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BY THREE DERIVATIVES OF A FUNCTION
INVOLVING TRIGAMMA FUNCTION**

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MONOTONICITY AND COMPLETE MONOTONICITY OF TWO FUNCTIONS DEFINED BY THREE DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION

FENG QI, LING-XIONG HAN, AND HONG-PING YIN

Dedicated to people facing and battling COVID-19

ABSTRACT. In the paper, by convolution theorem of the Laplace transforms, a monotonicity rule for the ratio of two Laplace transforms, Bernstein's theorem for completely monotonic functions, and other analytic techniques, the authors

- (1) verify decreasing monotonicity of a ratio between three derivatives of a function involving trigamma function;
- (2) find necessary and sufficient conditions for a function defined by three derivatives of a function involving trigamma function to be completely monotonic.

These results confirm previous guesses posed by the first author and generalize corresponding known conclusions.

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1. INTRODUCTION

In the literature [1, Section 6.4], the function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

and its logarithmic derivative $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ are called Euler's gamma function and digamma function respectively. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as the trigamma, tetragamma, pentagamma, and

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hexagamma functions respectively. All the derivatives $\psi^{(k)}(z)$ for $k \geq 0$ are known as polygamma functions.

Recall from Chapter XIII in [4], Chapter 1 in [11], and Chapter IV in [12] that, if a function $f(t)$ on an interval I has derivatives of all orders on I and satisfies inequalities $(-1)^n f^{(n)}(t) \geq 0$ for $t \in I$ and $n \in \{0\} \cup \mathbb{N}$, then we call $f(t)$ a completely monotonic function on I .

Let $\Phi(x) = x\psi'(x) - 1 = x[\psi'(x) - \frac{1}{x}]$ on $(0, \infty)$. In [7, Theorem 4.1] and [9, Theorem 4], the first author turned out the following necessary and sufficient conditions and double inequality:

- (1) if and only if $\alpha \geq 2$, the function $\mathfrak{H}_\alpha(x) = \Phi'(x) + \alpha\Phi^2(x)$ is completely monotonic on $(0, \infty)$;
- (2) if and only if $\alpha \leq 1$, the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0, \infty)$;
- (3) the double inequality $-2 < \frac{\Phi'(x)}{\Phi^2(x)} < -1$ is valid and sharp in the sense that the lower and upper bounds -2 and -1 cannot be replaced by any bigger and smaller ones respectively.

In [7, Theorem 1.1], the first author found the following necessary and sufficient conditions and limits:

- (1) if and only if $\beta \geq 2$, the function $H_\beta(x) = \frac{\Phi'(x)}{\Phi^\beta(x)}$ is decreasing on $(0, \infty)$, with the limits

$$\lim_{x \rightarrow 0^+} H_\beta(x) = \begin{cases} -1, & \beta = 2 \\ 0, & \beta > 2 \end{cases} \quad \text{and} \quad \lim_{x \rightarrow \infty} H_\beta(x) = \begin{cases} -2, & \beta = 2 \\ -\infty, & \beta > 2; \end{cases}$$

- (2) if $\beta \leq 1$, the function $H_\beta(x)$ is increasing on $(0, \infty)$, with the limits

$$H_\beta(x) \rightarrow \begin{cases} -\infty, & x \rightarrow 0^+ \\ 0, & x \rightarrow \infty. \end{cases}$$

For $k \in \{0\} \cup \mathbb{N}$ and $\lambda_k, \mu_k \in \mathbb{R}$, let

$$\mathfrak{J}_{k, \lambda_k}(x) = \Phi^{(2k+1)}(x) + \lambda_k [\Phi^{(k)}(x)]^2 \quad \text{and} \quad J_{k, \mu_k}(x) = \frac{\Phi^{(2k+1)}(x)}{[(-1)^k \Phi^{(k)}(x)]^{\mu_k}}$$

on $(0, \infty)$. In [5, Theorems 3.1 and 4.1], the first author presented the following necessary and sufficient conditions, limits, and double inequality:

- (1) if and only if $\lambda_k \geq \frac{(2k+2)!}{k!(k+1)!}$, the function $\mathfrak{J}_{k, \lambda_k}(x)$ is completely monotonic on $(0, \infty)$;
- (2) if and only if $\lambda_k \leq \frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}$, the function $-\mathfrak{J}_{k, \lambda_k}(x)$ is completely monotonic on $(0, \infty)$;
- (3) if and only if $\mu_k \geq 2$, the function $J_{k, \mu_k}(x)$ is decreasing on $(0, \infty)$, with the limits

$$\lim_{x \rightarrow 0^+} J_{k, \mu_k}(x) = \begin{cases} -\frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2 \\ 0, & \mu_k > 2 \end{cases}$$

and

$$\lim_{x \rightarrow \infty} J_{k, \mu_k}(x) = \begin{cases} -\frac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2 \\ -\infty, & \mu_k > 2; \end{cases}$$

(4) if $\mu_k \leq 1$, the function $J_{k,\mu_k}(x)$ is increasing on $(0, \infty)$, with the limits

$$J_{k,\mu_k}(x) \rightarrow \begin{cases} -\infty, & x \rightarrow 0^+ \\ 0, & x \rightarrow \infty; \end{cases}$$

(5) the double inequality

$$-\frac{(2k+2)!}{k!(k+1)!} < \frac{\Phi^{(2k+1)}(x)}{[\Phi^{(k)}(x)]^2} < -\frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

For $k \geq m \geq 0$, let

$$\mathcal{J}_{k,m}(x) = \frac{\Phi^{(2k+2)}(x)}{\Phi^{(k-m)}(x)\Phi^{(k+m+1)}(x)}$$

on $(0, \infty)$. In [5, Remark 5.3], the first author guessed that the function $\mathcal{J}_{k,m}(x)$ for $k \geq m \geq 0$ should be decreasing on $(0, \infty)$ and that the double inequality

$$-\frac{2(2k+2)!}{k!(k+1)!} < \mathcal{J}_{k,0}(x) < -\frac{(2k+2)!}{k!(k+1)!} \quad (1.1)$$

for $k \geq 0$ should be valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

For $m, n \in \{0\} \cup \mathbb{N}$ and $\omega_{m,n} \in \mathbb{R}$, let

$$Y_{m,n}(x) = \frac{\Phi^{(m+n+1)}(x)}{\Phi^{(m)}(x)\Phi^{(n)}(x)} \quad (1.2)$$

and

$$\mathcal{Y}_{m,n;\omega_{m,n}}(x) = \Phi^{(m+n+1)}(x) + \omega_{m,n}\Phi^{(m)}(x)\Phi^{(n)}(x). \quad (1.3)$$

It is clear that

$$Y_{m,n}(x) = Y_{n,m}(x), \quad \mathcal{Y}_{m,n;\omega_{m,n}}(x) = \mathcal{Y}_{n,m;\omega_{n,m}}(x), \\ Y_{k-m,k+m+1}(x) = \mathcal{J}_{k,m}(x), \quad \mathcal{Y}_{k,k;\omega_{k,k}}(x) = \mathfrak{J}_{\omega_{k,k}}(x), \quad \mathcal{Y}_{0,0;\omega_{0,0}}(x) = \mathfrak{H}_{\omega_{0,0}}(x).$$

In this paper, we will prove decreasing monotonicity of the function $Y_{m,n}(x)$ and find necessary and sufficient conditions on $\omega_{m,n}$ for $\pm(-1)^{m+n+1}\mathcal{Y}_{m,n;\omega_{m,n}}(x)$ to be completely monotonic on $(0, \infty)$. These results confirm the above guesses and generalize corresponding ones in [5, 7, 9] mentioned above.

2. LEMMAS

The following lemmas are necessary in this paper.

Lemma 2.1 (Convolution theorem for the Laplace transforms [12, pp. 91–92]). *Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \geq 0$ such that $|f_k(t)| \leq M_k e^{c_k t}$ for $k = 1, 2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u)f_2(t-u)du \right] e^{-st} dt = \int_0^\infty f_1(u)e^{-su} du \int_0^\infty f_2(v)e^{-sv} dv.$$

Lemma 2.2 ([13, Lemma 4]). *Let the functions $A(t)$ and $B(t) \neq 0$ be defined on $(0, \infty)$ such that their Laplace transforms $\int_0^\infty A(t)e^{-xt} dt$ and $\int_0^\infty B(t)e^{-xt} dt$ exist. If the ratio $\frac{A(t)}{B(t)}$ is increasing, then the ratio $\frac{\int_0^\infty A(t)e^{-xt} dt}{\int_0^\infty B(t)e^{-xt} dt}$ is decreasing on $(0, \infty)$.*

Lemma 2.3. *Let $x, y \in \mathbb{R}$ such that $0 < 2x < y$.*

(1) *When $y > 2x > 2(2 + \frac{1}{\ln 2}) = 6.885390\dots$, the function*

$$F(x, y) = 2\left(\frac{1}{x} - \frac{1}{y-x}\right) + \frac{1}{2}\left(\frac{2^{y-x}}{y-x} - \frac{2^x}{x}\right) - \frac{2^{y-x} - 2^x}{(y-x)x}$$

is positive.

(2) *For $k, m \in \mathbb{N}$ such that $6 \leq 2m < k$, the sequence $F(m, k)$ is positive.*

Proof. The function $F(x, y)$ can be rearranged as

$$F(x, y) = \frac{2(y-2x) + 2^{x-1}[2-y+x+(x-2)2^{y-2x}]}{x(y-x)}.$$

Therefore, it suffices to prove $2-y+x+(x-2)2^{y-2x} > 0$, that is,

$$2^{y-2x} > \frac{y-x-2}{x-2}. \quad (2.1)$$

Replacing $y-2x$ by t in (2.1) leads to

$$2^t > \frac{t+x-2}{x-2} = 1 + \frac{t}{x-2} \quad (2.2)$$

for $t > 0$ and $x > 2$. The inequality (2.2) can be reformulated as $x > 2 + \frac{t}{2^t-1}$. Since the function $\frac{t}{2^t-1}$ is decreasing from $(0, \infty)$ onto $(0, \frac{1}{\ln 2})$, it is sufficient for $x > 2 + \frac{1}{\ln 2} = 3.442695\dots$

Repeating those arguments before the inequality (2.1) hints us that, for proving $F(m, k) > 0$, it is sufficient to show

$$2^{k-2m} > \frac{k-m-2}{m-2} = 1 + \frac{k-2m}{m-2}$$

which can be rewritten as

$$\frac{k-2m}{2^{k-2m}-1} < m-2 \quad (2.3)$$

Since $\frac{t}{2^t-1}$ is decreasing in $t \in (0, \infty)$ and $k-2m \geq 1$, the largest value of the left hand side in the inequality (2.3) is $\frac{1}{2^1-1} = 1$ which means that the strict inequality (2.3) is valid for all $m \geq 4$. As a result, the sequence $F(m, k)$ is positive for all $m \geq 4$.

When $m = 3$, the sequence $F(3, k)$ is

$$F(3, k) = \frac{2^k - 32k + 128}{48(k-3)} = \frac{2^5 [2^{k-5} - (k-4)]}{48(k-3)}$$

which is positive for all $k > 2 \cdot 3 = 6$. The proof of Lemma 2.3 is complete. \square

Lemma 2.4. *Let*

$$h(t) = \begin{cases} \frac{e^t(e^t - 1 - t)}{(e^t - 1)^2}, & t \neq 0 \\ \frac{1}{2}, & t = 0 \end{cases}$$

on $(-\infty, \infty)$. Then, for any fixed $s \in (0, 1)$, the ratio $\frac{h(st)}{h^s(t)}$ is increasing in t from $(0, \infty)$ onto $(\frac{1}{2^{1-s}}, 1)$.

Proof. It is easy to see that

$$\lim_{t \rightarrow 0} \frac{h(st)}{h^s(t)} = \frac{\lim_{t \rightarrow 0} h(st)}{\lim_{t \rightarrow 0} h^s(t)} = \frac{\frac{1}{2}}{\frac{1}{2^s}} = \frac{1}{2^{1-s}}$$

and

$$\lim_{t \rightarrow \infty} H_s(t) = \frac{\lim_{t \rightarrow \infty} h(st)}{\lim_{t \rightarrow 0} h^s(t)} = \frac{1}{1^s} = 1.$$

Direct differentiating and expanding to power series give

$$\begin{aligned} \frac{d}{dt} \left[\frac{h(st)}{h^s(t)} \right] &= - \frac{se^{(1+s)t} \left[\begin{array}{l} (t-2)e^{(1+2s)t} + (t+2)e^{2st} + (2-st)e^{(2+s)t} \\ +4(s-1)te^{(1+s)t} - (2st^2+3st+2)e^{st} \\ -(st+2)e^{2t} + (2st^2+3t+2)e^t + (s-1)t \end{array} \right]}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left(\begin{array}{l} (3k+2)s^k + 2(2s+1)^k + ks(s+2)^{k-1} \\ +2k^2s^{k-1} + 4k(s+1)^{k-1} \\ +2k(1+2^{k-2})s + 2^{k+1} \\ - [2^{k+1}s^k + 2(s+2)^k \\ +4ks(s+1)^{k-1} + 2k(1+2^{k-2})s^{k-1} \\ +k(2s+1)^{k-1} + 2k^2s + 3k + 2] \end{array} \right) \frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left(\begin{array}{l} \sum_{m=1}^{k-1} [k2^{k-m} \binom{k-1}{m-1} + 4k \binom{k-1}{m} + 2^{m+1} \binom{k}{m}] s^m \\ +2k(1-k+2^{k-2})(s-s^{k-1}) \\ - \sum_{m=1}^{k-1} [k2^m \binom{k-1}{m} + 4k \binom{k-1}{m-1} + 2^{k-m+1} \binom{k}{m}] s^m \end{array} \right) \frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left(\begin{array}{l} \sum_{m=1}^{k-1} [k2^{k-m} \binom{k-1}{m-1} + 4k \binom{k-1}{m} + 2^{m+1} \binom{k}{m}] (s^m - s^{k-m}) \\ +2k(1-k+2^{k-2})(s-s^{k-1}) \end{array} \right) \frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left(\begin{array}{l} \sum_{3 \leq m < \frac{k}{2}} [k2^{k-m} \binom{k-1}{m-1} - k2^m \binom{k-1}{k-m-1}] \\ +4k \binom{k-1}{m} - 4k \binom{k-1}{k-m} \\ +2^{m+1} \binom{k}{m} - 2^{k-m+1} \binom{k}{k-m} \end{array} \right) \frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left(\begin{array}{l} 2 \sum_{3 \leq m < \frac{k}{2}} \frac{k!}{(m-1)!(k-m-1)!} \left[2 \left(\frac{1}{m} - \frac{1}{k-m} \right) \right] \\ + \frac{1}{2} \left(\frac{2^{k-m}}{k-m} - \frac{2^m}{m} - \frac{2^{k-m-2m}}{(k-m)m} \right) \end{array} \right) \frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)} \\ &= \frac{s^4 e^{(1+s)t} \left(\begin{array}{l} \frac{1-s}{36} t^7 + \frac{1-s^2}{45} t^8 + \frac{22(1-s^3)+15(s-s^2)}{2160} t^9 \\ + \frac{52(1-s^4)+63(s-s^3)}{15120} t^{10} \\ + \frac{285(1-s^5)+470(s-s^4)+238(s^2-s^3)}{302400} t^{11} + \dots \end{array} \right)}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}. \end{aligned}$$

Utilizing Lemma 2.3 reveals that the derivative $\frac{d}{dt} \left[\frac{h(st)}{h^s(t)} \right]$ is positive for $s \in (0, 1)$ and $t > 0$. Consequently, for $s \in (0, 1)$, the ratio $\frac{h(st)}{h^s(t)}$ is increasing in $t > 0$. The proof of Lemma 2.4 is complete. \square

Lemma 2.5 ([5, Lemma 2.2]). *For $k \geq 0$, the function $(-1)^k \Phi^{(k)}(x)$ is completely monotonic on $(0, \infty)$, with the limits*

$$(-1)^k x^{k+1} \Phi^{(k)}(x) \rightarrow \begin{cases} k!, & x \rightarrow 0^+; \\ \frac{k!}{2}, & x \rightarrow \infty. \end{cases} \quad (2.4)$$

Lemma 2.6 (Bernstein's theorem [12, p. 161, Theorem 12b]). *A function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\sigma(t), \quad x \in (0, \infty), \quad (2.5)$$

where $\sigma(s)$ is non-decreasing and the integral in (2.5) converges for $x \in (0, \infty)$.

3. DECREASING MONOTONICITY

In this section, we prove that the function $Y_{m,n}(x)$ defined in (1.2) is decreasing.

Theorem 3.1. *For $m, n \in \{0\} \cup \mathbb{N}$, the function $Y_{m,n}(x)$ defined in (1.2) is decreasing in x from $(0, \infty)$ onto the interval $\left(-\frac{2(m+n+1)!}{m!n!}, -\frac{(m+n+1)!}{m!n!}\right)$. Consequently, for $m, n \in \{0\} \cup \mathbb{N}$, the double inequality*

$$-\frac{2(m+n+1)!}{m!n!} < Y_{m,n}(x) < -\frac{(m+n+1)!}{m!n!} \quad (3.1)$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

Proof. In the proof of [9, Theorem 4], the first author established that

$$\Phi(x) = \int_0^\infty h(t) e^{-xt} dt. \quad (3.2)$$

Then the ratio $Y_{m,n}(x)$ can be rewritten as

$$\begin{aligned} Y_{m,n}(x) &= -\frac{\int_0^\infty t^{m+n+1} h(t) e^{-xt} dt}{\int_0^\infty t^m h(t) e^{-xt} dt \int_0^\infty t^n h(t) e^{-xt} dt} \\ &= -\frac{\int_0^\infty t^{m+n+1} h(t) e^{-xt} dt}{\int_0^\infty \left[\int_0^t u^m (t-u)^n h(u) h(t-u) du \right] e^{-xt} dt}, \end{aligned}$$

where we used Lemma 2.1. Basing on Lemma 2.2, in order to prove decreasing monotonicity of $Y_{m,n}(x)$, it suffices to show that the ratio

$$\mathfrak{Y}_{m,n}(t) = \frac{t^{m+n+1} h(t)}{\int_0^t u^m (t-u)^n h(u) h(t-u) du} \quad (3.3)$$

is decreasing in $t \in (0, \infty)$. By changing the variable $u = \frac{1+v}{2}t$, the denominator of $\mathfrak{Y}_{m,n}(t)$ becomes

$$\left(\frac{t}{2}\right)^{m+n+1} \int_{-1}^1 (1+v)^m (1-v)^n h\left(\frac{1+v}{2}t\right) h\left(\frac{1-v}{2}t\right) dv.$$

Accordingly, we obtain

$$\begin{aligned}
\frac{1}{\mathfrak{Y}_{m,n}(t)} &= \frac{\int_{-1}^1 (1+v)^m (1-v)^n h\left(\frac{1+v}{2}t\right) h\left(\frac{1-v}{2}t\right) dv}{2^{m+n+1} h(t)} \\
&= \frac{1}{2^{m+n+1}} \int_{-1}^1 (1+v)^m (1-v)^n \frac{h\left(\frac{1+v}{2}t\right) h\left(\frac{1-v}{2}t\right)}{h(t)} dv \quad (3.4) \\
&= \frac{1}{2^{m+n+1}} \int_{-1}^1 (1+v)^m (1-v)^n \frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)} dv,
\end{aligned}$$

where $s = \frac{1+v}{2} \in (0, 1)$. From Lemma 2.4, we find that the function $\frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)}$ is increasing in $t \in (0, \infty)$ for any fixed $s \in (0, 1)$. Hence, the function $\mathfrak{Y}_{m,n}(t)$ is decreasing on $(0, \infty)$. Therefore, the function $Y_{m,n}(x)$ for $m, n \in \{0\} \cup \mathbb{N}$ is decreasing on $(0, \infty)$.

Making use of the limits in (2.4) in Lemma 2.5 yields

$$\begin{aligned}
Y_{m,n}(x) &= -\frac{(-1)^{m+n+1} x^{m+n+2} \Phi^{(m+n+1)}(x)}{[(-1)^m x^{m+1} \Phi^{(m)}(x)][(-1)^n x^{k+m+2} \Phi^{(n)}(x)]} \\
&\rightarrow \begin{cases} -\frac{(m+n+1)!}{m!n!}, & x \rightarrow 0^+; \\ -\frac{2(m+n+1)!}{m!n!}, & x \rightarrow \infty. \end{cases}
\end{aligned}$$

The proof of Theorem 3.1 is complete. \square

4. NECESSARY AND SUFFICIENT CONDITIONS OF COMPLETE MONOTONICITY

In this section, we discover necessary and sufficient conditions on $\omega_{m,n}$ for the function $\pm(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ defined in (1.3) to be completely monotonic.

Theorem 4.1. *For $m, n \in \{0\} \cup \mathbb{N}$ and $\omega_{m,n} \in \mathbb{R}$,*

- (1) *if and only if $\omega_{m,n} \leq \frac{(m+n+1)!}{m!n!}$, the function $(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0, \infty)$;*
- (2) *if and only if $\omega_{m,n} \geq \frac{2(m+n+1)!}{m!n!}$, the function $(-1)^{m+n} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0, \infty)$;*
- (3) *the double inequality (3.1) is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.*

Proof. As did in the proof of Theorem 3.1, by virtue of the integral representation (3.2) and Lemma 2.1, we acquire

$$\begin{aligned}
(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x) &= \left[\int_0^\infty t^{m+n+1} h(t) e^{-xt} dt \right. \\
&\quad \left. - \omega_{m,n} \int_0^\infty t^m h(t) e^{-xt} dt \int_0^\infty t^n h(t) e^{-xt} dt \right] \\
&= \int_0^\infty \left[t^{m+n+1} h(t) - \omega_{m,n} \int_0^t u^m (t-u)^n h(u) h(t-u) du \right] e^{-xt} dt \\
&= \int_0^\infty \left[1 - \frac{\omega_{m,n}}{\mathfrak{Y}_{m,n}(t)} \right] t^{m+n+1} h(t) e^{-xt} dt,
\end{aligned}$$

where $\mathfrak{Y}_{m,n}(t)$ is defined by (3.3) and it has been proved in the proof of Theorem 3.1 to be decreasing on $(0, \infty)$. From Lemma 2.4, we conclude that the function $\frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)}$ is increasing in t from $(0, \infty)$ onto $(\frac{1}{2}, 1)$. Accordingly, by virtue of (3.4), we arrive at the sharp inequalities

$$\frac{1}{2^{m+n+2}} \int_{-1}^1 (1+v)^m (1-v)^n dv < \frac{1}{\mathfrak{Y}_{m,n}(t)} < \frac{1}{2^{m+n+1}} \int_{-1}^1 (1+v)^m (1-v)^n dv.$$

Since

$$\begin{aligned} \int_{-1}^1 (1+v)^m (1-v)^n dv &= \int_0^1 [(1+v)^m (1-v)^n + (1-v)^m (1+v)^n] dv \\ &= 2^{m+n+1} B(m+1, n+1) \\ &= 2^{m+n+1} \frac{m!n!}{(m+n+1)!}, \end{aligned}$$

where we used the formula

$$\begin{aligned} \int_0^1 [(1+x)^{\mu-1} (1-x)^{\nu-1} + (1+x)^{\nu-1} (1-x)^{\mu-1}] dx &= 2^{\mu+\nu-1} B(\mu, \nu) \\ &= 2^{\mu+\nu-1} \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \end{aligned}$$

for $\Re(\mu), \Re(\nu) > 0$ in [3, p. 321, 3.214], the double inequality

$$\frac{1}{2} \frac{m!n!}{(m+n+1)!} < \frac{1}{\mathfrak{Y}_{m,n}(t)} < \frac{m!n!}{(m+n+1)!}$$

is valid and sharp on $(0, \infty)$. Consequently, by virtue of Lemma 2.6, if and only if $\omega_{m,n} \leq \frac{(m+n+1)!}{m!n!}$, the function $(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0, \infty)$; if and only if $\omega_{m,n} \geq \frac{2(m+n+1)!}{m!n!}$, the function $(-1)^{m+n} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0, \infty)$.

The double inequality (3.1) follows from complete monotonicity of the functions $\pm(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$. The proof of the sharpness of the double inequality (3.1) is the same as did in the proof of Theorem 3.1. The proof of Theorem 4.1 is complete. \square

5. REMARKS

In this section, we list several remarks related to our main results and their proofs in this paper.

Remark 5.1. Lemma 2.4 in this paper generalizes a conclusion in [7, Lemma 2.3]: the function $\frac{h(2t)}{h^2(t)}$ is decreasing from $(0, \infty)$ onto $(1, 2)$.

Remark 5.2. The function $F(x, y)$ discussed in Lemma 2.3 can be reformulated as

$$F(x, y) = \left(\frac{1}{x} - \frac{1}{y-x} \right) \left[2 - \frac{1}{2} \frac{\frac{2^{y-x}}{y-x} - \frac{2^x}{x}}{\frac{1}{y-x} - \frac{1}{x}} - \frac{2^{y-x} - 2^x}{(y-x) - x} \right]$$

in which the functions

$$\frac{\frac{2^{y-x}}{y-x} - \frac{2^x}{x}}{\frac{1}{y-x} - \frac{1}{x}} \quad \text{and} \quad \frac{2^{y-x} - 2^x}{(y-x) - x}$$

can be regarded as special means [2, 10].

Let $x, y \in \mathbb{R}$ such that $0 < x < \frac{y}{2}$. Motivated by Lemma 2.3, we guess that,

- (1) when $2 < x < \frac{y}{2}$, the function $F(x, y)$ is positive;
- (2) when $y > 4$ and $0 < x < 2$, the function $F(x, y)$ is negative.

Furthermore, one can discuss positivity and negativity of the function $F(x, y)$ for all x, y satisfying $0 < x < \frac{y}{2}$.

Remark 5.3. When taking $m = k$ and $n = k + 1$, the double inequality (3.1) in Theorem 3.1 becomes the double inequality (1.1) guessed by the first author in [5, Remark 5.3].

Remark 5.4. For $m, n \in \{0\} \cup \mathbb{N}$, direct differentiation gives

$$Y'_{m,n}(x) = \frac{\Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)] - \Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]'}{[\Phi^{(m)}(x)\Phi^{(n)}(x)]^2}$$

on $(0, \infty)$. The decreasing monotonicity of $Y_{m,n}(x)$ in Theorem 3.1 implies that, for $m, n \in \{0\} \cup \mathbb{N}$, the inequality

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' > \Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)],$$

equivalently,

$$\frac{[\Phi^{(m)}(x)\Phi^{(n)}(x)]'}{\Phi^{(m)}(x)\Phi^{(n)}(x)} < \frac{\Phi^{(m+n+2)}(x)}{\Phi^{(m+n+1)}(x)},$$

is valid on $(0, \infty)$.

We guess that, for $m, n \in \{0\} \cup \mathbb{N}$, the function

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' - \Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]$$

is completely monotonic in $x \in (0, \infty)$.

One can also consider necessary and sufficient conditions on $\Lambda_{m,n} \in \mathbb{R}$ for $m, n \in \{0\} \cup \mathbb{N}$ such that the function

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' - \Lambda_{m,n}\Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]$$

and its negativity are respectively completely monotonic on $(0, \infty)$.

Remark 5.5. This paper is the sixth one in a series of articles including [5, 6, 7, 8, 9].

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