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Existence and uniqueness result for an hyperbolic scalar conservation law with a stochastic force using a finite volume approximation

C. Bauzet*, V. Castel† and J. Charrier†

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Abstract

We are interested here in multi-dimensional nonlinear scalar conservation laws forced by a multiplicative stochastic noise with a general time and space dependent flux-function. We address simultaneously theoretical and numerical issues in a general framework since we consider a large class of flux functions than the ones considered in the literature. More precisely we establish existence and uniqueness of a stochastic entropy solution together with the convergence of a finite volume scheme. The most significant novelty of this work is the use of a numerical approximation (instead of a viscous one) to get both the existence and the uniqueness of such a solution. Moreover the quantitative bounds obtained here to establish our uniqueness result constitute an important preliminary work to the establishment of strong error estimates. We also provide a L^∞ stability result on the stochastic entropy solution to complete this study.

Keywords : Stochastic PDE • first-order hyperbolic equation • multiplicative noise • finite volume method • monotone scheme • Young measures.

Mathematics Subject Classification (2000) : 35L60 • 60H15 • 35L60

1 Introduction

In this paper we study multi-dimensional nonlinear scalar conservation laws forced by a multiplicative noise with a time and space dependent flux-function and a given initial data in $L^2(\mathbb{R}^d)$. More precisely, we are interested in the following Cauchy problem in d space dimensions:

$$\begin{cases} du + \operatorname{div}_x [f(x, t, u)] dt &= g(u) dW & \text{in } \Omega \times \mathbb{R}^d \times (0, T) \\ u(\omega, x, 0) &= u_0(x), & \omega \in \Omega, x \in \mathbb{R}^d, \end{cases} \quad (1)$$

where div_x is the divergence operator with respect to the space variable (which belongs to \mathbb{R}^d), d is a positive integer, $T > 0$ and $W = \{W_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a standard adapted one-dimensional continuous Brownian motion defined on the classical Wiener space (Ω, \mathcal{F}, P) . This equation has to be understood in the following way: for almost any ω in Ω and for all φ in $\mathcal{D}(\mathbb{R}^d \times [0, T])$

$$\begin{aligned} & \int_{\mathbb{R}^d} u_0(x) \varphi(x, 0) dx + \int_Q \left[u(\omega, x, t) \partial_t \varphi(x, t) + f(x, t, u(\omega, x, t)) \cdot \nabla_x \varphi(x, t) \right] dx dt \\ &= \int_Q \left(\int_0^t g(u(\omega, x, s)) dW(s) \right) \partial_t \varphi(x, t) dx dt. \end{aligned} \quad (2)$$

In order to relieve the presentation of the paper, we omit in the sequel the variables ω, x, t and write u instead of $u(\omega, x, t)$ and we denote by x the space variable of f , t the time variable and v the third one.

The theoretical study of stochastic scalar conservation laws has been the subject of many works recently, let us mention in chronological order the contributions of [EKMS00], [Ki03], [FN08], [DeV10], [CDK12], [BVW12], [BVW14], [BM14], [KN16], [KS17]. A time-discretization of the equation has been studied by the use of an operator-splitting method in [HR91] and generalized to the multidimensional-case in [B14], see also [KS17] for a similar approach. Let us also mention the paper of [KR12] where a space-discretization of the equation is investigated by considering monotone numerical fluxes. Full discretizations through a finite volume approach have been studied in [BCG16-1], [BCG16-2], [BCG17], [DoV19], [DoV18], [FGH18] and [M18].

The present article extends the results of [BCG16-2] to a general flux function $f(x, t, v)$. More importantly, it provides a new proof of uniqueness which does not rely on the use of viscous approximations at all. Indeed, in [BCG16-1], the authors used the existence and uniqueness result proposed in [BVW12] to show the convergence of

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their flux-splitting finite volume approximation through the unique stochastic entropy solution of the hyperbolic problem. More precisely in [BVW12], the authors obtained their well-posedness result by studying a viscous parabolic approximation of the hyperbolic problem in the case where the flux function f was independent of the time and space variable. Then, in [BCG16-2], the authors generalized such an existence and uniqueness result to a particular time and space dependent flux function taking the form $\vec{v}(x, t)f(u)$, still by considering a viscous parabolic regularization. It allowed them to show a result of convergence for a general class of monotone finite volume schemes through the unique stochastic entropy solution of the hyperbolic problem. In [BCG17], the authors used the existence and uniqueness result of [KN16] obtained with a viscous approximation (and by the way of a kinetic approach) to get the convergence of their finite volume scheme in a bounded domain under Dirichlet boundary conditions. In this paper, both existence and uniqueness of a solution are established by using only the numerical approximation. To do so, we propose a new way to adapt the Kruzhkov doubling variable technique to the stochastic framework by comparing any generalized solution to a limit of the finite volume approximation. It is important to note that the estimates obtained in such a comparison technique are expected to constitute a crucial step towards the establishment of error estimates (as in the deterministic case, see [EGH00] for instance). Therefore this work provides an original proof of a theoretical result, but it also constitutes a first and important part of the work to establish strong error estimates. Additionally we provide a L^∞ stability result satisfied by the stochastic entropy solution in the case where u_0 is essentially bounded and g admits a compact support.

Remind that the main difficulties brought by the stochastic noise for the study of this kind of problems are the following ones : the use of smooth entropies for the entropy formulation (mainly due to the Itô derivation formula), the $L^2(\Omega \times \mathbb{R}^d \times (0, T))$ Hilbert framework required by the stochastic integral and the predictable measurability property of the solution due to the multiplicative nature of the noise.

The paper is organized as follows. In Section 2, we will present the problem, the assumptions made on the data, introduce the notion of stochastic entropy solution, as well as a generalized notion of solution (namely measure-valued entropy solution). The finite volume approximation constructed with general monotone flux will then be introduced, as well as some useful related notations. We will conclude this first section with the statement of the main results. The remainder of the paper will be devoted to the proof of this main result, which states the existence and uniqueness of the stochastic entropy solution as well as the convergence of the finite volume approximation under a stability condition on the time step. This proof relies on two key results. The first one is the convergence of the numerical approximation towards a measure-valued entropy solution (up to a subsequence) and will be established in Section 3. In order to prove it, we will start with classical preliminary results for this kind of equations : stability of the scheme, Weak-BV estimates and derivation of approximate entropy inequalities. Note that, contrary to [BCG16-2], it is crucial to have here explicit bounds at each step since they will be essential in what follows. The second key result is a uniqueness one and it will be established in Section 4 by adapting the Kruzhkov doubling variable technique to the stochastic case. Indeed we will compare a general measure-valued solution to the numerical approximation. At the end of Section 4, we will be able to prove the main result by gathering these two keys results. Finally, the proof of the L^∞ stability result will be given in the appendix.

2 Main result

2.1 Assumptions

In this work, we assume the following hypotheses:

H₁: $u_0 \in L^2(\mathbb{R}^d)$.

H₂: $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous bounded function with $g(0) = 0$. We denote by C_g a Lipschitz constant of g .

H₃: $f \in C^1(\mathbb{R}^d \times [0, T] \times \mathbb{R}, \mathbb{R}^d)$ and there exists a constant $C_f > 0$ such that for any $(x, t, v) \in \mathbb{R}^d \times [0, T] \times \mathbb{R}$,

$$\left| \frac{\partial f}{\partial v}(x, t, v) \right| \leq C_f.$$

H₄: For all $x \in \mathbb{R}^d$, there exist two reals $C_f^x > 0$ and $R^x > 0$ such that for all $(t, v) \in [0, T] \times \mathbb{R}$ and for all $y \in B(x, R^x)$,

$$|f(x, t, v) - f(y, t, v)| \leq C_f^x |x - y|.$$

H₅: There exists a real $C_f^T > 0$ such that for all $(x, v) \in \mathbb{R}^d \times \mathbb{R}$ and for all $(t, t') \in [0, T]^2$,

$$|f(x, t, v) - f(x, t', v)| \leq C_f^T |t - t'|.$$

H₆: For all $(x, t, v) \in \mathbb{R}^d \times [0, T] \times \mathbb{R}$, $\operatorname{div}_x [f(x, t, v)] = \sum_{i=1}^d \frac{\partial f_i}{\partial x_i}(x_1, \dots, x_d, t, v) = 0$.

H₇: For all $x \in \mathbb{R}^d$, there exist two reals $\tilde{C}_f^x > 0$ and $\tilde{R}^x > 0$ such that for all $(t, v) \in [0, T] \times \mathbb{R}$ and for all $y \in B(x, \tilde{R}^x)$,

$$\left| \frac{\partial f}{\partial v}(x, t, v) - \frac{\partial f}{\partial v}(y, t, v) \right| \leq \tilde{C}_f^x |x - y|.$$

Remark 1 (On these assumptions)

- Note that, as it is classically done in the deterministic setting for hyperbolic scalar conservation laws, for convenience one can assume that for all (x, t) in $\mathbb{R}^d \times [0, T]$, $f(x, t, 0) = \bar{0}$ without loss of generality thanks to H_6 .
- The existence and uniqueness result still holds without assuming that g is bounded (see [BVW12], [BCG16-2]), but we use this assumption to get the convergence of the numerical approximation.
- $g(0) = 0$ is an important condition. Indeed, if $g(0) \neq 0$, the problem is generally not well-posed in $L^2(\Omega \times \mathbb{R}^d \times (0, T))$.
- Note that the present study can be extended to the case where $\operatorname{div}_x[f(x, t, v)] \neq 0$ (which only brings technical difficulties) following for example the work of [CH00] in the deterministic case.
- Finally, if additionally g has compact support and $u_0 \in L^\infty(\mathbb{R}^d)$, then, as we will see in Corollary 1, the solution is also bounded. Thus, our result still holds if we suppose that we only have local bounds with respect to v in assumptions H_3 , H_4 and H_5 (i.e. if the constants C_f, C_f^x and C_f^T depend on v). It enables in particular to treat a larger class of flux functions such as Burgers' one.

2.2 The continuous problem: definitions and notations

First of all, let us introduce some notations and make precise the functional setting.

- We define the following sets $Q = \mathbb{R}^d \times (0, T)$ and for all $R > 0$, $Q_R = B(0, R) \times (0, T)$, where $B(0, R)$ denotes the ball of \mathbb{R}^d centered at 0 of radius R .
- $|x|$ denotes the Euclidean norm of x in \mathbb{R}^d and $x \cdot y$ the usual scalar product of x and y in \mathbb{R}^d .
- \mathcal{D}^+ ($\mathbb{R}^d \times [0, T]$) denotes the subset of non-negative elements of $\mathcal{D}(\mathbb{R}^d \times [0, T])$.
- For a given separable Banach space X , we denote by $\mathcal{N}_w^2(0, T, X)$ the space of the predictable X -valued processes endowed with the norm

$$\|\phi\|_{\mathcal{N}_w^2(0, T, X)}^2 = E \left[\int_0^T \|\phi\|_X^2 dt \right]$$

(see [DPZ92] p.94).

- \mathcal{A} denotes the set of any $C^3(\mathbb{R})$ convex functions such that η', η'' and η''' are bounded functions.
- For any entropy function $\eta \in \mathcal{A}$ we introduce the function Φ_η defined for any $a, b \in \mathbb{R}$ by

$$\Phi_\eta(x, t, a, b) = \int_b^a \eta'(\sigma - b) \frac{\partial f}{\partial v}(x, t, \sigma) d\sigma,$$

for all $(x, t) \in \mathbb{R}^d \times [0, T]$ and for all $v \in \mathbb{R}$. Note that $\Phi_\eta(x, t, \cdot, b)$ is then the entropy flux associated with $\eta(\cdot - b)$ which vanishes at b . This notation will be very useful in the proof of the uniqueness result in section 4. For any entropy $\eta \in \mathcal{A}$, $\Phi_\eta(\cdot, \cdot, \cdot, 0)$ is then an entropy flux associated with η .

- In the sequel, we will consider a ball $B(0, R) \subset \mathbb{R}^d$ for some $R > 0$. Using the assumptions H_4 and H_7 we get the existence of two constants $C_f^R > 0$ and \tilde{C}_f^R such that for all $(t, v) \in [0, T] \times \mathbb{R}$ and for all $(x, y) \in B(0, R)^2$,
- $$|f(x, t, v) - f(y, t, v)| \leq C_f^R |x - y| \text{ and } \left| \frac{\partial f}{\partial v}(x, t, v) - \frac{\partial f}{\partial v}(y, t, v) \right| \leq \tilde{C}_f^R |x - y|$$

Let us now introduce the concept of solution we are interested in for Problem (1).

Definition 1 (Stochastic entropy solution)

A function u of $\mathcal{N}_w^2(0, T, L^2(\mathbb{R}^d)) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d))$ is an entropy solution of the stochastic scalar conservation law (1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$, if P-a.s in Ω , for any $\eta \in \mathcal{A}$, and for any $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx + \int_Q \eta(u) \partial_t \varphi(x, t) dx dt + \int_Q \Phi_\eta(x, t, u, 0) \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \eta'(u) g(u) \varphi(x, t) dx dW(t) + \frac{1}{2} \int_Q g^2(u) \eta''(u) \varphi(x, t) dx dt. \end{aligned} \quad (3)$$

As it is classically done in the deterministic and stochastic setting with an entropy approach (see [BVW12], [BCG16-1] and [BCG16-2]), we also need to consider a generalized notion of entropy solution. In fact, in a first step, we will only prove the convergence (up to a subsequence) of the finite volume approximate solution to a measure-valued entropy solution. Then, thanks to the result of uniqueness stated in Section 4 and established by using the finite volume approximation, we will be able to deduce simultaneously the existence of an entropy solution of Problem (1) and the convergence of our approximate solution to the unique stochastic entropy solution of (1).

Definition 2 (Measure-valued entropy solution)

A function μ of $\mathcal{N}_w^2(0, T, L^2(\mathbb{R}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d \times (0, 1)))$ is a measure-valued entropy solution of

the stochastic scalar conservation law (1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$, if P-a.s in Ω , for any $\eta \in \mathcal{A}$, and for any $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$

$$0 \leq \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx + \int_Q \int_0^1 \eta(\mu(\cdot, \alpha)) \partial_t \varphi(x, t) d\alpha dx dt + \int_Q \int_0^1 \Phi_\eta(x, t, \mu(\cdot, \alpha), 0) \cdot \nabla_x \varphi(x, t) d\alpha dx dt \\ + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mu(\cdot, \alpha)) g(\mu(\cdot, \alpha)) \varphi(x, t) d\alpha dx dW(t) + \frac{1}{2} \int_Q \int_0^1 g^2(\mu(\cdot, \alpha)) \eta''(\mu(\cdot, \alpha)) \varphi(x, t) d\alpha dx dt.$$

2.3 The finite volume approximation: definitions and notations

Definition 3 (Admissible mesh) An admissible mesh \mathcal{T} of \mathbb{R}^d for the discretization of Problem (1) is given by a family of disjoint polygonal connected subsets of \mathbb{R}^d such that \mathbb{R}^d is the union of the closure of the elements of \mathcal{T} (which are called control volumes in the following) and such that the common interface between two control volumes is included in a hyperplane of \mathbb{R}^d . It is assumed that $h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty$ and that, for some $\bar{\alpha} \in \mathbb{R}_+^*$, we have

$$\bar{\alpha} h^d \leq |K|, \quad \text{and} \quad |\partial K| \leq \frac{1}{\bar{\alpha}} h^{d-1}, \quad \forall K \in \mathcal{T}, \quad (4)$$

where we denote by

- . ∂K the boundary of the control volume K .
- . $|K|$ the d -dimensional Lebesgue measure of K .
- . $|\partial K|$ the $(d-1)$ -dimensional Lebesgue measure of ∂K .
- . $\mathcal{N}(K)$ the set of control volumes neighbours of the control volume K .
- . $\sigma_{K,L}$ the common interface between K and L for any $L \in \mathcal{N}(K)$.
- . $n_{K,L}$ the unit normal vector to interface $\sigma_{K,L}$, oriented from K to L , for any $L \in \mathcal{N}(K)$.

From (4), we get the following inequality, which will be used several times later:

$$\frac{|\partial K|}{|K|} \leq \frac{1}{\bar{\alpha}^2 h}. \quad (5)$$

We now define monotone schemes. Consider an admissible mesh \mathcal{T} in the sense of Definition 3. In order to compute an approximation of u on $[0, T]$, we take $N \in \mathbb{N}^*$ and define the time step $k = \frac{T}{N} \in \mathbb{R}_+^*$. In this way:

$$[0, T] = \bigcup_{n=0}^{N-1} [nk, (n+1)k].$$

The equations satisfied by the discrete unknowns denoted by u_K^n , $n \in \{0, \dots, N-1\}$, $K \in \mathcal{T}$, are obtained by discretizing Problem (1). In order to do this, we introduce a family of numerical fluxes for the discretization of the divergence term. We also define for any $n \in \mathbb{N}^*$ and $K, L \in \mathcal{T}$ such that $L \in \mathcal{N}(K)$ and for all $v \in \mathbb{R}$ the following quantity:

$$f_{K,L}^n(v) = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} f(x, t, v) \cdot n_{K,L} d\gamma(x) dt, \quad (6)$$

where γ denotes the Lebesgue measure on the interfaces. Note that we have, for any $v, v' \in \mathbb{R}$:

$$\left| f_{K,L}^n(v) - f_{K,L}^n(v') \right| \leq C_f |v - v'|. \quad (7)$$

Definition 4 (Family of monotone numerical fluxes associated with a flux f)

A family of functions $(F_{K,L}^n)_{n \in \mathbb{N}, K \in \mathcal{T}, L \in \mathcal{N}(K)}$ from \mathbb{R}^2 to \mathbb{R} is said to be a family of monotone numerical fluxes associated with a flux f if for any $n \in \mathbb{N}$, $K \in \mathcal{T}$ and $L \in \mathcal{N}(K)$, $F_{K,L}^n$ satisfies the following assumptions of consistency, monotony, conservativity and regularity:

- (i) *Consistency* : for any $v \in \mathbb{R}$, $F_{K,L}^n(v, v) = f_{K,L}^n(v)$.
- (ii) *Monotonicity* : for any $(a, b) \in \mathbb{R}^2$, $v \mapsto F_{K,L}^n(v, b)$ is non-decreasing and $v \mapsto F_{K,L}^n(a, v)$ is non-increasing.
- (iii) *Conservativity* : for any $(a, b) \in \mathbb{R}^2$, $F_{K,L}^n(a, b) = -F_{L,K}^n(b, a)$.
- (iv) *Lipschitz-diagonal property* : $F_{K,L}^n$ is Lipschitz-continuous w.r.t the diagonal (Lip-diag) which means that there exist $F_1 > 0$ and $F_2 > 0$ such that for any $n \in \mathbb{N}$ and any $K \in \mathcal{T}$, $L \in \mathcal{N}(K)$, $F_{K,L}^n$ verifies for all $(a, b) \in \mathbb{R}^2$,

$$|F_{K,L}^n(b, a) - F_{K,L}^n(a, a)| \leq F_1 |b - a| \quad \text{and} \quad |F_{K,L}^n(a, b) - F_{K,L}^n(a, a)| \leq F_2 |b - a|.$$

The set $\{u_K^0, K \in \mathcal{T}\}$ is given by the mean value of the initial condition on each control volume:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \forall K \in \mathcal{T}. \quad (8)$$

Once a family of numerical fluxes $F_{K,L}^n$ is chosen, the equations satisfied by the discrete unknowns u_K^n , $n \in \{0, \dots, N-1\}$, $K \in \mathcal{T}$ are given by the following explicit scheme: for any $K \in \mathcal{T}$, any $n \in \{0, \dots, N-1\}$:

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k}, \quad (9)$$

where $W^n = W(nk)$ for $n \in \{0, \dots, N-1\}$.

The approximate finite volume solution $u_{\mathcal{T},k}$ may be defined on $\Omega \times \mathbb{R}^d \times [0, T]$ from the discrete unknowns u_K^n , $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$ which are computed in (9) by:

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n \text{ for } \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k). \quad (10)$$

Remark 2 Note that assumption H_6 implies that: $\forall s \in \mathbb{R}, \forall K \in \mathcal{T}$,

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(s, s) = 0. \quad (11)$$

Remark 3 (On the measurability of the approximate finite volume solution) Let us mention that using properties of the Brownian motion, for all K in \mathcal{T} and all n in $\{0, \dots, N-1\}$, u_K^n is \mathcal{F}_{nk} -measurable and so, as an elementary process adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, $u_{\mathcal{T},k}$ is predictable with values in $L^2(\mathbb{R}^d)$.

In what follows, we will use the following sets: for any $n \in \{0, \dots, N-1\}$ and $R > 0$

$$\mathcal{I}_n = \{(K, L) \in \mathcal{T}^2; L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}, \quad (12)$$

$$\mathcal{T}_R = \{K \in \mathcal{T} \text{ such that } K \subset B(0, R)\}, \quad (13)$$

$$\mathcal{I}_R^n = \{(K, L) \in \mathcal{T}_R^2 \text{ such that } L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}. \quad (14)$$

2.4 Statement of the main result

We now state the main result of this paper.

Theorem 1 (Existence, uniqueness of the entropy solution and convergence of the scheme)

Assume that hypotheses H_1 to H_7 hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^*$ and $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step. Let $u_{\mathcal{T},k}$ be the finite volume approximation defined by the finite volume scheme (8)-(9)-(10). Then Problem (1) admits a unique stochastic entropy solution in the sense of Definition 1, and the finite volume approximation $u_{\mathcal{T},k}$ converges to this solution, in $L_{loc}^m(\Omega \times Q)$ for any $1 \leq m < 2$ as $(h, k/h)$ tends to $(0, 0)$.

The proof of Theorem 1 constitutes the remainder of this work and will be achieved through two steps. Firstly, Section 3 will be devoted to the proof of Proposition 7, which states that the finite volume approximation $u_{\mathcal{T},k}$ converges, up to a subsequence, to a measure-valued entropy solution. Secondly in Section 4, we will establish in Proposition 9 that Problem (1) admits a unique measure-valued entropy solution which happens to be a stochastic entropy solution.

Remark 4 Under the CFL Condition

$$k \leq (1 - \xi) \frac{\bar{\alpha}^2 h}{F_1 + F_2} \quad (15)$$

one gets the "weak BV" estimate stated in Proposition 2. In the deterministic case, condition (15) for some $\xi \in (0, 1)$ is sufficient to show the convergence of $u_{\mathcal{T},k}$ to the unique entropy solution of the problem, whereas in the stochastic case, as in the work [BCG16-2], we are only able to prove it under the slightly stronger assumption $(h, k/h) \rightarrow (0, 0)$. Note that in [DoV19], convergence of finite volume approximations is established without this additional assumption, by using a kinetic approach.

Remark 5 This theorem can easily be generalized to the case of a stochastic finite dimensional perturbation of the form $g(u).d\mathbf{W}$ where g takes values into \mathbb{R}^p and \mathbf{W} is a p -dimensional Brownian motion. The study of the convergence of a full discretization in the case of an infinite dimensional noise will be the subject of a further work. Such noises are considered in [DoV19] or [FGH18], but in these works the question of the noise truncation is not or not totally addressed.

Remark 6 *The present work could be easily extended (under natural assumptions) to the case of random initial value u_0 or to the case of a space dependent noise $g(x, u)$, but we don't present this general case here for the sake of readability.*

Let us now state an additional result proved in the appendix.

Corollary 1 (L^∞ stability of the entropy solution) *Under H_1 to H_7 and the additional assumptions $u_0 \in L^\infty(\mathbb{R}^d)$ and g compactly supported, the unique stochastic entropy solution u of Problem (1) in the sense of Definition 1 belongs to $L^\infty(\Omega \times Q)$. More precisely for any $a, b \in \mathbb{R}$ with $a < b$, if for almost every x in \mathbb{R}^d $u_0(x) \in [a, b]$ and $\text{supp}(g) \subset [a, b]$, then $u(x, t)$ belongs to $[a, b]$ for almost all $(x, t) \in Q$ and P -almost surely in Ω .*

3 Convergence to a measure-valued solution $u_{\mathcal{T}, k}$

In this section, we will establish the convergence (up to a subsequence) of the finite volume approximation $u_{\mathcal{T}, k}$, defined by (9) and (10), towards a measure-valued entropy solution of Problem (1). Following [BCG16-2], we will start with classical preliminary results : stability estimate on $(u_{\mathcal{T}, k})$ (used at the end of this section to get compactness in the sense of Young measures), weak BV estimates on the discrete unknowns u_K^n (as in the deterministic setting to get convergence of the scheme) and finally an error bound of a time-continuous approximation. We will then be able to provide successively discrete and continuous entropy inequalities, which, together with the compactness property in the sense of Young measures, will enable us to prove the convergence, up to a subsequence, of the finite volume approximation $(u_{\mathcal{T}, k})$ to a measure-valued solution of Problem (1). Note that we will extend here to the case of a general flux function $f(x, t, v)$ the results established in [BCG16-2] with a flux function of the form $\bar{v}(x, t)f(v)$. To get such a generalization, we will adapt some technics from [CH00]. The results and proofs of this section follow closely the work [BCG16-2], but as we will need later (Section 4) to know explicitly the dependance of the constants appearing in the majorations of terms involving the entropy η and the test function φ , we will give here the main steps of the proofs.

3.1 Stability estimate

Proposition 1 ($L_t^\infty L_{\omega, x}^2$ estimate) *Let $T > 0$, $u_0 \in L^2(\mathbb{R}^d)$, \mathcal{T} be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^*$ and $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the Courant-Friedrichs-Levy (CFL) condition:*

$$k \leq \frac{\bar{\alpha}^2 h}{F_1 + F_2}. \quad (16)$$

Let $u_{\mathcal{T}, k}$ be the finite volume approximate solution defined by (9) and (10). Then, we have the following bound:

$$\|u_{\mathcal{T}, k}\|_{L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d))} \leq e^{C_g^2 T/2} \|u_0\|_{L^2(\mathbb{R}^d)}.$$

As a consequence, we get:

$$\|u_{\mathcal{T}, k}\|_{L^2(\Omega \times Q)}^2 \leq T e^{C_g^2 T} \|u_0\|_{L^2(\mathbb{R}^d)}^2.$$

Proof. Following [BCG16-2], we show by induction on $n \in \{0, \dots, N-1\}$ the following property:

$$\sum_{K \in \mathcal{T}} |K| E[(u_K^n)^2] \leq (1 + k C_g^2)^n \|u_0\|_{L^2(\mathbb{R}^d)}^2. \quad (P_n)$$

Set $n \in \{0, \dots, N-1\}$ and assume that (P_n) holds. Let us multiply the numerical scheme (9) by u_K^n , use the formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1} - u_K^n$ and $b = u_K^n$, replace $(u_K^{n+1} - u_K^n)^2$ by using the finite volume scheme (9) and take the expectation. Then, thanks to the independence between the random variables $(W^{n+1} - W^n)$ and u_K^n , we get:

$$\frac{|K|}{2} E[(u_K^{n+1})^2 - (u_K^n)^2] = B_1 - B_2 + D, \quad (17)$$

where:

$$B_1 = \frac{k^2}{2|K|} E \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) \right)^2 \right], B_2 = kE \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) u_K^n \right] \text{ and } D = \frac{|K|k}{2} E[g^2(u_K^n)].$$

Let us now define

$$B_3 = k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| E \left[u_K^n \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\} - u_L^n \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_L^n, u_L^n) \right\} \right].$$

Using the fact that for any $(a, b) \in \mathbb{R}^2$, $F_{K,L}^n(a, b) = -F_{L,K}^n(b, a)$, and (11), one gets that $\sum_{K \in \mathcal{T}} B_2 = B_3$.

We denote by $\phi_{K,L}^n$ the function defined for any $a \in \mathbb{R}$ by $\phi_{K,L}^n(a) = \int_0^a v \frac{d}{dv} \left(F_{K,L}^n(v, v) \right) dv$. Using this function we can rewrite the terms of the sum in B_3 as follows:

$$\begin{aligned} u_K^n \left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right) - u_L^n \left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_L^n, u_L^n) \right) \\ = \int_{u_K^n}^{u_L^n} \left(F_{K,L}^n(s, s) - F_{K,L}^n(u_K^n, u_L^n) \right) ds - \left(\phi_{K,L}^n(u_K^n) - \phi_{K,L}^n(u_L^n) \right). \end{aligned}$$

Thus, the term B_3 can be decomposed as $B_3 = B_4 - B_5$ where:

$$B_4 = E \left[\sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \int_{u_K^n}^{u_L^n} \left(F_{K,L}^n(s, s) - F_{K,L}^n(u_K^n, u_L^n) \right) ds \right]$$

and

$$B_5 = E \left[\sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \left\{ \phi_{K,L}^n(u_K^n) - \phi_{K,L}^n(u_L^n) \right\} \right] = 0,$$

thanks to (11).

So, $B_3 = B_4$. In order to get an estimate of B_4 , we now use the following technical lemma, which is a slight generalization of a result from [EGH00] (Lemma 4.5 p.107) in the sense that we don't need Lipschitz-continuous assumption on the considered function.

Lemma 1 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone function such that there exists $G > 0$ and $d \in \mathbb{R}$ such that for all $s \in \mathbb{R}$, $|g(d) - g(s)| \leq G|d - s|$. Then, for any $c \in \mathbb{R}$:*

$$\left| \int_c^d (g(s) - g(c)) ds \right| \geq \frac{1}{2G} (g(d) - g(c))^2, \forall c \in \mathbb{R}.$$

The proof of this lemma is exactly the same as in [EGH00].

Using the monotony of $F_{K,L}^n$ w.r.t its variables and its regularity, we apply this lemma with $d = u_K^n$, $c = u_L^n$, $G = F_2$ and $g(v) = F_{K,L}^n(u_K^n, v)$ and then with $d = u_L^n$, $c = u_K^n$, $G = F_1$ and $g(v) = F_{K,L}^n(v, u_L^n)$. We deduce that

$$B_3 = B_4 \geq E \left[\sum_{(K,L) \in \mathcal{I}_n} \frac{k |\sigma_{K,L}|}{2(F_1 + F_2)} \left\{ \left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right)^2 + \left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_L^n, u_L^n) \right)^2 \right\} \right].$$

Let us now turn to the study of B_1 . Using Cauchy-Schwarz inequality and (5), we have:

$$\begin{aligned} \sum_{K \in \mathcal{T}} B_1 &= E \left[\frac{k^2}{2} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\} \right)^2 \right] \\ &\leq E \left[\frac{k^2}{2\bar{\alpha}^2 h} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\}^2 \right]. \end{aligned}$$

Finally, using the CFL Condition (16), we get

$$\sum_{K \in \mathcal{T}} B_1 \leq E \left[\sum_{(K,L) \in \mathcal{I}_n} \frac{k |\sigma_{K,L}|}{2(F_1 + F_2)} \left\{ \left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right)^2 + \left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_L^n, u_L^n) \right)^2 \right\} \right],$$

and so, $\sum_{K \in \mathcal{T}} B_1 \leq B_3 = \sum_{K \in \mathcal{T}} B_2$, which provides by induction the first stability estimate and the second estimate is a direct consequence of the first one. ■

3.2 Weak BV estimates

In what follows, we take two real numbers $R > 0$ and $h > 0$ such that $h < R$.

Proposition 2 (Weak BV estimates) *Let \mathcal{T} be an admissible mesh in the sense of Definition 3, $T > 0$, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the CFL Condition:*

$$k \leq (1 - \xi) \frac{\bar{\alpha}^2 h}{F_1 + F_2}, \text{ for some } \xi \in (0, 1). \quad (18)$$

Let $u_{\mathcal{T},k}$ be the finite volume approximate solution defined by (9) and (10). Then, we have the three following bounds:

1. There exists $C_{BV,1} \in \mathbb{R}_+^*$, only depending on T, u_0, ξ, F_1, F_2 and C_g such that:

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| E \left[\left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right)^2 \right] \leq C_{BV,1}.$$

2. There exists $C_{BV,2} \in \mathbb{R}_+^*$, only depending on $R, d, T, \bar{\alpha}, u_0, \xi, F_1, F_2$ and C_g such that:

$$\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| E \left[\max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right) + \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right) \right] \leq C_{BV,2} h^{-1/2}.$$

3. There exists $C_{BV,3} \in \mathbb{R}_+^*$, only depending on $R, d, T, \bar{\alpha}, u_0, \xi, F_1, F_2$ and C_g such that:

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} |K| \times E \left[|u_K^{n+1} - u_K^n| \right] \leq C_{BV,2} \times h^{-1/2} + C_{BV,3} \times k^{-1/2}.$$

Proof. We give here the main steps of the proof, adapted from the deterministic case (see for example [CLF93] or [EGH00]). Note that points 1. and 2. are generalizations of results from [BCG16-2], and point 3. is a time weak BV estimate. Similarly to the proof of Proposition 1, multiplying the numerical scheme by ku_K^n , taking the expectation, summing over $K \in \mathcal{T}$ and $n \in \{0, \dots, N-1\}$, using the independence properties of the Brownian motion yields:

$$A_1 + A_2 + B = 0,$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left\{ k|K| E \left[g^2(u_K^n) \right] + \frac{k^2}{|K|} E \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) \right)^2 \right] \right\}, \\ A_2 &= \frac{1}{2} \sum_{K \in \mathcal{T}} |K| E \left[\left(u_K^N \right)^2 - \left(u_K^0 \right)^2 \right] \\ \text{and } B &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} k |\sigma_{K,L}| E \left[F_{K,L}^n(u_K^n, u_L^n) u_K^n \right]. \end{aligned}$$

Using Cauchy-Schwarz inequality, the CFL Condition (18), inequality (5) and the stability result in Proposition 1 one gets:

$$\begin{aligned} A_1 + A_2 &\geq \\ &- \frac{1}{2} \left(1 + TC_g^2 e^{TC_g^2} \right) \|u_0\|_{L^2(\mathbb{R}^d)}^2 \\ &+ \frac{(\xi - 1)}{2(F_1 + F_2)} \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| E \left[\max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right)^2 + \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right)^2 \right]. \end{aligned}$$

We now study the term B . As in the proof of Proposition 1, by using the function $\phi_{K,L}^n$, one has:

$$B = \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| E \left[\int_{u_K^n}^{u_L^n} \left(F_{K,L}^n(v, v) - F_{K,L}^n(u_K^n, u_L^n) \right) dv \right].$$

We now use again Lemma 1, the monotonicity of $F_{K,L}^n$, and deduce:

$$\begin{aligned} \int_{u_K^n}^{u_L^n} \left(F_{K,L}^n(v, v) - F_{K,L}^n(u_K^n, u_L^n) \right) dv &\geq \frac{1}{2(F_1 + F_2)} \left[\max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(c, c) - F_{K,L}^n(d, d) \right)^2 \right. \\ &\quad \left. + \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, d) - F_{K,L}^n(d, c) \right)^2 \right]. \end{aligned}$$

Now, since $A_1 + A_2 + B = 0$, we obtain:

$$\begin{aligned} \frac{1}{2} \left(1 + TC_g^2 e^{TC_g^2} \right) \|u_0\|_{L^2(\mathbb{R}^d)}^2 &\geq \frac{\xi}{2(F_1 + F_2)} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| E \left[\max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right)^2 \right. \\ &\quad \left. + \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right)^2 \right]. \end{aligned}$$

By reordering the summation and defining $C_{BV,1} = \frac{F_1 + F_2}{\xi} \left(1 + TC_g^2 e^{TC_g^2} \right) \|u_0\|_{L^2(\mathbb{R}^d)}^2$, we have in particular:

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| E \left[\left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right)^2 \right] \leq C_{BV,1},$$

which proves the first point of the proposition.

Let us now turn to the second point. Let us note:

$$T_1 = \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right) \text{ and } T_2 = \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right).$$

Using Cauchy-Schwarz inequality and the mesh properties (5), one gets:

$$\left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| E [T_1 + T_2] \right)^2 \leq \frac{T|B(0, R)|}{\bar{\alpha}^2 h} \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| E \left[(T_1 + T_2)^2 \right] \right), \quad (19)$$

where the set \mathcal{I}_R^n is defined by (14). Using the first point of Proposition 2, one gets:

$$\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| E \left[(T_1 + T_2)^2 \right] \leq 2C_{BV,1}.$$

Thus, we deduce from the point 1. of Proposition 2:

$$\left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| E [T_1 + T_2] \right)^2 \leq \frac{2T|B(0, R)|C_{BV,1}}{\bar{\alpha}^2 h}.$$

Therefore, we have:

$$\begin{aligned} & \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| E \left[\max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right) + \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right) \right] \\ & \leq C_{BV,2} \times h^{-1/2}, \end{aligned} \quad (20)$$

with $C_{BV,2} = \sqrt{\frac{2T|B(0, R)|C_{BV,1}}{\bar{\alpha}^2}}$, which concludes the proof of point 2. of the Proposition.

Finally, we are going to prove the last point of the Proposition: the time weak BV estimate. Using the numerical scheme (9), we have:

$$\begin{aligned} |K| \times |u_K^{n+1} - u_K^n| &= \left| -k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) + |K|g(u_K^n)(W^{n+1} - W^n) \right| \\ &\leq k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left| F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right| + |K| |g(u_K^n)| |W^{n+1} - W^n|. \end{aligned}$$

We sum this inequality for $K \in \mathcal{T}_R$ (where the set \mathcal{T}_R is defined by (13)), $n \in \{0, \dots, N-1\}$ and take the expectation.

On the one hand, we get:

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| E \left[\left| F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right| \right] &= \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| E \left[\left| F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right| \right. \\ &\quad \left. + \left| F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_L^n, u_L^n) \right| \right] \\ &\leq C_{BV,2} \times h^{-1/2}, \end{aligned}$$

where $C_{BV,2}$ (resp. \mathcal{I}_R^n) is defined in inequality (20) (resp. (14)). On the other hand, using Cauchy-Schwarz inequality, one has:

$$\begin{aligned} \sum_{n=0}^{N-1} E \left[\sum_{K \in \mathcal{T}_R} |K| |g(u_K^n)| |W^{n+1} - W^n| \right] &\leq \sum_{n=0}^{N-1} \left(\sum_{K \in \mathcal{T}_R} |K| \right)^{1/2} \left(\sum_{K \in \mathcal{T}_R} |K| E \left[g^2(u_K^n) (W^{n+1} - W^n)^2 \right] \right)^{1/2} \\ &\leq \sqrt{|B(0, R)|} C_g \times k^{1/2} \sum_{n=0}^{N-1} \left(\sum_{K \in \mathcal{T}_R} |K| E [(u_K^n)^2] \right)^{1/2} \\ &\leq C_{BV,3} \times k^{-1/2}, \end{aligned}$$

with $C_{BV,3} = \sqrt{|B(0, R)|} C_g T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)}$; then, we get that:

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} |K| \times E [u_K^{n+1} - u_K^n] \leq C_{BV,2} \times h^{-1/2} + C_{BV,3} \times k^{-1/2},$$

which concludes the proof. ■

3.3 A time-continuous approximation

Set $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$, as in [BCG16-2], in view of applying Itô' formula, we introduce a time-continuous process \bar{u}_K^n defined on $\Omega \times [nk, (n+1)k]$ from the discrete unknowns u_K^n by:

$$\bar{u}_K^n(s) = u_K^n - \int_{nk}^s \frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) dt + \int_{nk}^s g(u_K^n) dW(t) \quad (21)$$

$$= u_K^n - \frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\} + g(u_K^n) (W(s) - W^{nk}). \quad (22)$$

In this way, we can define a time-continuous approximate solution $\bar{u}_{\mathcal{T},k}$ on $\Omega \times \mathbb{R}^d \times [0, T]$ by

$$\bar{u}_{\mathcal{T},k}(\omega, x, t) = \bar{u}_K^n(\omega, t), \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k]. \quad (23)$$

Naturally, our aim is to estimate in what follows the error between the approximations $\bar{u}_{\mathcal{T},k}$ and $u_{\mathcal{T},k}$.

Proposition 3 (A time-continuous approximation) *Let $u_0 \in L^2(\mathbb{R}^d)$ and \mathcal{T} be an admissible mesh in the sense of Definition 3, $T > 0$, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step satisfying the CFL Condition (18). Let $\bar{u}_{\mathcal{T},k}$ be the time-continuous approximate solution defined by (21), and $u_{\mathcal{T},k}$ be the finite volume approximate solution defined by (9) and (10). Then, there exists a constant $C_{TA} \in \mathbb{R}_+^*$ depending only on $T, C_g, F_1, F_2, \bar{\alpha}, u_0$ and ξ such that:*

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 \leq C_{TA} k.$$

Note that this estimate is sharper than the one given in [BCG16-2], where the bound was equal, up to a constant, to $h + k$.

Proof. Using the equivalent definition (22) of $\bar{u}_{\mathcal{T},k}$, we have

$$\begin{aligned} & \|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 \\ &= \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} |K| E \left[g^2(u_K^n) (W(s) - W^n)^2 \right] ds \\ & \quad + \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} |K| E \left[\left(\frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\} \right)^2 \right] ds \\ & \leq \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} |K| k^2 C_g^2 E \left[(u_K^n)^2 \right] + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} k |K| \frac{k^2}{|K|^2} E \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\} \right)^2 \right] \\ & \leq k C_g^2 \|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 + k^2 \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \frac{|\partial K|}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| E \left[\left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right)^2 \right] \right). \end{aligned}$$

Due to the properties of the mesh (5) and the CFL Condition (18), one shows that $\frac{|\partial K|}{|K|} \leq \frac{1}{\bar{\alpha}^2 h} \leq \frac{(1-\xi)}{(F_1 + F_2)k}$.

Moreover, thanks to the first point of Proposition 2 and the $L_{\omega,x,t}^2$ estimate given by Proposition 1, one gets:

$$\begin{aligned} \|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 & \leq k C_g^2 \|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 + \frac{(1-\xi)}{(F_1 + F_2)k} C_{BV,1} k^2 \\ & \leq C_{TA} k, \end{aligned}$$

where $C_{TA} = \max \left\{ T C_g^2 e^{TC_g^2} \|u_0\|_{L^2(\mathbb{R}^d)}^2; \frac{(1-\xi)}{(F_1 + F_2)} C_{BV,1} \right\}$, which concludes the proof. ■

3.4 Discrete entropy inequalities on the approximate solution

In order to obtain entropy inequalities for $(u_{\mathcal{T},k})$, some discrete entropy inequalities satisfied by the discrete unknowns u_K^n are first derived in the upcoming propositions. Following [BCG16-2], we start in a first step (for technical reasons due to the stochastic noise) with the particular case of a Godunov numerical scheme. Then, we will generalize in a second step the obtained inequality to any monotone finite volume scheme. Indeed, we will use the fact that any monotone numerical flux can be decomposed as a convex combination between a Godunov numerical flux and a modified Lax-Friedrichs' one (the proof of such a splitting can be found in [CH00] for example).

3.4.1 The case of a family of Godunov's numerical fluxes

We assume in this subsection that, for all $K \in \mathcal{T}$, for all $L \in \mathcal{N}(K)$ and for all $n \in \{0, \dots, N-1\}$, $F_{K,L}^n$ is the Godunov's numerical flux defined as follows: for all $a, b \in \mathbb{R}$, if $a > b$ (resp. $a < b$), there exists $s_{K,L}^n(a, b) \in [b, a]$ (resp. $[a, b]$) such that:

$$F_{K,L}^n(a, b) = f_{K,L}^n(s_{K,L}^n(a, b)) = \max \left\{ f_{K,L}^n(v), v \in [b, a] \right\} \quad \left(\text{resp. } \min \left\{ f_{K,L}^n(v), v \in [a, b] \right\} \right). \quad (24)$$

Then, fixing $\eta \in \mathcal{A}$, and denoting for all $v \in \mathbb{R}$:

$$\Phi_{K,L}^n(v) = \int_0^v \eta'(\sigma) (f_{K,L}^n)'(\sigma) d\sigma,$$

we can define the entropy numerical flux $G_{K,L}^n$ by $G_{K,L}^n(a, b) = \Phi_{K,L}^n(s_{K,L}^n(a, b))$, for all $a, b \in \mathbb{R}$.

Remark 7 From (11), we deduce that for all $a \in \mathbb{R}$, $G_{K,L}^n(a, a) = \Phi_{K,L}^n(a)$ and for all $s \in \mathbb{R}$,

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \Phi_{K,L}^n(s) = \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| G_{K,L}^n(s, s) = 0.$$

Proposition 4 (Discrete entropy inequalities) Assume that hypotheses H_1 to H_7 hold and that for all $n \in \{0, \dots, N-1\}$ and $(K, L) \in \mathcal{T}_R$, $(F_{K,L}^n)_{n,K,L}$ is the family of Godunov's fluxes associated with the flux f . Let $u_0 \in L^2(\mathbb{R}^d)$ and \mathcal{T} be an admissible mesh in the sense of Definition 3, $T > 0$, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step satisfying the CFL Condition (18) with $F_1 = F_2 = C_f$. Then, there exists a constant $C_{DE,G}$ depending only on $R, d, \bar{\alpha}, u_0, \xi, F_1, F_2, C_f^R, C_f^T, \|g\|_\infty$ and C_g such that P-a.s in Ω , for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \frac{\eta(u_K^{n+1}) - \eta(u_K^n)}{k} \varphi(x, nk) dx dt + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \Phi_\eta(x, t, u_K^n, 0) \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dx dW(t) + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) dx dt \\ & \geq \tilde{R}_G^{h,k,\eta,\varphi}, \end{aligned} \quad (25)$$

where for any P-measurable set A :

$$\begin{aligned} \left| E \left[\mathbb{1}_A \tilde{R}_G^{h,k,\eta,\varphi} \right] \right| & \leq C_{DE,G} \left[\|\eta''\|_\infty \|\varphi\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty \times h^{1/2} \right. \\ & \left. + (\|\partial_t \varphi\|_\infty \|\eta'\|_\infty + \|\varphi\|_\infty \|\eta''\|_\infty) \times k^{1/2} + \|\varphi\|_\infty \|\eta'''\|_\infty \times k \right]. \end{aligned}$$

Proof. In order to prove this proposition, we are going to show firstly that inequality (38) holds for a certain $\tilde{R}_G^{h,k,\eta,\varphi}$ and in a second time, we will estimate the quantity $E[\mathbb{1}_A \tilde{R}_G^{h,k,\eta,\varphi}]$ for a given P-measurable set A . We will use in particular some techniques developed in the deterministic framework (see for example [EGH00], [CH00] or [CCLF95]) and deepen their adaptation to our stochastic case.

Let $T > 0$, $u_0 \in L^2(\mathbb{R}^d)$, \mathcal{T} be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^*$ and $k = \frac{T}{N} \in \mathbb{R}_+^*$. We assume that the CFL Condition (18) holds for some $\xi \in (0, 1)$. In this manner, the estimates given by Proposition 1 and 2 hold. Consider $\eta \in \mathcal{A}$ and $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$, thus there exists $R > h$ such that $\text{supp } \varphi \subset B(0, R-h) \times [0, T]$.

STEP I: Existence of $\tilde{R}_G^{h,k,\eta,\varphi}$

Let $K \in \mathcal{T}$ and $n \in \{0, \dots, N-1\}$. The application of Itô's formula to the process \bar{u}_K^n and the function $v \in \mathbb{R} \mapsto \eta(v) \in \mathbb{R}$ on the interval $[nk, (n+1)k]$ yields P-a.s in Ω :

$$\begin{aligned} \eta(\bar{u}_K^n((n+1)k)) - \eta(\bar{u}_K^n(nk)) & = - \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) \frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) dt \\ & \quad + \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) g(u_K^n) dW(t) + \frac{1}{2} \int_{nk}^{(n+1)k} \eta''(\bar{u}_K^n(t)) g^2(u_K^n) dt. \end{aligned}$$

Let us multiply this equation by $|K| \varphi_K^n$, where $\varphi_K^n = \frac{1}{|K|} \int_K \varphi(x, nk) dx$, and sum for all $K \in \mathcal{T}_R$ and $n \in \{0, \dots, N-1\}$. Thus, one gets P-a.s in Ω :

$$0 = A^{h,k,\eta,\varphi} - B^{h,k,\eta,\varphi} + C^{h,k,\eta,\varphi} + D^{h,k,\eta,\varphi},$$

where:

$$\begin{aligned}
A^{h,k,\eta,\varphi} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} |K| \varphi_K^n \left\{ \eta(u_K^{n+1}) - \eta(u_K^n) \right\} = - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \frac{\eta(u_K^{n+1}) - \eta(u_K^n)}{k} \varphi(x, nk) dx dt \\
B^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) \varphi_K^n dt, \\
C^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} |K| \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) g(u_K^n) \varphi_K^n dW(t) \\
\text{and } D^{h,k,\eta,\varphi} &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} |K| \int_{nk}^{(n+1)k} \eta''(\bar{u}_K^n(t)) g^2(u_K^n) \varphi_K^n dt.
\end{aligned}$$

Let us analyze separately the last three terms. We decompose $B^{h,k,\eta,\varphi}$ in the following way:

$$B^{h,k,\eta,\varphi} = \left(B^{h,k,\eta,\varphi} - \tilde{B}^{h,k,\eta,\varphi} \right) + \left(\tilde{B}^{h,k,\eta,\varphi} - B_G^{h,k,\eta,\varphi} \right) + \left(B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi} \right) + B_\Phi^{h,k,\eta,\varphi},$$

where:

$$\begin{aligned}
\tilde{B}^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta'(u_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) \varphi(x, nk) dx dt, \\
B_G^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| G_{K,L}^n(u_K^n, u_L^n) \varphi(x, nk) dx dt \\
\text{and } B_\Phi^{h,k,\eta,\varphi} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \Phi_\eta(x, t, u_K^n, 0) \cdot \nabla_x \varphi(x, nk) dx dt.
\end{aligned}$$

We first show that $\tilde{B}^{h,k,\eta,\varphi} - B_G^{h,k,\eta,\varphi} \geq 0$ almost surely.

Using Remarks 2 and 7, we can rewrite $\tilde{B}^{h,k,\eta,\varphi} - B_G^{h,k,\eta,\varphi}$ in the following way:

$$\begin{aligned}
\tilde{B}^{h,k,\eta,\varphi} - B_G^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \frac{k}{|K|} \int_K \varphi(x, nk) dx \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ \eta'(u_K^n) \left(F_{K,L}^n(u_K^n, u_L^n) - G_{K,L}^n(u_K^n, u_K^n) \right) \right. \\
&\quad \left. - \left(G_{K,L}^n(u_K^n, u_L^n) - G_{K,L}^n(u_K^n, u_K^n) \right) \right\}.
\end{aligned}$$

Following [BCG16-2], we use the fact that our numerical fluxes are the Godunov's one. Indeed, thanks to their explicit definitions given by (24), one can affirm that the numerical entropy fluxes $(G_{K,L}^n)_{n,K,L}$, satisfy for any $(K, L) \in \mathcal{I}_n$:

$$\eta'(u_K^n) \left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right) - \left(G_{K,L}^n(u_K^n, u_L^n) - G_{K,L}^n(u_K^n, u_K^n) \right) \geq 0, \quad (26)$$

and, thus:

$$B^{h,k,\eta,\varphi} \geq B^{h,k,\eta,\varphi} - \tilde{B}^{h,k,\eta,\varphi} + B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi} + B_\Phi^{h,k,\eta,\varphi}.$$

Conclusion of STEP I: Since P-a.s in Ω , $A^{h,k,\eta,\varphi} - B^{h,k,\eta,\varphi} + C^{h,k,\eta,\varphi} + D^{h,k,\eta,\varphi} = 0$, we get with the study of $B^{h,k,\eta,\varphi}$:

$$\begin{aligned}
A^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi} + \tilde{C}^{h,k,\eta,\varphi} + \tilde{D}^{h,k,\eta,\varphi} &\geq (B^{h,k,\eta,\varphi} - \tilde{B}^{h,k,\eta,\varphi}) + (B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi}) \\
&\quad + (\tilde{C}^{h,k,\eta,\varphi} - C^{h,k,\eta,\varphi}) + (\tilde{D}^{h,k,\eta,\varphi} - D^{h,k,\eta,\varphi}),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{C}^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_K \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW(t) dx \\
\text{and } \tilde{D}^{h,k,\eta,\varphi} &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_K \int_{nk}^{(n+1)k} \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) dt dx.
\end{aligned}$$

In this way, we get inequality (38), with:

$$\tilde{R}_G^{h,k,\eta,\varphi} = (B^{h,k,\eta,\varphi} - \tilde{B}^{h,k,\eta,\varphi}) + (B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi}) + (\tilde{C}^{h,k,\eta,\varphi} - C^{h,k,\eta,\varphi}) + (\tilde{D}^{h,k,\eta,\varphi} - D^{h,k,\eta,\varphi}). \quad (27)$$

STEP II: Estimate of $E \left[\mathbb{1}_A \tilde{R}_G^{h,k,\eta,\varphi} \right]$

Let A be a P-measurable set. In this second step, we estimate the term $E \left[\mathbb{1}_A \tilde{R}_G^{h,k,\eta,\varphi} \right]$.

II.1 Study of $E \left[\mathbb{1}_A(B^{h,k,\eta,\varphi} - \tilde{B}^{h,k,\eta,\varphi}) \right]$

For almost all $\omega \in \Omega$, $t \in (nk, (n+1)k)$, any $K \in \mathcal{T}$ and any $n \in \{0, \dots, N-1\}$, there exists $v_K^n(w, t) \in \mathbb{R}$ such that:

$$\eta'(\bar{u}_K^n(t)) - \eta'(u_K^n) = \eta''(v_K^n(w, t))(\bar{u}_{\mathcal{T},k}(t) - u_K^n).$$

So, $B^{h,k,\eta,\varphi} - \tilde{B}^{h,k,\eta,\varphi} = T_1^{h,k,\eta,\varphi} - T_2^{h,k,\eta,\varphi}$ where:

$$T_1^{h,k,\eta,\varphi} = - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(v_K^n(t)) \frac{t-nk}{|K|} \varphi(x, nk) dx dt \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) \right)^2$$

and

$$T_2^{h,k,\eta,\varphi} = \frac{1}{|K|} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta''(v_K^n(t)) g(u_K^n) (W(t) - W(nk)) \varphi(x, nk) dx dt \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) \right).$$

We first study $E \left[\mathbb{1}_A T_1^{h,k,\eta,\varphi} \right]$. Using Remark 2, Cauchy-Schwarz inequality, the properties of the mesh (5) and the point 1. of Proposition 2 (which gives the existence of a constant $C_{BV,1}$), we get:

$$\begin{aligned} \left| E \left[\mathbb{1}_A T_1^{h,k,\eta,\varphi} \right] \right| &= \left| E \left[\mathbb{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(v_K^n(t)) \frac{t-nk}{|K|} \varphi(x, nk) dx dt \right. \right. \\ &\quad \left. \left. \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\} \right)^2 \right] \right| \\ &\leq \|\eta''\|_\infty \|\varphi\|_\infty \frac{k}{\bar{\alpha}^2 h} C_{BV,1}. \end{aligned}$$

Let us now estimate $E \left[\mathbb{1}_A T_2^{h,k,\eta,\varphi} \right]$. Using the same arguments as previously, one gets

$$\begin{aligned} \left(E \left[\mathbb{1}_A T_2^{h,k,\eta,\varphi} \right] \right)^2 &= \left(E \left[\mathbb{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(v_K^n(t)) g(u_K^n) (W(t) - W(nk)) \right. \right. \\ &\quad \left. \left. \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) \right) \varphi(x, nk) dx dt \right] \right)^2 \\ &\leq \|\eta''\|_\infty^2 \|\varphi\|_\infty^2 C_g^2 \|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \frac{k^2}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \right) \\ &\quad \times E \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\}^2 \right] \\ &\leq \left(\|\eta''\|_\infty^2 \|\varphi\|_\infty^2 C_g^2 T e^{TC_g^2} \|u_0\|_{L^2(\mathbb{R}^d)}^2 \frac{C_{BV,1}}{\bar{\alpha}^2} \right) \times \frac{k}{h}. \end{aligned}$$

Gathering these two estimates, we get

$$\left| E \left[\mathbb{1}_A(B^{h,k,\eta,\varphi} - \tilde{B}^{h,k,\eta,\varphi}) \right] \right| \leq \|\eta''\|_\infty \|\varphi\|_\infty \frac{C_{BV,1} k}{\bar{\alpha}^2 h} + \left(\|\eta''\|_\infty \|\varphi\|_\infty C_g T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} \frac{\sqrt{C_{BV,1}}}{\bar{\alpha}} \right) \sqrt{\frac{k}{h}}.$$

II.2 Study of $E \left[\mathbb{1}_A(B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi}) \right]$

To begin with, we split $B_G^{h,k,\eta,\varphi}$ and $B_\Phi^{h,k,\eta,\varphi}$ into the sum of two terms. Using Remark 7 and the anti-symmetry property of the numerical flux, we can rewrite:

$$B_G^{h,k,\eta,\varphi} = \bar{T}_1^{h,k,\eta,\varphi} - \bar{T}_2^{h,k,\eta,\varphi},$$

where:

$$\begin{aligned} \bar{T}_1^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \frac{k}{|K|} |\sigma_{K,L}| \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right\} \int_K \varphi(x, nk) dx \\ \text{and } \bar{T}_2^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \frac{k}{|L|} |\sigma_{K,L}| \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_L^n) \right\} \int_L \varphi(x, nk) dx. \end{aligned}$$

Let us note that using Assumption H_6 , we have for all $(x, t) \in Q$ and $a, b \in \mathbb{R}$:

$$\operatorname{div}_x \left[\Phi_\eta(x, t, a, b) \right] = 0. \quad (28)$$

Then, $B_\Phi^{h,k,\eta,\varphi}$ can be rewritten in the following way:

$$\begin{aligned} B_\Phi^{h,k,\eta,\varphi} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \operatorname{div}_x \left[\Phi_\eta(x, t, u_K^n, 0) \varphi(x, nk) \right] dx dt \\ &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \sum_{L \in \mathcal{N}(K)} \int_{\sigma_{K,L}} \left(\Phi_\eta(x, t, u_K^n, 0) \cdot n_{K,L} \right) \varphi(x, nk) d\gamma(x) dt \\ &= T_1^{h,k,\eta,\varphi} - T_2^{h,k,\eta,\varphi}, \end{aligned}$$

where:

$$\begin{aligned} T_1^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{T}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_\eta(x, t, u_K^n, 0) \cdot n_{K,L} \right\} \varphi(x, nk) d\gamma(x) dt \\ \text{and } T_2^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{T}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_\eta(x, t, u_L^n, 0) \cdot n_{K,L} \right\} \varphi(x, nk) d\gamma(x) dt. \end{aligned}$$

Now, our aim is to estimate $|\bar{T}_1^{h,k,\eta,\varphi} - T_1^{h,k,\eta,\varphi}|$ and $|\bar{T}_2^{h,k,\eta,\varphi} - T_2^{h,k,\eta,\varphi}|$. To do this, we first note that we have:

$$\begin{aligned} \bar{T}_1^{h,k,\eta,\varphi} - T_1^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{T}_R^n} k |\sigma_{K,L}| \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right\} \left(\frac{1}{|K|} \int_K \varphi(x, nk) dx - \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) \right) \\ &\quad + \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{T}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_\eta(x, t, u_K^n, 0) \cdot n_{K,L} - \Phi_{K,L}^n(u_K^n) \right\} \varphi(x, nk) d\gamma(x) dt. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \bar{T}_2^{h,k,\eta,\varphi} - T_2^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{T}_R^n} k |\sigma_{K,L}| \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_L^n) \right\} \left(\frac{1}{|L|} \int_L \varphi(x, nk) dx - \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) \right) \\ &\quad + \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{T}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_\eta(x, t, u_L^n, 0) \cdot n_{K,L} - \Phi_{K,L}^n(u_L^n) \right\} \varphi(x, nk) d\gamma(x) dt. \end{aligned}$$

Let us now bound $|\bar{T}_1^{h,k,\eta,\varphi} - T_1^{h,k,\eta,\varphi}|$ and $|\bar{T}_2^{h,k,\eta,\varphi} - T_2^{h,k,\eta,\varphi}|$.

- Let us begin with the estimate of $G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n)$ and $G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_L^n)$. Set $(K, L) \in \mathcal{I}_R^n$, we then have $u_K^n > u_L^n$ which implies that there exists $s_{K,L}^n(u_K^n, u_L^n) \in [u_L^n, u_K^n]$ such that

$$f_{K,L}^n(s_{K,L}^n(u_K^n, u_L^n)) = \max_{s \in [u_L^n, u_K^n]} f_{K,L}^n(s)$$

and hence:

$$G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) = \Phi_{K,L}^n(s_{K,L}^n(u_K^n, u_L^n)) - \Phi_{K,L}^n(u_K^n) = \int_{u_K^n}^{s_{K,L}^n(u_K^n, u_L^n)} \eta'(v) \left(f_{K,L}^n \right)'(v) dv.$$

Using an integration by parts formula, the convexity of η and the fact that $(F_{K,L}^n)$ is a family of Godunov fluxes, we have for any $(K, L) \in \mathcal{I}_R^n$:

$$\left| G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right| \leq 2 \|\eta'\|_\infty \max_{u_L^n \leq c \leq d \leq u_K^n} \left| F_{K,L}^n(d, c) - F_{K,L}^n(L, d) \right|. \quad (29)$$

$$\left| G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_L^n) \right| \leq 2 \|\eta'\|_\infty \max_{u_L^n \leq c \leq d \leq u_K^n} \left| F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right|. \quad (30)$$

- We get easily that for any $K \in \mathcal{T}$:

$$\left| \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) - \frac{1}{|K|} \int_K \varphi(x, nk) dx \right| \leq h \|\nabla_x \varphi\|_\infty. \quad (31)$$

- Let us now bound $\sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_\eta(x, t, u_K^n, 0) \cdot n_{K,L} - \Phi_{K,L}^n(u_K^n) \right\} \varphi(x, nk) d\gamma(x) dt$.

First, note that, thanks to the definition of $\Phi_{K,L}^n$, this term is equal to:

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_\eta(x, t, u_K^n, 0) \cdot n_{K,L} - \Phi_{K,L}^n(u_K^n) \right\} \varphi(x, nk) d\gamma(x) dt \\ = & \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \frac{1}{k|\sigma_{K,L}|} \left(\int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_\eta(x, t, u_K^n, 0) - \Phi_\eta(y, s, u_K^n, 0) \right\} \cdot n_{K,L} d\gamma(y) ds \right) \varphi(x, nk) d\gamma(x) dt. \end{aligned}$$

Using Assumptions H₄ and H₅, we deduce that for any $x, y \in B(0, R)$, $t, s \in [0, T]$ and $a, b \in \mathbb{R}$

$$|\Phi_\eta(x, t, a, b) - \Phi_\eta(x, s, a, b)| \leq 2\|\eta'\|_\infty C_f^T |t - s|, \quad (32)$$

$$|\Phi_\eta(x, t, a, b) - \Phi_\eta(y, t, a, b)| \leq 2\|\eta'\|_\infty C_f^R |x - y|. \quad (33)$$

Indeed, thanks to an integration par parts, we have

$$\begin{aligned} |\Phi_\eta(x, t, a, b) - \Phi_\eta(x, s, a, b)| & \leq |\eta'(a - b)(f(x, t, a) - f(x, s, a)) + \left| \int_b^a \eta''(\sigma - b)(f(x, t, \sigma) - f(x, s, \sigma)) d\sigma \right| \\ & \leq 2C_f^T |t - s| |\eta'(a - b)|. \end{aligned}$$

The proof of (33) is similar.

Hence, by denoting x_σ the center of the edge $\sigma_{K,L}$, we deduce that P-a.s. in Ω we have:

$$\begin{aligned} & \left| \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_\eta(x, t, u_K^n, 0) \cdot n_{K,L} - \Phi_{K,L}^n(u_K^n) \right\} \varphi(x, nk) d\gamma(x) dt \right| \\ & = \left| \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_\eta(x, t, u_K^n, 0) - \Phi_\eta(y, s, u_K^n, 0) \right\} \cdot n_{K,L} \right. \\ & \quad \times \left. \left\{ \varphi(x, nk) - \varphi(x_\sigma, nk) \right\} d\gamma(y) ds d\gamma(x) dt \right| \\ & \leq \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \sum_{L \in \mathcal{N}(K)} k|\sigma_{K,L}| 2(C_f^T k + C_f^R h) \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty \\ & \leq \|\nabla_x \varphi\|_\infty \|\eta'\|_\infty \frac{1}{\bar{\alpha}^2} T |B(0, R)| \times 2(C_f^T k + C_f^R h). \end{aligned} \quad (34)$$

- We are now ready to compare $B_G^{h,k,\eta,\varphi}$ to $B_\Phi^{h,k,\eta,\varphi}$. We first recall that:

$$B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi} = (\bar{T}_1^{h,k,\eta,\varphi} - T_1^{h,k,\eta,\varphi}) - (\bar{T}_2^{h,k,\eta,\varphi} - T_2^{h,k,\eta,\varphi}).$$

Using (29), (30), (31) and (34), a.s. in Ω , we get:

$$\begin{aligned} \left| \bar{T}_1^{h,k,\eta,\varphi} - T_1^{h,k,\eta,\varphi} \right| & \leq 2\|\eta'\|_\infty \|\nabla_x \varphi\|_\infty h \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k|\sigma_{K,L}| \left\{ \max_{u_L^n \leq c \leq d \leq u_K^n} \left| F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right| \right\} \\ & \quad + 2T |B(0, R)| (C_f^T k + C_f^R h) \frac{\|\nabla_x \varphi\|_\infty}{\bar{\alpha}^2} \\ \text{and } \left| \bar{T}_2^{h,k,\eta,\varphi} - T_2^{h,k,\eta,\varphi} \right| & \leq 2\|\eta'\|_\infty \|\nabla_x \varphi\|_\infty h \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k|\sigma_{K,L}| \left\{ \max_{u_L^n \leq c \leq d \leq u_K^n} \left| F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right| \right\} \\ & \quad + 2T |B(0, R)| (C_f^T k + C_f^R h) \frac{\|\nabla_x \varphi\|_\infty}{\bar{\alpha}^2}. \end{aligned}$$

So, combining these two inequalities and using the weak BV estimate of Proposition 2, we have:

$$\left| E \left[\mathbb{1}_A (B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi}) \right] \right| \leq 2\|\eta'\|_\infty \|\nabla_x \varphi\|_\infty C_{BV,2} h^{1/2} + \frac{2T |B(0, R)| (C_f^T k + C_f^R h) \|\nabla_x \varphi\|_\infty}{\bar{\alpha}^2}. \quad (35)$$

II.3 Study of $E \left[\mathbb{1}_A (\tilde{C}^{h,k,\eta,\varphi} - C^{h,k,\eta,\varphi}) \right]$

$$\left| E \left[\mathbb{1}_A (\tilde{C}^{h,k,\eta,\varphi} - C^{h,k,\eta,\varphi}) \right] \right| = S_1^{h,k,\eta,\varphi} + S_2^{h,k,\eta,\varphi},$$

where

$$S_1^{h,k,\eta,\varphi} = \left| E \left[\sum_{K \in \mathcal{T}_R} \sum_{n=0}^{N-1} \int_K \mathbb{1}_A \int_{nk}^{(n+1)k} \left\{ \eta'(\bar{u}_K^n(t)) - \eta'(u_K^n) \right\} g(u_K^n) \left\{ \varphi(x, nk) - \varphi(x, t) \right\} dW(t) dx \right] \right|$$

and

$$S_2^{h,k,\eta,\varphi} = \left| E \left[\sum_{K \in \mathcal{T}_R} \sum_{n=0}^{N-1} \int_K \mathbb{1}_A \int_{nk}^{(n+1)k} \left\{ \eta'(\bar{u}_K^n(t)) - \eta'(u_K^n) \right\} g(u_K^n) \varphi(x, t) dW(t) dx \right] \right|.$$

Using successively Cauchy-Schwarz inequality on $\Omega \times B(0, R)$, Itô isometry and Proposition 1, one gets:

$$\begin{aligned} S_1^{h,k,\eta,\varphi} &= \left| E \left[\sum_{K \in \mathcal{T}_R} \sum_{n=0}^{N-1} \int_K \mathbb{1}_A \int_{nk}^{(n+1)k} \left\{ \eta'(\bar{u}_K^n(t)) - \eta'(u_K^n) \right\} g(u_K^n) \left\{ \varphi(x, nk) - \varphi(x, t) \right\} dW(t) dx \right] \right| \\ &\leq \sqrt{|B(0, R)|} \sum_{n=0}^{N-1} \left\{ \sum_{K \in \mathcal{T}_R} \int_K E \left[\left(\int_{nk}^{(n+1)k} \left\{ \eta'(\bar{u}_K^n(t)) - \eta'(u_K^n) \right\} g(u_K^n) \left\{ \varphi(x, nk) - \varphi(x, t) \right\} dW(t) \right)^2 \right] dx \right\}^{1/2} \\ &\leq \sqrt{k} \sqrt{|B(0, R)|} 2C_g \|\partial_t \varphi\|_\infty \|\eta'\|_\infty T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Note that here the assumption on the boundedness of g is crucial. Using Proposition 3, we have:

$$\begin{aligned} (S_2^{h,k,\eta,\varphi})^2 &= \left| E \left[\mathbb{1}_A \int_{B(0,R)} \int_0^T \left\{ \eta'(\bar{u}_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k}) \right\} g(u_{\mathcal{T},k}) \varphi(x, t) dW(t) dx \right] \right|^2 \\ &\leq |B(0, R)| \int_{B(0,R)} E \left[\left(\int_0^T \left\{ \eta'(\bar{u}_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k}) \right\} g(u_{\mathcal{T},k}) \varphi(x, t) dW(t) \right)^2 \right] dx \\ &\leq |B(0, R)| \|\varphi\|_\infty^2 \|\eta''\|_\infty^2 \|g\|_\infty^2 C_{TA} \times k. \end{aligned}$$

So, we deduce that:

$$\left| E \left[\mathbb{1}_A (\tilde{C}^{h,k,\eta,\varphi} - C^{h,k,\eta,\varphi}) \right] \right| \leq \sqrt{|B(0, R)|} 2C_g \|\partial_t \varphi\|_\infty \|\eta'\|_\infty T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} k^{1/2} + \sqrt{C_{TA} |B(0, R)|} \|\varphi\|_\infty \|\eta''\|_\infty \|g\|_\infty k^{1/2}.$$

II.4 Study of $E \left[\mathbb{1}_A (\tilde{D}^{h,k,\eta,\varphi} - D^{h,k,\eta,\varphi}) \right]$

Using again the boundedness of g , Cauchy-Schwarz inequality and Proposition 3, one gets:

$$\begin{aligned} \left| E \left[\mathbb{1}_A (\tilde{D}^{h,k,\eta,\varphi} - D^{h,k,\eta,\varphi}) \right] \right| &= \left| \frac{1}{2} E \left[\sum_{K \in \mathcal{T}_R} \sum_{n=0}^{N-1} \int_K \int_{nk}^{(n+1)k} \mathbb{1}_A \left[\eta''(u_K^n) - \eta''(\bar{u}_K^n(t)) \right] g^2(u_K^n) \varphi(x, nk) dx dt \right] \right| \\ &\leq \frac{\sqrt{T} |B(0, R)|}{2} \|g\|_\infty^2 \|\varphi\|_\infty \|\eta'''\|_\infty C_{TA} \times k. \end{aligned}$$

Conclusion of STEP II:

By gathering the results obtained previously, one gets that for any P-measurable set A :

$$\begin{aligned} \left| E \left[\mathbb{1}_A \tilde{R}_G^{h,k,\eta,\varphi} \right] \right| &\leq \|\eta''\|_\infty \|\varphi\|_\infty \frac{C_{BV,1} k}{\alpha^2 h} + \left(\|\eta''\|_\infty \|\varphi\|_\infty C_g T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} \frac{\sqrt{C_{BV,1}}}{\bar{\alpha}} \right) \sqrt{\frac{k}{h}} + 2 \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty C_{BV,2} h^{1/2} \\ &\quad + \frac{2 \|\nabla_x \varphi\|_\infty}{\bar{\alpha}^2} T |B(0, R)| (C_f^R h + C_f^R k) \\ &\quad + \sqrt{|B(0, R)|} 2C_g \|\partial_t \varphi\|_\infty \|\eta'\|_\infty T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} k^{1/2} + \sqrt{C_{TA} |B(0, R)|} \|\varphi\|_\infty \|\eta''\|_\infty \|g\|_\infty k^{1/2} \\ &\quad + \frac{\sqrt{T} |B(0, R)|}{2} \|g\|_\infty^2 \|\varphi\|_\infty \|\eta'''\|_\infty C_{TA} \times k, \end{aligned}$$

which concludes the proof of the proposition by using the CFL Condition (18). ■

3.4.2 The general case

Similarly to what has been done in [BCG16-2], we prove the case of general monotone numerical fluxes by using the previous study on Godunov's numerical fluxes and the following decomposition lemma (see [CH00]).

Lemma 2 Consider a family of monotone numerical fluxes $(F_{K,L}^n)_{n,K,L}$ in the sense of Definition 4 with constants \bar{F}_1, \bar{F}_2 . Then, there exists a family of functions $(\theta_{K,L}^n)_{n,K,L}$ taking values in $[0, 1]$ such that for any $n \in \mathbb{N}$, $K \in \mathcal{T}$ and $L \in \mathcal{N}(K)$, for any $(a, b) \in \mathbb{R}^2$:

$$F_{K,L}^n(a, b) = \theta_{K,L}^n(a, b) F_{K,L}^{n,G}(a, b) + (1 - \theta_{K,L}^n(a, b)) F_{K,L}^{n,LF}(a, b),$$

where $(F_{K,L}^{n,G})_{n,K,L}$ is the family of Godunov's numerical fluxes associated with the flux function f and $(F_{K,L}^{n,LF})_{n,K,L}$ is the family of modified Lax-Friedrichs' numerical fluxes with parameter $D = \max(2\bar{F}_1, 2\bar{F}_2, C_f)$ associated with the flux function f .

Remark 8 Note that the numerical fluxes $(F_{K,L}^{n,G})_{n,K,L}$ and $(F_{K,L}^{n,LF})_{n,K,L}$ satisfy Definition 4 with the following respective parameters $F_1 = F_2 = C_f$ and $F_1 = F_2 = D$.

Let us recall the definition of a family of modified Lax-Friedrichs' numerical fluxes associated with the flux function f with parameter D (where $D \geq C_f$). Since f is a Lipschitz-continuous function, it can be decomposed in the following way: $f = f_1 + f_2$ where

$$f_1 : (x, t, v) \mapsto \frac{f(x, t, v) + Dv}{2} \text{ and } f_2 : (x, t, v) \mapsto \frac{f(x, t, v) - Dv}{2}$$

are respectively non-decreasing and non-increasing. Then, for any $n \in \mathbb{N}^*$ and any $K, L \in \mathcal{T}$, the discretization $f_{K,L}^n$ of the flux function f defined by (6) can be decomposed in the following manner: $f_{K,L}^n = f_{K,L}^{n,(1)} + f_{K,L}^{n,(2)}$ where for all $s \in \mathbb{R}$

$$f_{K,L}^{n,(1)}(s) = \frac{f_{K,L}^n(s) + Ds}{2} \text{ and } f_{K,L}^{n,(2)}(s) = \frac{f_{K,L}^n(s) - Ds}{2}.$$

The modified Lax-Friedrichs numerical fluxes with parameter D are then defined for any $a, b \in \mathbb{R}$ by:

$$F_{K,L}^{n,LF}(a, b) = F_{K,L}^{n,(LF,1)}(a, b) + F_{K,L}^{n,(LF,2)}(a, b),$$

where

$$F_{K,L}^{n,(LF,1)}(a, b) = f_{K,L}^{n,(1)}(a) \text{ and } F_{K,L}^{n,(LF,2)}(a, b) = f_{K,L}^{n,(2)}(b). \quad (36)$$

Note that for $i = 1, 2$, $F_{K,L}^{n,(LF,i)}$ belongs to the class of Godunov numerical fluxes as upwind numerical fluxes associated with the monotonic functions $f_{K,L}^{n,(i)}$. Hence we can naturally define the entropy numerical flux $G_{K,L}^{n,LF}$ associated with $F_{K,L}^{n,LF}$ by:

$$G_{K,L}^{n,LF}(a, b) = G_{K,L}^{n,(LF,1)}(a, b) + G_{K,L}^{n,(LF,2)}(a, b),$$

where for any $a, b \in \mathbb{R}$,

$$G_{K,L}^{n,(LF,1)}(a, b) = \int_0^a \eta'(v)(f_{K,L}^{n,(1)})'(v)dv \text{ and } G_{K,L}^{n,(LF,2)}(a, b) = \int_0^b \eta'(v)(f_{K,L}^{n,(2)})'(v)dv. \quad (37)$$

Proposition 5 (Discrete entropy inequalities) Assume that Assumptions H_1 to H_7 hold and that $(F_{K,L}^n)_{n,K,L}$ is a family of monotone numerical fluxes associated with the flux function f . Let $u_0 \in L^2(\mathbb{R}^d)$ and \mathcal{T} be an admissible mesh in the sense of Definition 3, $T > 0$, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step satisfying the CFL Condition (18). Then, there exists a constant C_{DE} depending only on $R, d, \bar{\alpha}, u_0, \xi, F_1, F_2, C_f^R, C_f^T, \|g\|_\infty$ and C_g such that $P - a.s$ in Ω , for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$:

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \frac{\eta(u_K^{n+1}) - \eta(u_K^n)}{k} \varphi(x, nk) dx dt + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \Phi_\eta(x, t, u_K^n, 0) \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dx dW(t) + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) dx dt \\ & \geq \tilde{R}^{h,k,\eta,\varphi}, \end{aligned} \quad (38)$$

where for any P -measurable set A :

$$\begin{aligned} \left| E \left[\mathbf{1}_A \tilde{R}^{h,k,\eta,\varphi} \right] \right| & \leq C_{DE} \left[\|\eta''\|_\infty \|\varphi\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty \times h^{1/2} \right. \\ & \left. + (\|\partial_t \varphi\|_\infty \|\eta'\|_\infty + \|\varphi\|_\infty \|\eta''\|_\infty) \times k^{1/2} + \|\varphi\|_\infty \|\eta'''\|_\infty \times k \right]. \end{aligned} \quad (39)$$

Proof. Most of the proof of Proposition 4 can be applied without any modifications, since we only used the fact that $(F_{K,L}^n)$ was a family of Godunov numerical fluxes in the following two steps: firstly when we proven in the point **I** that $\tilde{B}^{h,k,\eta,\varphi} - B_G^{h,k,\eta,\varphi} \geq 0$ a.s. in Ω and secondly when we established (29)-(30) in the point **II.2**. For these two points, we have truly exploited the fact that $F_{K,L}^n$ was of Godunov's type through the use of the numerical entropy fluxes $(G_{K,L}^n)_{n,K,L}$. In order to get these inequalities in the general case, we use the decomposition given by Lemma 2 to define the family of entropy numerical fluxes $(G_{K,L}^n)_{n,K,L}$ associated with $(F_{K,L}^n)_{n,K,L}$ in the following way: for any $a, b \in \mathbb{R}$,

$$G_{K,L}^n(a, b) = \theta_{K,L}^n(a, b) G_{K,L}^{n,G}(a, b) + (1 - \theta_{K,L}^n(a, b)) G_{K,L}^{n,LF}(a, b),$$

where $G_{K,L}^{n,G}(a, b)$ and $G_{K,L}^{n,LF}(a, b)$ denote respectively the entropy numerical fluxes associated with $F_{K,L}^{n,G}$ and $F_{K,L}^{n,LF}$.

I In order to show that $\tilde{B}^{h,k,\eta,\varphi} - B_G^{h,k,\eta,\varphi} \geq 0$ almost surely, we split the sum into two terms:

$$\begin{aligned} \tilde{B}^{h,k,\eta,\varphi} - B_G^{h,k,\eta,\varphi} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \frac{k}{|K|} \int_K \varphi(x, nk) dx \sum_{L \in \mathcal{N}(K)} \theta_{K,L}^n(u_K^n, u_L^n) |\sigma_{K,L}| \left\{ \eta'(u_K^n) F_{K,L}^{n,G}(u_K^n, u_L^n) - G_{K,L}^{n,G}(u_K^n, u_L^n) \right\} \\ &\quad + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \frac{k}{|K|} \int_K \varphi(x, nk) dx \sum_{L \in \mathcal{N}(K)} (1 - \theta_{K,L}^n(u_K^n, u_L^n)) |\sigma_{K,L}| \left\{ \eta'(u_K^n) F_{K,L}^{n,LF}(u_K^n, u_L^n) - G_{K,L}^{n,LF}(u_K^n, u_L^n) \right\}. \end{aligned}$$

Exactly as in the point **I** of the proof of Proposition 4, we get $\eta'(u_K^n) F_{K,L}^{n,G}(u_K^n, u_L^n) - G_{K,L}^{n,G}(u_K^n, u_L^n) \geq 0$ for any $(K, L) \in \mathcal{I}_R^n$. The second term can be decomposed as:

$$\begin{aligned} \eta'(u_K^n) F_{K,L}^{n,LF}(u_K^n, u_L^n) - G_{K,L}^{n,LF}(u_K^n, u_L^n) &= \eta'(u_K^n) F_{K,L}^{n,(LF,1)}(u_K^n, u_L^n) - G_{K,L}^{n,(LF,1)}(u_K^n, u_L^n) \\ &\quad + \eta'(u_K^n) F_{K,L}^{n,(LF,2)}(u_K^n, u_L^n) - G_{K,L}^{n,(LF,2)}(u_K^n, u_L^n), \end{aligned}$$

where $F_{K,L}^{n,(LF,1)}$, $F_{K,L}^{n,(LF,2)}$, $G_{K,L}^{n,(LF,1)}$, $G_{K,L}^{n,(LF,2)}$ are respectively defined by (36)-(37). Hence the proof given in the point **I** of the proof of Proposition 4 can be applied directly to $F_{K,L}^{n,(LF,1)}$ and $F_{K,L}^{n,(LF,2)}$ since any upwind numerical flux is a Godunov's one. We have then for any $(K, L) \in \mathcal{I}_R^n$

$$\eta'(u_K^n) F_{K,L}^{n,LF}(u_K^n, u_L^n) - G_{K,L}^{n,LF}(u_K^n, u_L^n) \geq 0.$$

II.2 Let us now establish inequalities (29)-(30) in order to get bounds for the terms $|\bar{T}_1^{h,k} - T_1^{h,k}|$ and $|\bar{T}_2^{h,k} - T_2^{h,k}|$. Set $(K, L) \in \mathcal{I}_R^n$. In order to prove (29) for a family of general numerical fluxes, we use the splitting:

$$\begin{aligned} G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) &= \theta_{K,L}^n(u_K^n, u_L^n) \left(G_{K,L}^{n,G}(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right) \\ &\quad + (1 - \theta_{K,L}^n(u_K^n, u_L^n)) \left(G_{K,L}^{n,LF}(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right). \end{aligned}$$

Once again, we apply directly the proof of Proposition 4 for the Godunov part and get

$$\left| G_{K,L}^{n,G}(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right| \leq 2 \|\eta'\|_\infty \max_{u_L^n \leq c \leq d \leq u_K^n} \left| F_{K,L}^{n,G}(d, c) - F_{K,L}^{n,G}(d, d) \right|.$$

As above, we deal with the Lax-Friedrichs part by using the decomposition of $F_{K,L}^{n,LF}$ involving the upwind numerical fluxes $F_{K,L}^{n,(LF,1)}$ and $F_{K,L}^{n,(LF,2)}$. For $i = 1, 2$, we set for any v in \mathbb{R}

$$\Phi_{K,L}^{n,(LF,i)}(v) = \int_0^v \eta'(\sigma) (f_{K,L}^{n,(i)})'(\sigma) d\sigma$$

and one gets the decomposition:

$$G_{K,L}^{n,LF}(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) = \left(G_{K,L}^{n,(LF,1)}(u_K^n, u_L^n) - \Phi_{K,L}^{n,(LF,1)}(u_K^n) \right) + \left(G_{K,L}^{n,(LF,2)}(u_K^n, u_L^n) - \Phi_{K,L}^{n,(LF,2)}(u_K^n) \right).$$

Using again the fact that any upwind numerical flux belongs to the class of Godunov's ones, we can apply the proof of Proposition 4 directly and get for $i = 1, 2$:

$$\left| G_{K,L}^{n,(LF,i)}(u_K^n, u_L^n) - \Phi_{K,L}^{n,(LF,i)}(u_K^n) \right| \leq 2 \|\eta'\|_\infty \max_{u_L^n \leq c \leq d \leq u_K^n} \left| F_{K,L}^{n,(LF,i)}(d, c) - F_{K,L}^{n,(LF,i)}(d, d) \right|.$$

Using the same technique, we obtain similar bounds for $G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_L^n)$. Then, by applying Proposition 2 with the numerical fluxes $F_{K,L}^{n,G}$, $F_{K,L}^{n,(LF,1)}$ and $F_{K,L}^{n,(LF,2)}$, one gets the existence of constants $C_{BV,2}^G$, $C_{BV,2}^{LF,1}$ and $C_{BV,2}^{LF,2}$ such that

$$\left| E \left[\mathbf{1}_A (B_G^{h,k,\eta,\varphi} - B_\Phi^{h,k,\eta,\varphi}) \right] \right| \leq 2 \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty (C_{BV,2}^G + C_{BV,2}^{LF,1} + C_{BV,2}^{LF,2}) h^{1/2} + \frac{2T |B(0, R)| (C_f^T k + C_f^R h) \|\nabla_x \varphi\|_\infty}{\bar{\alpha}^2},$$

which achieves the proof since these constants depend only on $R, d, \bar{\alpha}, u_0, \xi, F_1, F_2, C_f^R, C_f^T, \|g\|_\infty$ and C_g .

■

3.5 Continuous entropy inequalities on the approximate solution $u_{\mathcal{T},k}$

The following proposition investigates the entropy inequalities which are satisfied by the approximate solution $u_{\mathcal{T},k}$.

Proposition 6 (Continuous entropy inequalities on the discrete solution) Assume that hypotheses H_1 to H_7 hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^*$, $T \in \mathbb{R}_+^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step such that the CFL Condition (18) holds for some $\xi \in (0, 1)$. Then, there exists a constant C_{CE} , which depends only on $R, d, \bar{\alpha}, u_0, \xi, F_1, F_2, C_f^R, C_f^T, \|g\|_\infty$ and C_g , such that P-a.s in Ω , for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$:

$$\begin{aligned} & \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx + \int_Q \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt + \int_Q \Phi_\eta(x, t, u_{\mathcal{T},k}, 0) \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dx dW(t) + \frac{1}{2} \int_Q \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) dx dt \geq R^{h,k,\eta,\varphi}, \end{aligned} \quad (40)$$

where for any P-measurable set A :

$$\begin{aligned} |E[\mathbb{1}_A R^{h,k,\eta,\varphi}]| & \leq C_{CE} \left[\|\eta''\|_\infty \|\varphi\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty h^{1/2} + (\|\partial_t \varphi\|_\infty \|\eta'\|_\infty + \|\varphi\|_\infty \|\eta''\|_\infty) k^{1/2} \right. \\ & \left. + \|\eta'''\|_\infty \|\varphi\|_\infty k + \int_{B(0,R)} |u_0(x) - u_{\mathcal{T},0}(x)| dx + (\|\eta'\|_\infty \|\partial_t \nabla_x \varphi\|_\infty + \|\eta''\|_\infty \|\partial_t \varphi\|_\infty) k \right]. \end{aligned} \quad (41)$$

From here we will use the notation $u_{\mathcal{T},0}(x) = u_K^0$ for $x \in K$.

Proof. Using Proposition 5, we have:

$$A^{h,k,\eta,\varphi} + B^{h,k,\eta,\varphi} + C^{h,k,\eta,\varphi} + D^{h,k,\eta,\varphi} \geq \tilde{R}^{h,k,\eta,\varphi},$$

where:

$$\begin{aligned} A^{h,k,\eta,\varphi} & = - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \frac{\eta(u_K^{n+1}) - \eta(u_K^n)}{k} \varphi(x, nk) dx dt, \\ B^{h,k,\eta,\varphi} & = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \Phi_\eta(x, t, u_K^n, 0) \cdot \nabla_x \varphi(x, nk) dx dt, \\ C^{h,k,\eta,\varphi} & = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dx dW(t), \\ D^{h,k,\eta,\varphi} & = \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) dx dt \end{aligned}$$

and $\tilde{R}^{h,k,\eta,\varphi}$ verifies inequality (39).

Let us show that inequality (40) holds for a convenient $R^{h,k,\eta,\varphi}$. To do this, we introduce the following quantities:

$$\begin{aligned} A_1^{h,k,\eta,\varphi} & = \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_Q \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt. \\ B_1^{h,k,\eta,\varphi} & = \int_Q \Phi_\eta(x, t, u_{\mathcal{T},k}, 0) \cdot \nabla_x \varphi(x, t) dx dt. \\ C_1^{h,k,\eta,\varphi} & = \int_Q \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dx dW(t). \\ D_1^{h,k,\eta,\varphi} & = \frac{1}{2} \int_Q \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) dx dt. \end{aligned}$$

Therefore, if we set:

$$R^{h,k,\eta,\varphi} = \tilde{R}^{h,k,\eta,\varphi} + (A_1^{h,k,\eta,\varphi} - A^{h,k,\eta,\varphi}) + (B_1^{h,k,\eta,\varphi} - B^{h,k,\eta,\varphi}) + (C_1^{h,k,\eta,\varphi} - C^{h,k,\eta,\varphi}) + (D_1^{h,k,\eta,\varphi} - D^{h,k,\eta,\varphi}),$$

one obtains that P-a.s in Ω :

$$\begin{aligned} & \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_Q \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt + \int_Q \Phi_\eta(x, t, u_{\mathcal{T},k}, 0) \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_Q \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dx dW(t) + \frac{1}{2} \int_Q \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) dx dt \geq R^{h,k,\eta,\varphi}. \end{aligned}$$

Now, let A be a P-measurable set. It remains to estimate $E[\mathbb{1}_A R^{h,k,\eta,\varphi}]$. Thanks to Proposition 5 we have a bound on $E[\mathbb{1}_A \tilde{R}^{h,k,\eta,\varphi}]$. Let us analyze separately each other term. Using a discrete integration by parts we get :

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x,t) dx dt &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \eta(u_K^n) \int_K \left\{ \varphi(x, (n+1)k) - \varphi(x, nk) \right\} dx \\
&= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \left\{ \eta(u_K^{n+1}) - \eta(u_K^n) \right\} \int_K \varphi(x, (n+1)k) dx - \sum_{K \in \mathcal{T}_R} \eta(u_K^0) \int_K \varphi(x, 0) dx.
\end{aligned}$$

Then, using the CFL Condition (18) and the point 3. of Proposition 2, we deduce that:

$$\begin{aligned}
|E[\mathbf{1}_A(A^{h,k,\eta,\varphi} - A_1^{h,k,\eta,\varphi})]| &\leq \left| \left[\int_K (\eta(u_0(x)) - \eta(u_{\mathcal{T},0}(x))) \varphi(x, 0) dx \right] \right| \\
&+ \left| E \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \frac{\eta(u_K^{n+1}) - \eta(u_K^n)}{k} \int_{nk}^{(n+1)k} \int_K (\varphi(x, (n+1)k) - \varphi(x, nk)) dx dt \right] \right| \\
&\leq \int_{\mathbb{R}^d} |\eta(u_0(x)) - \eta(u_{\mathcal{T},0}(x))| \varphi(x, 0) dx + \|\eta'\|_{\infty} \|\partial_t \varphi\|_{\infty} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} k |K| E[|\eta(u_K^{n+1}) - \eta(u_K^n)|] \\
&\leq \|\eta'\|_{\infty} \|\varphi\|_{\infty} \int_{B(0,R)} |u_0(x) - u_{\mathcal{T},0}(x)| dx + \|\eta'\|_{\infty} \|\partial_t \varphi\|_{\infty} \left(\sqrt{\frac{(1-\xi)}{F_1 + F_2}} \bar{\alpha} C_{BV,2} + C_{BV,3} \right) k^{1/2}.
\end{aligned}$$

Let us now notice that for all $(x, t) \in Q$ and $a, b \in \mathbb{R}$, we have:

$$|\Phi_{\eta}(x, t, a, b)| = \left| \int_b^a \eta'(\sigma - b) \partial_v f(x, t, \sigma) d\sigma \right| \leq \|\eta'\|_{\infty} C_f |a - b|. \quad (42)$$

So, one gets:

$$\left| \Phi_{\eta}(x, t, u_K^n, 0) \int_{nk}^{(n+1)k} \int_K \left\{ \nabla_x \varphi(x, nk) - \nabla_x \varphi(x, t) \right\} dx dt \right| \leq \|\eta'\|_{\infty} C_f |u_K^n| \times \|\partial_t \nabla_x \varphi\|_{\infty} |K| k^2.$$

Then, using the stability result of Proposition 1, we obtain:

$$\begin{aligned}
|E[\mathbf{1}_A(B^{h,k,\eta,\varphi} - B_1^{h,k,\eta,\varphi})]| &\leq \|\eta'\|_{\infty} C_f \|\partial_t \nabla_x \varphi\|_{\infty} k \times \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}_R} |K| E[|u_K^n|] \\
&\leq \|\eta'\|_{\infty} C_f \|\partial_t \nabla_x \varphi\|_{\infty} \sqrt{|B(0, R)|} T e^{C_g^2 T/2} \|u_0\|_{L^2(\mathbb{R}^d)} k.
\end{aligned}$$

Using Cauchy-Schwarz inequality, the isometric property of Itô integral and finally Proposition 1, we get :

$$\begin{aligned}
|E[\mathbf{1}_A(C^{h,k,\eta,\varphi} - C_1^{h,k,\eta,\varphi})]| &= \left| E \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \eta'(u_K^n) g(u_K^n) \int_{nk}^{(n+1)k} \int_K \left\{ \varphi(x, nk) - \varphi(x, t) \right\} dx dW(t) \right] \right| \\
&\leq \sum_{n=0}^{N-1} \sqrt{|B(0, R)|} \left(\sum_{K \in \mathcal{T}_R} \int_K \int_{nk}^{(n+1)k} E \left[\left(\eta'(u_K^n) g(u_K^n) \left\{ \varphi(x, nk) - \varphi(x, t) \right\} \right)^2 \right] dt dx \right)^{1/2} \\
&\leq \|\eta'\|_{\infty} C_g \|\partial_t \varphi\|_{\infty} \sqrt{|B(0, R)|} T e^{C_g^2 T/2} \|u_0\|_{L^2(\mathbb{R}^d)} k^{1/2}.
\end{aligned}$$

We bound the last term by using again Proposition 1:

$$\left| E[\mathbf{1}_A(D^{h,k,\eta,\varphi} - D_1^{h,k,\eta,\varphi})] \right| \leq \frac{1}{2} \|\eta''\|_{\infty} C_g^2 \|\partial_t \varphi\|_{\infty} T e^{C_g^2 T} \|u_0\|_{L^2(\mathbb{R}^d)}^2 k.$$

Therefore, thanks to the estimate (39) on $\tilde{R}^{h,k,\eta,\varphi}$, for any P-measurable set A , we have:

$$\begin{aligned}
|E[\mathbf{1}_A R^{h,k,\eta,\varphi}]| &\leq C_{CE} \left[\|\eta''\|_{\infty} \|\varphi\| \frac{k^{1/2}}{h^{1/2}} + \|\eta'\|_{\infty} \|\nabla_x \varphi\|_{\infty} h^{1/2} + (\|\partial_t \varphi\|_{\infty} \|\eta'\|_{\infty} + \|\varphi\|_{\infty} \|\eta''\|_{\infty}) k^{1/2} + \|\eta'''\|_{\infty} \|\varphi\|_{\infty} k \right] \\
&+ \|\eta'\|_{\infty} \|\varphi\|_{\infty} \int_{B(0,R)} |u_0(x) - u_{\mathcal{T},0}(x)| dx + \|\eta'\|_{\infty} \|\partial_t \varphi\|_{\infty} \left(\sqrt{\frac{(1-\xi)}{F_1 + F_2}} \bar{\alpha} C_{BV,2} + C_{BV,3} \right) k^{1/2} \\
&+ \|\eta'\|_{\infty} C_f \|\partial_t \nabla_x \varphi\|_{\infty} \sqrt{|B(0, R)|} T e^{C_g^2 T/2} \|u_0\|_{L^2(\mathbb{R}^d)} k + \|\eta'\|_{\infty} C_g \|\partial_t \varphi\|_{\infty} \sqrt{|B(0, R)|} T e^{C_g^2 T/2} \|u_0\|_{L^2(\mathbb{R}^d)} k^{1/2} \\
&+ \frac{1}{2} \|\eta''\|_{\infty} C_g^2 \|\partial_t \varphi\|_{\infty} T e^{C_g^2 T} \|u_0\|_{L^2(\mathbb{R}^d)}^2 k.
\end{aligned}$$

■

3.6 Convergence to a measure-valued entropy solution

In order to pass to the limit in the non linear terms, we use the estimate stated in Proposition 1 which yields compactness in the sense of Young measures. Following [BVW12, BCG16-2], we get the following result of convergence.

Proposition 7 (Convergence to a measure-valued entropy solution) *Assume that hypotheses H_1 to H_7 hold. Let $N \in \mathbb{N}^*$ and $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step. Let $(u_{\mathcal{T},k})$ be the finite volume approximation defined by (9). Then $(u_{\mathcal{T},k})$ converges (up to a subsequence) in the sense of Young measures to a measure-valued entropy solution μ of (1) in the sense of Definition 2 as $(h, k/h) \rightarrow (0, 0)$.*

Proof. Let \mathcal{T} be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step. Since we are interested in the limit as $(h, k/h) \rightarrow (0, 0)$ we can suppose that the CFL Condition (18) holds for some $\xi \in (0, 1)$. Consider a P-measurable set A , $\eta \in \mathcal{A}$, $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and apply Proposition 6. Multiplying the obtained inequality (40) by $\mathbb{1}_A$ and taking the expectation gives

$$\begin{aligned} & E \left[\mathbb{1}_A \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx \right] + E \left[\mathbb{1}_A \int_Q \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] + E \left[\mathbb{1}_A \int_Q \Phi_\eta(x, t, u_{\mathcal{T},k}, 0) \cdot \nabla_x \varphi(x, t) dx dt \right] \\ & + E \left[\mathbb{1}_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dx dW(t) \right] + \frac{1}{2} E \left[\mathbb{1}_A \int_Q \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) dx dt \right] \\ & \geq E \left[\mathbb{1}_A R^{h,k,\eta,\varphi} \right]. \end{aligned} \quad (43)$$

To show the convergence of $u_{\mathcal{T},k}$ towards a measure-valued entropy solution of our problem, we aim to pass to the limit in the above inequality. We deduce from Proposition 6 that $E \left[\mathbb{1}_A R^{h,k,\eta,\varphi} \right] \rightarrow 0$ as $(h, \frac{k}{h}) \rightarrow (0, 0)$. Thus, it remains to study the convergence of the left-hand side of (43). Recall that thanks to the estimate stated in Proposition 1, $u_{\mathcal{T},k}$ converges (up to a subsequence denoted in the same way) in the sense of Young measures to an "entropy process" denoted by μ in $L^2(\Omega \times Q \times (0, 1))$. Following [BCG16-2], one gets the following convergences as $(h, k/h) \rightarrow (0, 0)$:

$$\begin{aligned} & E \left[\mathbb{1}_A \int_Q \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] \rightarrow E \left[\mathbb{1}_A \int_Q \left(\int_0^1 \eta(\mu(x, t, \alpha)) d\alpha \right) \partial_t \varphi(x, t) dx dt \right], \\ & \frac{1}{2} E \left[\mathbb{1}_A \int_Q \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) dx dt \right] \rightarrow \frac{1}{2} E \left[\mathbb{1}_A \int_Q \left(\int_0^1 \eta''(\mu(x, t, \alpha)) g^2(\mu(x, t, \alpha)) d\alpha \right) \varphi(x, t) dx dt \right], \\ & E \left[\mathbb{1}_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dx dW(t) \right] \rightarrow E \left[\mathbb{1}_A \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mu(x, t, \alpha)) g(\mu(x, t, \alpha)) \varphi(x, t) d\alpha dx dW(t) \right]. \end{aligned}$$

Let us now study the term associated with the flux function f : we introduce

$$\Psi_1 : (\omega, x, t, u) \in \Omega \times Q \times \mathbb{R} \mapsto \mathbb{1}_A(\omega) \Phi_\eta(x, t, u, 0) \cdot \nabla_x \varphi(x, t) \in \mathbb{R}$$

which is a Carathéodory function. We deduce from Proposition 1 and inequality (42) that $\Psi_1(\cdot, u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q)$. Hence we have, as $(h, k/h) \rightarrow (0, 0)$:

$$E \left[\mathbb{1}_A \int_Q \Phi_\eta(x, t, u_{\mathcal{T},k}, 0) \cdot \nabla_x \varphi(x, t) dx dt \right] \rightarrow E \left[\mathbb{1}_A \int_Q \left(\int_0^1 \Phi_\eta(x, t, \mu(x, t, \alpha), 0) d\alpha \right) \cdot \nabla_x \varphi(x, t) dx dt \right].$$

By passing to the limit in (43), we obtain that for any P-measurable set A , for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$

$$\begin{aligned} 0 & \leq E \left[\mathbb{1}_A \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx \right] + E \left[\mathbb{1}_A \int_Q \int_0^1 \eta(\mu(x, t, \alpha)) \partial_t \varphi(x, t) d\alpha dx dt \right] \\ & + E \left[\mathbb{1}_A \int_Q \int_0^1 \Phi_\eta(x, t, \mu(x, t, \alpha), 0) \cdot \nabla_x \varphi(x, t) d\alpha dx dt \right] \\ & + E \left[\mathbb{1}_A \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mu(x, t, \alpha)) g(\mu(x, t, \alpha)) \varphi(x, t) d\alpha dx dW(t) \right] \\ & + E \left[\mathbb{1}_A \frac{1}{2} \int_Q \int_0^1 \eta''(\mu(x, t, \alpha)) g^2(\mu(x, t, \alpha)) \varphi(x, t) d\alpha dx dt \right]. \end{aligned}$$

Hence, μ is a measure-valued entropy solution in the sense of Definition 2, which concludes the proof. \blacksquare

4 Uniqueness of the solution

In the previous section, we have proven the existence of a measure-valued entropy solution of Problem (1) through the convergence (up to a subsequence) of the finite volume approximation $(u_{\mathcal{T},k})$ as $(h, k/h) \rightarrow (0, 0)$. In order to prove Theorem 1, we establish the uniqueness of such a measure-valued entropy solution, and that it has moreover the property to be an entropy solution in the sense of Definition 1. The idea to get such a uniqueness result is to adapt the method of doubling variables introduced by Kruzhkov (see [EGH00] for example) which provides a Kato inequality. But, as explained in [BVW12], in the stochastic case, the use of Kruzhkov's entropy is out of range and it is also not possible to compare directly two generic measure-valued entropy solutions. For this last reason, the uniqueness was obtained by comparing a parabolic approximation to a generic solution in [BVW12, BCG16-2]. In this work we chose a different approach : the uniqueness is obtained by comparing a generic measure-valued entropy solution to the finite volume approximation $(u_{\mathcal{T},k})$. With this method, we do not need to introduce a viscous approximation and moreover, we expect that the estimates obtained will enable us to get an error bound in a further work.

4.1 Notations and preliminary results

We first introduce a family of smooth entropies in order to approach Kruzhkov's entropies.

Definition 5 Let $\delta > 0$. We define the function $\eta_\delta: \mathbb{R} \rightarrow \mathbb{R}_+$ by $\eta_\delta(0) = 0$ and:

$$\eta'_\delta(a) = \begin{cases} 1 & \text{if } a > \delta \\ \sin\left(\frac{\pi}{2\delta}a\right) & \text{if } |a| \leq \delta \\ -1 & \text{if } a < -\delta. \end{cases}$$

Note that we can then easily show that for all $a \in \mathbb{R}$, we have:

$$|\eta_\delta(a) - |a|| \leq \delta. \quad (44)$$

Using the notation introduced in Section 2.2, we then have

$$\Phi_{\eta_\delta}(x, t, a, b) = \int_b^a \eta'_\delta(\sigma - b) \partial_\sigma f(x, t, \sigma) d\sigma.$$

As noticed previously, $\Phi_{\eta_\delta}(\cdot, \cdot, \cdot, b)$ is then the entropy flux associated with $\eta_\delta(\cdot - b)$ which vanishes at $a = b$. We will also denote by $\Phi(\cdot, \cdot, \cdot, b)$ the entropy flux associated with the Kruzhkov's entropy $|\cdot - b|$ which vanishes at $a = b$. We then have classically

$$\Phi(x, t, a, b) = f(x, t, a \uparrow b) - f(x, t, a \downarrow b) = \text{sgn}(a - b) [f(x, t, a) - f(x, t, b)]. \quad (45)$$

This entropy flux is also such that for any $x, x' \in B(0, R)$, $t, t' \in [0, T]$ and $a, a', b \in \mathbb{R}$:

$$|\Phi_{\eta_\delta}(x, t, a, b) - \Phi(x, t, a, b)| \leq 4C_f \delta, \quad (46)$$

$$|\Phi(x, t, a, b)| \leq C_f |a - b|, \quad (47)$$

$$|\Phi(x, t, a, b) - \Phi(x, t, a', b)| \leq C_f |a - a'|, \quad (48)$$

and

$$|\Phi(x, t, a, b) - \Phi(x, t', a', b)| \leq 2C_f^T |t - t'|, \quad (49)$$

$$|\Phi(x, t, a, b) - \Phi(x', t, a', b)| \leq 2C_f^R |x - x'|. \quad (50)$$

Definition 6 (Mollifier sequences) Let $\rho, \bar{\rho} \in \mathcal{D}(\mathbb{R})$ and $\tilde{\rho} \in \mathcal{D}(\mathbb{R}^d)$ be such that: for all $(a, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$,

$$\text{Supp}(\rho) \subset [-1; 1], \quad \rho(a) \geq 0, \quad \rho(-a) = \rho(a) \quad \text{and} \quad \int_{\mathbb{R}} \rho(a) da = 1.$$

$$\text{Supp}(\bar{\rho}) \subset [-2; 0], \quad \bar{\rho}(t) \geq 0, \quad \bar{\rho}(-t - 1) = \bar{\rho}(t - 1) \quad \text{and} \quad \int_{\mathbb{R}} \bar{\rho}(t) dt = 1.$$

$$\text{Supp}(\tilde{\rho}) \subset B(0, 1), \quad \tilde{\rho}(x) \geq 0, \quad \tilde{\rho}(-x) = \tilde{\rho}(x) \quad \text{and} \quad \int_{\mathbb{R}^d} \tilde{\rho}(x) dx = 1.$$

We introduce $\rho_l, \bar{\rho}_p$ and $\tilde{\rho}_q$ the mollifiers sequences defined for $(a, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ and $l, p, q \in \mathbb{N}^*$ by

$$\rho_l(a) = l\rho(la), \quad \bar{\rho}_p(t) = p\bar{\rho}(pt) \quad \text{and} \quad \tilde{\rho}_q(x) = q^d \tilde{\rho}(qx).$$

We then have the following elementary properties on these mollifiers.

Lemma 3 For any $l, p, q \in \mathbb{N}^*$, the functions $\rho_l, \bar{\rho}_p$ and $\tilde{\rho}_q$ verify the following properties:

1. $\text{Supp}(\rho_l) \subset [-1/l, 1/l]$, $\text{Supp}(\bar{\rho}_p) \subset [-2/p, 0]$ and $\text{Supp}(\tilde{\rho}_q) \subset B(0, 1/q)$.
2. There exists a constant $C_\rho > 0$ such that:

$$\|\rho_l\|_\infty \leq l \times C_\rho, \quad \|\bar{\rho}_p\|_\infty \leq p \times C_\rho, \quad \|\tilde{\rho}_q\|_\infty \leq q^d \times C_\rho,$$

$$\text{and } \|\rho_l'\|_\infty \leq l^2 \times C_\rho, \quad \|\bar{\rho}_p'\|_\infty \leq p^2 \times C_\rho, \quad \|\nabla \tilde{\rho}_q\|_\infty \leq q^{d+1} \times C_\rho.$$

3. The previous constant $C_\rho > 0$ also satisfies for any $a, t, x \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$:

$$\int_{\mathbb{R}} |\rho_l'(a-b)| db \leq l \times C_\rho, \quad \int_{\mathbb{R}} |\bar{\rho}_p'(t-s)| ds \leq p \times C_\rho, \quad \int_{\mathbb{R}^d} |\nabla \tilde{\rho}_q(x-y)| dy \leq q \times C_\rho,$$

$$\int_{\mathbb{R}} |a| \rho_l(a) da \leq C_\rho \times \frac{1}{l}, \quad \int_{\mathbb{R}} |t| \bar{\rho}_p(t) dt \leq C_\rho \times \frac{1}{p} \quad \text{and} \quad \int_{\mathbb{R}^d} |x| \tilde{\rho}_q(x) dx \leq C_\rho \times \frac{1}{q}.$$

4.2 Kato inequality

Proposition 8 *Let ν be any measure-valued entropy solution in the sense of Definition 2 and μ be the limit in the sense of Young measures of a subsequence of $(u_{\mathcal{T},k})$ given by Proposition 7. Then for all $\varphi \in \mathcal{D}^+(Q)$,*

$$E \left[\int_Q \int_{(0,1)^2} \left\{ |\mu(x,t,\alpha) - \nu(x,t,\beta)| \partial_t \varphi(x,t) + \Phi(x,t, \mu(x,t,\alpha), \nu(x,t,\beta)) \cdot \nabla_x \varphi(x,t) \right\} d\alpha d\beta dx dt \right] \geq 0. \quad (51)$$

Proof. Let ν be any measure-valued entropy solution in the sense of Definition 2, $\kappa \in \mathbb{R}$, $\varphi \in \mathcal{D}^+(Q)$ and R such that $\text{Supp}(\varphi) \subset B(0, R) \times [0, T]$. For any $(y, s) \in Q$, we take the test function $\tilde{\Psi}_{y,s} : (x, t) \mapsto \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y)$ and the entropy $\eta_\delta(\cdot - \kappa)$ in Definition 2, then by multiplying it by $\rho_l(u_{\mathcal{T},k}(y, s) - \kappa)$, integrating w.r.t (κ, y, s) on $\mathbb{R} \times Q$, applying Fubini and stochastic Fubini theorems and taking the expectation, one has:

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 \geq 0 \quad (52)$$

where:

$$A_1 = E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x,t,\alpha) - \kappa) \partial_t \varphi(x,t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y,s) - \kappa) d\alpha dx dt d\kappa dy ds \right],$$

$$A_2 = E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x,t,\alpha) - \kappa) \varphi(x,t) \bar{\rho}_p'(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y,s) - \kappa) d\alpha dx dt d\kappa dy ds \right],$$

$$A_3 = E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \Phi_{\eta_\delta}(x,t,\nu(x,t,\alpha), \kappa) \cdot \nabla_x \varphi(x,t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y,s) - \kappa) d\alpha dx dt d\kappa dy ds \right],$$

$$A_4 = E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \Phi_{\eta_\delta}(x,t,\nu(x,t,\alpha), \kappa) \cdot \nabla \tilde{\rho}_q(x-y) \varphi(x,t) \bar{\rho}_p(t-s) \rho_l(u_{\mathcal{T},k}(y,s) - \kappa) d\alpha dx dt d\kappa dy ds \right],$$

$$A_5 = E \left[\int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta_\delta'(\nu(x,t,\alpha) - \kappa) g(\nu(x,t,\alpha)) \varphi(x,t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) d\alpha dx dW(t) \rho_l(u_{\mathcal{T},k}(y,s) - \kappa) d\kappa dy ds \right],$$

$$A_6 = \frac{1}{2} E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta''(\nu(x,t,\alpha) - \kappa) g^2(\nu(x,t,\alpha)) \varphi(x,t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y,s) - \kappa) d\alpha dx dt d\kappa dy ds \right]$$

and $A_7 = E \left[\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(u_0(x) - \kappa) \varphi(x,0) \bar{\rho}_p(-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y,s) - \kappa) d\alpha dx d\kappa dy ds \right].$

Indeed, we can notice that $\Phi_{\eta_\delta(\cdot - \kappa)}(\cdot, \cdot, \cdot, 0)$ is equal, up to a constant, to $\Phi_{\eta_\delta}(\cdot, \cdot, \cdot, \kappa)$.

Now, by fixing $(x, t, \alpha) \in Q \times (0, 1)$ and considering the test function $\Psi_{x,t} : (y, s) \mapsto \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y)$ in the entropy formulation (40) satisfied by $u_{\mathcal{T},k}$, multiplying it by $\rho_l(\nu(x, t, \alpha) - \kappa)$, then integrating it with respect to (κ, x, t, α) on $\mathbb{R} \times Q \times (0, 1)$, applying Fubini and stochastic Fubini theorems and taking the expectation, we get:

$$B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 \geq \mathcal{R}, \quad (53)$$

where:

$$\begin{aligned}
B_2 &= E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, t) \times -\tilde{\rho}'_p(t-s) \times \tilde{\rho}_q(x-y) \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_4 &= E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \varphi(x, t) \tilde{\rho}_p(t-s) \Phi_{\eta_\delta}(y, s, u_{\mathcal{T},k}(y, s), \kappa) \cdot \left(-\nabla \tilde{\rho}_m(x-y) \right) \times \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_5 &= E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \eta'_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, t) g(u_{\mathcal{T},k}(y, s)) \tilde{\rho}_p(t-s) \tilde{\rho}_q(x-y) dy dW(s) \rho_l(\nu(x, t, \alpha) - \kappa) d\kappa dx dt d\alpha \right], \\
B_6 &= \frac{1}{2} E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \eta''_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, t) g^2(u_{\mathcal{T},k}(y, s)) \tilde{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_1 &= B_3 = B_7 = 0, \text{ and} \\
\mathcal{R} &= E \left[\int_0^1 \int_Q \int_{\mathbb{R}} R^{h,k,\eta_\delta(\cdot-\kappa), \Psi_{x,t}} \rho_l(\nu(x, t, \alpha) - \kappa) d\kappa d\alpha dx dt \right].
\end{aligned}$$

Note that Proposition 6 gives the existence of the term $R^{h,k,\eta_\delta(\cdot-\kappa), \Psi_{x,t}}$ which satisfies for any $(x, t) \in Q$

$$\begin{aligned}
\left| E \left[R^{h,k,\eta_\delta(\cdot-\kappa), \Psi_{x,t}} \right] \right| &\leq C_{CE} \left[\|\eta''_\delta\|_\infty \|\Psi_{x,t}\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'_\delta\|_\infty \|\nabla_y \Psi_{x,t}\|_\infty h^{1/2} + (\|\partial_s \Psi_{x,t}\|_\infty \|\eta'\|_\infty + \|\Psi_{x,t}\|_\infty \|\eta''_\delta\|_\infty) k^{1/2} \right. \\
&\quad \left. + \|\eta''_\delta\|_\infty \|\Psi_{x,t}\|_\infty k + \int_{B(0,R)} |u_0(x) - u_{\mathcal{T},0}(x)| dx + (\|\eta'_\delta\|_\infty \|\partial_s \nabla_y \Psi_{x,t}\|_\infty + \|\eta''_\delta\|_\infty \|\partial_s \Psi_{x,t}\|_\infty) k \right]. \quad (54)
\end{aligned}$$

In order to conclude, we aim to pass to the limit as h, k, δ go to 0 and p, q, l tend to $+\infty$, but we need to be careful: the order of the passages to the limit is crucial and h, k, p cannot be chosen independently. Hence, in this proof, we take $(h_r)_{r \in \mathbb{N}}$ a sequence of mesh sizes such that $\lim_{r \rightarrow \infty} h_r = 0$. We define then a sequence of time steps $(k_r)_{r \in \mathbb{N}}$ by $k_r = (h_r)^{21}$. Note that $(h_r, k_r/h_r) \rightarrow (0, 0)$ as $r \rightarrow +\infty$ and for r large enough, there exists $\xi \in (0, 1)$ such that the CFL Condition (18) is fulfilled. We also define a sequence $(p_r)_{r \in \mathbb{N}}$ by $p_r = (h_r)^{-5}$. For the sake of readability, the index r will be omitted when it is not important and from now on, we will denote by C a constant whose value may change from one line to another but which only depends on $T, R, d, u_0, \nu, C_f^T, C_f^R, C_g, \varphi, F_1, F_2, \bar{\alpha}$ and C_ρ . In order to simplify the expressions of the bounds, we suppose from now on that $h, k, \delta \leq 1$ and $p, q, l \geq 1$. Summing up the preceding two inequalities, our aim is now to estimate each terms $(A_i + B_i)$ for $i \in \{0, \dots, 7\}$ and to pass to the limit with respect to r, l, δ, q in this order. More precisely, we prove successively that:

$$\begin{aligned}
\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_1 + B_1 &= E \left[\int_Q \int_{(0,1)^2} |\nu(x, t, \alpha) - \mu(x, t, \beta)| \partial_t \varphi(x, t) d\alpha d\beta dx dt \right], \\
\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_2 + B_2 &= 0, \\
\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_3 + B_3 &= E \left[\int_Q \int_{(0,1)^2} \Phi(x, t, \nu(x, t, \alpha), \mu(x, t, \beta)) \cdot \nabla_x \varphi(x, t) d\alpha d\beta dx dt \right], \\
\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_4 + B_4 &= 0, \\
\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_5 + B_5 + A_6 + B_6 &= 0, \\
\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_7 + B_7 &= 0, \\
\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} \mathcal{R} &= 0.
\end{aligned}$$

4.2.1 Study of $A_1 + B_1$

Firstly, since $B_1 = 0$, we only analyze the term A_1 . We rewrite it in the following way:

$$A_1 = (A_1 - A_{1,1}) + (A_{1,1} - A_{1,2}) + (A_{1,2} - A_{1,3}) + (A_{1,3} - A_{1,4}) + (A_{1,4} - A_{1,5}) + A_{1,5},$$

where:

$$\begin{aligned}
A_{1,1} &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x, t, \alpha) - \kappa) \partial_t \varphi(x, t) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, t) - \kappa) d\alpha dx dt d\kappa dy \right], \\
A_{1,2} &= E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) \partial_t \varphi(x, t) \tilde{\rho}_q(x-y) d\alpha dx dt dy \right], \\
A_{1,3} &= E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 |\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)| \partial_t \varphi(x, t) \tilde{\rho}_q(x-y) d\alpha dx dt dy \right], \\
A_{1,4} &= E \left[\int_Q \int_0^1 |\nu(x, t, \alpha) - u_{\mathcal{T},k}(x, t)| \partial_t \varphi(x, t) d\alpha dx dt \right], \\
\text{and } A_{1,5} &= E \left[\int_Q \int_0^1 \int_0^1 |\nu(x, t, \alpha) - \mu(x, t, \beta)| \partial_t \varphi(x, t) d\beta d\alpha dx dt \right].
\end{aligned}$$

Using a change of variables, we get

$$A_{1,1} = E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \partial_s \varphi(x, s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right] = \tilde{A}_{1,1} + \bar{A}_{1,1},$$

where:

$$\begin{aligned} \tilde{A}_{1,1} &= E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \partial_s \varphi(x, s) \tilde{\rho}_q(x-y) \rho_l(\xi) \bar{\rho}_p(t-s) d\alpha dx ds d\xi dy dt \right] \\ \text{and } \bar{A}_{1,1} &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \partial_s \varphi(x, s) \tilde{\rho}_q(x-y) \rho_l(\xi) \left(1 - \int_0^T \bar{\rho}_p(t-s) dt \right) d\alpha dx ds d\xi dy \right]. \end{aligned}$$

Let us bound $|A_1 - \tilde{A}_{1,1}|$ and $|\bar{A}_{1,1}|$. Using the fact that $\|\eta'_\delta\|_\infty = 1$, one has:

$$\begin{aligned} |A_1 - \tilde{A}_{1,1}| &\leq \left| E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \left\{ \eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) - \eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \right\} \right. \right. \\ &\quad \left. \left. \times \partial_t \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy dt \right] \right| \\ &\quad + \left| E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \left\{ \partial_t \varphi(x, t) - \partial_t \varphi(x, s) \right\} \right. \right. \\ &\quad \left. \left. \times \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy dt \right] \right| \\ &\leq \|\partial_t \varphi\|_\infty \|\eta'_\delta\|_\infty E \left[\int_Q \int_{\mathbb{R}} \int_{Q_R} \int_0^1 \left| \nu(x, t, \alpha) - \nu(x, s, \alpha) \right| \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy dt \right] \\ &\quad + \|\partial_{tt} \varphi\|_\infty E \left[\int_Q \int_{\mathbb{R}} \int_{Q_R} \int_0^1 |t-s| \|\eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi)\| \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy dt \right] \\ &\leq \|\partial_t \varphi\|_\infty E \left[\int_Q \int_{Q_R} \int_0^1 \left| \nu(x, t, \alpha) - \nu(x, s, \alpha) \right| \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) d\alpha dx ds dy dt \right] \\ &\quad + \|\partial_{tt} \varphi\|_\infty \times \frac{2}{p} E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi)| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right]. \end{aligned}$$

The first term can be bounded as follows:

$$\begin{aligned} &E \left[\int_Q \int_{Q_R} \int_0^1 \left| \nu(x, t, \alpha) - \nu(x, s, \alpha) \right| \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) d\alpha dx ds dy dt \right] \\ &\leq E \left[\int_0^T \int_{Q_R} \int_0^1 \left| \nu(x, t, \alpha) - \nu(x, s, \alpha) \right| \bar{\rho}_p(t-s) d\alpha dx ds dt \right] \\ &\leq \varepsilon(2/p, \nu, Q_R), \end{aligned}$$

where $\varepsilon(2/p, \nu, Q_R) = \sup \left\{ \|\nu - \nu(\cdot, \cdot - \tau, \cdot)\|_{L^1(\Omega \times Q_R \times (0,1))}; |\tau| \leq 2/p \right\}$. Since $\nu \in L^1_{loc}(\Omega \times Q \times (0, 1))$ and $Q_R \times (0, 1)$ is bounded, we have $\nu \in L^1(\Omega \times Q_R \times (0, 1))$ and $\varepsilon(2/p, \nu, Q_R) \longrightarrow 0$ when $p \longrightarrow \infty$.

Using (44), Proposition 1 and Lemma 3, we can bound the second term of the sum as follows:

$$\begin{aligned} &E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi)| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy dt \right] \\ &= E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi)| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy dt \right] \\ &\leq E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 \left| \eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi) - |\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi| \right| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy dt \right] \\ &\quad + E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy dt \right] \\ &\leq \delta \times E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy dt \right] + E \left[\int_0^T \int_{B(0,R)} \int_0^1 |\nu(x, t, \alpha)| d\alpha dx dt \right] \\ &\quad + E \left[\int_Q \int_{B(0,R)} |u_{\mathcal{T},k}(y, t)| \tilde{\rho}_q(x-y) dx dy dt \right] + E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} |\xi| \tilde{\rho}_q(x-y) \rho_l(\xi) dx d\xi dy dt \right] \\ &\leq \delta \times T |B(0, R)| + \|\nu\|_{L^1(\Omega \times Q_R \times (0,1))} + \|u_{\mathcal{T},k}\|_{L^1(\Omega \times B(0,R+1) \times (0,T))} + T |B(0, R)| C_\rho \frac{1}{l} \\ &\leq \delta \times T |B(0, R)| + \sqrt{T |B(0, R)|} \|\nu\|_{L^2(\Omega \times Q_R \times (0,1))} + \sqrt{|B(0, R+1)|} T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} + T |B(0, R)| C_\rho \frac{1}{l}. \end{aligned}$$

Therefore, one gets:

$$\begin{aligned} |A_1 - \bar{A}_{1,1}| &\leq \|\partial_t \varphi\|_\infty \times \varepsilon(2/p, \nu, Q_R) + \frac{2}{p} \|\partial_{tt} \varphi\|_\infty \left(\delta \times T |B(0, R)| + \sqrt{T |B(0, R)|} \|\nu\|_{L^2(\Omega \times Q_R \times (0,1))} \right) \\ &\quad + \sqrt{|B(0, R+1)|} T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} + T |B(0, R)| C_\rho \frac{1}{l} \\ &\leq C \left(\varepsilon(2/p, \nu, Q_R) + \frac{1}{p} \right). \end{aligned}$$

Now, we estimate $|\bar{A}_{1,1}|$. Since $1 - \int_0^T \bar{\rho}_p(t-s) dt$ is equal to 0 for $s \geq 2/p$ and using again (44), we have:

$$\begin{aligned} |\bar{A}_{1,1}| &\leq \left| E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_0^{2/p} \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \partial_s \varphi(x, s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right] \right| \\ &\leq \|\partial_s \varphi\|_\infty \left\{ E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_0^{2/p} \int_{B(0,R)} \int_0^1 \left| \eta_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) - |\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi| \right| \tilde{\rho}_q(x-y) \right. \right. \\ &\quad \left. \left. \times \rho_l(\xi) d\alpha dx ds d\xi dy \right] + E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_0^{2/p} \int_{B(0,R)} \int_0^1 |\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right] \right\} \\ &\leq \|\partial_s \varphi\|_\infty \left\{ \frac{2\delta}{p} |B(0, R)| + \|\nu\|_{L^\infty(0,T;L^1(\Omega \times B(0,R) \times (0,1)))} \times \frac{2}{p} \right. \\ &\quad \left. + \|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^1(\Omega \times B(0,R+1)))} \times \frac{2}{p} + |B(0, R)| C_\rho \times \frac{2}{lp} \right\} \\ &\leq \|\partial_s \varphi\|_\infty \left\{ \frac{2\delta}{p} |B(0, R)| + \sqrt{|B(0, R)|} \|\nu\|_{L^\infty(0,T;L^2(\Omega \times B(0,R) \times (0,1)))} \times \frac{2}{p} \right. \\ &\quad \left. + \sqrt{|B(0, R+1)|} e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} \times \frac{2}{p} + |B(0, R)| C_\rho \times \frac{2}{lp} \right\} \\ &\leq \frac{C}{p}. \end{aligned}$$

So, one gets:

$$|A_1 - A_{1,1}| \leq C \left(\varepsilon(2/p, \nu, Q_R) + \frac{1}{p} \right).$$

Let us now bound $|A_{1,1} - A_{1,2}|$. Using a change of variables and Lemma 3, we have:

$$\begin{aligned} |A_{1,1} - A_{1,2}| &= \left| E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \left\{ \eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi) - \eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) \right\} \right. \right. \\ &\quad \left. \left. \times \partial_t \varphi(x, t) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy \right] \right| \\ &\leq \|\partial_t \varphi\|_\infty \|\eta'_\delta\|_\infty E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{Q_R} \int_0^1 |\xi| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy \right] \\ &\leq \|\partial_t \varphi\|_\infty \|\eta'_\delta\|_\infty T |B(0, R)| C_\rho \times \frac{1}{l}. \end{aligned}$$

So, since $\|\eta'_\delta\|_\infty = 1$, we obtain:

$$|A_{1,1} - A_{1,2}| \leq C \frac{1}{l}.$$

Using again (44), we have:

$$\begin{aligned} |A_{1,2} - A_{1,3}| &= \left| E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \left\{ \eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) - |\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)| \right\} \partial_t \varphi(x, t) \tilde{\rho}_q(x-y) d\alpha dx dt dy \right] \right| \\ &\leq \delta \|\partial_t \varphi\|_\infty E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \tilde{\rho}_q(x-y) d\alpha dx dt dy \right]. \end{aligned}$$

So, we obtain:

$$|A_{1,2} - A_{1,3}| \leq C\delta.$$

In order to bound $|A_{1,3} - A_{1,4}|$, we start by rewriting $A_{1,4}$:

$$A_{1,4} = E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 |\nu(y, t, \alpha) - u_{\mathcal{T},k}(y, t)| \partial_t \varphi(y, t) \tilde{\rho}_q(x-y) d\alpha dt dy dx \right].$$

Using the reverse triangular inequality and Proposition 1 , one gets:

$$\begin{aligned}
|A_{1,3} - A_{1,4}| &\leq \left| E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \left\{ |\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)| - |\nu(y, t, \alpha) - u_{\mathcal{T},k}(y, t)| \right\} \partial_t \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \right| \\
&\quad + \left| E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 |\nu(y, t, \alpha) - u_{\mathcal{T},k}(y, t)| \left\{ \partial_t \varphi(x, t) - \partial_t \varphi(y, t) \right\} \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \right| \\
&\leq \|\partial_t \varphi\|_\infty E \left[\int_{\mathbb{R}^d} \int_{Q_R} \int_0^1 |\nu(x, t, \alpha) - \nu(y, t, \alpha)| \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \\
&\quad + \|\nabla_x \partial_t \varphi\|_\infty E \left[\int_{B(0, R+1)} \int_0^T \int_{|x-y| < 1/q} \int_0^1 |\nu(y, t, \alpha) - u_{\mathcal{T},k}(y, t)| |x - y| \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \\
&\leq \|\partial_t \varphi\|_\infty \times \bar{\varepsilon}(1/q, \nu, Q_R) + \|\nabla_x \partial_t \varphi\|_\infty \times \frac{1}{q} E \left[\int_{B(0, R+1)} \int_0^T \int_0^1 |\nu(y, t, \alpha) - u_{\mathcal{T},k}(y, t)| d\alpha dt dy \right] \\
&\leq \|\partial_t \varphi\|_\infty \times \bar{\varepsilon}(1/q, \nu, Q_R) + \|\nabla_x \partial_t \varphi\|_\infty \times \frac{1}{q} \times \left(\|\nu\|_{L^1(\Omega \times B(0, R+1) \times (0, T) \times (0, 1))} + \|u_{\mathcal{T},k}\|_{L^1(\Omega \times B(0, R+1) \times (0, T))} \right),
\end{aligned}$$

where $\bar{\varepsilon}(1/q, \nu, Q_R) = \sup \left\{ \|\nu - \nu(\cdot - \xi, \cdot, \cdot)\|_{L^1(\Omega \times Q_R \times (0, 1))}; |\xi| \leq 1/q \right\}$. As previously, we have $\bar{\varepsilon}(1/q, \nu, Q_R) \longrightarrow 0$ when $q \longrightarrow \infty$. Then, we deduce that:

$$|A_{1,3} - A_{1,4}| \leq C \left(\bar{\varepsilon}(1/q, \nu, Q_R) + \frac{1}{q} \right).$$

In order to bound $|A_{1,4} - A_{1,5}|$, we define

$$\gamma(k, h, Q_R) = \left| E \left[\int_Q \int_0^1 \int_0^1 \left\{ |\nu(x, t, \alpha) - u_{\mathcal{T},k}(x, t)| - |\nu(x, t, \alpha) - \mu(x, t, \beta)| \right\} \partial_t \varphi(x, t) d\beta d\alpha dx dt \right] \right|.$$

Since $H : (v, \omega, x, t) \mapsto \int_0^1 |\nu(x, t, \alpha) - v| |\partial_t \varphi(x, t)| d\alpha$ is a Carathéodory function such that $H(u_{\mathcal{T},k}, \cdot, \cdot, \cdot)$ is bounded in $L^2(\Omega \times Q)$, thus it is uniformly integrable. So, we have:

$$E \left[\int_Q H(u_{\mathcal{T},k}, \omega, x, t) dx dt \right] \longrightarrow E \left[\int_Q \int_0^1 \int_0^1 |\nu(x, t, \alpha) - \mu(x, t, \beta)| |\partial_t \varphi(x, t)| d\beta d\alpha dx dt \right],$$

when $r \longrightarrow +\infty$, which means that $\gamma(k_r, h_r, Q_R) = \gamma(h_r^{21}, h_r, Q_R) \longrightarrow 0$ when $r \longrightarrow +\infty$. Therefore, one obtains finally from the above bounds that:

$$\begin{aligned}
|A_1 - A_{1,5}| &\leq C \left(\varepsilon(2/p, \nu, Q_R) + \frac{1}{p} + \frac{1}{l} + \delta + \bar{\varepsilon}(1/q, \nu, Q_R) + \frac{1}{q} + \gamma(k, h, Q_R) \right) \\
&\leq C \left(\varepsilon(2h_r^5, \nu, Q_R) + h_r^5 + \frac{1}{l} + \delta + \bar{\varepsilon}(1/q, \nu, Q_R) + \frac{1}{q} + \gamma(h_r^{21}, h_r, Q_R) \right).
\end{aligned}$$

Finally, we can pass to the limit in the term $A_1 + B_1$ and we have:

$$\limsup_{q \rightarrow +\infty} \limsup_{\delta \rightarrow 0} \limsup_{l \rightarrow +\infty} \limsup_{r \rightarrow +\infty} \left| (A_1 + B_1) - A_{1,5} \right| \leq 0,$$

hence we have proven that

$$\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} (A_1 + B_1) = A_{1,5} = E \left[\int_Q \int_{(0,1)^2} |\nu(x, t, \alpha) - \mu(x, t, \beta)| |\partial_t \varphi(x, t)| d\alpha d\beta dx dt \right].$$

4.2.2 Study of $A_2 + B_2$

Since η_δ and ρ_l are even functions, using Fubini's theorem, we obtain:

$$\begin{aligned}
A_2 + B_2 &= E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta (\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \varphi(x, t) \tilde{\rho}'_p(t - s) \tilde{\rho}_q(x - y) \rho_l(\xi) dx dt d\xi dy ds \right] \\
&\quad - E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta (u_{\mathcal{T},k}(y, s) - \nu(x, t, \alpha) - \tau) \varphi(x, t) \tilde{\rho}'_p(t - s) \tilde{\rho}_q(x - y) \rho_l(-\tau) dx dt d\tau dy ds \right] \\
&= 0.
\end{aligned}$$

4.2.3 Study of $A_3 + B_3$

Since $B_3 = 0$, we only consider A_3 and we start by rewriting it in the following way:

$$A_3 = (A_3 - A_{3,1}) + (A_{3,1} - A_{3,2}) + (A_{3,2} - A_{3,3}) + (A_{3,3} - A_{3,4}) + (A_{3,4} - A_{3,5}) + A_{3,5},$$

where:

$$\begin{aligned} A_3 &= E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \Phi_{\eta_\delta}(x, t, \nu(x, t, \alpha), \kappa) \cdot \nabla_x \varphi(x, t) \bar{\rho}_p(t-s) \bar{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\alpha dx dt d\kappa dy ds \right], \\ A_{3,1} &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \Phi_{\eta_\delta}(x, t, \nu(x, t, \alpha), \kappa) \cdot \nabla_x \varphi(x, t) \bar{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, t) - \kappa) d\alpha dx dt d\kappa dy \right], \\ A_{3,2} &= E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \Phi_{\eta_\delta}(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(y, t)) \cdot \nabla_x \varphi(x, t) \bar{\rho}_q(x-y) d\alpha dx dt dy \right], \\ A_{3,3} &= E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \Phi(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(y, t)) \cdot \nabla_x \varphi(x, t) \bar{\rho}_q(x-y) d\alpha dx dt dy \right], \\ A_{3,4} &= E \left[\int_Q \int_0^1 \Phi(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(x, t)) \cdot \nabla_x \varphi(x, t) d\alpha dx dt \right] \\ \text{and } A_{3,5} &= E \left[\int_Q \int_0^1 \int_0^1 \Phi(x, t, \nu(x, t, \alpha), \mu(x, t, \beta)) \cdot \nabla_x \varphi(x, t) d\beta d\alpha dx dt \right]. \end{aligned}$$

Let us compare A_3 and $A_{3,5}$. We start by bounding $|A_3 - A_{3,1}|$ thanks to the following splitting of $A_{3,1}$:

$$A_{3,1} = \tilde{A}_{3,1} + \bar{A}_{3,1},$$

where:

$$\begin{aligned} \tilde{A}_{3,1} &= E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi) \cdot \nabla_x \varphi(x, s) \bar{\rho}_p(t-s) \bar{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy ds \right], \\ \bar{A}_{3,1} &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi) \cdot \nabla_x \varphi(x, s) \left(1 - \int_0^T \bar{\rho}_p(t-s) dt \right) \bar{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right]. \end{aligned}$$

Then, using (32), we have:

$$\begin{aligned} |A_3 - \tilde{A}_{3,1}| &\leq \left| E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \left\{ \Phi_{\eta_\delta}(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(y, s) - \xi) - \Phi_{\eta_\delta}(x, t, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi) \right\} \cdot \nabla_x \varphi(x, t) \right. \right. \\ &\quad \left. \left. \times \bar{\rho}_p(t-s) \bar{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy ds \right] \right| \\ &\quad + \left| E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \left\{ \Phi_{\eta_\delta}(x, t, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi) - \Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi) \right\} \cdot \nabla_x \varphi(x, t) \right. \right. \\ &\quad \left. \left. \times \bar{\rho}_p(t-s) \bar{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy ds \right] \right| \\ &\quad + \left| E \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi) \cdot \left\{ \nabla_x \varphi(x, t) - \nabla_x \varphi(x, s) \right\} \right. \right. \\ &\quad \left. \left. \times \bar{\rho}_p(t-s) \bar{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy ds \right] \right| \\ &\leq \|\nabla_x \varphi\|_\infty C_f E \left[\int_0^T \int_{Q_R} \int_0^1 |\nu(x, t, \alpha) - \nu(x, s, \alpha)| \bar{\rho}_p(t-s) d\alpha dx dt ds \right] \\ &\quad + \|\nabla_x \varphi\|_\infty 2C_f |B(0, R)| E \left[\int_0^T \int_0^T |t-s| \bar{\rho}_p(t-s) dt ds \right] \\ &\quad + \|\partial_t \nabla_x \varphi\|_\infty \times \frac{2}{p} E \left[\int_Q \int_{\mathbb{R}} \int_{B(0, R)} \int_0^1 |\Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi)| \bar{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right] \\ &\leq \|\nabla_x \varphi\|_\infty C_f \times \varepsilon(2/p, \nu, Q_R) + \|\nabla_x \varphi\|_\infty C_f |B(0, R)| \times \frac{4T}{p} \\ &\quad + \|\partial_t \nabla_x \varphi\|_\infty \times \frac{2}{p} E \left[\int_Q \int_{\mathbb{R}} \int_{B(0, R)} \int_0^1 |\Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi)| \bar{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right], \end{aligned}$$

where $\varepsilon(2/p, \nu, Q_R) = \sup \left\{ \|\nu - \nu(\cdot, \cdot - \tau, \cdot)\|_{L^1(\Omega \times Q_R \times (0,1))}; |\tau| \leq 2/p \right\}$, and $\varepsilon(2/p, \nu, Q_R) \longrightarrow 0$ when $p \longrightarrow \infty$.

Moreover, using (46) and (47), one gets:

$$\begin{aligned}
& E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi)| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right] \\
& \leq E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi) - \Phi(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi)| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right] \\
& \quad + E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\Phi(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi)| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right] \\
& \leq 4C_f \delta \times E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right] \\
& \quad + C_f \times E \left[\int_Q \int_{\mathbb{R}} \int_{B(0,R)} \int_0^1 |\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx d\xi dy ds \right] \\
& \leq 4C_f \delta T |B(0, R)| + \|\nu\|_{L^1(\Omega \times B(0,R) \times (0,T) \times (0,1))} + \|u_{\mathcal{T},k}\|_{L^1(\Omega \times B(0,R+1) \times (0,T))} + T |B(0, R)| C_\rho \frac{1}{l}.
\end{aligned}$$

Therefore, we obtain that:

$$\begin{aligned}
|A_3 - \tilde{A}_{3,1}| & \leq \|\nabla_x \varphi\|_\infty C_f \varepsilon(2/p, \nu, Q_R) + \|\nabla_x \varphi\|_\infty |B(0, R)| \times \frac{4T}{p} + \|\partial_t \nabla_x \varphi\|_\infty \times \frac{2}{p} \left(4C_f T |B(0, R)| \times \delta \right. \\
& \quad \left. + \sqrt{T |B(0, R)|} \|\nu\|_{L^2(\Omega \times B(0,R) \times (0,T) \times (0,1))} + \sqrt{|B(0, R+1)|} T e^{TC_g^2/2} \|u_0\|_{L^2(\mathbb{R}^d)} + T |B(0, R)| \frac{C_\rho}{l} \right) \\
& \leq C \left(\varepsilon(2/p, \nu, Q_R) + \frac{1}{p} \right).
\end{aligned}$$

Now, let us estimate $|\bar{A}_{3,1}|$. Using the inequality (42), we have:

$$\begin{aligned}
|\bar{A}_{3,1}| & \leq E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_0^{2/p} \int_{\mathbb{R}^d} \int_0^1 |\Phi_{\eta_\delta}(x, s, \nu(x, s, \alpha), u_{\mathcal{T},k}(y, s) - \xi)| \|\nabla_x \varphi(x, s)\| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right] \\
& \leq \|\nabla_x \varphi\|_\infty C_f \frac{2}{p} \times \left\{ \|\nu\|_{L^\infty(0,T;L^1(\Omega \times B(0,R) \times (0,1)))} + \|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^1(\Omega \times B(0,R+1)))} + |B(0, R)| \frac{C_\rho}{l} \right\} \\
& \leq C \frac{1}{p}.
\end{aligned}$$

Finally, we get:

$$|A_3 - A_{3,1}| \leq C \left(\varepsilon(2/p, \nu, Q_R) + \frac{1}{p} \right).$$

Since for any $(x, t) \in Q$ and $a, b, b' \in \mathbb{R}$ we have:

$$|\Phi_{\eta_\delta}(x, t, a, b) - \Phi_{\eta_\delta}(x, t, a, b')| \leq 5C_f |b - b'|, \quad (55)$$

hence, we get:

$$\begin{aligned}
|A_{3,1} - A_{3,2}| & \leq \|\nabla_x \varphi\|_\infty E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{Q_R} \int_0^1 5C_f |\xi| \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right] \\
& \leq \|\nabla_x \varphi\|_\infty 5C_f T |B(0, R)| \frac{C_\rho}{l}.
\end{aligned}$$

Using (42), we obtain:

$$|A_{3,2} - A_{3,3}| \leq \|\nabla_x \varphi\|_\infty E \left[\int_{\mathbb{R}^d} \int_{Q_R} (4C_f \delta) \tilde{\rho}_q(x-y) dx dt dy \right] \leq C \delta.$$

Thanks to (47), (48), (49), one gets:

$$\begin{aligned}
& |A_{3,3} - A_{3,4}| \\
& \leq \left| E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \left\{ \Phi(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(y, t)) - \Phi(x, t, \nu(y, t, \alpha), u_{\mathcal{T},k}(y, t)) \right\} \cdot \nabla_x \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dy dt dx \right] \right| \\
& \quad + \left| E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \left\{ \Phi(x, t, \nu(y, t, \alpha), u_{\mathcal{T},k}(y, t)) - \Phi(y, t, \nu(y, t, \alpha), u_{\mathcal{T},k}(y, t)) \right\} \cdot \nabla_x \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dy dt dx \right] \right| \\
& \quad + \left| E \left[E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \Phi(y, t, \nu(y, t, \alpha), u_{\mathcal{T},k}(y, t)) \cdot \left\{ \nabla_x \varphi(x, t) - \nabla_x \varphi(y, t) \right\} \tilde{\rho}_q(x - y) d\alpha dy dt dx \right] \right] \right| \\
& \leq \|\nabla_x \varphi\|_\infty C_f^R E \left[\int_{\mathbb{R}^d} \int_{Q_R} \int_0^1 |\nu(x, t, \alpha) - \nu(y, t, \alpha)| \tilde{\rho}_q(x - y) d\alpha dy dt dx \right] \\
& \quad + 2\|\nabla_x \varphi\|_\infty C_f E \left[\int_{\mathbb{R}^d} \int_{Q_R} \int_0^1 |x - y| \tilde{\rho}_q(x - y) d\alpha dy dt dx \right] \\
& \quad + \|\nabla_{xx} \varphi\|_\infty C_f E \left[\int_{B(0, R+1)} \int_0^T \int_{|x-y| < 1/q} \int_0^1 |\nu(y, t, \alpha) - u_{\mathcal{T},k}(y, t)| |x - y| \tilde{\rho}_q(x - y) d\alpha dy dt dx \right] \\
& \leq \|\nabla_x \varphi\|_\infty C_f \bar{\varepsilon}(1/q, \nu, Q_R) + 2\|\nabla_x \varphi\|_\infty C_f^R T |B(0, R)| \times \frac{C_\rho}{q} \\
& \quad + \|\nabla_{xx} \varphi\|_\infty C_f \times \frac{1}{q} \left(\|\nu\|_{L^1(\Omega \times Q_{R+1} \times (0,1))} + \|u_{\mathcal{T},k}\|_{L^1(\Omega \times Q_{R+1})} \right),
\end{aligned}$$

where $\bar{\varepsilon}(1/q, \nu, Q_R) = \sup \left\{ \|\nu - \nu(\cdot - \xi, \cdot, \cdot)\|_{L^1(\Omega \times Q_R \times (0,1))}; |\xi| \leq 1/q \right\}$, and $\bar{\varepsilon}(1/q, \nu, Q_R) \longrightarrow 0$ when $q \longrightarrow \infty$. Therefore, we deduce that:

$$|A_{3,3} - A_{3,4}| \leq C \left(\bar{\varepsilon}(1/q, \nu, Q_R) + \frac{1}{q} \right).$$

Since $u_{\mathcal{T},k}$ tends to μ in the sense of Young measures and $\tilde{H} : (v, \omega, x, t) \mapsto \int_0^1 \Phi(x, t, \nu(x, t, \alpha), v) \cdot \nabla_x \varphi(x, t) d\alpha$ is a Carathéodory function such that $\tilde{H}(u_{\mathcal{T},k}, \cdot, \cdot)$ is bounded in $L^2(\Omega \times Q)$, if one denotes

$$\bar{\gamma}(k, h, Q_R) = \left| E \left[\int_Q \int_0^1 \int_0^1 \left\{ \Phi(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(x, t)) - \Phi(x, t, \nu(x, t, \alpha), \mu(x, t, \beta)) \right\} \cdot \nabla_x \varphi(x, t) d\beta d\alpha dx dt \right] \right|,$$

one gets that $\bar{\gamma}(k_r, h_r, Q_R) = \bar{\gamma}(h_r^{21}, h_r, Q_R) \longrightarrow 0$ when $r \longrightarrow +\infty$. Therefore, we deduce from the above estimates that we have:

$$\begin{aligned}
|A_3 - A_{3,5}| & \leq C \left(\varepsilon(2/p, \nu, Q_R) + \frac{1}{p} + \delta + \frac{1}{l} + \bar{\varepsilon}(1/q, \nu, Q_R) + \frac{1}{q} + \bar{\gamma}(k_r, h_r, Q_R) \right) \\
& \leq C \left(\varepsilon(2h_r^5, \nu, Q_R) + h_r^5 + \delta + \frac{1}{l} + \bar{\varepsilon}(1/q, \nu, Q_R) + \frac{1}{q} + \bar{\gamma}(h_r^{21}, h_r, Q_R) \right).
\end{aligned}$$

Finally, we can pass to the limit in the term $A_3 + B_3$ and we obtain:

$$\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} (A_3 + B_3) = A_{3,5} = E \left[\int_Q \int_{(0,1)^2} \Phi(x, t, \nu(x, t, \alpha), \mu(x, t, \beta)) \cdot \nabla_x \varphi(x, t) d\alpha d\beta dx dt \right].$$

4.2.4 Study of $A_4 + B_4$

Using the change of variables $\xi = u_{\mathcal{T},k}(y, s) - \kappa$ in A_4 and $\xi = \nu(x, t, \alpha) - \kappa$ in B_4 , one gets

$$A_4 = (A_4 - A_{4,1}) + (A_{4,1} - A_{4,2}) + A_{4,2} \text{ and } B_4 = (B_4 - B_{4,1}) + (B_{4,1} - B_{4,2}) + (B_{4,2} - B_{4,3}) + B_{4,3},$$

$$\text{where } A_{4,1} = E \left[\int_Q \int_Q \int_0^1 \Phi_{\eta_\delta}(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(y, s)) \cdot \nabla \tilde{\rho}_q(x - y) \varphi(x, t) \tilde{\rho}_p(t - s) d\alpha dx dt dy ds \right],$$

$$A_{4,2} = E \left[\int_Q \int_Q \int_0^1 \Phi(x, t, \nu(x, t, \alpha), u_{\mathcal{T},k}(y, s)) \cdot \nabla \tilde{\rho}_q(x - y) \varphi(x, t) \tilde{\rho}_p(t - s) d\alpha dx dt dy ds \right],$$

$$B_{4,1} = -E \left[\int_Q \int_Q \int_0^1 \Phi_{\eta_\delta}(y, s, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) \cdot \nabla \tilde{\rho}_q(x - y) \varphi(x, t) \tilde{\rho}_p(t - s) d\alpha dx dt dy ds \right],$$

$$B_{4,2} = -E \left[\int_Q \int_Q \int_0^1 \Phi(y, s, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) \cdot \nabla \tilde{\rho}_q(x - y) \varphi(x, t) \tilde{\rho}_p(t - s) d\alpha dx dt dy ds \right]$$

$$\text{and } B_{4,3} = -E \left[\int_Q \int_Q \int_0^1 \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) \cdot \nabla \tilde{\rho}_q(x - y) \varphi(x, t) \tilde{\rho}_p(t - s) d\alpha dx dt dy ds \right].$$

Let us estimate separately $(A_4 - A_{4,2})$ and $(B_4 - B_{4,3})$. Using (55) and Lemma 3, we obtain:

$$|A_4 - A_{4,1}| \leq 5\|\varphi\|_\infty C_f T E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{B(0,R)} |\xi| |\nabla \tilde{\rho}_q(x - y)| \rho_l(\xi) dx d\xi dy \right] \leq C \frac{q}{l}.$$

Similarly, we get:

$$|B_4 - B_{4,1}| \leq \|\varphi\|_\infty E \left[\int_Q \int_{\mathbb{R}} \int_{Q_R} \int_0^1 5C_f |\xi| |\nabla \tilde{\rho}_q(x-y)| \bar{\rho}_p(t-s) \rho_l(\xi) d\alpha dx dt d\xi dy ds \right] \leq C \frac{q}{l}.$$

Using (46), one has:

$$|A_{4,1} - A_{4,2}| \leq \|\varphi\|_\infty E \left[\int_Q \int_{Q_R} 4C_f \delta |\nabla \tilde{\rho}_q(x-y)| \bar{\rho}_p(t-s) dx dt dy ds \right] \leq C q \delta.$$

In the same way, we obtain that:

$$|B_{4,1} - B_{4,2}| \leq \|\varphi\|_\infty E \left[\int_Q \int_{Q_R} \{4C_f \delta\} |\nabla \tilde{\rho}_q(x-y)| \bar{\rho}_p(t-s) dx dt dy ds \right] \leq C q \delta.$$

Thanks to (49), we have:

$$\begin{aligned} |B_{4,2} - B_{4,3}| &= \left| E \left[\int_Q \int_Q \int_0^1 \left\{ \Phi(y, s, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) - \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) \right\} \cdot \nabla \tilde{\rho}_q(x-y) \right. \right. \\ &\quad \left. \left. \times \varphi(x, t) \bar{\rho}_p(t-s) d\alpha dx dt dy ds \right] \right| \\ &\leq \|\varphi\|_\infty E \left[\int_Q \int_{Q_R} 2C_f^T |t-s| |\nabla \tilde{\rho}_q(x-y)| \bar{\rho}_p(t-s) dx dt dy ds \right] \leq C \frac{q}{p}. \end{aligned}$$

Now, using the equality $\Phi(x, t, a, b) = \Phi(x, t, b, a)$ for all (x, t, a, b) in $Q \times \mathbb{R}^2$, we have:

$$\begin{aligned} A_{4,2} + B_{4,3} &= E \left[\int_Q \int_Q \int_0^1 \left\{ \Phi(x, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) - \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) \right\} \cdot \nabla \tilde{\rho}_q(x-y) \right. \\ &\quad \left. \times \varphi(x, t) \bar{\rho}_p(t-s) d\alpha dx dt dy ds \right]. \end{aligned}$$

Moreover, using (28) and (33), we obtain:

$$\begin{aligned} &\left| E \left[\int_Q \int_Q \int_0^1 \left\{ \Phi(x, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) - \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) \right\} \cdot \nabla \tilde{\rho}_q(x-y) \varphi(x, t) \bar{\rho}_p(t-s) d\alpha dx dt dy ds \right] \right| \\ &= \left| E \left[\int_Q \int_Q \int_0^1 \left\{ \Phi(x, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) - \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) \right\} \tilde{\rho}_q(x-y) \cdot \nabla_x \varphi(x, t) \bar{\rho}_p(t-s) d\alpha dx dt dy ds \right] \right| \\ &\leq \frac{C}{q}. \end{aligned}$$

And so

$$\begin{aligned} |A_{4,2} + B_{4,3}| &\leq \frac{C}{q} + \left| E \left[\int_Q \int_Q \int_0^1 \left(\left\{ \Phi(x, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) - \Phi(x, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) \right\} \right. \right. \right. \\ &\quad \left. \left. - \left\{ \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) - \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) \right\} \right) \cdot \nabla \tilde{\rho}_q(x-y) \varphi(x, t) \bar{\rho}_p(t-s) d\alpha dx dt dy ds \right] \right|. \end{aligned}$$

Using the expression (45) of Φ we have:

$$\begin{aligned} &\Phi(x, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) - \Phi(x, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) \\ &= \left(f(x, t, u_{\mathcal{T},k}(y, s), \top \nu(x, t, \alpha)) - f(x, t, u_{\mathcal{T},k}(y, s), \top \nu(y, t, \alpha)) \right) \\ &\quad - \left(f(x, t, u_{\mathcal{T},k}(y, s), \perp \nu(x, t, \alpha)) - f(x, t, u_{\mathcal{T},k}(y, s), \perp \nu(y, t, \alpha)) \right) \\ &= \left(u_{\mathcal{T},k}(y, s), \top \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s), \top \nu(y, t, \alpha) \right) \\ &\quad \times \int_0^1 \partial_v f \left(x, t, u_{\mathcal{T},k}(y, s), \top \nu(y, t, \alpha) + \theta (u_{\mathcal{T},k}(y, s), \top \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s), \top \nu(y, t, \alpha)) \right) d\theta \\ &\quad + \left(u_{\mathcal{T},k}(y, s), \perp \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s), \perp \nu(y, t, \alpha) \right) \\ &\quad \times \int_0^1 \partial_v f \left(x, t, u_{\mathcal{T},k}(y, s), \perp \nu(y, t, \alpha) + \theta (u_{\mathcal{T},k}(y, s), \perp \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s), \perp \nu(y, t, \alpha)) \right) d\theta, \end{aligned}$$

and, similarly:

$$\begin{aligned}
& \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(x, t, \alpha)) - \Phi(y, t, u_{\mathcal{T},k}(y, s), \nu(y, t, \alpha)) \\
&= \left(u_{\mathcal{T},k}(y, s) \top \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \top \nu(y, t, \alpha) \right) \\
&\quad \times \int_0^1 \partial_v f \left(y, t, u_{\mathcal{T},k}(y, s) \top \nu(y, t, \alpha) + \theta(u_{\mathcal{T},k}(y, s) \top \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \top \nu(y, t, \alpha)) \right) d\theta \\
&\quad + \left(u_{\mathcal{T},k}(y, s) \perp \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \perp \nu(y, t, \alpha) \right) \\
&\quad \times \int_0^1 \partial_v f \left(y, t, u_{\mathcal{T},k}(y, s) \perp \nu(y, t, \alpha) + \theta(u_{\mathcal{T},k}(y, s) \perp \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \perp \nu(y, t, \alpha)) \right) d\theta.
\end{aligned}$$

Since for all $a, b, b' \in \mathbb{R}$, $|a \top b - a \top b'| \leq |b - b'|$ and $|a \perp b - a \perp b'| \leq |b - b'|$, we deduce that:

$$\begin{aligned}
|A_{4,2} + B_{4,3}| &\leq \frac{C}{q} + E \left[\int_Q \int_{Q_R} \int_0^1 |\nu(x, t, \alpha) - \nu(y, t, \alpha)| |\nabla \tilde{\rho}_q(x - y)| \varphi(x, t) \tilde{\rho}_p(t - s) \right. \\
&\quad \times \left(\left| \int_0^1 \partial_v f \left(x, t, u_{\mathcal{T},k}(y, s) \top \nu(y, t, \alpha) + \theta(u_{\mathcal{T},k}(y, s) \top \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \top \nu(y, t, \alpha)) \right) d\theta \right. \right. \\
&\quad \left. \left. - \int_0^1 \partial_v f \left(y, t, u_{\mathcal{T},k}(y, s) \top \nu(y, t, \alpha) + \theta(u_{\mathcal{T},k}(y, s) \top \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \top \nu(y, t, \alpha)) \right) d\theta \right| \right. \\
&\quad \left. + \left| \int_0^1 \partial_v f \left(x, t, u_{\mathcal{T},k}(y, s) \perp \nu(y, t, \alpha) + \theta(u_{\mathcal{T},k}(y, s) \perp \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \perp \nu(y, t, \alpha)) \right) d\theta \right. \right. \\
&\quad \left. \left. - \int_0^1 \partial_v f \left(y, t, u_{\mathcal{T},k}(y, s) \perp \nu(y, t, \alpha) + \theta(u_{\mathcal{T},k}(y, s) \perp \nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) \perp \nu(y, t, \alpha)) \right) d\theta \right| \right) d\alpha dx dt dy ds \Big].
\end{aligned}$$

Using Assumption H7, we get:

$$\begin{aligned}
|A_{4,2} + B_{4,3}| &\leq \frac{C}{q} + \|\varphi\|_\infty 2\tilde{C}_f^R E \left[\int_Q \int_{Q_R} \int_0^1 |\nu(x, t, \alpha) - \nu(y, t, \alpha)| |\nabla \tilde{\rho}_q(x - y)| \tilde{\rho}_p(t - s) |x - y| d\alpha dx dt dy ds \right] \\
&\leq C \left(\frac{1}{q} + \bar{\varepsilon}(1/q, \nu, Q_R) \right),
\end{aligned}$$

where $\bar{\varepsilon}(1/q, \nu, Q_R) = \sup \left\{ \|\nu - \nu(\cdot - \xi, \cdot, \cdot)\|_{L^1(\Omega \times Q_R \times (0,1))}; |\xi| \leq 1/q \right\}$.

Therefore, thanks to the above bounds, we deduce that:

$$|A_4 + B_4| \leq C \left(q\delta + \frac{q}{l} + qh_r^5 + \frac{1}{q} + \bar{\varepsilon}(1/q, \nu, Q_R) \right).$$

Finally, we obtain:

$$\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_4 + B_4 = 0.$$

4.2.5 Study of $A_7 + B_7$

Since $B_7 = 0$, we consider:

$$A_7 = E \left[\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(u_0(x) - u_{\mathcal{T},k}(y, s) + \xi) \varphi(x, 0) \tilde{\rho}_p(-s) \tilde{\rho}_q(x - y) \rho_l(\xi) d\alpha dx d\xi dy ds \right].$$

In order to show the convergence of A_7 to 0, we use a new entropy formulation. We fix $x \in \mathbb{R}^d$ and the idea is to apply Proposition 6 to the entropy $\eta_\delta(\cdot - \kappa)$ and the following test function:

$$(y, s) \mapsto \bar{\Psi}_x(y, s) = \varphi(x, 0) \tilde{\rho}_q(x - y) \int_s^\infty \tilde{\rho}_p(-\tau) d\tau.$$

Due to Proposition 6, for any x in \mathbb{R}^d there exists $R^{h,k,\eta_\delta(\cdot - \kappa), \bar{\Psi}_x}$ which satisfies (41) and such that P almost surely in Ω :

$$\begin{aligned}
& \int_{\mathbb{R}^d} \eta_\delta(u_0(y) - \kappa) \varphi(x, 0) \tilde{\rho}_q(x - y) dy + \int_Q \eta_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, 0) \tilde{\rho}_q(x - y) \left(\int_s^\infty \tilde{\rho}_p(-\tau) d\tau \right)' dy ds \\
&+ \int_Q \Phi_{\eta_\delta}(y, s, u_{\mathcal{T},k}(y, s), \kappa) \varphi(x, 0) \left(-\nabla \tilde{\rho}_q(x - y) \right) \left(\int_s^\infty \tilde{\rho}_p(-\tau) d\tau \right) dy ds \\
&+ \int_0^T \int_{\mathbb{R}^d} \eta'_\delta(u_{\mathcal{T},k}(y, s) - \kappa) g(u_{\mathcal{T},k}(y, s)) \varphi(x, 0) \tilde{\rho}_q(x - y) \left(\int_s^\infty \tilde{\rho}_p(-\tau) d\tau \right) dy dW(s) \\
&+ \frac{1}{2} \int_Q \eta''_\delta(u_{\mathcal{T},k}(y, s) - \kappa) g^2(u_{\mathcal{T},k}(y, s)) \varphi(x, 0) \tilde{\rho}_q(x - y) \left(\int_s^\infty \tilde{\rho}_p(-\tau) d\tau \right) dy ds \geq R^{h,k,\eta_\delta(\cdot - \kappa), \bar{\Psi}_x}.
\end{aligned}$$

Multiplying it by $\rho_l(u_0(x) - \kappa)$ then integrating w.r.t x on \mathbb{R}^d , κ on \mathbb{R} and taking the expectation, we get:

$$D_2 + D_4 + D_5 + D_6 + D_7 \geq \tilde{\mathcal{R}}, \quad (56)$$

where:

$$\begin{aligned} D_2 &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \eta_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, 0) \tilde{\rho}_q(x - y) \rho_l(u_0(x) - \kappa) \left(\int_s^\infty \bar{\rho}_p(-\tau) d\tau \right)' dy ds d\kappa dx \right], \\ D_4 &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \Phi_{\eta_\delta}(y, s, u_{\mathcal{T},k}(y, s), \kappa) \cdot \nabla \tilde{\rho}_q(x - y) \varphi(x, 0) \rho_l(u_0(x) - \kappa) \left(\int_s^\infty \bar{\rho}_p(-\tau) d\tau \right) dy ds d\kappa dx \right], \\ D_5 &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^T \eta'_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, 0) g(u_{\mathcal{T},k}(y, s)) \tilde{\rho}_q(x - y) \rho_l(u_0(x) - \kappa) \left(\int_s^\infty \bar{\rho}_p(-\tau) d\tau \right) dW(s) dy d\kappa dx \right], \\ D_6 &= \frac{1}{2} E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \eta''_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, 0) g^2(u_{\mathcal{T},k}(y, s)) \tilde{\rho}_q(x - y) \rho_l(u_0(x) - \kappa) \left(\int_s^\infty \bar{\rho}_p(-\tau) d\tau \right) dy ds d\kappa dx \right], \\ D_7 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \eta_\delta(u_0(y) - \kappa) \varphi(x, 0) \tilde{\rho}_q(x - y) \rho_l(u_0(x) - \kappa) dy d\kappa dx \end{aligned}$$

$$\text{and } \tilde{\mathcal{R}} = E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} R^{h,k,\eta_\delta(\cdot - \kappa), \bar{\Psi}_x} \rho_l(u_0(x) - \kappa) d\kappa dx \right].$$

Since η_δ and ρ_l are even functions, one shows that $D_2 = -A_7$, then

$$0 \leq A_7 \leq D_4 + D_5 + D_6 + D_7 - \tilde{\mathcal{R}}.$$

In view to show the convergence of A_7 to 0, we prove that each term in the right hand side of inequality (56) converges to 0 when passing to the limit successively with respect to r, l, δ and q .

Using the fact that $\text{Supp}(\bar{\rho}_p) \subset [-2/p, 0]$, (42) and Proposition 1, we have:

$$|D_4| \leq E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_0^{2/p} \int_{\mathbb{R}^d} |\Phi_{\eta_\delta}(y, s, u_{\mathcal{T},k}(y, s), u_0(x) - \xi)| |\nabla \tilde{\rho}_q(x - y)| |\varphi(x, 0)| \rho_l(\xi) dy ds d\xi dx \right] \leq C \frac{q}{p}.$$

Thanks to Cauchy-Schwarz inequality, the isometric property of the Itô integral, Jensen inequality, the fact that g is a bounded function and that $\|\eta'_\delta\|_\infty = 1$, one gets:

$$\begin{aligned} |D_5| &\leq \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left\{ E \left[\left(\int_0^T \eta'_\delta(u_{\mathcal{T},k}(y, s) - u_0(x) + \xi) g(u_{\mathcal{T},k}(y, s)) \left(\int_s^\infty \bar{\rho}_p(-\tau) d\tau \right) dW(s) \right)^2 \right] \right\}^{1/2} \right. \\ &\quad \times \left. \left(\int_{\mathbb{R}^d} \varphi(x, 0) \tilde{\rho}_q(x - y) \rho_l(\xi) dx \right) d\xi dy \right| \\ &\leq \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}} E \left\{ \left[\int_0^T \eta_\delta'^2(u_{\mathcal{T},k}(y, s) - u_0(x) + \xi) g^2(u_{\mathcal{T},k}(y, s)) \left(\int_s^\infty \bar{\rho}_p(-\tau) d\tau \right)^2 ds \right] \right\}^{1/2} \right. \\ &\quad \times \left. \left(\int_{\mathbb{R}^d} \varphi(x, 0) \tilde{\rho}_q(x - y) \rho_l(\xi) dx \right) d\xi dy \right| \\ &\leq C \frac{1}{p^{1/2}}. \end{aligned}$$

Moreover, since g is a bounded function and $\|\eta''_\delta\|_\infty = \frac{\pi}{2\delta}$, one has:

$$\begin{aligned} |D_6| &= \left| \frac{1}{2} E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \eta''_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, 0) g^2(u_{\mathcal{T},k}(y, s)) \tilde{\rho}_q(x - y) \rho_l(u_0(x) - \kappa) \left(\int_s^\infty \bar{\rho}_p(-\tau) d\tau \right) dy ds d\kappa dx \right] \right| \\ &\leq C \frac{1}{p\delta}. \end{aligned}$$

Thanks to (44) and the triangular inequality, we get:

$$\begin{aligned} |D_7| &\leq \|\varphi(\cdot, 0)\|_\infty \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{B(0,R)} \left| \eta_\delta(u_0(y) - u_0(x) + \xi) - |u_0(y) - u_0(x) + \xi| \right| \tilde{\rho}_q(x - y) \rho_l(\xi) dx d\xi dy \right. \\ &\quad \left. + \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{B(0,R)} |u_0(y) - u_0(x) + \xi| \tilde{\rho}_q(x - y) \rho_l(\xi) dx d\xi dy \right\} \\ &\leq \|\varphi(\cdot, 0)\|_\infty \left\{ \delta |B(0, R)| + \frac{1}{l} |B(0, R)| + \bar{\varepsilon}(1/q, u_0, B(0, R)) \right\} \\ &\leq C \left(\delta + \frac{1}{l} + \bar{\varepsilon}(1/q, u_0, B(0, R)) \right), \end{aligned}$$

where $\bar{\varepsilon}(1/q, u_0, B(0, R)) = \sup \left\{ \|u_0 - u_0(\cdot - \xi)\|_{L^1(B(0, R))}; |\xi| \leq 1/q \right\}$.

Let us bound $\tilde{\mathcal{R}}$. For each x in \mathbb{R}^d , due to Proposition 6, the expression of $\bar{\Psi}_x$ and Lemma 3 we have

$$\begin{aligned} |E [R^{h, k, \eta_\delta(\cdot - \kappa), \bar{\Psi}_x}]| &\leq C_{CE} \left[\|\eta'_\delta\|_\infty \|\bar{\Psi}_x\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'_\delta\|_\infty \|\nabla_y \bar{\Psi}_x\|_\infty h^{1/2} + (\|\partial_s \bar{\Psi}_x\|_\infty \|\eta'_\delta\|_\infty + \|\bar{\Psi}_x\|_\infty \|\eta''_\delta\|_\infty) k^{1/2} \right. \\ &\quad \left. + \|\eta'''_\delta\|_\infty \|\bar{\Psi}_x\|_\infty k + \int_{B(0, R)} |u_0(x) - u_{\mathcal{T}, 0}(x)| dx + (\|\eta'_\delta\|_\infty \|\partial_s \nabla_y \bar{\Psi}_x\|_\infty + \|\eta''_\delta\|_\infty \|\partial_s \bar{\Psi}_x\|_\infty) k \right] \\ &\leq C \left(\frac{q^d k^{1/2}}{\delta h^{1/2}} + q^{d+1} h^{1/2} + pq^d k^{1/2} + \frac{q^d}{\delta^2} k + \int_{B(0, R)} |u_0(x) - u_{\mathcal{T}, 0}(x)| dx + pq^{d+1} k + \frac{pq^d}{\delta} k \right). \end{aligned}$$

We deduce the following bound for $\tilde{\mathcal{R}}$ by integrating with respect to κ and x the above inequality:

$$\begin{aligned} |\tilde{\mathcal{R}}| &= E \left[\int_{B(0, R)} \int_{\mathbb{R}} \tilde{R}^{h, k, \eta_\delta(\cdot - \kappa), \bar{\Psi}_x} \rho_l(u_0(x) - \kappa) d\kappa dx \right] \\ &\leq C \left(\frac{q^d k^{1/2}}{\delta h^{1/2}} + q^{d+1} h^{1/2} + pq^d k^{1/2} + \frac{q^d}{\delta^2} k + \int_{B(0, R)} |u_0(x) - u_{\mathcal{T}, 0}(x)| dx + pq^{d+1} k + \frac{pq^d}{\delta} k \right). \end{aligned}$$

Finally, since

$$0 \leq A_7 + B_7 \leq |D_4 + D_5 + D_6 + D_7| + |\tilde{\mathcal{R}}|,$$

and since we have chosen to take $p = p_r = (h_r)^{-5}$ and $k = k_r = h_r^{21}$, one gets that

$$\begin{aligned} |A_7 + B_7| &\leq C \left(\frac{q}{p} + \frac{1}{p^{1/2}} + \frac{1}{p\delta} + \delta + \frac{1}{l} + \bar{\varepsilon}(1/q, u_0, B(0, R)) \right. \\ &\quad \left. + \frac{q^d k^{1/2}}{\delta h^{1/2}} + q^{d+1} h^{1/2} + pq^d k^{1/2} + \frac{q^d}{\delta^2} k + \int_{B(0, R)} |u_0(x) - u_{\mathcal{T}, 0}(x)| dx + pq^{d+1} k + \frac{pq^d}{\delta} k \right) \\ &\leq C \left(qh_r^5 + h_r^{5/2} + \frac{h_r^5}{\delta} + \delta + \frac{1}{l} + \bar{\varepsilon}(1/q, u_0, B(0, R)) + \frac{h_r^{10} q^d}{\delta} + h_r^{1/2} q^{d+1} \right. \\ &\quad \left. + q^d h_r^{11/2} + \frac{h_r^{21} q^d}{\delta^2} + q^{d+1} h_r^{16} + q^d \frac{h_r^{16}}{\delta} + \int_{B(0, R)} |u_{\mathcal{T}, 0}(y) - u_0(y)| dy \right). \end{aligned}$$

Using the fact that $u_0 \in L^2(\mathbb{R}^d)$ and passing successively to the superior limit with respect to r, l, δ and q , one concludes that

$$\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_7 + B_7 = 0.$$

4.2.6 Study of the stochastic terms: $A_5 + B_5 + A_6 + B_6$

Let us now turn to the study of terms coming from the stochastic integral. Since $\text{Supp}(\bar{\rho}_p) \subset \mathbb{R}_-$, and using the martingale property of the stochastic integral, we have:

$$\begin{aligned} B_5 &= E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^T \eta'_\delta(u_{\mathcal{T}, k}(y, s) - \kappa) g(u_{\mathcal{T}, k}(y, s)) \varphi(x, t) \bar{\rho}_p(t - s) \bar{\rho}_q(x - y) dW(s) dy \rho_l(\nu(x, t, \alpha) - \kappa) d\kappa d\alpha dx dt \right] \\ &= \int_Q \int_{\mathbb{R}} E \left[\int_t^T \int_{\mathbb{R}^d} \eta'_\delta(u_{\mathcal{T}, k}(y, s) - \kappa) g(u_{\mathcal{T}, k}(y, s)) \bar{\rho}_p(t - s) \bar{\rho}_q(x - y) dy dW(s) \int_0^1 \varphi(x, t) \rho_l(\nu(x, t, \alpha) - \kappa) d\alpha \right] d\kappa dx dt \\ &= 0. \end{aligned}$$

By the same type of martingale arguments and since $\text{Supp}(\bar{\rho}_p(\cdot - s)) \cap [0, T] \subset [(s - 2/p)^+, s]$, one gets that:

$$\begin{aligned} A_5 &= E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{(s-2/p)^+}^s \eta'_\delta(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t - s) \bar{\rho}_q(x - y) dW(t) \rho_l(u_{\mathcal{T}, k}(y, s) - \kappa) d\kappa dx d\alpha ds dy \right] \\ &= E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{(s-2/p)^+}^s \eta'_\delta(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t - s) \bar{\rho}_q(x - y) dW(t) \right] \\ &\quad \times \left[\rho_l(u_{\mathcal{T}, k}(y, s) - \kappa) - \rho_l(u_{\mathcal{T}, k}(y, (s - 2/p)^+) - \kappa) \right] d\kappa dx d\alpha ds dy. \end{aligned}$$

Set $s \in (0, T)$ and $p \in \mathbb{N}^*$, thus there exist $n_1, n_2 \in \{0, \dots, N - 1\}$ such that $s \in [n_1 k; (n_1 + 1)k)$ and $(s - 2/p)^+ \in [n_2 k; (n_2 + 1)k)$. One has then $u_{\mathcal{T}, k}(y, s) = \bar{u}_{\mathcal{T}, k}(y, n_1 k)$ and $u_{\mathcal{T}, k}(y, (s - 2/p)^+) = \bar{u}_{\mathcal{T}, k}(y, n_2 k)$. Remind at this point that $\bar{u}_{\mathcal{T}, k}$ is the solution of the following equation:

$$d\bar{u}_{\mathcal{T}, k}(y, t) = A(y, t) dt + g(u_{\mathcal{T}, k}(y, t)) dW(t),$$

where A is defined on $Q \times \Omega$ by:

$$A(y, t) = -\frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K, L}| F_{K, L}^n(u_K^n, u_L^n) = -\frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K, L}| \left\{ F_{K, L}^n(u_K^n, u_L^n) - F_{K, L}^n(u_K^n, u_K^n) \right\},$$

for $y \in K$ and $t \in [nk; (n+1)k]$.

Note that if $n_1 = n_2$, thus $u_{\mathcal{T},k}(y, s) = u_{\mathcal{T},k}(y, (s - 2/p)^+)$, so $A_5 = B_5 = 0$. Else, if $n_1 > n_2$, we have $(n_1 - n_2)k < \frac{2}{p} + k$, and by applying Itô's formula to $\rho_l(\bar{u}_{\mathcal{T},k}(y, \cdot) - \kappa)$ between n_2k and n_1k and using a theorem of derivation of stochastic integrals with respect to parameters (see for example [Ku82] Theorem 7.6), one gets:

$$\begin{aligned}
& \rho_l(\bar{u}_{\mathcal{T},k}(y, n_1k) - \kappa) - \rho_l(\bar{u}_{\mathcal{T},k}(y, n_2k) - \kappa) \\
&= \int_{n_2k}^{n_1k} \rho_l'(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) A(y, \tau) d\tau + \int_{n_2k}^{n_1k} \rho_l'(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g(u_{\mathcal{T},k}(y, \tau)) dW(\tau) \\
&\quad + \frac{1}{2} \int_{n_2k}^{n_1k} \rho_l''(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g^2(u_{\mathcal{T},k}(y, \tau)) d\tau \\
&= -\frac{d}{d\kappa} \left(\int_{n_2k}^{n_1k} \rho_l(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) A(y, \tau) d\tau + \int_{n_2k}^{n_1k} \rho_l(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g(u_{\mathcal{T},k}(y, \tau)) dW(\tau) \right. \\
&\quad \left. + \frac{1}{2} \int_{n_2k}^{n_1k} \rho_l'(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g^2(u_{\mathcal{T},k}(y, \tau)) d\tau \right).
\end{aligned}$$

So, we obtain that:

$$\begin{aligned}
A_5 + B_5 &= -E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{(s-2/p)^+}^s \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) dW(t) \right. \\
&\quad \times \left(\int_{n_2k}^{n_1k} \rho_l(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) A(y, \tau) d\tau + \int_{n_2k}^{n_1k} \rho_l(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g(u_{\mathcal{T},k}(y, \tau)) dW(\tau) \right. \\
&\quad \left. \left. + \frac{1}{2} \int_{n_2k}^{n_1k} \rho_l'(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g^2(u_{\mathcal{T},k}(y, \tau)) d\tau \right) dx d\kappa d\alpha ds dy \right] \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where:

$$\begin{aligned}
I_1 &= -E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{(s-2/p)^+}^s \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) dW(t) \right. \\
&\quad \left. \times \int_{n_2k}^{n_1k} \rho_l(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) A(y, \tau) d\tau dx d\kappa d\alpha ds dy \right],
\end{aligned}$$

$$\begin{aligned}
I_2 &= -E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{(s-2/p)^+}^s \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) dW(t) \right. \\
&\quad \left. \times \int_{n_2k}^{n_1k} \rho_l(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g(u_{\mathcal{T},k}(y, \tau)) dW(\tau) dx d\kappa d\alpha ds dy \right]
\end{aligned}$$

$$\begin{aligned}
\text{and } I_3 &= -E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{(s-2/p)^+}^s \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) dW(t) \right. \\
&\quad \left. \times \frac{1}{2} \int_{n_2k}^{n_1k} \rho_l'(\bar{u}_{\mathcal{T},k}(y, \tau) - \kappa) g^2(u_{\mathcal{T},k}(y, \tau)) d\tau dx d\kappa d\alpha ds dy \right].
\end{aligned}$$

Using the isometric property of the Itô integral and the fact that $\text{Supp}(\eta''_\delta) \subset [-\delta; \delta]$, we find

$$\begin{aligned}
|I_1| &\leq \int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left\{ E \left[\left(\int_{(s-2/p)^+}^s \eta''_\delta(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \bar{\rho}_q(x-y) d\alpha dW(t) \right)^2 \right] \right\}^{1/2} \\
&\quad \times \left\{ E \left[\left(\int_{n_2 k}^{n_1 k} \rho_l(\bar{u}_{\tau, k}(y, \tau) - \kappa) A(y, \tau) d\tau \right)^2 \right] \right\}^{1/2} dx d\kappa ds dy \\
&\leq \int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left\{ E \left[\int_{(s-2/p)^+}^s \eta''_\delta{}^2(\nu(x, t, \alpha) - \kappa) g^2(\nu(x, t, \alpha)) \varphi^2(x, t) \bar{\rho}_p^2(t-s) \bar{\rho}_q^2(x-y) d\alpha dt \right] \right\}^{1/2} \\
&\quad \times \sqrt{(n_1 - n_2)k} \left\{ E \left[\int_{n_2 k}^{n_1 k} \rho_l^2(\bar{u}_{\tau, k}(y, \tau) - \kappa) A^2(y, \tau) d\tau \right] \right\}^{1/2} dx d\kappa ds dy \\
&\leq C p q^d l \sqrt{k + \frac{2}{p}} \left(\int_0^T \int_0^1 \int_{B(0, R)} \int_{|x-y| < 1/q} \int_{\mathbb{R}} E \left[\int_{(s-2/p)^+}^s \eta''_\delta{}^2(\nu(x, t, \alpha) - \kappa) g^2(\nu(x, t, \alpha)) dt \right] d\kappa dy dx d\alpha ds \right. \\
&\quad \left. + \int_0^T \int_{B(0, R)} \int_{|x-y| < 1/q} \int_{\mathbb{R}} E \left[\int_{n_2 k_r}^{n_1 k_r} \mathbb{1}_{\{|\bar{u}_{\tau, k}(y, \tau) - \kappa| < 1/l\}} A^2(y, \tau) d\tau \right] d\kappa dy dx ds \right) \\
&\leq C \frac{p q^d l}{\delta^2} \delta \sqrt{k + \frac{2}{p}} \int_0^T \int_0^1 \int_{B(0, R)} \int_{|x-y| < 1/q} E \left[\int_{(s-2/p)^+}^s g^2(\nu(x, t, \alpha)) dt \right] dy dx d\alpha ds \\
&\quad + C p q^d \frac{1}{q^d} \sqrt{k + \frac{2}{p}} \int_0^T \int_{B(0, R+1)} E \left[\int_{n_2 k}^{n_1 k} A^2(y, \tau) d\tau \right] dy ds.
\end{aligned}$$

The first right-hand side term can be dominated in the following way:

$$\begin{aligned}
C \frac{p q^d l}{\delta} \sqrt{k + \frac{2}{p}} \int_0^T \int_0^1 \int_{B(0, R)} \int_{|x-y| < 1/q} E \left[\int_{(s-2/p)^+}^s g^2(\nu(x, t, \alpha)) dt \right] dy dx d\alpha ds \\
\leq C \frac{p q^d l}{\delta} \sqrt{k + \frac{2}{p}} \frac{2}{p q^d} \int_0^T \int_0^1 \int_{B(0, R)} E \left[g^2(\nu(x, s, \alpha)) \right] dx d\alpha ds \\
\leq C \frac{l}{\delta} \sqrt{k + \frac{2}{p}}.
\end{aligned}$$

For the second one, we propose to decompose the interval $(n_2 k, n_1 k)$. Using Cauchy-Schwarz inequality, the Lipdiag property of $(F_{K, L}^n)$ (stated in Definition 4), the property of the mesh (5), the stability result of Proposition 1, and the inequality $(n_1 - n_2)k < \frac{2}{p} + k$, we obtain that:

$$\begin{aligned}
&\int_0^T \int_{B(0, R+1)} E \left[\int_{n_2 k}^{n_1 k} A^2(y, \tau) d\tau \right] dy ds \\
&= E \left[\int_0^T \sum_{K \in \mathcal{T}_{R+1}} \int_K \sum_{n=n_2}^{n_1-1} \int_{n k_r}^{(n+1)k_r} \left(\frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K, L}| \left\{ F_{K, L}^n(u_K^n, u_L^n) - F_{K, L}^n(u_K^n, u_K^n) \right\} \right)^2 d\tau dy ds \right] \\
&\leq \frac{T}{\bar{\alpha}^2 h} E \left[\sum_{n=n_2}^{n_1-1} k \sum_{K \in \mathcal{T}_{R+1}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K, L}| \left\{ F_{K, L}^n(u_K^n, u_L^n) - F_{K, L}^n(u_K^n, u_K^n) \right\}^2 \right] \\
&= \frac{T}{\bar{\alpha}^2 h} \sum_{n=n_2}^{n_1-1} k \sum_{(K, L) \in \mathcal{I}_{R+1}^n} |\sigma_{K, L}| E \left[\left\{ F_{K, L}^n(u_K^n, u_L^n) - F_{K, L}^n(u_K^n, u_K^n) \right\}^2 + \left\{ F_{K, L}^n(u_K^n, u_L^n) - F_{K, L}^n(u_L^n, u_L^n) \right\}^2 \right] \\
&\leq \frac{T}{\bar{\alpha}^2 h} \sum_{n=n_2}^{n_1-1} k \sum_{(K, L) \in \mathcal{I}_{R+1}^n} |\sigma_{K, L}| E \left[(F_2)^2 |u_K^n - u_L^n|^2 + (F_1)^2 |u_K^n - u_L^n|^2 \right] \\
&\leq 2T \frac{(F_1)^2 + (F_2)^2}{\bar{\alpha}^2 h} \sum_{n=n_2}^{n_1-1} k \sum_{K \in \mathcal{T}_{R+1}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K, L}| E \left[(u_K^n)^2 \right] \\
&\leq 8T \frac{(F_1)^2 + (F_2)^2}{\bar{\alpha}^4} e^{C_g T} \|u_0\|_{L^2(\mathbb{R}^d)}^2 \frac{1}{h^2} \left(\frac{2}{p} + k \right).
\end{aligned}$$

Therefore, we have:

$$|I_1| \leq C \left(l \frac{h_r^{5/2}}{\delta} + h_r^{1/2} \right).$$

Using the same type of arguments, we obtain the following estimate on I_3 :

$$|I_3| \leq C \left(\frac{l^2}{\delta} + l \right) p^{1/2} \left(\frac{1}{p} + k \right) \leq C h_r^{5/2} \left(\frac{l^2}{\delta} + l \right).$$

It remains to consider I_2 . The idea is to compare it later on with $A_6 + B_6$. We start by decomposing I_2 in the following way

$$\begin{aligned} I_2 &= -E \left[\int_Q \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{(s-2/p)^+}^s \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) dW(t) \right. \\ &\quad \left. \times \int_{n_2 k_r}^{n_1 k_r} \rho_l(\bar{u}_{\mathcal{T}, k}(y, \tau) - \kappa) g(u_{\mathcal{T}, k}(y, \tau)) dW(\tau) dx d\kappa d\alpha ds dy \right] \\ &= (I_2 - \tilde{I}_2) + \tilde{I}_2, \end{aligned}$$

where:

$$\begin{aligned} \tilde{I}_2 &= - \int_0^1 \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E \left[\left(\int_{(s-2/p)^+}^s \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) dW(t) \right) \right. \\ &\quad \left. \times \left(\int_{n_2 k_r}^{n_1 k_r} \rho_l(u_{\mathcal{T}, k}(y, \tau) - \kappa) g(u_{\mathcal{T}, k}(y, \tau)) dW(\tau) \right) \right] \tilde{\rho}_q(x-y) dx d\kappa dy ds d\alpha. \end{aligned}$$

Using the martingale property of the stochastic integral and Itô isometry, since $n_2 k_r \leq (s-2/p)^+ \leq n_1 k_r \leq s$, one gets thanks to Proposition 3

$$\begin{aligned} |I_2 - \tilde{I}_2| &= \left| \int_0^1 \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E \left[\int_{(s-2/p)^+}^{n_1 k_r} \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \right. \right. \\ &\quad \left. \left. \times \left\{ \rho_l(\bar{u}_{\mathcal{T}, k}(y, t) - \kappa) - \rho_l(u_{\mathcal{T}, k}(y, t) - \kappa) \right\} g(u_{\mathcal{T}, k}(y, t)) dt \right] dy d\kappa dx ds d\alpha \right| \\ &\leq C \delta l^2 k_r^{1/2}. \end{aligned}$$

By denoting

$$\tilde{X}(x, t, \alpha, y, s) = \eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \rho_l(u_{\mathcal{T}, k}(y, t) - \kappa) g(u_{\mathcal{T}, k}(y, t)) \tilde{\rho}_q(x-y),$$

and using again the isometry and martingale properties of Itô integral, the fact that $\text{supp}(\rho_p) \subset [-\frac{2}{p}, 0]$, one can decompose \tilde{I}_2 in the following manner:

$$\begin{aligned} \tilde{I}_2 &= - \int_0^1 \int_Q \int_{\mathbb{R}} \int_{B(0, R)} E \left[\int_0^T \tilde{X}(x, t, \alpha, y, s) dt \right] dx d\kappa dy ds d\alpha + \int_0^1 \int_Q \int_{\mathbb{R}} \int_{B(0, R)} E \left[\int_{n_1 k_r}^s \tilde{X}(x, t, \alpha, y, s) dt \right] dx d\kappa dy ds d\alpha \\ &= (\tilde{I}_2 - \bar{I}_2) + (\bar{I}_2 - \bar{I}_{2,1}) + \bar{I}_{2,1}, \end{aligned}$$

where:

$$\begin{aligned} \bar{I}_2 &= - \int_0^1 \int_Q \int_{\mathbb{R}} \int_{B(0, R)} \int_0^T E \left[\tilde{X}(x, t, \alpha, y, s) \right] dt dx d\kappa dy ds d\alpha \\ &= - \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{Q_R} E \left[\eta_\delta''(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \rho_l(u_{\mathcal{T}, k}(y, t) - \kappa) g(u_{\mathcal{T}, k}(y, t)) \tilde{\rho}_q(x-y) \right] dt dx d\kappa dy d\alpha \end{aligned}$$

$$\text{and } \bar{I}_{2,1} = - \int_0^1 \int_{\mathbb{R}^d} \int_{Q_R} E \left[\eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T}, k}(y, t)) g(\nu(x, t, \alpha)) g(u_{\mathcal{T}, k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x-y) \right] dt dx dy d\alpha.$$

Since $0 \leq s - n_1 k \leq k$, we have:

$$|\tilde{I}_2 - \bar{I}_2| = \left| \int_0^1 \int_Q \int_{\mathbb{R}} \int_{B(0, R)} E \left[\int_{n_1 k_r}^s \tilde{X}(x, t, \alpha, y, s) dt \right] dx d\kappa dy ds d\alpha \right| \leq C \frac{pk}{\delta}.$$

Using the fact that $\|\eta_\delta'''\|_\infty = \frac{\pi^2}{4\delta^2}$, we get:

$$\begin{aligned} |\bar{I}_2 - \bar{I}_{2,1}| &= \left| \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{Q_R} E \left[\eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T}, k}(y, t) + \xi) g(\nu(x, t, \alpha)) g(u_{\mathcal{T}, k}(y, t)) \varphi(x, t) \rho_l(\xi) \tilde{\rho}_q(x-y) \right] dt dx d\xi dy d\alpha \right. \\ &\quad \left. - \int_0^1 \int_{\mathbb{R}^d} \int_{Q_R} E \left[\eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T}, k}(y, t)) g(\nu(x, t, \alpha)) g(u_{\mathcal{T}, k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x-y) \right] dt dx dy d\alpha \right| \\ &\leq C \frac{1}{l\delta^2}. \end{aligned}$$

Therefore, one has:

$$|I_2 - \bar{I}_{2,1}| \leq C \left(l^2 k^{1/2} \delta + \frac{pk}{\delta} + \frac{1}{l\delta^2} \right).$$

Now let us show that $|A_6 + B_6 + \bar{I}_{2,1}| \rightarrow 0$ when passing to the limit successively with respect to r, l, δ, q . Using the parity of η_δ'' and ρ_l we decompose A_6 and B_6 in the following way:

$$A_6 = (A_6 - A_{6,1}) + (A_{6,1} - A_{6,2}) + A_{6,2} \quad \text{and} \quad B_6 = (B_6 - B_{6,1}) + (B_{6,1} - B_{6,2}) + B_{6,2},$$

where:

$$\begin{aligned} A_{6,1} &= \frac{1}{2}E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi) g^2(\nu(x, t, \alpha)) \varphi(x, t) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy \right], \\ A_{6,2} &= \frac{1}{2}E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) g^2(\nu(x, t, \alpha)) \varphi(x, t) \tilde{\rho}_q(x-y) d\alpha dx dt dy \right] \\ \text{and } B_{6,1} &= \frac{1}{2}E \left[\int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi) g^2(u_{\mathcal{T},k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x-y) \rho_l(\xi) dx dt d\xi dy d\alpha \right], \\ B_{6,2} &= \frac{1}{2}E \left[\int_0^1 \int_{\mathbb{R}^d} \int_Q \eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) g^2(u_{\mathcal{T},k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x-y) dx dt dy d\alpha \right]. \end{aligned}$$

The idea is to show that $|A_6 - A_{6,2}|$, $|B_6 - B_{6,2}|$ and $|A_{6,2} + B_{6,2} + \bar{I}_{2,1}|$ tend to 0.

We start by rewriting $A_{6,1} = \tilde{A}_{6,1} + \bar{A}_{6,1}$, where

$$\begin{aligned} \tilde{A}_{6,1} &= \frac{1}{2}E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \int_0^T \eta_\delta''(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) g^2(\nu(x, s, \alpha)) \varphi(x, s) \tilde{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) dt d\alpha dx ds d\xi dy \right] \\ \text{and} \\ \bar{A}_{6,1} &= \frac{1}{2}E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta''(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) g^2(\nu(x, s, \alpha)) \varphi(x, s) \left(1 - \int_0^T \tilde{\rho}_p(t-s) dt\right) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right]. \end{aligned}$$

Using the same type of arguments as previously, the change of variables $\kappa = \xi - u_{\mathcal{T},k}(y, s)$ in A_6 and the fact that $\|\eta_\delta'''\|_\infty = \frac{\pi^2}{4\delta^2}$, we obtain the following estimates:

$$\begin{aligned} 2|A_6 - \tilde{A}_{6,1}| &\leq E \left[\int_Q \int_{\mathbb{R}} \int_{Q_R} \int_0^1 \left| \eta_\delta''(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) - \eta_\delta''(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \right| \varphi(x, t) g^2(\nu(x, t, \alpha)) \right. \\ &\quad \times \tilde{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy ds \left. \right] \\ &\quad + E \left[\int_Q \int_{\mathbb{R}} \int_{Q_R} \int_0^1 \left| \eta_\delta''(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \right| \left| \varphi(x, t) - \varphi(x, s) \right| g^2(\nu(x, t, \alpha)) \right. \\ &\quad \times \tilde{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy ds \left. \right] \\ &\quad + E \left[\int_Q \int_{\mathbb{R}} \int_{Q_R} \int_0^1 \left| \eta_\delta''(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \right| \varphi(x, s) \left| g^2(\nu(x, t, \alpha)) - g^2(\nu(x, s, \alpha)) \right| \right. \\ &\quad \times \tilde{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx dt d\xi dy ds \left. \right] \\ &\leq \frac{\pi^2}{4\delta^2} \|\varphi\|_\infty \|g\|_\infty^2 E \left[\int_0^T \int_0^1 \int_{Q_R} \left| \nu(x, t, \alpha) - \nu(x, s, \alpha) \right| \tilde{\rho}_p(t-s) dx dt d\alpha ds \right] \\ &\quad + \frac{\pi}{2\delta} \|\partial_t \varphi\|_\infty \|g\|_\infty^2 |B(0, R)| E \left[\int_0^T \int_0^T |t-s| \tilde{\rho}_p(t-s) dt ds \right] \\ &\quad + \frac{\pi}{\delta} \|\varphi\|_\infty \|g\|_\infty C_g E \left[\int_0^T \int_0^1 \int_{Q_R} \left| \nu(x, t, \alpha) - \nu(x, s, \alpha) \right| \tilde{\rho}_p(t-s) dx dt d\alpha ds \right] \\ &\leq C \left(\frac{\varepsilon(2/p, \nu, Q_R)}{\delta^2} + \frac{1}{p\delta} \right), \end{aligned}$$

where $\varepsilon(2/p, \nu, Q_R) = \sup \left\{ \|\nu - \nu(\cdot, \cdot - \tau, \cdot)\|_{L^1(\Omega \times Q_R \times (0,1))}; |\tau| \leq 2/p \right\}$.

Since $\int_0^T \tilde{\rho}_p(t-s) dt = 1$ if $s \geq 2/p$, one gets:

$$\begin{aligned} 2|\bar{A}_{6,1}| &\leq \left| E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{B(0,R)} \int_0^{2/p} \int_0^1 \eta_\delta''(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) g^2(\nu(x, s, \alpha)) \varphi(x, s) \tilde{\rho}_q(x-y) \rho_l(\xi) d\alpha dx ds d\xi dy \right] \right| \\ &\leq \|\varphi\|_\infty \|g\|_\infty^2 \|\eta_\delta''\|_\infty E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_0^{2/p} \int_{B(0,R)} \tilde{\rho}_q(x-y) \rho_l(\xi) dx ds d\xi dy \right] \\ &\leq C \frac{1}{p\delta}. \end{aligned}$$

Moreover:

$$\begin{aligned}
2|A_{6,1} - A_{6,2}| &= \left| E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi) g^2(\nu(x, t, \alpha)) \varphi(x, t) \tilde{\rho}_q(x - y) \rho_l(\xi) d\alpha dx dt d\xi dy \right] \right. \\
&\quad \left. - E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) g^2(\nu(x, t, \alpha)) \varphi(x, t) \tilde{\rho}_q(x - y) \rho_l(\xi) d\alpha dx dt d\xi dy \right] \right| \\
&\leq \frac{\pi^2}{4} \|\varphi\|_\infty \|g\|_\infty^2 T |B(0, R)| \times \frac{1}{l\delta^2}.
\end{aligned}$$

Finally,

$$|A_6 - A_{6,2}| \leq |A_6 - \tilde{A}_{6,1}| + |\bar{A}_{6,1}| + |A_{6,1} - A_{6,2}| \leq C \left(\frac{\varepsilon(2/p, \nu, Q_R)}{\delta^2} + \frac{1}{p\delta} + \frac{1}{l\delta^2} \right).$$

As previously, we start by decomposing $B_{6,1}$ in the following way: $B_{6,1} = \tilde{B}_{6,1} + \bar{B}_{6,1}$, where:

$$\tilde{B}_{6,1} = \frac{1}{2} E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta''_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) g^2(u_{\mathcal{T},k}(y, s)) \varphi(x, s) \tilde{\rho}_p(t - s) \tilde{\rho}_q(x - y) \rho_l(\xi) dx ds d\xi dy d\alpha \right]$$

and

$$\bar{B}_{6,1} = \frac{1}{2} E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta''_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) g^2(u_{\mathcal{T},k}(y, s)) \varphi(x, s) \left(1 - \int_0^T \tilde{\rho}_p(t - s) dt \right) \tilde{\rho}_q(x - y) \rho_l(\xi) dx ds d\xi dy d\alpha \right].$$

We have then:

$$\begin{aligned}
2|B_6 - \tilde{B}_{6,1}| &\leq E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_{Q_R} \left| \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) - \eta''_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \right| g^2(u_{\mathcal{T},k}(y, s)) \varphi(x, t) \right. \\
&\quad \left. \times \tilde{\rho}_p(t - s) \tilde{\rho}_q(x - y) \rho_l(\xi) dx dt d\xi dy ds d\alpha \right] \\
&\quad + E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_{Q_R} \left| \eta''_\delta(\nu(x, s, \alpha) - u_{\mathcal{T},k}(y, s) + \xi) \right| g^2(u_{\mathcal{T},k}(y, s)) \left| \varphi(x, t) - \varphi(x, s) \right| \right. \\
&\quad \left. \times \tilde{\rho}_p(t - s) \tilde{\rho}_q(x - y) \rho_l(\xi) dx dt d\xi dy ds d\alpha \right] \\
&\leq \|\eta''_\delta\|_\infty \|\varphi\|_\infty \|g\|_\infty^2 E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_{Q_R} \left| \nu(x, t, \alpha) - \nu(x, s, \alpha) \right| \tilde{\rho}_p(t - s) \tilde{\rho}_q(x - y) \rho_l(\xi) dx dt d\xi dy ds d\alpha \right] \\
&\quad + \|\eta''_\delta\|_\infty \|\partial_t \varphi\|_\infty \|g\|_\infty^2 E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_{Q_R} |t - s| \tilde{\rho}_p(t - s) \tilde{\rho}_q(x - y) \rho_l(\xi) dx dt d\xi dy ds d\alpha \right] \\
&\leq C \left(\frac{\varepsilon(2/p, \nu, Q_R)}{\delta^2} + \frac{1}{p\delta} \right),
\end{aligned}$$

and, as for the estimate of $|\bar{A}_{6,1}|$, we obtain: $|\bar{B}_{6,1}| \leq C \frac{1}{p\delta}$. Moreover,

$$\begin{aligned}
2|B_{6,1} - B_{6,2}| &= \left| E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t) + \xi) g^2(u_{\mathcal{T},k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x - y) \rho_l(\xi) d\alpha dx dt d\xi dy \right] \right. \\
&\quad \left. - E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_Q \int_0^1 \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) g^2(u_{\mathcal{T},k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x - y) \rho_l(\xi) d\alpha dx dt d\xi dy \right] \right| \\
&\leq \frac{\pi^2}{4} \|\varphi\|_\infty \|g\|_\infty^2 T |B(0, R)| \times \frac{1}{l\delta^2}.
\end{aligned}$$

Finally,

$$|B_6 - B_{6,2}| \leq |B_6 - \tilde{B}_{6,1}| + |\bar{B}_{6,1}| + |B_{6,1} - B_{6,2}| \leq C \left(\frac{\varepsilon(2/p, \nu, Q_R)}{\delta^2} + \frac{1}{p\delta} + \frac{1}{l\delta^2} \right).$$

In order to study $|A_{6,2} + B_{6,2} + \bar{I}_{2,1}|$, we start by noticing that:

$$\begin{aligned}
A_{6,2} + B_{6,2} + \bar{I}_{2,1} &= - E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \int_0^T \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) g(\nu(x, t, \alpha)) g(u_{\mathcal{T},k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \\
&\quad + \frac{1}{2} E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \int_0^T \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) g^2(\nu(x, t, \alpha)) \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \\
&\quad + \frac{1}{2} E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \int_0^T \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) g^2(u_{\mathcal{T},k}(y, t)) \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \\
&= \frac{1}{2} E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \int_0^T \eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) \left(g(\nu(x, t, \alpha)) - g(u_{\mathcal{T},k}(y, t)) \right)^2 \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dx dt dy \right].
\end{aligned}$$

Then, using the compact support of η''_δ , we obtain:

$$\begin{aligned} & \left| A_{62} + B_{62} + \bar{I}_{2,1} \right| \\ & \leq \frac{1}{2} \|\varphi\|_\infty C_g^2 E \left[\int_{\mathbb{R}^d} \int_Q \int_0^1 \int_0^T |\eta''_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t))| |\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)|^2 \tilde{\rho}_q(x - y) d\alpha dx dt dy \right] \\ & \leq \frac{\pi}{4} \|\varphi\|_\infty C_g^2 T |B(0, R)| \times \delta. \end{aligned}$$

Finally, one has:

$$\begin{aligned} |A_5 + B_5 + A_6 + B_6| & \leq |I_1| + |I_3| + |I_2 - \bar{I}_{2,1}| + |A_6 - A_{6,2}| + |B_6 - B_{6,2}| + |A_{6,2} + B_{6,2} + \bar{I}_{2,1}| \\ & \leq C \times \left(h_r^{1/2} + l^2 h_r^{21/2} \delta + \frac{h_r^{16}}{\delta} + \frac{1}{l\delta^2} + \frac{\varepsilon(2h_r^5, \nu, Q_R)}{\delta^2} + \frac{h_r^5}{\delta} + \delta \right), \end{aligned}$$

which gives, after passing successively to the superior limits with respect to r, l, δ, q :

$$\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_5 + B_5 + A_6 + B_6 = 0.$$

4.2.7 Study of the rest \mathcal{R}

Remind that \mathcal{R} is defined by:

$$\mathcal{R} = E \left[\int_0^1 \int_Q \int_{\mathbb{R}} R^{h,k,\eta_\delta(\cdot-\kappa), \Psi_{x,t}} \rho_l(\nu(x, t, \alpha) - \kappa) d\kappa d\alpha dx dt \right].$$

Using (54), we deduce that:

$$\begin{aligned} |\mathcal{R}| & \leq C_{CE} \int_Q \left[\|\eta''_\delta\|_\infty \|\Psi_{x,t}\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'_\delta\|_\infty \|\nabla_y \Psi_{x,t}\|_\infty h^{1/2} + (\|\partial_s \Psi_{x,t}\|_\infty \|\eta'\|_\infty + \|\Psi_{x,t}\|_\infty \|\eta''_\delta\|_\infty) k^{1/2} \right. \\ & \quad \left. + \|\eta'''_\delta\|_\infty \|\Psi_{x,t}\|_\infty k + \int_{B(0,R)} |u_0(z) - u_{\mathcal{T},0}(z)| dz + (\|\eta'_\delta\|_\infty \|\partial_s \nabla_y \Psi_{x,t}\|_\infty + \|\eta''_\delta\|_\infty \|\partial_s \Psi_{x,t}\|_\infty) k \right] dx dt. \quad (57) \end{aligned}$$

Note that if we combine directly this estimate on $|\mathcal{R}|$ with standard ones on $\bar{\rho}_p, \tilde{\rho}_q$ and η_δ , the right hand-side term $\left| \int_Q \|\nabla_y \Psi_{x,t}\|_\infty \|\eta'_\delta\|_\infty h^{1/2} dx dt \right|$ will be bounded by $C \frac{q^{d+1}}{h_r^{9/2}}$ which blows up when r goes to infinity, then it is, in our case, a useless estimate. For each $(x, t) \in Q$, this term comes from the estimation (35) in the proof of Proposition 4. Let us improve such an estimate in the particular case where the function φ in (35) is replaced by the function $\Psi_{x,t}$: for each $(x, t) \in Q$, using the notations introduced in the proof of Proposition 4, we get:

$$\begin{aligned} & B_G^{h_r, k_r, \eta_\delta, \Psi_{x,t}} - B_\Phi^{h_r, k_r, \eta_\delta, \Psi_{x,t}} = \bar{T}_1 - T_1 - (\bar{T}_2 - T_2) \\ & = \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k_r |\sigma_{K,L}| \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right\} \varphi(x, t) \bar{\rho}_p(t - nk_r) \left(\frac{1}{|K|} \int_K \tilde{\rho}_q(x - y) dy - \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \tilde{\rho}_q(x - y) d\gamma(y) \right) \\ & \quad + \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \int_{nk_r}^{(n+1)k_r} \int_{\sigma_{K,L}} \left\{ \Phi_{\eta_\delta(\cdot-\kappa)}(y, s, u_K^n, 0) \cdot n_{K,L} - \Phi_{K,L}^n(u_K^n) \right\} \varphi(x, t) \bar{\rho}_p(t - nk) \tilde{\rho}_q(x - y) d\gamma(y) ds \\ & \quad - \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k_r |\sigma_{K,L}| \left\{ G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_L^n) \right\} \varphi(x, t) \bar{\rho}_p(t - nk) \left(\frac{1}{|L|} \int_L \tilde{\rho}_q(x - y) dy - \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \tilde{\rho}_q(x - y) d\gamma(y) \right) \\ & \quad + \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \int_{nk_r}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_{\eta_\delta(\cdot-\kappa)}(y, s, u_L^n, 0) \cdot n_{K,L} - \Phi_{K,L}^n(u_L^n) \right\} \varphi(x, t) \bar{\rho}_p(t - nk) \tilde{\rho}_q(x - y) d\gamma(y) ds \\ & = U_1^{h_r, k_r, p, q, \eta_\delta(\cdot-\kappa)}(x, t) + V_1^{h_r, k_r, p, q, \eta_\delta(\cdot-\kappa)}(x, t) + U_2^{h_r, k_r, p, q, \eta_\delta(\cdot-\kappa)}(x, t) + V_2^{h_r, k_r, p, q, \eta_\delta(\cdot-\kappa)}(x, t). \end{aligned}$$

Let us study each term of this sum separately. By multiplying them by $\rho_l(\nu(x, t, \alpha) - \kappa)$ then integrating with respect to κ, x, t, α and using the estimates (29)-(31), we have:

$$\begin{aligned} & \left| E \left[\int_0^1 \int_Q \int_{\mathbb{R}} U_1^{h,k,p,q,\eta_\delta(\cdot-\kappa)}(x, t) \rho_l(\nu(x, t, \alpha) - \kappa) d\kappa dx dt d\alpha \right] \right| \\ & \leq h \|\nabla \tilde{\rho}_q\|_\infty E \left[\int_Q \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k |\sigma_{K,L}| \left| G_{K,L}^n(u_K^n, u_L^n) - \Phi_{K,L}^n(u_K^n) \right| \varphi(x, t) \bar{\rho}_p(t - nk) dx dt \right] \\ & \leq Ch q^{d+1} E \left[\sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k |\sigma_{K,L}| \max_{u_L^n \leq c \leq d \leq u_K^n} \left| F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right| \right]. \end{aligned}$$

Similarly, one gets:

$$\begin{aligned} & \left| E \left[\int_0^1 \int_Q \int_{\mathbb{R}} U_2^{h,k,p,q,\eta_\delta(\cdot-\kappa)}(x,t) \rho_l(\nu(x,t,\alpha) - \kappa) d\kappa dx dt d\alpha \right] \right| \\ & \leq Chq^{d+1} E \left[\sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k |\sigma_{K,L}| \max_{u_K^n \leq c \leq d \leq u_L^n} \left| F_{K,L}^n(d,c) - F_{K,L}^n(c,c) \right| \right]. \end{aligned}$$

Using the same arguments as for the obtention of the estimates (32)-(34), which gives in our particular case:

$$\left| \Phi_{\eta_\delta(\cdot-\kappa)}(x,t,a,0) - \Phi_{\eta_\delta(\cdot-\kappa)}(y,s,a,0) \right| \leq 2(C_f^T k + C_f^R h) \leq Ch,$$

for all $(x,y) \in \sigma_{K,L}^2$, $(t,s) \in (nk, (n+1)k)$, $a \in \mathbb{R}$, and if we denote by y_σ the center of the edge $\sigma_{K,L}$, we have:

$$\begin{aligned} & \left| E \left[\int_0^1 \int_Q \int_{\mathbb{R}} V_1^{h,k,p,q,\eta_\delta(\cdot-\kappa)}(x,t) \rho_l(\nu(x,t,\alpha) - \kappa) d\kappa dx dt d\alpha \right] \right| \\ & = \left| E \left[\int_{Q_R} \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left\{ \Phi_{\eta_\delta(\cdot-\kappa)}(y,s,u_K^n,0) \cdot n_{K,L} - \Phi_{K,L}^n(u_K^n) \right\} \varphi(x,t) \bar{\rho}_p(t-nk) \bar{\rho}_q(x-y) d\gamma(y) ds dx dt \right] \right| \\ & = \left| E \left[\int_{Q_R} \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} \frac{\varphi(x,t) \bar{\rho}_p(t-nk)}{k |\sigma_{K,L}|} \int_{[nk, (n+1)k]^2} \int_{(\sigma_{K,L})^2} \left\{ \Phi_{\eta_\delta(\cdot-\kappa)}(y,s,u_K^n,0) - \Phi_{\eta_\delta(\cdot-\kappa)}(z,\tilde{s},u_K^n,0) \right\} \cdot n_{K,L} \right. \right. \\ & \quad \left. \left. \times \left\{ \bar{\rho}_q(x-y) - \bar{\rho}_q(x-y_\sigma) \right\} d\gamma(z) d\tilde{s} d\gamma(y) ds dx dt \right] \right| \\ & \leq \|\varphi\|_\infty |B(0,R)| C_\rho q^{d+1} h \times Ch \times E \left[\sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_R^n} k |\sigma_{K,L}| \right] \\ & \leq Cq^{d+1} h^2. \end{aligned}$$

Using the same arguments one gets

$$\left| E \left[\int_0^1 \int_Q \int_{\mathbb{R}} V_2^{h,k,p,q,\eta_\delta(\cdot-\kappa)}(x,t) \rho_l(\nu(x,t,\alpha) - \kappa) d\kappa dx dt d\alpha \right] \right| \leq Cq^{d+1} h^2.$$

Therefore, using Proposition 2, we find that:

$$\left| E \left[\int_0^1 \int_Q \int_{\mathbb{R}} \left(B_G^{h_r, k_r, \eta_\delta(\cdot-\kappa)}(x,t) - B_\Phi^{h_r, k_r, \eta_\delta(\cdot-\kappa)}(x,t) \right) \rho_l(\nu(x,t,\alpha) - \kappa) d\kappa dx dt d\alpha \right] \right| \leq Cq^{d+1} h^{1/2}.$$

Finally, using this estimate which enables us to improve (57), Lemma 3, the bound (44) and the fact that we have chosen to take $p = h_r^{-5}$, $k_r = h_r^{21}$, one gets:

$$\begin{aligned} |\mathcal{R}| & \leq C_{CE} \int_{Q_R} \left[\|\eta_\delta''\|_\infty \|\Psi_{x,t}\|_\infty \frac{k^{1/2}}{h^{1/2}} + (\|\partial_s \Psi_{x,t}\|_\infty \|\eta'\|_\infty + \|\Psi_{x,t}\|_\infty \|\eta_\delta''\|_\infty) k^{1/2} \right. \\ & \quad \left. + \|\eta_\delta'''\|_\infty \|\Psi_{x,t}\|_\infty k + \int_{B(0,R)} |u_0(z) - u_{\mathcal{T},0}(z)| dz + (\|\eta_\delta'\|_\infty \|\partial_s \nabla_y \Psi_{x,t}\|_\infty + \|\eta_\delta''\|_\infty \|\partial_s \Psi_{x,t}\|_\infty) k \right] dx dt + Cq^{d+1} h^{1/2} \\ & \leq C \left(\frac{k_r^{1/2} p_r q^d}{h_r^{1/2} \delta} + q^{d+1} h^{1/2} + p_r^2 q^d k_r^{1/2} + \frac{p_r q^d k_r^{1/2}}{\delta} + \frac{p_r q^d k_r}{\delta^2} + p_r^2 q^{d+1} k_r + \frac{p_r^2 q^d}{\delta} k_r + \int_{B(0,R)} |u_{\mathcal{T},0}(z) - u_0(z)| dz \right) \\ & \leq C \left(\frac{h_r^5 q^d}{\delta} + q^{d+1} h_r^{1/2} + q^d h_r^{1/2} + \frac{q^{d+1} h_r^2}{\delta} + \frac{q^d h_r^{16}}{\delta^2} + q^{d+1} h_r^{11} + \frac{q^d h_r^{11}}{\delta} + \int_{B(0,R)} |u_{\mathcal{T},0}(z) - u_0(z)| dz \right). \end{aligned}$$

Using the fact that $u_0 \in L^2(\mathbb{R}^d)$ and the compactness of $B(0,R)$, we deduce that $\lim_{r \rightarrow +\infty} \mathcal{R} = 0$.

4.2.8 End of the proof of Kato inequality

By summing (52) and (53), one gets:

$$(A_1 + B_1) + (A_2 + B_2) + (A_3 + B_3) + (A_4 + B_4) + (A_5 + B_5) + (A_6 + B_6) + (A_7 + B_7) \geq \mathcal{R},$$

by passing to the limit successively with respect to r, l, δ, q and due to the previous computations, we get

$$E \left[\int_Q \int_{(0,1)^2} \left\{ \left| \nu(x,t,\alpha) - \mu(x,t,\beta) \right| \partial_t \varphi(x,t) + \Phi(x,t, \nu(x,t,\alpha), \mu(x,t,\beta)) \cdot \nabla_x \varphi(x,t) \right\} d\alpha d\beta dx dt \right] \geq 0.$$

■

4.3 Uniqueness result

Proposition 9 *Assume that hypotheses H_1 to H_7 hold, then Problem (1) admits a unique measure-valued entropy solution, which is moreover a stochastic entropy solution in the sense of Definition 1.*

Proof. Arguing as in [BVW12], we take $R > C_f T$ and set $\psi \in C_c^1(\mathbb{R}_+, [0, 1])$ such that $\psi(r) = 1$ if $r \in [0, R + C_f T]$, $\psi(r) = 0$ if $r \in [R + C_f T + 1, \infty)$ and $\psi'(r) \leq 0$ on \mathbb{R}_+ . Thanks to a regularization procedure, we apply Proposition 8 to the following test function:

$$\varphi(x, t) = \begin{cases} \psi(|x| + C_f t) \frac{T-t}{T} & \text{if } x \in \mathbb{R}^d \text{ and } t \leq T, \\ 0 & \text{if } x \in \mathbb{R}^d \text{ and } t > T. \end{cases} \quad (58)$$

We obtain then:

$$E \left[\int_Q \int_{(0,1)^2} \left\{ |\mu(x, t, \alpha) - \nu(x, t, \beta)| \left(\frac{T-t}{T} C_f \psi'(|x| + C_f t) - \frac{1}{T} \psi(|x| + C_f t) \right) + \Phi(x, t, \mu(x, t, \alpha), \nu(x, t, \beta)) \frac{T-t}{T} \psi'(|x| + C_f t) \frac{x}{|x|} \right\} d\alpha d\beta dx dt \right] \geq 0.$$

Since $\psi' \leq 0$ and thanks to (47), we deduce that:

$$E \left[\int_Q \int_{(0,1)^2} |\mu(x, t, \alpha) - \nu(x, t, \beta)| \left(-\frac{1}{T} \psi(|x| + C_f t) \right) d\alpha d\beta dx dt \right] \geq 0$$

and hence

$$E \left[\int_0^T \int_{B(0, R+C_f T)} \int_{(0,1)^2} |\mu(x, t, \alpha) - \nu(x, t, \beta)| d\alpha d\beta dx dt \right] = 0.$$

Then, since R is arbitrary large, one concludes thanks to Proposition 8 that on the one hand $\mu(x, t, \alpha) = \nu(x, t, \beta)$ for a.a. x, t, α, β and P-a.s, which proves that there exists a unique measure-valued entropy solution. On the other hand, setting

$$u(x, t) = \int_0^1 \mu(x, t, \beta) d\beta = \int_0^1 \mu(x, t, \alpha) d\beta,$$

then P-a.s and for almost all α , we have $\mu(x, t, \alpha) = u(x, t)$, thus μ is independent of α , hence u is the unique stochastic entropy solution in the sense of Definition 1, which concludes the proof of Theorem 9. ■

Proof of Theorem 1. The proof of the main result is a direct consequence of Proposition 7 and Proposition 9. Firstly, the uniqueness of the stochastic entropy solution is given by Proposition 9. Secondly, we know from Proposition 7 that, up to a subsequence, the finite volume approximation converges (in the sense of Young measures) to a measure-valued entropy solution. Using again Proposition 9 (and arguing as in [BVW12]), we deduce that the whole sequence converges to a stochastic entropy solution in $L^1_{loc}(\Omega \times Q)$, which gives in particular the existence of a stochastic entropy solution. To conclude, since $(u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q)$, we deduce that it converges to the unique stochastic entropy solution in $L^p_{loc}(\Omega \times Q)$ for any $1 \leq p < 2$.

A Appendix

A.1 Proof of Corollary 1

Proof. Set $a, b \in \mathbb{R}$ such that $a < b$. Let u be the unique stochastic entropy solution of (1) and suppose that $u_0(x) \in [a, b]$ for almost every x in \mathbb{R}^d and that $\text{supp}(g) \subset [a, b]$. Since u is a stochastic entropy solution, we have by definition for any $\eta \in \mathcal{A}$ and any $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$

$$0 \leq \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx + \int_Q \eta(u) \partial_t \varphi(x, t) dx dt + \int_Q \Phi_\eta(x, t, u, 0) \cdot \nabla_x \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{R}^d} \eta'(u) g(u) \varphi(x, t) dx dW(t) + \frac{1}{2} \int_Q g^2(u) \eta''(u) \varphi(x, t) dx dt. \quad (59)$$

We consider an entropy $\eta \in \mathcal{A}$ such that for any $x \in [a, b]$, $\eta(x) = 0$ and for any $x \notin [a, b]$, $\eta(x) > 0$. We also consider $R > C_f T$, and the test function φ defined by (58). Note that for any $(x, t) \in Q$ and $v \in \mathbb{R}$ we have

$$0 \leq C_f \eta(v) + \Phi_\eta(x, t, v, 0) \cdot \frac{x}{|x|}.$$

Then using the assumptions on g and u_0 and the properties of the chosen entropy and test functions, by applying the above inequality (59) (thanks to a regularization procedure since φ is not smooth enough), one gets:

$$\begin{aligned} 0 &\leq \int_Q \eta(u(x,t)) \partial_t \varphi(x,t) + \Phi_\eta(x,t,u(x,t),0) \cdot \nabla_x \varphi(x,t) dx dt \\ &\leq \int_Q \left(\eta(u(x,t)) C_f + \Phi_\eta(x,t,u(x,t),0) \cdot \frac{x}{|x|} \right) \psi'(|x| + C_f t) \frac{T-t}{t} - \frac{1}{T} \eta(u(x,t)) \psi(|x| + C_f t) dx dt \\ &\leq -\frac{1}{T} \int_Q \eta(u(x,t)) \psi(|x| + C_f t) dx dt \leq -\frac{1}{T} \int_0^T \int_{B(0,R)} \eta(u(x,t)) dx dt \leq 0. \end{aligned}$$

We deduce that, for arbitrary large R and for any $(x,t) \in B(0,R) \times [0,T]$, we have $\eta(u(x,t)) = 0$ which implies that $u(x,t) \in [a,b]$ for almost all $(\omega,x,t) \in \Omega \times Q$. ■

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