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BY THREE DERIVATIVES OF A FUNCTION  
INVOLVING TRIGAMMA FUNCTION**

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# MONOTONICITY AND COMPLETE MONOTONICITY OF TWO FUNCTIONS DEFINED BY THREE DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION

FENG QI, LING-XIONG HAN, AND HONG-PING YIN

*Dedicated to people facing and battling COVID-19*

ABSTRACT. In the paper, by convolution theorem of the Laplace transforms, a monotonicity rule for the ratio of two Laplace transforms, Bernstein's theorem for completely monotonic functions, and other analytic techniques, the authors

- (1) verify decreasing monotonicity of a ratio between three derivatives of a function involving trigamma function;
- (2) find necessary and sufficient conditions for a function defined by three derivatives of a function involving trigamma function to be completely monotonic.

These results confirm previous guesses posed by the first author and generalize corresponding known conclusions.

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## 1. INTRODUCTION

In the literature [1, Section 6.4], the function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

and its logarithmic derivative  $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$  are called Euler's gamma function and digamma function respectively. Further, the functions  $\psi'(z)$ ,  $\psi''(z)$ ,  $\psi'''(z)$ , and  $\psi^{(4)}(z)$  are known as the trigamma, tetragamma, pentagamma, and

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*Key words and phrases*. monotonicity; complete monotonicity; completely monotonic function; trigamma function; derivative; ratio; convolution theorem; inequality; monotonicity rule; Laplace transform; Bernstein's theorem; exponential function; guess.

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hexagamma functions respectively. All the derivatives  $\psi^{(k)}(z)$  for  $k \geq 0$  are known as polygamma functions.

Recall from Chapter XIII in [4], Chapter 1 in [11], and Chapter IV in [12] that, if a function  $f(t)$  on an interval  $I$  has derivatives of all orders on  $I$  and satisfies inequalities  $(-1)^n f^{(n)}(t) \geq 0$  for  $t \in I$  and  $n \in \{0\} \cup \mathbb{N}$ , then we call  $f(t)$  a completely monotonic function on  $I$ .

Let  $\Phi(x) = x\psi'(x) - 1 = x[\psi'(x) - \frac{1}{x}]$  on  $(0, \infty)$ . In [7, Theorem 4.1] and [9, Theorem 4], the first author turned out the following necessary and sufficient conditions and double inequality:

- (1) if and only if  $\alpha \geq 2$ , the function  $\mathfrak{H}_\alpha(x) = \Phi'(x) + \alpha\Phi^2(x)$  is completely monotonic on  $(0, \infty)$ ;
- (2) if and only if  $\alpha \leq 1$ , the function  $-\mathfrak{H}_\alpha(x)$  is completely monotonic on  $(0, \infty)$ ;
- (3) the double inequality  $-2 < \frac{\Phi'(x)}{\Phi^2(x)} < -1$  is valid and sharp in the sense that the lower and upper bounds  $-2$  and  $-1$  cannot be replaced by any bigger and smaller ones respectively.

In [7, Theorem 1.1], the first author found the following necessary and sufficient conditions and limits:

- (1) if and only if  $\beta \geq 2$ , the function  $H_\beta(x) = \frac{\Phi'(x)}{\Phi^\beta(x)}$  is decreasing on  $(0, \infty)$ , with the limits

$$\lim_{x \rightarrow 0^+} H_\beta(x) = \begin{cases} -1, & \beta = 2 \\ 0, & \beta > 2 \end{cases} \quad \text{and} \quad \lim_{x \rightarrow \infty} H_\beta(x) = \begin{cases} -2, & \beta = 2 \\ -\infty, & \beta > 2; \end{cases}$$

- (2) if  $\beta \leq 1$ , the function  $H_\beta(x)$  is increasing on  $(0, \infty)$ , with the limits

$$H_\beta(x) \rightarrow \begin{cases} -\infty, & x \rightarrow 0^+ \\ 0, & x \rightarrow \infty. \end{cases}$$

For  $k \in \{0\} \cup \mathbb{N}$  and  $\lambda_k, \mu_k \in \mathbb{R}$ , let

$$\mathfrak{J}_{k, \lambda_k}(x) = \Phi^{(2k+1)}(x) + \lambda_k [\Phi^{(k)}(x)]^2 \quad \text{and} \quad J_{k, \mu_k}(x) = \frac{\Phi^{(2k+1)}(x)}{[(-1)^k \Phi^{(k)}(x)]^{\mu_k}}$$

on  $(0, \infty)$ . In [5, Theorems 3.1 and 4.1], the first author presented the following necessary and sufficient conditions, limits, and double inequality:

- (1) if and only if  $\lambda_k \geq \frac{(2k+2)!}{k!(k+1)!}$ , the function  $\mathfrak{J}_{k, \lambda_k}(x)$  is completely monotonic on  $(0, \infty)$ ;
- (2) if and only if  $\lambda_k \leq \frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}$ , the function  $-\mathfrak{J}_{k, \lambda_k}(x)$  is completely monotonic on  $(0, \infty)$ ;
- (3) if and only if  $\mu_k \geq 2$ , the function  $J_{k, \mu_k}(x)$  is decreasing on  $(0, \infty)$ , with the limits

$$\lim_{x \rightarrow 0^+} J_{k, \mu_k}(x) = \begin{cases} -\frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2 \\ 0, & \mu_k > 2 \end{cases}$$

and

$$\lim_{x \rightarrow \infty} J_{k, \mu_k}(x) = \begin{cases} -\frac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2 \\ -\infty, & \mu_k > 2; \end{cases}$$

(4) if  $\mu_k \leq 1$ , the function  $J_{k,\mu_k}(x)$  is increasing on  $(0, \infty)$ , with the limits

$$J_{k,\mu_k}(x) \rightarrow \begin{cases} -\infty, & x \rightarrow 0^+ \\ 0, & x \rightarrow \infty; \end{cases}$$

(5) the double inequality

$$-\frac{(2k+2)!}{k!(k+1)!} < \frac{\Phi^{(2k+1)}(x)}{[\Phi^{(k)}(x)]^2} < -\frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}$$

is valid on  $(0, \infty)$  and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

For  $k \geq m \geq 0$ , let

$$\mathcal{J}_{k,m}(x) = \frac{\Phi^{(2k+2)}(x)}{\Phi^{(k-m)}(x)\Phi^{(k+m+1)}(x)}$$

on  $(0, \infty)$ . In [5, Remark 5.3], the first author guessed that the function  $\mathcal{J}_{k,m}(x)$  for  $k \geq m \geq 0$  should be decreasing on  $(0, \infty)$  and that the double inequality

$$-\frac{2(2k+2)!}{k!(k+1)!} < \mathcal{J}_{k,0}(x) < -\frac{(2k+2)!}{k!(k+1)!} \quad (1.1)$$

for  $k \geq 0$  should be valid on  $(0, \infty)$  and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

For  $m, n \in \{0\} \cup \mathbb{N}$  and  $\omega_{m,n} \in \mathbb{R}$ , let

$$Y_{m,n}(x) = \frac{\Phi^{(m+n+1)}(x)}{\Phi^{(m)}(x)\Phi^{(n)}(x)} \quad (1.2)$$

and

$$\mathcal{Y}_{m,n;\omega_{m,n}}(x) = \Phi^{(m+n+1)}(x) + \omega_{m,n} \Phi^{(m)}(x)\Phi^{(n)}(x). \quad (1.3)$$

It is clear that

$$Y_{m,n}(x) = Y_{n,m}(x), \quad \mathcal{Y}_{m,n;\omega_{m,n}}(x) = \mathcal{Y}_{n,m;\omega_{n,m}}(x), \\ Y_{k-m,k+m+1}(x) = \mathcal{J}_{k,m}(x), \quad \mathcal{Y}_{k,k;\omega_{k,k}}(x) = \mathfrak{J}_{\omega_{k,k}}(x), \quad \mathcal{Y}_{0,0;\omega_{0,0}}(x) = \mathfrak{H}_{\omega_{0,0}}(x).$$

In this paper, we will prove decreasing monotonicity of the function  $Y_{m,n}(x)$  and find necessary and sufficient conditions on  $\omega_{m,n}$  for  $\pm(-1)^{m+n+1}\mathcal{Y}_{m,n;\omega_{m,n}}(x)$  to be completely monotonic on  $(0, \infty)$ . These results confirm the above guesses and generalize corresponding ones in [5, 7, 9] mentioned above.

## 2. LEMMAS

The following lemmas are necessary in this paper.

**Lemma 2.1** (Convolution theorem for the Laplace transforms [12, pp. 91–92]). *Let  $f_k(t)$  for  $k = 1, 2$  be piecewise continuous in arbitrary finite intervals included in  $(0, \infty)$ . If there exist some constants  $M_k > 0$  and  $c_k \geq 0$  such that  $|f_k(t)| \leq M_k e^{c_k t}$  for  $k = 1, 2$ , then*

$$\int_0^\infty \left[ \int_0^t f_1(u)f_2(t-u)du \right] e^{-st} dt = \int_0^\infty f_1(u)e^{-su} du \int_0^\infty f_2(v)e^{-sv} dv.$$

**Lemma 2.2** ([13, Lemma 4]). *Let the functions  $A(t)$  and  $B(t) \neq 0$  be defined on  $(0, \infty)$  such that their Laplace transforms  $\int_0^\infty A(t)e^{-xt} dt$  and  $\int_0^\infty B(t)e^{-xt} dt$  exist.*

*If the ratio  $\frac{A(t)}{B(t)}$  is increasing, then the ratio  $\frac{\int_0^\infty A(t)e^{-xt} dt}{\int_0^\infty B(t)e^{-xt} dt}$  is decreasing on  $(0, \infty)$ .*

**Lemma 2.3.** *Let  $x, y \in \mathbb{R}$  such that  $0 < 2x < y$ .*

(1) *When  $y > 2x > 2(2 + \frac{1}{\ln 2}) = 6.885390\dots$ , the function*

$$F(x, y) = 2\left(\frac{1}{x} - \frac{1}{y-x}\right) + \frac{1}{2}\left(\frac{2^{y-x}}{y-x} - \frac{2^x}{x}\right) - \frac{2^{y-x} - 2^x}{(y-x)x}$$

*is positive.*

(2) *For  $k, m \in \mathbb{N}$  such that  $6 \leq 2m < k$ , the sequence  $F(m, k)$  is positive.*

*Proof.* The function  $F(x, y)$  can be rearranged as

$$F(x, y) = \frac{2(y-2x) + 2^{x-1}[2-y+x+(x-2)2^{y-2x}]}{x(y-x)}.$$

Therefore, it suffices to prove  $2-y+x+(x-2)2^{y-2x} > 0$ , that is,

$$2^{y-2x} > \frac{y-x-2}{x-2}. \quad (2.1)$$

Replacing  $y-2x$  by  $t$  in (2.1) leads to

$$2^t > \frac{t+x-2}{x-2} = 1 + \frac{t}{x-2} \quad (2.2)$$

for  $t > 0$  and  $x > 2$ . The inequality (2.2) can be reformulated as  $x > 2 + \frac{t}{2^t-1}$ . Since the function  $\frac{t}{2^t-1}$  is decreasing from  $(0, \infty)$  onto  $(0, \frac{1}{\ln 2})$ , it is sufficient for  $x > 2 + \frac{1}{\ln 2} = 3.442695\dots$

Repeating those arguments before the inequality (2.1) hints us that, for proving  $F(m, k) > 0$ , it is sufficient to show

$$2^{k-2m} > \frac{k-m-2}{m-2} = 1 + \frac{k-2m}{m-2}$$

which can be rewritten as

$$\frac{k-2m}{2^{k-2m}-1} < m-2 \quad (2.3)$$

Since  $\frac{t}{2^t-1}$  is decreasing in  $t \in (0, \infty)$  and  $k-2m \geq 1$ , the largest value of the left hand side in the inequality (2.3) is  $\frac{1}{2^1-1} = 1$  which means that the strict inequality (2.3) is valid for all  $m \geq 4$ . As a result, the sequence  $F(m, k)$  is positive for all  $m \geq 4$ .

When  $m = 3$ , the sequence  $F(3, k)$  is

$$F(3, k) = \frac{2^k - 32k + 128}{48(k-3)} = \frac{2^5 [2^{k-5} - (k-4)]}{48(k-3)}$$

which is positive for all  $k > 2 \cdot 3 = 6$ . The proof of Lemma 2.3 is complete.  $\square$

**Lemma 2.4.** *Let*

$$h(t) = \begin{cases} \frac{e^t(e^t - 1 - t)}{(e^t - 1)^2}, & t \neq 0 \\ \frac{1}{2}, & t = 0 \end{cases}$$

*on  $(-\infty, \infty)$ . Then, for any fixed  $s \in (0, 1)$ , the ratio  $\frac{h(st)}{h^s(t)}$  is increasing in  $t$  from  $(0, \infty)$  onto  $(\frac{1}{2^{1-s}}, 1)$ .*

*Proof.* It is easy to see that

$$\lim_{t \rightarrow 0} \frac{h(st)}{h^s(t)} = \frac{\lim_{t \rightarrow 0} h(st)}{\lim_{t \rightarrow 0} h^s(t)} = \frac{\frac{1}{2}}{\frac{1}{2^s}} = \frac{1}{2^{1-s}}$$

and

$$\lim_{t \rightarrow \infty} H_s(t) = \frac{\lim_{t \rightarrow \infty} h(st)}{\lim_{t \rightarrow 0} h^s(t)} = \frac{1}{1^s} = 1.$$

Direct differentiating and expanding to power series give

$$\begin{aligned} \frac{d}{dt} \left[ \frac{h(st)}{h^s(t)} \right] &= - \frac{se^{(1+s)t} \left[ \begin{array}{l} (t-2)e^{(1+2s)t} + (t+2)e^{2st} + (2-st)e^{(2+s)t} \\ + 4(s-1)te^{(1+s)t} - (2st^2 + 3st + 2)e^{st} \\ - (st+2)e^{2t} + (2st^2 + 3t + 2)e^t + (s-1)t \end{array} \right]}{(e^t - 1)^3 (e^{st} - 1)^3 h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left( \begin{array}{l} (3k+2)s^k + 2(2s+1)^k + ks(s+2)^{k-1} \\ + 2k^2 s^{k-1} + 4k(s+1)^{k-1} \\ + 2k(1+2^{k-2})s + 2^{k+1} \\ - [2^{k+1}s^k + 2(s+2)^k \\ + 4ks(s+1)^{k-1} + 2k(1+2^{k-2})s^{k-1} \\ + k(2s+1)^{k-1} + 2k^2s + 3k + 2] \end{array} \right) \frac{t^k}{k!}}{(e^t - 1)^3 (e^{st} - 1)^3 h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left( \begin{array}{l} \sum_{m=1}^{k-1} [k2^{k-m} \binom{k-1}{m-1} + 4k \binom{k-1}{m} + 2^{m+1} \binom{k}{m}] s^m \\ + 2k(1-k+2^{k-2})(s-s^{k-1}) \\ - \sum_{m=1}^{k-1} [k2^m \binom{k-1}{m} + 4k \binom{k-1}{m-1} + 2^{k-m+1} \binom{k}{m}] s^m \end{array} \right) \frac{t^k}{k!}}{(e^t - 1)^3 (e^{st} - 1)^3 h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left( \begin{array}{l} \sum_{m=1}^{k-1} [k2^{k-m} \binom{k-1}{m-1} + 4k \binom{k-1}{m} + 2^{m+1} \binom{k}{m}] (s^m - s^{k-m}) \\ + 2k(1-k+2^{k-2})(s-s^{k-1}) \end{array} \right) \frac{t^k}{k!}}{(e^t - 1)^3 (e^{st} - 1)^3 h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left( \begin{array}{l} \sum_{3 \leq m < \frac{k}{2}} [k2^{k-m} \binom{k-1}{m-1} - k2^m \binom{k-1}{k-m-1}] \\ + 4k \binom{k-1}{m} - 4k \binom{k-1}{k-m} \\ + 2^{m+1} \binom{k}{m} - 2^{k-m+1} \binom{k}{k-m} \end{array} \right) (s^m - s^{k-m}) \frac{t^k}{k!}}{(e^t - 1)^3 (e^{st} - 1)^3 h^{s+1}(t)} \\ &= \frac{se^{(1+s)t} \sum_{k=7}^{\infty} \left( \begin{array}{l} 2 \sum_{3 \leq m < \frac{k}{2}} \frac{k!}{(m-1)!(k-m-1)!} \left[ 2 \left( \frac{1}{m} - \frac{1}{k-m} \right) \right] \\ + \frac{1}{2} \left( \frac{2^{k-m}}{k-m} - \frac{2^m}{m} - \frac{2^{k-m-2m}}{(k-m)m} \right) \end{array} \right) (s^m - s^{k-m}) \frac{t^k}{k!}}{(e^t - 1)^3 (e^{st} - 1)^3 h^{s+1}(t)} \\ &= \frac{s^4 e^{(1+s)t} \left( \begin{array}{l} \frac{1-s}{36} t^7 + \frac{1-s^2}{45} t^8 + \frac{22(1-s^3)+15(s-s^2)}{2160} t^9 \\ + \frac{52(1-s^4)+63(s-s^3)}{15120} t^{10} \\ + \frac{285(1-s^5)+470(s-s^4)+238(s^2-s^3)}{302400} t^{11} + \dots \end{array} \right)}{(e^t - 1)^3 (e^{st} - 1)^3 h^{s+1}(t)}. \end{aligned}$$

Utilizing Lemma 2.3 reveals that the derivative  $\frac{d}{dt} \left[ \frac{h(st)}{h^s(t)} \right]$  is positive for  $s \in (0, 1)$  and  $t > 0$ . Consequently, for  $s \in (0, 1)$ , the ratio  $\frac{h(st)}{h^s(t)}$  is increasing in  $t > 0$ . The proof of Lemma 2.4 is complete.  $\square$

**Lemma 2.5** ([5, Lemma 2.2]). *For  $k \geq 0$ , the function  $(-1)^k \Phi^{(k)}(x)$  is completely monotonic on  $(0, \infty)$ , with the limits*

$$(-1)^k x^{k+1} \Phi^{(k)}(x) \rightarrow \begin{cases} k!, & x \rightarrow 0^+; \\ \frac{k!}{2}, & x \rightarrow \infty. \end{cases} \quad (2.4)$$

**Lemma 2.6** (Bernstein's theorem [12, p. 161, Theorem 12b]). *A function  $f(x)$  is completely monotonic on  $(0, \infty)$  if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\sigma(t), \quad x \in (0, \infty), \quad (2.5)$$

where  $\sigma(s)$  is non-decreasing and the integral in (2.5) converges for  $x \in (0, \infty)$ .

### 3. DECREASING MONOTONICITY

In this section, we prove that the function  $Y_{m,n}(x)$  defined in (1.2) is decreasing.

**Theorem 3.1.** *For  $m, n \in \{0\} \cup \mathbb{N}$ , the function  $Y_{m,n}(x)$  defined in (1.2) is decreasing in  $x$  from  $(0, \infty)$  onto the interval  $\left(-\frac{2(m+n+1)!}{m!n!}, -\frac{(m+n+1)!}{m!n!}\right)$ . Consequently, for  $m, n \in \{0\} \cup \mathbb{N}$ , the double inequality*

$$-\frac{2(m+n+1)!}{m!n!} < Y_{m,n}(x) < -\frac{(m+n+1)!}{m!n!} \quad (3.1)$$

is valid on  $(0, \infty)$  and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

*Proof.* In the proof of [9, Theorem 4], the first author established that

$$\Phi(x) = \int_0^\infty h(t) e^{-xt} dt. \quad (3.2)$$

Then the ratio  $Y_{m,n}(x)$  can be rewritten as

$$\begin{aligned} Y_{m,n}(x) &= -\frac{\int_0^\infty t^{m+n+1} h(t) e^{-xt} dt}{\int_0^\infty t^m h(t) e^{-xt} dt \int_0^\infty t^n h(t) e^{-xt} dt} \\ &= -\frac{\int_0^\infty t^{m+n+1} h(t) e^{-xt} dt}{\int_0^\infty \left[ \int_0^t u^m (t-u)^n h(u) h(t-u) du \right] e^{-xt} dt}, \end{aligned}$$

where we used Lemma 2.1. Basing on Lemma 2.2, in order to prove decreasing monotonicity of  $Y_{m,n}(x)$ , it suffices to show that the ratio

$$\mathfrak{Y}_{m,n}(t) = \frac{t^{m+n+1} h(t)}{\int_0^t u^m (t-u)^n h(u) h(t-u) du} \quad (3.3)$$

is decreasing in  $t \in (0, \infty)$ . By changing the variable  $u = \frac{1+v}{2}t$ , the denominator of  $\mathfrak{Y}_{m,n}(t)$  becomes

$$\left(\frac{t}{2}\right)^{m+n+1} \int_{-1}^1 (1+v)^m (1-v)^n h\left(\frac{1+v}{2}t\right) h\left(\frac{1-v}{2}t\right) dv.$$

Accordingly, we obtain

$$\begin{aligned}
\frac{1}{\mathfrak{Y}_{m,n}(t)} &= \frac{\int_{-1}^1 (1+v)^m (1-v)^n h\left(\frac{1+v}{2}t\right) h\left(\frac{1-v}{2}t\right) dv}{2^{m+n+1} h(t)} \\
&= \frac{1}{2^{m+n+1}} \int_{-1}^1 (1+v)^m (1-v)^n \frac{h\left(\frac{1+v}{2}t\right) h\left(\frac{1-v}{2}t\right)}{h(t)} dv \quad (3.4) \\
&= \frac{1}{2^{m+n+1}} \int_{-1}^1 (1+v)^m (1-v)^n \frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)} dv,
\end{aligned}$$

where  $s = \frac{1+v}{2} \in (0, 1)$ . From Lemma 2.4, we find that the function  $\frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)}$  is increasing in  $t \in (0, \infty)$  for any fixed  $s \in (0, 1)$ . Hence, the function  $\mathfrak{Y}_{m,n}(t)$  is decreasing on  $(0, \infty)$ . Therefore, the function  $Y_{m,n}(x)$  for  $m, n \in \{0\} \cup \mathbb{N}$  is decreasing on  $(0, \infty)$ .

Making use of the limits in (2.4) in Lemma 2.5 yields

$$\begin{aligned}
Y_{m,n}(x) &= -\frac{(-1)^{m+n+1} x^{m+n+2} \Phi^{(m+n+1)}(x)}{[(-1)^m x^{m+1} \Phi^{(m)}(x)][(-1)^n x^{k+m+2} \Phi^{(n)}(x)]} \\
&\rightarrow \begin{cases} -\frac{(m+n+1)!}{m!n!}, & x \rightarrow 0^+; \\ -\frac{2(m+n+1)!}{m!n!}, & x \rightarrow \infty. \end{cases}
\end{aligned}$$

The proof of Theorem 3.1 is complete.  $\square$

#### 4. NECESSARY AND SUFFICIENT CONDITIONS OF COMPLETE MONOTONICITY

In this section, we discover necessary and sufficient conditions on  $\omega_{m,n}$  for the function  $\pm(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$  defined in (1.3) to be completely monotonic.

**Theorem 4.1.** For  $m, n \in \{0\} \cup \mathbb{N}$  and  $\omega_{m,n} \in \mathbb{R}$ ,

- (1) if and only if  $\omega_{m,n} \leq \frac{(m+n+1)!}{m!n!}$ , the function  $(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$  is completely monotonic on  $(0, \infty)$ ;
- (2) if and only if  $\omega_{m,n} \geq \frac{2(m+n+1)!}{m!n!}$ , the function  $(-1)^{m+n} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$  is completely monotonic on  $(0, \infty)$ ;
- (3) the double inequality (3.1) is valid on  $(0, \infty)$  and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

*Proof.* As did in the proof of Theorem 3.1, by virtue of the integral representation (3.2) and Lemma 2.1, we acquire

$$\begin{aligned}
(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x) &= \left[ \int_0^\infty t^{m+n+1} h(t) e^{-xt} dt \right. \\
&\quad \left. - \omega_{m,n} \int_0^\infty t^m h(t) e^{-xt} dt \int_0^\infty t^n h(t) e^{-xt} dt \right] \\
&= \int_0^\infty \left[ t^{m+n+1} h(t) - \omega_{m,n} \int_0^t u^m (t-u)^n h(u) h(t-u) du \right] e^{-xt} dt \\
&= \int_0^\infty \left[ 1 - \frac{\omega_{m,n}}{\mathfrak{Y}_{m,n}(t)} \right] t^{m+n+1} h(t) e^{-xt} dt,
\end{aligned}$$



where  $\mathfrak{Y}_{m,n}(t)$  is defined by (3.3) and it has been proved in the proof of Theorem 3.1 to be decreasing on  $(0, \infty)$ . From Lemma 2.4, we conclude that the function  $\frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)}$  is increasing in  $t$  from  $(0, \infty)$  onto  $(\frac{1}{2}, 1)$ . Accordingly, by virtue of (3.4), we arrive at the sharp inequalities

$$\frac{1}{2^{m+n+2}} \int_{-1}^1 (1+v)^m (1-v)^n dv < \frac{1}{\mathfrak{Y}_{m,n}(t)} < \frac{1}{2^{m+n+1}} \int_{-1}^1 (1+v)^m (1-v)^n dv.$$

Since

$$\begin{aligned} \int_{-1}^1 (1+v)^m (1-v)^n dv &= \int_0^1 [(1+v)^m (1-v)^n + (1-v)^m (1+v)^n] dv \\ &= 2^{m+n+1} B(m+1, n+1) \\ &= 2^{m+n+1} \frac{m!n!}{(m+n+1)!}, \end{aligned}$$

where we used the formula

$$\begin{aligned} \int_0^1 [(1+x)^{\mu-1} (1-x)^{\nu-1} + (1+x)^{\nu-1} (1-x)^{\mu-1}] dx &= 2^{\mu+\nu-1} B(\mu, \nu) \\ &= 2^{\mu+\nu-1} \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \end{aligned}$$

for  $\Re(\mu), \Re(\nu) > 0$  in [3, p. 321, 3.214], the double inequality

$$\frac{1}{2} \frac{m!n!}{(m+n+1)!} < \frac{1}{\mathfrak{Y}_{m,n}(t)} < \frac{m!n!}{(m+n+1)!}$$

is valid and sharp on  $(0, \infty)$ . Consequently, by virtue of Lemma 2.6, if and only if  $\omega_{m,n} \leq \frac{(m+n+1)!}{m!n!}$ , the function  $(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$  is completely monotonic on  $(0, \infty)$ ; if and only if  $\omega_{m,n} \geq \frac{2(m+n+1)!}{m!n!}$ , the function  $(-1)^{m+n} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$  is completely monotonic on  $(0, \infty)$ .

The double inequality (3.1) follows from complete monotonicity of the functions  $\pm(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ . The proof of the sharpness of the double inequality (3.1) is the same as did in the proof of Theorem 3.1. The proof of Theorem 4.1 is complete.  $\square$

## 5. REMARKS

In this section, we list several remarks related to our main results and their proofs in this paper.

*Remark 5.1.* Lemma 2.4 in this paper generalizes a conclusion in [7, Lemma 2.3]: the function  $\frac{h(2t)}{h^2(t)}$  is decreasing from  $(0, \infty)$  onto  $(1, 2)$ .

*Remark 5.2.* The function  $F(x, y)$  discussed in Lemma 2.3 can be reformulated as

$$F(x, y) = \left( \frac{1}{x} - \frac{1}{y-x} \right) \left[ 2 - \frac{1}{2} \frac{\frac{2^{y-x}}{y-x} - \frac{2^x}{x}}{\frac{1}{y-x} - \frac{1}{x}} - \frac{2^{y-x} - 2^x}{(y-x) - x} \right]$$

in which the functions

$$\frac{\frac{2^{y-x}}{y-x} - \frac{2^x}{x}}{\frac{1}{y-x} - \frac{1}{x}} \quad \text{and} \quad \frac{2^{y-x} - 2^x}{(y-x) - x}$$

can be regarded as special means [2, 10].

Let  $x, y \in \mathbb{R}$  such that  $0 < x < \frac{y}{2}$ . Motivated by Lemma 2.3, we guess that,

- (1) when  $2 < x < \frac{y}{2}$ , the function  $F(x, y)$  is positive;
- (2) when  $y > 4$  and  $0 < x < 2$ , the function  $F(x, y)$  is negative.

Furthermore, one can discuss positivity and negativity of the function  $F(x, y)$  for all  $x, y$  satisfying  $0 < x < \frac{y}{2}$ .

*Remark 5.3.* When taking  $m = k$  and  $n = k + 1$ , the double inequality (3.1) in Theorem 3.1 becomes the double inequality (1.1) guessed by the first author in [5, Remark 5.3].

*Remark 5.4.* For  $m, n \in \{0\} \cup \mathbb{N}$ , direct differentiation gives

$$Y'_{m,n}(x) = \frac{\Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)] - \Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]'}{[\Phi^{(m)}(x)\Phi^{(n)}(x)]^2}$$

on  $(0, \infty)$ . The decreasing monotonicity of  $Y_{m,n}(x)$  in Theorem 3.1 implies that, for  $m, n \in \{0\} \cup \mathbb{N}$ , the inequality

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' > \Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)],$$

equivalently,

$$\frac{[\Phi^{(m)}(x)\Phi^{(n)}(x)]'}{\Phi^{(m)}(x)\Phi^{(n)}(x)} < \frac{\Phi^{(m+n+2)}(x)}{\Phi^{(m+n+1)}(x)},$$

is valid on  $(0, \infty)$ .

We guess that, for  $m, n \in \{0\} \cup \mathbb{N}$ , the function

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' - \Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]$$

is completely monotonic in  $x \in (0, \infty)$ .

One can also consider necessary and sufficient conditions on  $\Lambda_{m,n} \in \mathbb{R}$  for  $m, n \in \{0\} \cup \mathbb{N}$  such that the function

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' - \Lambda_{m,n}\Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]$$

and its negativity are respectively completely monotonic on  $(0, \infty)$ .

*Remark 5.5.* This paper is the sixth one in a series of articles including [5, 6, 7, 8, 9].

#### REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
- [2] P. S. Bullen, *Handbook of Means and Their Inequalities*, Revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and Their Inequalities*, Reidel, Dordrecht], Mathematics and its Applications **560**, Kluwer Academic Publishers Group, Dordrecht, 2003; available online at <https://doi.org/10.1007/978-94-017-0399-4>.
- [3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015; available online at <https://doi.org/10.1016/B978-0-12-384933-5.00013-8>.
- [4] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; available online at <https://doi.org/10.1007/978-94-017-1043-5>.
- [5] F. Qi, *Complete monotonicity and monotonicity of two functions defined by two derivatives of a function involving trigamma function*, HAL preprint (2020), available online at <https://hal.archives-ouvertes.fr/hal-02993723>.

- [6] F. Qi, *Lower bound of sectional curvature of manifold of beta distributions and complete monotonicity of functions involving polygamma functions*, MDPI Preprints **2020**, 2020110315, 22 pages; available online at <https://doi.org/10.20944/preprints202011.0315.v1>.
- [7] F. Qi, *Monotonicity of a ratio involving trigamma and tetragamma functions*, OSF Preprints (2020), available online at <https://doi.org/10.31219/osf.io/5rfb8>.
- [8] F. Qi, *Necessary and sufficient conditions for two functions defined by two derivatives of a function involving trigamma function to be completely monotonic or monotonic*, OSF Preprints (2020), available online at <https://doi.org/10.31219/osf.io/6ar4p>.
- [9] F. Qi, *Some properties of several functions involving polygamma functions and originating from the sectional curvature of the beta manifold*, São Paulo J. Math. Sci. **15** (2021), in press; available online at <https://doi.org/10.1007/s40863-020-00193-1>.
- [10] F. Qi and D. Lim, *Integral representations of bivariate complex geometric mean and their applications*, J. Comput. Appl. Math. **330** (2018), 41–58; available online at <https://doi.org/10.1016/j.cam.2017.08.005>.
- [11] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at <https://doi.org/10.1515/9783110269338>.
- [12] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
- [13] Z.-H. Yang and J.-F. Tian, *Monotonicity and inequalities for the gamma function*, J. Inequal. Appl. **2017**, Paper No. 317, 15 pages; available online at <https://doi.org/10.1186/s13660-017-1591-9>.

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