{|a|^{1/\beta} e^{-\frac{\pi i \operatorname{sgn} a}{2\beta}}}{\beta^{\lambda/\beta+1}} \times \begin{cases} -(\lambda+1)^{-1}, & \beta = 1, \\ (4\lambda+2)^{-1}, & \beta = 1/2, \\ \Gamma(-1/\beta), & \beta \notin \{1, 1/2\}. \end{cases}$$

and  $c_1 = \int \phi \, d\mu$  if  $\beta < 1$  and  $c_2 = -\frac{1}{2} \int |\phi|^2 \, d\mu$  if  $\beta < \frac{1}{2}$ .

*Proof.* First, we write  $d\mu(x) = f(0)\chi(x) \, dx + xg(x) \, dx$ , where  $\chi$  is the characteristic function of the interval  $[0, 1]$  and  $g \in \mathcal{C}([0, 1])$ . For the contribution of  $\chi \, dx$  we apply Proposition 2.2 with any fixed  $c < 1/\beta$ ,  $\alpha = 1/\beta$ ,  $\xi = \lambda/\beta + 1$ , any fixed  $\rho \in (0, 1)$  and  $\delta > 0$ . By [GR07, 4.272.6], for  $\operatorname{Re}(s) < 1/\beta$  and  $\eta \in [0, 1]$  we have

$$G_\eta(s) = e^{-s\frac{\pi i}{2}(1-\eta)\operatorname{sgn}(a)} |a|^s \int_0^1 x^{-\beta s} |\log x|^{\lambda s} \, dx = e^{-s\frac{\pi i}{2}(1-\eta)\operatorname{sgn}(a)} |a|^s \frac{\Gamma(\lambda s + 1)}{(1-\beta s)^{\lambda s + 1}}.$$

Notice also that by Stirling's formula  $G_\eta(s) \ll e^{\pi(\frac{1-\eta}{2})|\tau|} |\tau|^{-1/2}$  as  $|\tau| = |\operatorname{Im} s| \rightarrow \infty$ , so that in any case  $\Gamma(-s)G_\eta(s) \ll |\tau|^{-1-\operatorname{Re}(s)}$ . Therefore the hypotheses of Proposition 2.2 are easily verified with

$$\varrho = |a|^{1/\beta} e^{-\frac{\pi i \operatorname{sgn} a}{2\beta}} \frac{\Gamma(\lambda/\beta + 1)}{\beta^{\lambda/\beta+1}}.$$

Thus,

$$\int_0^1 (e^{it\phi(x)} - 1) \, dx = itc'_1 + c'_2 t^2 + c_* t^{1/\beta} |\log t|^{\lambda/\beta+v_{1/\beta}} + O(t^{1/\beta} |\log t|^{\lambda/\beta+v_{1/\beta}-\rho})$$

with coefficients as given in (2.4) with  $G_0(1) = -i \int \phi \chi \, dx$  and  $G_0(2) = -\int \phi^2 \chi \, dx$ . Finally, as in Proposition 2.1 we deduce

$$\int (e^{it\phi(x)} - 1) x g(x) \, dx = ic''_1 t + c''_2 t^2 + O(Kt^{\alpha'})$$

for any  $0 < \alpha' < \min(3, \frac{2}{\beta})$  and with  $c''_1 = \int \phi(x) x g(x) \, dx$  if  $\alpha' > 1$  and  $c''_2 = -\frac{1}{2} \int \phi(x)^2 x g(x) \, dx$  if  $\alpha' > 2$ . The result then follows.  $\square$

## 2.2. Addition.

**Lemma 2.4.** For  $j \in \{1, 2\}$ , let  $\delta_j(x) = e^{it\phi_j(x)} - 1$ . Then

$$(2.5) \quad \begin{aligned} \mathcal{I}[\phi_1 + \phi_2](t) &= \mathcal{I}[\phi_1](t) + \mathcal{I}[\phi_2](t) + \int \delta_1(x) \delta_2(x) \, d\mu(x) \\ &= \mathcal{I}[\phi_1](t) + \mathcal{I}[\phi_2](t) + O\left(\prod_{j \in \{1, 2\}} |\operatorname{Re} \mathcal{I}[\phi_j](t)|^{1/2}\right) \end{aligned}$$

*Proof.* The first equation is simply the relation  $e^{it(\phi_1(x)+\phi_2(x))} - 1 = \delta_1(x) + \delta_2(x) + \delta_1(x)\delta_2(x)$  integrated over  $x$ . The last term is bounded using the Cauchy-Schwarz inequality

$$\left( \int |\delta_1(x) \delta_2(x)| \, d\mu(x) \right)^2 \leq \prod_{j \in \{1, 2\}} \int |\delta_j(x)|^2 \, d\mu(x)$$

and expanding the square on the right-hand side.  $\square$

**Proposition 2.5.** For  $j \in \{1, 2\}$ , let  $\alpha_j \in (0, 2]$ , let  $L_j, R_j$  be positive functions defined on a neighborhood of 0 in  $\mathbb{R}_+^*$ , and  $\phi_j \in \mathcal{G}(\alpha_j, L_j, R_j)$ . If  $\alpha_1 \leq \alpha_2$ , and under the following assumptions:

- $R_j(t), L_j(t) = t^{o(1)}$  as  $t \rightarrow 0$ ,
- $R_j(t) = O(L_j(t))$ ,
- $t^2 = O(t^{\alpha_1} L_1(t))$ ,

we have

$$\phi_1 + \phi_2 \in \mathcal{G}(\alpha_1, L_1, R_+), \quad R_+ = \begin{cases} R_1 & \text{if } \alpha_1 < \alpha_2, \\ R_1 + L_2 + \sqrt{L_1 L_2} & \text{if } \alpha_1 = \alpha_2 < 2, \\ R_1 + L_2 + \sqrt{L_1}(\sqrt{L_2} + 1) & \text{if } \alpha_1 = \alpha_2 = 2. \end{cases}$$

Moreover,

$$\begin{aligned} c_1(\phi_1 + \phi_2) &= c_1(\phi_1) + c_1(\phi_2), \\ c_*(\phi_1 + \phi_2) &= c_*(\phi_1). \end{aligned}$$

*Proof.* We use Lemma 2.4; when computing the real part in (2.5), the term  $ic_1 t$  vanishes.  $\square$

*Remark.* Note that using this result might induce a slight quantitative loss in the two cases when  $\alpha_1 = \alpha_2$ . What is gained at this price is that we are only required to study each  $\phi_j$  separately, which simplifies the analysis.

We also remark that this estimate is useful only when the term  $c_2 t^2$  is not relevant in (1.3). In the complementary case, Proposition 2.1 can be used, although the ensuing error term will typically be worse than optimal by a factor of  $|\log t|$ .

It is straightforward to generalize Proposition 2.5, affecting to each  $\phi_j$  a different value of the frequency: under the same hypotheses and notations, and additionally that  $L_j, R_j$  tend monotonically to  $+\infty$  at 0,

$$\int e^{it_1 \phi_1(x) + it_2 \phi_2(x)} d\mu(x) = 1 + ic_1(\phi_1)t_1 + ic_1(\phi_2)t_2 + c_* t_1^{\alpha_1} L_1(t_1) + O(t_+^2 + t_+^{\alpha_1} R_+(t_+)),$$

where  $c_1, c_*$  are as in the conclusion of Proposition 2.5, and  $t_+ = \max\{t_1, t_2\}$ .

### 3. APPLICATIONS

We now describe the applications we will be interested in. The measure is the Gauss-Kuzmin distribution

$$d\mu(x) = \frac{dx}{(1+x)\log 2} \quad (x \in [0, 1]).$$

The measure  $\mu$  is invariant under the Gauss map  $T(x) = \{1/x\}$  on  $(0, 1)$ , in particular,

$$(3.1) \quad \mathcal{I}[\phi \circ T](t) = \mathcal{I}[\phi](t).$$

**3.1. Central values of the Estermann function.** The first application we discuss is the ‘‘period function’’  $\phi : \mathbb{R} \rightarrow \mathbb{C}$  associated with the Estermann function (1.4), namely

$$\phi(x) = D\left(\frac{1}{2}, 1/x\right) - D\left(\frac{1}{2}, x\right),$$

initially defined in  $\mathbb{Q} \cap (0, 1]$ . By [Bet16], this function can be extended to a continuous function on  $(0, 1]$ , more precisely given by an expression of the shape (3.2) below. Interpreting  $\phi$  to be  $\mathbb{R}^2$ -valued, the analogue of the integral (1.1) is estimated using the following.

**Corollary 3.1.** *Let  $\varepsilon > 0$ ,  $\mathcal{E} : [0, 1] \rightarrow \mathbb{C}$  be a bounded, continuous function, and*

$$(3.2) \quad \phi_j(x) := \left( \frac{1}{2} x^{-1/2} (\log(1/x) + \gamma_0 - \log(8\pi) - \frac{\pi}{2}) + \zeta\left(\frac{1}{2}\right)^2 + \operatorname{Re} \mathcal{E}((-1)^j x) \right) \\ \left( \frac{(-1)^{j-1}}{2} x^{-1/2} (\log(1/x) + \gamma_0 - \log(8\pi) + \frac{\pi}{2}) + \operatorname{Im} \mathcal{E}((-1)^j x) \right).$$

*Let also  $\mathbf{u}_j := \left( \frac{1}{(-1)^{j-1}} \right)$ . Then for some vector  $\boldsymbol{\mu} \in \mathbb{R}^2$ , and all  $\mathbf{t} \in \mathbb{R}^2$ , we have*

$$\begin{aligned} & \int_0^1 e^{i\langle \mathbf{t}, \phi_1(x) + \phi_2(T(x)) \rangle} d\mu(x) \\ &= 1 + i\langle \mathbf{t}, \boldsymbol{\mu} \rangle - \frac{1}{3 \log 2} \sum_{j \in \{1, 2\}} \langle \mathbf{t}, \mathbf{u}_j \rangle^2 |\log |\langle \mathbf{t}, \mathbf{u}_j \rangle||^3 + O_\varepsilon(\|\mathbf{t}\|^2 |\log \|\mathbf{t}\||^{2+\varepsilon}). \end{aligned}$$

*Proof.* Let  $\varepsilon \in (0, 1)$ . Using Corollary 2.3 with  $\beta = 1/2$  and  $\lambda \in \{0, 1\}$ , and Proposition 2.1, we obtain

$$\begin{aligned} (x \mapsto \pm \frac{1}{2}x^{-1/2}|\log x|) &\in \mathcal{G}(2, |\log|^3, |\log|^{2+\varepsilon}), \\ (x \mapsto (\gamma_0 - \log(8\pi) + \frac{\pi}{2})x^{-1/2}) &\in \mathcal{G}(2, |\log|, |\log|^\varepsilon), \\ (x \mapsto \operatorname{Im} \mathcal{E}(\pm x)) &\in \mathcal{G}(3, 1, 1), \end{aligned}$$

as well as  $c_*(x \mapsto \pm \frac{1}{2}x^{-1/2}|\log x|) = -\frac{1}{3\log 2}$ . From Proposition 2.5 and the ensuing remark, and using the property (3.1), we obtain for  $j \in \{1, 2\}$

$$\int_0^1 (e^{i\langle \mathbf{t}, \phi_j(x) \rangle} - 1) d\mu(x) = i\langle \mathbf{t}, \boldsymbol{\mu}_j \rangle + c_* \langle \mathbf{t}, \mathbf{u}_j \rangle^2 |\log |\langle \mathbf{t}, \mathbf{u}_j \rangle||^3 + O_\varepsilon(\|\mathbf{t}\|^2 |\log \|\mathbf{t}\||^{2+\varepsilon}),$$

where  $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{R}^2$ . On the other hand, we have

$$\Delta(\mathbf{t}) := \int_0^1 (e^{i\langle \mathbf{t}, \phi_1(x) \rangle} - 1)(e^{i\langle \mathbf{t}, \phi_2(T(x)) \rangle} - 1) d\mu(x) = \int_0^1 (e^{i\langle \mathbf{t}, \phi_2(x) \rangle} - 1) F_x(\mathbf{t}) dx,$$

where

$$F_x(\mathbf{t}) = \frac{1}{\log 2} \sum_{n \geq 1} \frac{e^{i\langle \mathbf{t}, \phi_1(1/(n+x)) \rangle} - 1}{(n+x)(n+x+1)}.$$

By a Taylor expansion at order 1, we have  $|F_x(\mathbf{t})| \ll \|\mathbf{t}\|$  uniformly in  $x$ , and therefore

$$|\Delta(\mathbf{t})| \ll \|\mathbf{t}\|^2 \int_0^1 \|\phi_2(x)\| dx \ll \|\mathbf{t}\|^2.$$

By (2.5), we deduce

$$\int_0^1 e^{i\langle \mathbf{t}, \phi_1(x) + \phi_2(T(x)) \rangle} d\mu(x) = 1 + \int_0^1 (e^{i\langle \mathbf{t}, \phi_1(x) \rangle} + e^{i\langle \mathbf{t}, \phi_2(T(x)) \rangle} - 2) d\mu(x) + O(\|\mathbf{t}\|^2),$$

whence the claimed estimate.  $\square$

**3.2. Moments of continued fractions coefficients.** The next application we consider pertains to the moments functions  $\Sigma_\lambda$  of continued fractions coefficients, where  $\lambda \geq 0$  is the order of the moment. The function of interest to us here is

$$\phi_\lambda(x) = \lfloor 1/x \rfloor^\lambda.$$

The case  $\lambda < 1/2$  can be easily dealt with using Proposition 2.1, so we do not focus on it here.

A first approach is to use Proposition 2.5 to approximate  $\lfloor 1/x \rfloor$  by  $1/x$ , and then use Corollary 2.3. This leads to the following.

**Corollary 3.2.** *Let  $\lambda \geq 1/2$ . The function  $\phi_\lambda$  given by  $\phi_\lambda(x) = \lfloor 1/x \rfloor^\lambda$  satisfies the following.*

– If  $\lambda = 1/2$ , then with  $c_* = -1/(\log 2)$ , we have

$$(3.3) \quad \mathcal{I}[\phi_{1/2}](t) = ic_1 t + c_* t^2 |\log t| + O_\varepsilon(t^2 |\log t|^\varepsilon).$$

– If  $\lambda > 1/2$  and  $\lambda \neq 1$ , then with  $c_* = -\exp(-\pi i/(2\lambda))\Gamma(1 - 1/\lambda)/\log 2$ , we have

$$\mathcal{I}[\phi_\lambda](t) = (\mathbf{1}_{\lambda < 1})ic_1 t + c_* t^{1/\lambda} + O_\varepsilon(t^{1/\lambda} |\log t|^{-1+\varepsilon})$$

When  $1/2 \leq \lambda < 1$ , we have  $c_1 = \int_0^1 \phi_\lambda(x) d\mu(x)$ .

*Proof.* We write  $\phi_\lambda(x) = p_\lambda(x) + r_\lambda(x)$ , where  $p_\lambda(x) = x^{-\lambda}$  and  $r_\lambda(x) \ll_\lambda \lfloor 1/x \rfloor^{\lambda-1}$ . By Proposition 2.1, we have  $r_\lambda \in \mathcal{G}(\min(3, \frac{1}{\lambda-1/3}), 1, 1)$ .

We consider first the case  $\lambda > 1/2$ ,  $\lambda \neq 1$ . By Corollary 2.3, we have  $p_\lambda \in \mathcal{G}(\frac{1}{\lambda}, 1, |\log|^{-1+\varepsilon})$ . We deduce, by Proposition 2.5, that  $\phi_\lambda \in \mathcal{G}(\frac{1}{\lambda}, 1, |\log|^{-1+\varepsilon})$ , and this yields the second and third cases.

If  $\lambda = 1/2$ , then Corollary 2.3 implies  $p_{1/2} \in \mathcal{G}(2, |\log|, |\log|^\varepsilon)$ , and by Proposition 2.1, for some  $c \in \mathbb{R}$ , we have

$$\mathcal{I}[r_{1/2}](t) = ict + O(t^2)$$



On the other hand, since  $\left| (e^{itp_{1/2}(x)} - 1)(e^{itr_{1/2}(x)} - 1) \right| \ll t^2 \left| p_{1/2}(x)r_{1/2}(x) \right| \ll t^2$ , we get

$$\int_0^1 (e^{itp_{1/2}(x)} - 1)(e^{itr_{1/2}(x)} - 1) d\mu(x) = O(t^2).$$

By (2.5), we conclude (3.3) as claimed.  $\square$

The case  $\lambda = 1$  could be analyzed by the same method, but we chose to study it separately to obtain a more precise error term by another approach, using Proposition 2.2 directly. The associated Mellin transform  $G_0(s)$  is related to the Riemann  $\zeta$ -function.

**Corollary 3.3.** *The function  $\phi$  given by  $\phi(x) = \lfloor 1/x \rfloor$  satisfies*

$$\mathcal{I}[\phi](t) = -\frac{it}{\log 2}(\log t + \gamma_0 - \frac{\pi i}{2}) + O_\varepsilon(t^{2-\varepsilon}).$$

*Proof.* The integral (2.2) converges for all  $c < 1$ . A quick computation shows that an analytic continuation of  $G_\eta(s)$  is given by

$$G_\eta(s) = \frac{\exp(-s\frac{\pi i}{2}(1-\eta))}{\log 2} \{\zeta(2-s) + H(s)\},$$

where  $H(s) = \sum_{n \geq 1} n^s (\log(1 + \frac{1}{n(n+2)}) - \frac{1}{n^2})$  is analytic and uniformly bounded in  $\operatorname{Re}(s) \leq 2 - \varepsilon$ . We have

$$\int_{\operatorname{Re}(s)=2-\varepsilon} |\Gamma(-s)G_\eta(s)| |ds| \ll_\varepsilon 1 + \int_0^\infty |\zeta(\varepsilon + i\tau)| \frac{d\tau}{1+\tau^2} \ll_\varepsilon 1$$

by the Stirling formula. The polar behaviour (2.3) is given by

$$G_0(s) = \frac{\exp(-s\frac{\pi i}{2})}{\log 2} \{\zeta(2-s) + H(s)\} = \frac{\exp(-s\frac{\pi i}{2})}{\log 2} \left\{ \frac{1}{1-s} + A + O(s-1) \right\}$$

for  $s$  in a neighborhood of 1, where

$$\begin{aligned} A &= \sum_{n \geq 1} \left( n \log \left( 1 + \frac{1}{n(n+2)} \right) - \log \left( 1 + \frac{1}{n} \right) \right) \\ &= - \lim_{N \rightarrow \infty} \sum_{n=1}^N \left( n \log \left( 1 + \frac{1}{n+1} \right) - (n-1) \log \left( 1 + \frac{1}{n} \right) \right) \\ &= -1. \end{aligned}$$

Applying Proposition 2.2 with  $\delta = 1/2$  and  $\alpha = 1$  yields the claimed result up to  $O(t)$ . Our more precise statement follows from noting that there is no branch cut along  $s \geq 1$  in this case, so that the residue theorem may be used. We obtain

$$\operatorname{Res}_{s=1} \Gamma(-s)G_0(s)t^s = \frac{it}{\log 2}(\gamma_0 - \frac{\pi i}{2} + \log t),$$

whence the claimed estimate. One could go further, isolating a pole of order 2 at  $s = 2$ , and this would give an error term  $O(t^2 |\log t|)$ .  $\square$

**3.3. Dedekind sums.** The final example we discuss is related to Dedekind sums, for the definition of which we refer to [BD, Section 2.4]. The ‘‘period function’’  $\phi$  relevant to us here is

$$\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor.$$

Compared with the case of  $x \mapsto \lfloor 1/x \rfloor$  studied in Corollary 3.3, the relevant exponent  $\alpha$  is again 1, but the leading term turns out to be  $t$  (the terms  $t \log t$  vanish).

**Corollary 3.4.** *The map  $\phi$  on  $(0, 1)$  given by  $\phi(x) = \lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor$  satisfies*

$$\mathcal{I}[\phi](t) = -\frac{\pi}{\log 2}t + O(t^2 |\log t|^2).$$

*Proof.* We consider

$$\begin{aligned} \Delta(t) &:= \int_0^1 (e^{-it\lfloor 1/T(x) \rfloor} - 1)(e^{it\lfloor 1/x \rfloor} - 1) d\mu(x) \\ &= \int_0^1 (e^{-it\lfloor 1/x \rfloor} - 1)F_x(t) dx, \end{aligned}$$

with  $F_x(t) = \frac{1}{\log 2} \sum_{n \geq 1} \frac{e^{itn} - 1}{(n+x)(n+1+x)}$ . Since  $|e^{iu} - 1| \ll |u|^{1-1/|\log t|}$  for all  $u \in \mathbb{R}$ , we find

$$F_x(t) \ll t \sum_{n \geq 1} \frac{1}{n^{1+1/|\log t|}} \ll t|\log t|.$$

Similarly,

$$\int_0^1 |e^{-it\lfloor 1/x \rfloor} - 1| dx \ll t \int_0^1 x^{-1+1/|\log t|} dx \ll t|\log t|.$$

We thus obtain  $\Delta(t) = O((t \log t)^2)$ . Using [Corollary 3.3](#) with the improved error term  $O(t^2|\log t|)$ , [\(3.1\)](#) and [\(2.5\)](#), we deduce

$$\int_0^1 e^{it(\lfloor 1/x \rfloor - \lfloor 1/T(x) \rfloor)} d\mu(x) = 1 + 2 \operatorname{Re} I(t) + O((t \log t)^2),$$

where  $I(t) = \int_0^1 (e^{it\lfloor 1/x \rfloor} - 1) d\mu(x)$ . [Corollary 3.3](#) allows us to conclude.  $\square$

#### REFERENCES

- [Bet16] S. Bettin, *On the reciprocity law for the twisted second moment of Dirichlet L-functions*, Trans. Amer. Math. Soc. **368** (2016), no. 10, 6887–6914.
- [BD] S. Bettin and S. Drappeau, *Limit laws for rational continued fractions and value distribution of quantum modular forms*, Preprint.
- [BH86] N. Bleistein and R. A. Handelsman, *Asymptotic expansions of integrals*, second ed., Dover Publications, Inc., New York, 1986. MR 863284
- [Fel71] W. Feller, *An introduction to probability theory and its applications. Vol. II*, Second edition, John Wiley & Sons, Inc., New York-London-Sydney, 1971. MR 0270403
- [GR07] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, seventh ed., Elsevier/Academic Press, Amsterdam, 2007, Translated from the Russian.
- [IL71] I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff Publishing, Groningen, 1971, With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman. MR 0322926
- [Lóp08] J. L. López, *Asymptotic expansions of Mellin convolution integrals*, SIAM Rev. **50** (2008), no. 2, 275–293. MR 2403051
- [Olv97] F. W. J. Olver, *Asymptotics and special functions*, AKP Classics, A K Peters, Ltd., Wellesley, MA, 1997, Reprint of the 1974 original [Academic Press, New York; MR0435697 (55 #8655)]. MR 1429619
- [Ten15] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015.
- [Var93] I. Vardi, *Dedekind sums have a limiting distribution*, Internat. Math. Res. Notices (1993), no. 1, 1–12. MR 1201746

SB: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, 16146 GENOVA, ITALY  
*Email address:* `bettin@dima.unige.it`

SD: AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M UMR 7373, 13453 MARSEILLE, FRANCE  
*Email address:* `sary-aurelien.drappeau@univ-amu.fr`