



HAL
open science

DECREASING MONOTONICITY OF TWO RATIOS DEFINED BY THREE OR FOUR POLYGAMMA FUNCTIONS

Feng Qi

► **To cite this version:**

Feng Qi. DECREASING MONOTONICITY OF TWO RATIOS DEFINED BY THREE OR FOUR POLYGAMMA FUNCTIONS. 2020. hal-02998414v1

HAL Id: hal-02998414

<https://hal.science/hal-02998414v1>

Preprint submitted on 10 Nov 2020 (v1), last revised 2 Feb 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

DECREASING MONOTONICITY OF TWO RATIOS DEFINED BY THREE OR FOUR POLYGAMMA FUNCTIONS

FENG QI

Dedicated to people facing and battling COVID-19

ABSTRACT. In the paper, by virtue of convolution theorem for the Laplace transforms, by three monotonicity rules for the ratios of two definite integrals, of two Laplace transforms, and of two functions, and by other analytic techniques, the author presents decreasing monotonicity of two ratios defined by three or four polygamma functions. These results confirm guesses posed by the author.

CONTENTS

1.	Preliminaries and motivations	1
2.	Lemmas	3
3.	Monotonicity of a ratio defined by three polygamma functions	6
4.	Monotonicity of a ratio defined by four polygamma functions	7
5.	An alternative proof of Theorem 1.1	8
6.	Remarks	10
	References	12

1. PRELIMINARIES AND MOTIVATIONS

In the literature [1, Section 6.4], the function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

and its logarithmic derivative $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ are called Euler's gamma function and digamma function respectively. Moreover, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as the trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives $\psi^{(k)}(z)$ for $k \in \{0\} \cup \mathbb{N}$ are known as polygamma functions.

Recall from Chapter XIII in [9], Chapter 1 in [20], and Chapter IV in [22] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies

2010 *Mathematics Subject Classification*. Primary 33B15; Secondary 26A48, 26A51, 26D07, 26D15, 26D20, 44A10, 60E15.

Key words and phrases. decreasing monotonicity; ratio; polygamma function; exponential function; majorization; inequality; convolution theorem for the Laplace transform; monotonicity rule for the ratio of two Laplace transforms; monotonicity rule for the ratio of two definite integrals; monotonicity rule for the ratio of two functions.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

$(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$, then we call $f(x)$ a completely monotonic function on I .

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{R}^n$. A n -tuple α is said to strictly majorize β (in symbols $\alpha \succ \beta$) if $(\alpha_{[1]}, \alpha_{[2]}, \dots, \alpha_{[n]}) \neq (\beta_{[1]}, \beta_{[2]}, \dots, \beta_{[n]})$, $\sum_{i=1}^k \alpha_{[i]} \geq \sum_{i=1}^k \beta_{[i]}$ for $1 \leq k \leq n-1$, and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$, where $\alpha_{[1]} \geq \alpha_{[2]} \geq \dots \geq \alpha_{[n]}$ and $\beta_{[1]} \geq \beta_{[2]} \geq \dots \geq \beta_{[n]}$ are rearrangements of α and β in a descending order. See [8, p. 8, Definition A.1] or closely related texts and references in the papers [5, 19, 21, 25].

Theorem 1.1 ([12, Theorem 3.1]). *Let $p, q, m, n \in \{0\} \cup \mathbb{N}$ satisfying $(p, q) \succ (m, n)$ and let*

$$F_{p,m,n,q;c_p,m,n,q}(x) = \begin{cases} |\psi^{(m)}(x)| |\psi^{(n)}(x)| - c_{p,m,n,q} |\psi^{(p)}(x)|, & q = 0 \\ |\psi^{(m)}(x)| |\psi^{(n)}(x)| - c_{p,m,n,q} |\psi^{(p)}(x)| |\psi^{(q)}(x)|, & q \geq 1 \end{cases}$$

for $c_{p,m,n,q} \in \mathbb{R}$ and $x \in (0, \infty)$. Then

(1) for $q \geq 0$, if and only if

$$c_{p,m,n,q} \leq \begin{cases} \frac{(m-1)!(n-1)!}{(p-1)!}, & q = 0 \\ \frac{(m-1)!(n-1)!}{(p-1)!(q-1)!}, & q \geq 1, \end{cases}$$

the function $F_{p,m,n,q;c_p,m,n,q}(x)$ is completely monotonic in $x \in (0, \infty)$;

(2) for $q \geq 1$, if and only if $c_{p,m,n,q} \geq \frac{m!n!}{p!q!}$, the function $-F_{p,m,n,q;c_p,m,n,q}(x)$ is completely monotonic in $x \in (0, \infty)$;

(3) the double inequality

$$-\frac{(m+n-1)!}{(m-1)!(n-1)!} < \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} < 0 \quad (1.1)$$

for $m, n \in \mathbb{N}$ and the double inequality

$$\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!} < \frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} < \frac{m!n!}{p!q!} \quad (1.2)$$

for $m, n, p, q \in \mathbb{N}$ with $p > m \geq n > q \geq 1$ and $m+n = p+q$ are valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars respectively.

Theorem 1.2 ([12, Theorem 3.2]). *For $k \in \mathbb{N}$ and $x \in (0, \infty)$, let*

$$\mathcal{F}_{k,\eta_k}(x) = \psi^{(2k)}(x) + \eta_k [\psi^{(k)}(x)]^2 \quad \text{and} \quad \mathfrak{F}_{k,\vartheta_k}(x) = \frac{\psi^{(2k)}(x)}{[(-1)^{k+1}\psi^{(k)}(x)]^{\vartheta_k}}.$$

Then the following conclusions are true:

- (1) if and only if $\eta_k \geq \frac{1}{2} \frac{(2k)!}{(k-1)!k!}$, the function $\mathcal{F}_{k,\eta_k}(x)$ is completely monotonic on $(0, \infty)$;
- (2) if and only if $\eta_k \leq 0$, the function $-\mathcal{F}_{k,\eta_k}(x)$ is completely monotonic on $(0, \infty)$;
- (3) if and only if $\vartheta_k \geq 2$, the function $\mathfrak{F}_{k,\vartheta_k}(x)$ is decreasing on $(0, \infty)$;
- (4) if and only if $\vartheta_k \leq \frac{2k+1}{k+1}$, the function $\mathfrak{F}_{k,\vartheta_k}(x)$ is increasing on $(0, \infty)$;

(5) the following limits are valid:

$$\lim_{x \rightarrow 0^+} \mathfrak{F}_{k, \vartheta_k}(x) = \begin{cases} -\frac{(2k)!}{[(k)!]^{\frac{2k+1}{k+1}}}, & \vartheta_k = \frac{2k+1}{k+1} \\ 0, & \vartheta_k > \frac{2k+1}{k+1} \\ -\infty, & \vartheta_k < \frac{2k+1}{k+1} \end{cases}$$

and

$$\lim_{x \rightarrow \infty} \mathfrak{F}_{k, \vartheta_k}(x) = \begin{cases} -\frac{(2k-1)!}{[(k-1)!]^2}, & \vartheta_k = 2 \\ -\infty, & \vartheta_k > 2 \\ 0, & \vartheta_k < 2; \end{cases}$$

(6) the double inequality

$$-\frac{1}{2} \frac{(2k)!}{(k-1)!k!} < \frac{\psi^{(2k)}(x)}{[(-1)^{k+1}\psi^{(k)}(x)]^2} < 0$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.

In [12, Remark 6.2], the author guessed that,

(1) for $m, n \in \mathbb{N}$, the function

$$Q_{m,n}(x) = \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} \quad (1.3)$$

should be decreasing on $(0, \infty)$;

(2) for $m, n, p, q \in \mathbb{N}$ such that $(p, q) \succ (m, n)$, the function

$$\mathcal{Q}_{m,n;p,q}(x) = \frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} \quad (1.4)$$

should be decreasing on $(0, \infty)$.

It is clear that $Q_{k,k}(x) = \mathfrak{F}_{k,2}(x)$ which is decreasing on $(0, \infty)$.

In this paper, we aim to confirm these two guesses. We also supply an alternative proof of Theorem 1.1.

2. LEMMAS

The following lemmas are necessary in this paper.

Lemma 2.1 ([1, p. 260, 6.4.1]). *The integral representation*

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-zt} dt \quad (2.1)$$

is valid for $\Re(z) > 0$ and $n \geq 1$.

Lemma 2.2 (Convolution theorem for the Laplace transforms [22, pp. 91–92]). *Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \geq 0$ such that $|f_k(t)| \leq M_k e^{c_k t}$ for $k = 1, 2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t-u) du \right] e^{-st} dt = \int_0^\infty f_1(u) e^{-su} du \int_0^\infty f_2(v) e^{-sv} dv.$$

Lemma 2.3 ([24, Lemma 4]). *Let the functions $A(x)$ and $B(x) \neq 0$ be defined on $(0, \infty)$ such that their Laplace transforms exist. If the ratio $\frac{A(x)}{B(x)}$ is increasing, then the ratio $\frac{\int_0^\infty A(x)e^{-xt} dx}{\int_0^\infty B(x)e^{-xt} dx}$ is decreasing on $(0, \infty)$.*

Lemma 2.4. *Let*

$$g(t) = \begin{cases} \frac{t}{1 - e^{-t}}, & t \neq 0; \\ 1, & t = 0. \end{cases}$$

Then the following conclusions are valid:

- (1) *The function $g(t)$ is infinitely differentiable on $(-\infty, \infty)$, increasing from $(-\infty, \infty)$ onto $(0, \infty)$, convex on $(-\infty, \infty)$, and logarithmically concave on $(-\infty, \infty)$.*
- (2) *For fixed $s \in (0, 1)$, the ratio $\frac{g^s(t)}{g(st)}$ is decreasing in t from $(0, \infty)$ onto $(0, 1)$.*
- (3) *For $s \in (0, \frac{1}{2})$ and $t \in (0, \infty)$, the partial derivative*

$$\frac{\partial^2 \ln[g(st)g((1-s)t)]}{\partial s \partial t} > 0. \quad (2.2)$$

Proof. The differentiability, monotonicity, convexity, and logarithmic concavity of the function $g(t)$ have been proved in [12, Lemma 2.3].

It is straightforward that

$$\lim_{t \rightarrow 0} \frac{g^s(t)}{g(st)} = \lim_{t \rightarrow 0} \frac{g^s(t)}{g(st)} = \frac{1^s}{1} = 1$$

and

$$\lim_{t \rightarrow \infty} \frac{g^s(t)}{g(st)} = \lim_{t \rightarrow \infty} \frac{[g(t)/t]^s}{[g(st)/st]} \lim_{t \rightarrow \infty} \frac{t^s}{st} = \frac{1}{s} \lim_{t \rightarrow \infty} \frac{1}{t^{1-s}} = 0.$$

The first derivative of the ratio $\frac{g^s(t)}{g(st)}$ is

$$\frac{d}{dt} \left[\frac{g^s(t)}{g(st)} \right] = \frac{sg^s(t)}{g(st)} \left[\frac{g'(t)}{g(t)} - \frac{g'(st)}{g(st)} \right].$$

Hence, for arriving at decreasing monotonicity of the ratio $\frac{g^s(t)}{g(st)}$, it is sufficient to show that the ratio $\frac{g'(t)}{g(t)}$ is decreasing on $(0, \infty)$. For this, it is sufficient to show that the function $g(t)$ is logarithmically concave on $(0, \infty)$. The logarithmic concavity of the function $g(t)$ on $(-\infty, \infty)$ was proved in [12, Lemma 2.3].

For $a, b \in \mathbb{R}$ with $a < b$, let $f(x)$ and $g(x)$ be continuous on $[a, b]$, differentiable on (a, b) , and $g'(x) \neq 0$ on (a, b) . In [2], the following monotonicity rule was discovered: if the ratio $\frac{f'(x)}{g'(x)}$ is increasing on (a, b) , then both $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are increasing with respect to $x \in (a, b)$. Straightforward differentiation gives

$$\begin{aligned} \frac{\partial \ln[g(st)g((1-s)t)]}{\partial t} &= \frac{1}{t} \frac{t + 2e^t + 2 - e^{st}[(1-s)t + 2] - e^{(1-s)t}(st + 2)}{(e^{st} - 1)[e^{(1-s)t} - 1]}, \\ \lim_{s \rightarrow 0} [t + 2e^t + 2 - e^{st}[(1-s)t + 2] - e^{(1-s)t}(st + 2)] &= \lim_{s \rightarrow 0} ((e^{st} - 1)[e^{(1-s)t} - 1]) \\ &= 0, \\ \frac{[t + 2e^t + 2 - e^{st}[(1-s)t + 2] - e^{(1-s)t}(st + 2)]'_s}{[(e^{st} - 1)(e^{(1-s)t} - 1)]'_s} &= \frac{e^t(st + 1) - e^{2st}[(1-s)t + 1]}{e^t - e^{2st}}, \\ \lim_{s \rightarrow 1/2} (e^t(st + 1) - e^{2st}[(1-s)t + 1]) &= \lim_{s \rightarrow 1/2} (e^t - e^{2st}) = 0, \end{aligned}$$

and

$$\frac{(e^t(st+1) - e^{2st}[(1-s)t+1])'_s}{(e^t - e^{2st})'_s} = \frac{1}{2} - \frac{e^{(1-2s)t} - (1-2s)t - 1}{2t}$$

is increasing in $s \in [0, \frac{1}{2}]$. This means that the partial derivative $\frac{\partial \ln[g(st)g((1-s)t)]}{\partial t}$ is increasing in $s \in [0, \frac{1}{2}]$ for fixed $t > 0$. As a result, the inequality (2.2) is valid. The proof of Lemma 2.4 is complete. \square

Lemma 2.5 ([12, Lemma 2.1]). *For $k \in \mathbb{N}$, we have the limits*

$$\lim_{x \rightarrow 0^+} [x^k \psi^{(k-1)}(x)] = (-1)^k (k-1)! \quad (2.3)$$

and

$$\lim_{x \rightarrow \infty} [x^k \psi^{(k)}(x)] = (-1)^{k-1} (k-1)! \quad (2.4)$$

Lemma 2.6. *For $m, n, p, q \in \mathbb{N}$ such that $(p, q) \succ (m, n)$, the function*

$$\frac{s^{m-1}(1-s)^{n-1} + (1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1} + (1-s)^{p-1}s^{q-1}}$$

is increasing in $s \in (0, \frac{1}{2})$.

Proof. Direct computation yields

$$\begin{aligned} \frac{s^{m-1}(1-s)^{n-1} + (1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1} + (1-s)^{p-1}s^{q-1}} &= \frac{(\frac{1}{s}-1)^{n-1} + (\frac{1}{s}-1)^{m-1}}{(\frac{1}{s}-1)^{q-1} + (\frac{1}{s}-1)^{p-1}} \\ &= \frac{z^{n-1} + z^{m-1}}{z^{q-1} + z^{p-1}} = \frac{z^{n-q} + z^{m-q}}{1 + z^{p-q}} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dz} \left(\frac{z^b + z^c}{1 + z^a} \right) &= \frac{(bz^b + cz^c)(1 + z^a) - az^a(z^b + z^c)}{z(1 + z^a)^2} \\ &= \frac{bz^b(1 - z^{2c}) + cz^c(1 - z^{2b})}{z(1 + z^a)^2} \\ &< 0, \end{aligned}$$

where $z = \frac{1}{s} - 1 \in (1, \infty)$ and $a = p - q > b = m - q \geq c = n - q > 0$ with $a = b + c$. The proof of Lemma 2.6 is complete. \square

Lemma 2.7. *Let the functions $U(x)$, $V(x) > 0$, and $W(x, t) > 0$ be integrable in $x \in (a, b)$. If the ratios $\frac{\partial W(x, t)/\partial t}{W(x, t)}$ and $\frac{U(x)}{V(x)}$ are both increasing or both decreasing in $x \in (a, b)$, then the ratio*

$$R(t) = \frac{\int_a^b U(x)W(x, t)dx}{\int_a^b V(x)W(x, t)dx}$$

is increasing in t ; if one of the ratios $\frac{\partial W(x, t)/\partial t}{W(x, t)}$ and $\frac{U(x)}{V(x)}$ are increasing and the other is decreasing in $x \in (a, b)$, then the ratio $R(t)$ is decreasing in t .

Proof. Direct differentiation gives

$$R'(t) = \frac{\left[\int_a^b U(x) \frac{\partial W(x, t)}{\partial t} dx \int_a^b V(x)W(x, t) dx - \int_a^b U(x)W(x, t) dx \int_a^b V(x) \frac{\partial W(x, t)}{\partial t} dx \right]}{\left[\int_a^b V(x)W(x, t) dx \right]^2}$$

$$\begin{aligned}
&= \frac{\left[\int_a^b \int_a^b U(x) \frac{\partial W(x,t)}{\partial t} V(y) W(y,t) dx dy \right]}{\left[- \int_a^b \int_a^b U(x) W(x,t) V(y) \frac{\partial W(y,t)}{\partial t} dx dy \right]} \\
&= \frac{\int_a^b V(x) W(x,t) dx}{\left[\int_a^b V(x) W(x,t) dx \right]^2} \\
&= \frac{\int_a^b \int_a^b U(x) V(y) W(x,t) W(y,t) \left[\frac{\partial W(x,t)/\partial t}{W(x,t)} - \frac{\partial W(y,t)/\partial t}{W(y,t)} \right] dx dy}{\left[\int_a^b V(x) W(x,t) dx \right]^2} \\
&= \frac{\int_a^b \int_a^b \left(\left[\frac{U(x)}{V(x)} - \frac{U(y)}{V(y)} \right] \left[\frac{\partial W(x,t)/\partial t}{W(x,t)} - \frac{\partial W(y,t)/\partial t}{W(y,t)} \right] \right.}{2 \left[\int_a^b V(x) W(x,t) dx \right]^2} \\
&\quad \left. \times V(x) V(y) W(x,t) W(y,t) \right) dx dy}{2 \left[\int_a^b V(x) W(x,t) dx \right]^2}.
\end{aligned}$$

The proof of Lemma 2.7 is complete. \square

Lemma 2.8 ([22, p. 161, Theorem 12b]). *A function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\sigma(t), \quad x \in (0, \infty), \quad (2.5)$$

where $\sigma(s)$ is non-decreasing and the integral in (2.5) converges for $x \in (0, \infty)$.

3. MONOTONICITY OF A RATIO DEFINED BY THREE POLYGAMMA FUNCTIONS

In this section, we prove that the function $Q_{m,n}(x)$ defined in (1.3) is decreasing.

Theorem 3.1. *For $m, n \in \mathbb{N}$, the function $Q_{m,n}(x)$ defined in (1.3) is decreasing from $(0, \infty)$ onto $(-\frac{(m+n-1)!}{(m-1)!(n-1)!}, 0)$. Consequently, the double inequality (1.1), that is,*

$$-\frac{(m+n-1)!}{(m-1)!(n-1)!} < Q_{m,n} < 0, \quad m, n \in \mathbb{N},$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

Proof. By virtue of the integral representation (2.1), we can rearranged $Q_{m,n}(x)$ as

$$Q_{m,n}(x) = -\frac{\int_0^\infty t^{m+n-1} g(t) e^{-xt} dt}{\int_0^\infty t^{m-1} g(t) e^{-xt} dt \int_0^\infty t^{n-1} g(t) e^{-xt} dt}.$$

By Lemma 2.2, we obtain

$$Q_{m,n}(x) = -\frac{\int_0^\infty t^{m+n-1} g(t) e^{-xt} dt}{\int_0^\infty \left[\int_0^t u^{m-1} (t-u)^{n-1} g(u) g(t-u) du \right] e^{-xt} dt}.$$

By Lemma 2.3, we only need to prove the ratio

$$\begin{aligned}
P_{m,n}(t) &= \frac{t^{m+n-1} g(t)}{\int_0^t u^{m-1} (t-u)^{n-1} g(u) g(t-u) du} \\
&= \frac{t^{m+n-1} g(t)}{t^{m+n-1} \int_0^1 s^{m-1} (1-s)^{n-1} g(st) g((1-s)t) ds} \\
&= \frac{1}{\int_0^1 s^{m-1} (1-s)^{n-1} \frac{g(st)}{g^s(t)} \frac{g((1-s)t)}{g^{1-s}(t)} ds}
\end{aligned}$$

is decreasing on $(0, \infty)$. Hence, it suffices to show that the ratio $\frac{g(st)}{g^s(t)}$ for fixed $s \in (0, 1)$ is increasing in $t \in (0, \infty)$. This increasing monotonicity has been proved in Lemma 2.4 in this paper. As a result, the function $Q_{m,n}(x)$ defined in (1.3) is decreasing on $(0, \infty)$.

Making use of the limits (2.3) and (2.4) in Lemma 2.5 yields

$$\begin{aligned} \lim_{x \rightarrow 0^+} Q_{m,n}(x) &= \frac{(\lim_{x \rightarrow 0^+} x) \lim_{x \rightarrow 0^+} [x^{m+n+1} \psi^{(m+n)}(x)]}{\lim_{x \rightarrow 0^+} [x^{m+1} \psi^{(m)}(x)] \lim_{x \rightarrow 0^+} [x^{n+1} \psi^{(n)}(x)]} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow \infty} Q_{m,n}(x) &= \frac{\lim_{x \rightarrow \infty} [x^{m+n} \psi^{(m+n)}(x)]}{\lim_{x \rightarrow \infty} [x^m \psi^{(m)}(x)] \lim_{x \rightarrow \infty} [x^n \psi^{(n)}(x)]} \\ &= \frac{(-1)^{m+n-1} (m+n-1)!}{(-1)^{m-1} (m-1)! (-1)^{n-1} (n-1)!} \\ &= -\frac{(m+n-1)!}{(m-1)! (n-1)!}. \end{aligned}$$

The proof of Theorem 3.1 is complete. \square

4. MONOTONICITY OF A RATIO DEFINED BY FOUR POLYGAMMA FUNCTIONS

In this section, we prove that the function $Q_{m,n;p,q}$ defined in (1.4) is decreasing.

Theorem 4.1. *For $m, n, p, q \in \mathbb{N}$ with the majorizing relation $(p, q) \succ (m, n)$, the ratio $Q_{m,n;p,q}(x)$ defined in (1.4) is decreasing from $(0, \infty)$ onto $(\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!}, \frac{m!n!}{p!q!})$. Consequently, for $m, n, p, q \in \mathbb{N}$ with $(p, q) \succ (m, n)$, the double inequality (1.2), that is,*

$$\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!} < Q_{m,n;p,q}(x) < \frac{m!n!}{p!q!},$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars respectively.

Proof. By the limits (2.3) and (2.4) in Lemma 2.5, we obtain

$$\begin{aligned} \lim_{x \rightarrow 0^+} Q_{m,n;p,q}(x) &= \frac{\lim_{x \rightarrow 0^+} [x^{m+1} \psi^{(m)}(x)] \lim_{x \rightarrow 0^+} [x^{n+1} \psi^{(n)}(x)]}{\lim_{x \rightarrow 0^+} [x^{p+1} \psi^{(p)}(x)] \lim_{x \rightarrow 0^+} [x^{q+1} \psi^{(q)}(x)]} \\ &= \frac{(-1)^{m+1} m! (-1)^{n+1} n!}{(-1)^{p+1} p! (-1)^{q+1} q!} \\ &= \frac{m!n!}{p!q!} \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow \infty} Q_{m,n;p,q}(x) &= \frac{\lim_{x \rightarrow \infty} [x^m \psi^{(m)}(x)] \lim_{x \rightarrow \infty} [x^n \psi^{(n)}(x)]}{\lim_{x \rightarrow \infty} [x^p \psi^{(p)}(x)] \lim_{x \rightarrow \infty} [x^q \psi^{(q)}(x)]} \\ &= \frac{(-1)^{m-1} (m-1)! (-1)^{n-1} (n-1)!}{(-1)^{p-1} (p-1)! (-1)^{q-1} (q-1)!} \\ &= \frac{(q-1)!(n-1)!}{(p-1)!(q-1)!}. \end{aligned}$$

Making use of the integral representation (2.1) yields

$$\mathcal{Q}_{m,n;p,q}(x) = \frac{\int_0^\infty t^{m-1}g(t)e^{-xt}dt \int_0^\infty t^{n-1}g(t)e^{-xt}dt}{\int_0^\infty t^{p-1}g(t)e^{-xt}dt \int_0^\infty t^{q-1}g(t)e^{-xt}dt}.$$

Utilizing Lemma 2.2 gives

$$\mathcal{Q}_{m,n;p,q}(x) = \frac{\int_0^\infty \left[\int_0^t u^{m-1}(t-u)^{n-1}g(u)g(t-u)du \right] e^{-xt}dt}{\int_0^\infty \left[\int_0^t u^{p-1}(t-u)^{q-1}g(u)g(t-u)du \right] e^{-xt}dt}.$$

Employing Lemma 2.3 tells us that, it suffices to prove the increasing monotonicity in t of the ratio

$$\begin{aligned} \frac{\int_0^t u^{m-1}(t-u)^{n-1}g(u)g(t-u)du}{\int_0^t u^{p-1}(t-u)^{q-1}g(u)g(t-u)du} &= \frac{\int_0^1 s^{m-1}(1-s)^{n-1}g(st)g((1-s)t)ds}{\int_0^1 s^{p-1}(1-s)^{q-1}g(st)g((1-s)t)ds} \\ &= \frac{\int_0^{1/2} [s^{m-1}(1-s)^{n-1} + (1-s)^{m-1}s^{n-1}]g(st)g((1-s)t)ds}{\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + (1-s)^{p-1}s^{q-1}]g(st)g((1-s)t)ds} \\ &= \frac{\int_0^{1/2} \phi_{m,n}(s)\varphi(s,t)ds}{\int_0^{1/2} \phi_{p,q}(s)\varphi(s,t)ds}, \end{aligned}$$

where

$$\phi_{i,j}(s) = s^{i-1}(1-s)^{j-1} + (1-s)^{i-1}s^{j-1} \quad \text{and} \quad \varphi(s,t) = g(st)g((1-s)t). \quad (4.1)$$

Lemma 2.6 implies that the ratio $\frac{\phi_{m,n}(s)}{\phi_{p,q}(s)}$ is increasing in $s \in (0, \frac{1}{2})$ for $(p, q) \succ (m, n)$. Further making use of the inequality (2.2) in Lemma 2.4 and utilizing Lemma 2.7 reveal that the function $\mathcal{Q}_{m,n;p,q}(x)$ is decreasing in $x \in (0, \infty)$. The proof of Theorem 4.1 is complete. \square

5. AN ALTERNATIVE PROOF OF THEOREM 1.1

In this section, we supply an alternative proof of Theorem 1.1.

An alternative proof of Theorem 1.1. For $q = 0$, we have

$$\begin{aligned} F_{p,m,n,0;c_{p,m,n,0}}(x) &= |\psi^{(m)}(x)| |\psi^{(n)}(x)| - c_{p,m,n,0} |\psi^{(p)}(x)| \\ &= \int_0^\infty \frac{t^m}{1-e^{-t}} e^{-xt} dt \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-xt} dt - c_{p,m,n,0} \int_0^\infty \frac{t^p}{1-e^{-t}} e^{-xt} dt \\ &= \int_0^\infty \left[\int_0^t u^{m-1}(t-u)^{n-1}g(u)g(t-u)du - c_{p,m,n,0} t^{m+n-1}g(t) \right] e^{-xt} dt \\ &= \int_0^\infty \left[\int_0^1 s^{m-1}(1-s)^{n-1} \frac{g(st)g((1-s)t)}{g(t)} ds - c_{p,m,n,0} \right] t^{m+n-1}g(t)e^{-xt} dt, \end{aligned}$$

where we used the integral representation (2.1) and Lemma 2.2. From the second property in Lemma 2.4, it follows that the function

$$\int_0^1 s^{m-1}(1-s)^{n-1} \frac{g(st)g((1-s)t)}{g(t)} ds$$

is decreasing in $t \in (0, \infty)$ and has the limits

$$\int_0^1 s^{m-1}(1-s)^{n-1} \frac{g(st)g((1-s)t)}{g(t)} ds \rightarrow \begin{cases} \int_0^1 s^{m-1}(1-s)^{n-1} ds, & t \rightarrow 0 \\ \infty, & t \rightarrow \infty. \end{cases}$$

Consequently, basing on Lemma 2.8, we see that, if and only if

$$c_{p,m,n,0} \leq \int_0^1 s^{m-1}(1-s)^{n-1} ds = B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

the function $F_{p,m,n,0;c_{p,m,n,0}}(x)$ is completely monotonic on $(0, \infty)$.

For $q \geq 1$, we have

$$\begin{aligned} F_{p,m,n,q;c_{p,m,n,q}}(x) &= |\psi^{(m)}(x)| |\psi^{(n)}(x)| - c_{p,m,n,q} |\psi^{(p)}(x)| |\psi^{(q)}(x)| \\ &= \int_0^\infty t^{m-1} g(t) e^{-xt} dt \int_0^\infty t^{n-1} g(t) e^{-xt} dt \\ &\quad - c_{p,m,n,q} \int_0^\infty t^{p-1} g(t) e^{-xt} dt \int_0^\infty t^{q-1} g(t) e^{-xt} dt \\ &= \int_0^\infty \left[\int_0^t u^{m-1} (t-u)^{n-1} g(u) g(t-u) du \right] e^{-xt} dt \\ &\quad - c_{p,m,n,q} \int_0^\infty \left[\int_0^t u^{p-1} (t-u)^{q-1} g(u) g(t-u) du \right] e^{-xt} dt \\ &= \int_0^\infty \left[t^{m+n-1} \int_0^1 s^{m-1} (1-s)^{n-1} g(st) g((1-s)t) ds \right. \\ &\quad \left. - c_{p,m,n,q} t^{p+q-1} \int_0^1 s^{p-1} (1-s)^{q-1} g(st) g((1-s)t) ds \right] e^{-xt} dt \\ &= \int_0^\infty \left[\frac{\int_0^1 s^{m-1} (1-s)^{n-1} g(st) g((1-s)t) ds}{\int_0^1 s^{p-1} (1-s)^{q-1} g(st) g((1-s)t) ds} - c_{p,m,n,q} \right] \\ &\quad \times \left[t^{m+n-1} \int_0^1 s^{p-1} (1-s)^{q-1} g(st) g((1-s)t) ds \right] e^{-xt} dt \\ &= \int_0^\infty \left[\frac{\int_0^{1/2} [s^{m-1} (1-s)^{n-1} + s^{n-1} (1-s)^{m-1}] g(st) g((1-s)t) ds}{\int_0^{1/2} [s^{p-1} (1-s)^{q-1} + s^{q-1} (1-s)^{p-1}] g(st) g((1-s)t) ds} \right. \\ &\quad \left. - c_{p,m,n,q} \right] \left[t^{m+n-1} \int_0^1 s^{p-1} (1-s)^{q-1} g(st) g((1-s)t) ds \right] e^{-xt} dt, \end{aligned}$$

where we used the integral representation (2.1) and Lemma 2.2. Employing the inequality (2.2) in Lemma 2.4 and applying Lemmas 2.6 and 2.7 reveal that the function

$$\frac{\int_0^{1/2} [s^{m-1} (1-s)^{n-1} + s^{n-1} (1-s)^{m-1}] g(st) g((1-s)t) ds}{\int_0^{1/2} [s^{p-1} (1-s)^{q-1} + s^{q-1} (1-s)^{p-1}] g(st) g((1-s)t) ds} = \frac{\int_0^{1/2} \phi_{m,n}(s) \varphi(s, t) ds}{\int_0^{1/2} \phi_{p,q}(s) \varphi(s, t) ds}$$

is increasing in $t \in (0, \infty)$, where $\phi_{i,j}(s)$ and $\varphi(s, t)$ are defined in (4.1). It is easy to see that

$$\lim_{t \rightarrow 0} \frac{\int_0^{1/2} \phi_{m,n}(s) \varphi(s, t) ds}{\int_0^{1/2} \phi_{p,q}(s) \varphi(s, t) ds} = \frac{\int_0^{1/2} [s^{m-1} (1-s)^{n-1} + s^{n-1} (1-s)^{m-1}] ds}{\int_0^{1/2} [s^{p-1} (1-s)^{q-1} + s^{q-1} (1-s)^{p-1}] ds}$$

$$= \frac{\int_0^1 s^{m-1}(1-s)^{n-1} ds}{\int_0^1 s^{p-1}(1-s)^{q-1} ds} = \frac{B(m, n)}{B(p, q)} = \frac{(m-1)!(n-1)!}{(p-1)!(q-1)!}.$$

Since $\lim_{t \rightarrow \infty} \frac{g(t)}{t} = 1$, we acquire

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\int_0^{1/2} \phi_{m,n}(s) \varphi(s, t) ds}{\int_0^{1/2} \phi_{p,q}(s) \varphi(s, t) ds} &= \frac{\int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}] s(1-s) ds}{\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}] s(1-s) ds} \\ &= \frac{\int_0^1 s^m(1-s)^n ds}{\int_0^1 s^p(1-s)^q ds} = \frac{B(m+1, n+1)}{B(p+1, q+1)} = \frac{m!n!}{p!q!}. \end{aligned}$$

Combining these with Lemma 2.8 concludes that,

(1) if and only if

$$c_{p,m,n,q} \leq \frac{(m-1)!(n-1)!}{(p-1)!(q-1)!},$$

the function $F_{p,m,n,q;c_{p,m,n,q}}(x)$ is completely monotonic in $x \in (0, \infty)$;

(2) if and only if $c_{p,m,n,q} \geq \frac{m!n!}{p!q!}$, the function $-F_{p,m,n,q;c_{p,m,n,q}}(x)$ is completely monotonic in $x \in (0, \infty)$.

The proof of Theorem 1.1 is complete. \square

6. REMARKS

Finally, we list several remarks on our main results and their proofs.

Remark 6.1. The papers [3, 4, 6, 7, 23] are related to Theorem 1.1. Theorems 3.1 and 4.1 in this paper are related to results reviewed and surveyed in [10, 17] and closely related references therein.

Remark 6.2. Lemma 2.4 in this paper generalizes a conclusion in [12, Lemma 2.3]: the function $\frac{g(2t)}{g^2(t)}$ is decreasing from $(0, \infty)$ onto $(0, 1)$.

Remark 6.3. Taking $W(x, t) = e^{-xt}$ in Lemma 2.7 gives

$$\frac{\partial W(x, t)/\partial t}{W(x, t)} = \frac{\partial e^{-xt}/\partial t}{e^{-xt}} = -x$$

which is decreasing in $x \in (-\infty, \infty)$. Further setting $U(x) = A(x)$, $V(x) = B(x)$, and $(a, b) = (0, \infty)$ in Lemma 2.7 leads to Lemma 2.3 which was established in [24, Lemma 4]. This means that Lemma 2.7 in this paper is a generalization of [24, Lemma 4].

Remark 6.4. Since the majorizing relation $(n+2, n) \succ (n+1, n+1)$, we see that Theorem 4.1 in this paper generalizes a conclusion in [23, Theorem 2]: the function $\frac{[\psi^{(n+1)}(x)]^2}{\psi^{(n)}(x)\psi^{(n+2)}(x)}$ for $n \geq 1$ is decreasing from $(0, \infty)$ onto $(\frac{n}{n+1}, \frac{n+1}{n+2})$.

Remark 6.5. Direct differentiation gives

$$Q'_{m,n}(x) = \frac{\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x) - \psi^{(m+n)}(x)[\psi^{(m)}(x)\psi^{(n)}(x)]'}{[\psi^{(m)}(x)\psi^{(n)}(x)]^2}.$$

The decreasing monotonicity of $Q_{m,n}(x)$ in Theorem 3.1 implies that the inequality

$$\psi^{(m+n)}(x)[\psi^{(m)}(x)\psi^{(n)}(x)]' - \psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x) > 0,$$

equivalently,

$$\frac{[\psi^{(m)}(x)\psi^{(n)}(x)]'}{\psi^{(m)}(x)\psi^{(n)}(x)} > \frac{\psi^{(m+n+1)}(x)}{\psi^{(m+n)}(x)},$$

is valid on $(0, \infty)$ for $m, n \in \mathbb{N}$.

We guess that, for $m, n \in \mathbb{N}$, the function

$$\psi^{(m+n)}(x)[\psi^{(m)}(x)\psi^{(n)}(x)]' - \psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)$$

should be completely monotonic in $x \in (0, \infty)$.

Generally, one can discuss necessary and sufficient conditions on $\Omega_{m,n} \in \mathbb{R}$ such that the function

$$\psi^{(m+n)}(x)[\psi^{(m)}(x)\psi^{(n)}(x)]' - \Omega_{m,n}\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)$$

and its negativity are respectively completely monotonic on $(0, \infty)$.

Remark 6.6. It is immediate that

$$\mathcal{Q}'_{m,n;p,q}(x) = \frac{\begin{pmatrix} [\psi^{(m)}(x)\psi^{(n)}(x)]'[\psi^{(p)}(x)\psi^{(q)}(x)] \\ -[\psi^{(m)}(x)\psi^{(n)}(x)][\psi^{(p)}(x)\psi^{(q)}(x)]' \end{pmatrix}}{[\psi^{(p)}(x)\psi^{(q)}(x)]^2}.$$

The monotonicity of $\mathcal{Q}_{m,n;p,q}(x)$ in Theorem 4.1 implies that the inequality

$$[\psi^{(m)}(x)\psi^{(n)}(x)][\psi^{(p)}(x)\psi^{(q)}(x)]' - [\psi^{(m)}(x)\psi^{(n)}(x)]'[\psi^{(p)}(x)\psi^{(q)}(x)] > 0,$$

equivalently,

$$\frac{[\psi^{(p)}(x)\psi^{(q)}(x)]'}{\psi^{(p)}(x)\psi^{(q)}(x)} > \frac{[\psi^{(m)}(x)\psi^{(n)}(x)]'}{\psi^{(m)}(x)\psi^{(n)}(x)},$$

is valid on $(0, \infty)$ for $(p, q) \succ (m, n)$.

We guess that, for $(p, q) \succ (m, n)$, the function

$$[\psi^{(m)}(x)\psi^{(n)}(x)][\psi^{(p)}(x)\psi^{(q)}(x)]' - [\psi^{(m)}(x)\psi^{(n)}(x)]'[\psi^{(p)}(x)\psi^{(q)}(x)]$$

should be completely monotonic in $x \in (0, \infty)$.

Generally, for $(p, q) \succ (m, n)$, one can discuss necessary and sufficient conditions on $\Omega_{m,n;p,q} \in \mathbb{R}$ such that the function

$$[\psi^{(m)}(x)\psi^{(n)}(x)][\psi^{(p)}(x)\psi^{(q)}(x)]' - \Omega_{m,n;p,q}[\psi^{(m)}(x)\psi^{(n)}(x)]'[\psi^{(p)}(x)\psi^{(q)}(x)]$$

and its negativity are respectively completely monotonic on $(0, \infty)$.

Remark 6.7. For $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $n \geq 2$, and two nonnegative integer tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}^n$, let

$$P_{\alpha,\beta;C_{\alpha,\beta}}(x) = \prod_{r=1}^n \psi^{(\alpha_r)}(x) - C_{\alpha,\beta} \prod_{r=1}^n \psi^{(\beta_r)}(x)$$

and

$$Q_{\alpha,\beta}(x) = \frac{\prod_{r=1}^n \psi^{(\alpha_r)}(x)}{\prod_{r=1}^n \psi^{(\beta_r)}(x)}$$

on $(0, \infty)$, where we denote $\psi^{(0)}(x) = -1$ for convenience. It is clear that

$$\begin{aligned} P_{(2k,0),(k,k);C_{(2k,0),(k,k)}}(x) &= \mathcal{F}_{k,-C_{(2k,0),(k,k)}}(x), & Q_{(2k,0),(k,k)}(x) &= \mathfrak{F}_{k,2}(x), \\ Q_{(m+n,0),(m,n)}(x) &= Q_{m,n}(x), & Q_{(m,n),(p,q)}(x) &= Q_{m,n;p,q}(x). \end{aligned}$$

We guess that, if $\alpha \succ \beta$, the function $Q_{\alpha,\beta}(x)$ is increasing from $(0, \infty)$ onto the interval

$$\left(\prod_{r=1}^n \frac{\alpha_r!}{\beta_r!}, \prod_{r=1}^n \frac{(\alpha_r - 1)!}{(\beta_r - 1)!} \right).$$

Generally, for $\alpha \succ \beta$, one can discuss necessary and sufficient conditions on $C_{\alpha,\beta} \in \mathbb{R}$ such that the function $P_{\alpha,\beta;C_{\alpha,\beta}}(x)$ and its negativity are respectively completely monotonic on $(0, \infty)$.

Remark 6.8. This paper is the eighth one in a series of articles including [11, 12, 13, 14, 15, 16, 18].

REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
- [2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.
- [3] H. Alzer and J. Wells, *Inequalities for the polygamma functions*, SIAM J. Math. Anal. **29** (1998), no. 6, 1459–1466; available online at <https://doi.org/10.1137/S0036141097325071>.
- [4] N. Batir, *On some properties of digamma and polygamma functions*, J. Math. Anal. Appl. **328** (2007), no. 1, 452–465; available online at <https://doi.org/10.1016/j.jmaa.2006.05.065>.
- [5] Y.-M. Chu and X.-H. Zhang, *Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave*, J. Math. Kyoto Univ. **48** (2008), no. 1, 229–238; available online at <https://doi.org/10.1215/kjm/1250280982>.
- [6] P. Gao, *Some completely monotonic functions involving the polygamma functions*, J. Inequal. Appl. **2019**, Paper No. 218, 9 pages; available online at <https://doi.org/10.1186/s13660-019-2172-x>.
- [7] P. Gao, *Some monotonicity properties of gamma and q-gamma functions*, ISRN Math. Anal. **2011**, Art. ID 375715, 15 pages; available online at <https://doi.org/10.5402/2011/375715>.
- [8] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and its Applications*, 2nd Ed., Springer Verlag, New York/Dordrecht/Heidelberg/London, 2011; available online at <http://dx.doi.org/10.1007/978-0-387-68276-1>.
- [9] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; available online at <https://doi.org/10.1007/978-94-017-1043-5>.
- [10] F. Qi, *Completely monotonic degree of a function involving trigamma and tetragamma functions*, AIMS Math. **5** (2020), no. 4, 3391–3407; available online at <https://doi.org/10.3934/math.2020219>.
- [11] F. Qi, *Complete monotonicity and monotonicity of two functions defined by two derivatives of a function involving trigamma function*, HAL preprint (2020), available online at <https://hal.archives-ouvertes.fr/hal-02993723>.
- [12] F. Qi, *Lower bound of sectional curvature of manifold of beta distributions and complete monotonicity of functions involving polygamma functions*, MDPI Preprints **2020**, 2020110315, 22 pages; available online at <https://doi.org/10.20944/preprints202011.0315.v1>.
- [13] F. Qi, *Monotonicity and complete monotonicity of two functions constituted via three derivatives of a function involving trigamma function*, OSF Preprints (2020), available online at <https://doi.org/10.31219/osf.io/whb2q>.
- [14] F. Qi, *Monotonicity of a ratio involving trigamma and tetragamma functions*, OSF Preprints (2020), available online at <https://doi.org/10.31219/osf.io/5rfb8>.
- [15] F. Qi, *Necessary and sufficient conditions for two functions defined by two derivatives of a function involving trigamma function to be completely monotonic or monotonic*, OSF Preprints (2020), available online at <https://doi.org/10.31219/osf.io/6ar4p>.

- [16] F. Qi, *Some properties of several functions involving polygamma functions and originating from the sectional curvature of the beta manifold*, São Paulo J. Math. Sci. **15** (2021), in press; available online at <https://doi.org/10.1007/s40863-020-00193-1>.
- [17] F. Qi and R. P. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. **2019**, Paper No. 36, 42 pages; available online at <https://doi.org/10.1186/s13660-019-1976-z>.
- [18] F. Qi, L.-X. Han, and H.-P. Yin, *Monotonicity and complete monotonicity of two functions defined by three derivatives of a function involving trigamma function*, HAL preprint (2020), available online at <https://hal.archives-ouvertes.fr/hal-02998203>.
- [19] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *Schur-convexity of the Catalan–Qi function related to the Catalan numbers*, Tbilisi Math. J. **9** (2016), no. 2, 141–150; available online at <http://dx.doi.org/10.1515/tmj-2016-0026>.
- [20] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at <https://doi.org/10.1515/9783110269338>.
- [21] H.-N. Shi, *Two Schur-convex functions related to Hadamard-type integral inequalities*, Publ. Math. Debrecen **78** (2011), no. 2, 393–403; available online at <https://doi.org/10.5486/PMD.2011.4777>.
- [22] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
- [23] Z.-H. Yang, *Some properties of the divided difference of psi and polygamma functions*, J. Math. Anal. Appl. **455** (2017), no. 1, 761–777; available online at <https://doi.org/10.1016/j.jmaa.2017.05.081>.
- [24] Z.-H. Yang and J.-F. Tian, *Monotonicity and inequalities for the gamma function*, J. Inequal. Appl. **2017**, Paper No. 317, 15 pages; available online at <https://doi.org/10.1186/s13660-017-1591-9>.
- [25] H.-P. Yin, X.-M. Liu, J.-Y. Wang, and B.-N. Guo, *Necessary and sufficient conditions on the Schur convexity of a bivariate mean*, AIMS Math. **6** (2021), no. 1, 296–303; available online at <https://doi.org/10.3934/math.2021018>.

INSTITUTE OF MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO 454010, HENAN, CHINA; COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO 028043, INNER MONGOLIA, CHINA; SCHOOL OF MATHEMATICAL SCIENCES, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300387, CHINA

Email address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: <https://qifeng618.wordpress.com>, <https://orcid.org/0000-0001-6239-2968>