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► **To cite this version:**

Mohamed Sghiar. Prime numbers, Goldbach's conjecture, De Polignac's conjecture, Legendre's conjecture, and Landau's conjecture. 2020. hal-02997473v3

HAL Id: hal-02997473

<https://hal.science/hal-02997473v3>

Preprint submitted on 24 Nov 2020 (v3), last revised 16 Dec 2020 (v4)

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**Prime numbers, Goldbach's conjecture, De Polignac's conjecture,
Legendre's conjecture, and Landau's conjecture**

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Abstract : Inspired by the article [3], the introduction of the function $\hat{\mathbb{S}}$ whose integer zeros are the prime numbers, will allow me to demonstrate analytically the Goldbach's conjecture [6], the De Polignac's Conjecture [7], the Legendre's conjecture [9], and Landau's conjecture [10] .

Résumé : Inspiré de l'article [3], l'introduction de la fonction $\hat{\mathbb{S}}$ dont les zéros entiers sont les nombres premiers, va me permettre de démontrer analytiquement la conjecture de Goldbach [6], la conjecture de De Polignac [7], la conjecture de Legendre [9] et la conjecture de Landau [10].

Keywords : Prime Number, the Goldbach conjectures, the Gamma function, the De Polignac's Conjecture, the Legendre's conjecture, the Landau's conjecture.

Code MSC : 11P32, 11N05, 11N36, 11A41

M. SGHIAR. Prime numbers, Goldbach's conjecture, De Polignac's conjecture, Legendre's conjecture, and Landau's conjecture

In this version I apply the same techniques to prove the Legendre's conjecture [9] and the Landau's conjecture [10].

I- INTRODUCTION

In 1742, in a letter to Leonhard Euler, Christian Goldbach conjectured [6] : Any even integer greater than 3 can be written as the sum of two prime numbers.

De Polignac's conjecture, stated by Alphonse De Polignac in 1849 [7], is as follows : Any non-zero even number $2m$ is equal to the difference of two consecutive prime numbers in an infinite number of ways.

Inspired by the function \mathbb{S} which I introduced in the article [3], I introduce the function $\hat{\mathbb{S}}$ which, also like \mathbb{S} , allows to find the prime numbers .

The study of the function $S_{2m}(z) = \hat{\mathbb{S}}^2(z) + \hat{\mathbb{S}}^2(2m - z)$ will allow me to demonstrate the Goldbach conjecture. And the study of the function : $S_{2m}(z) = \hat{\mathbb{S}}^2(z) + \hat{\mathbb{S}}^2(2m + z)$ will allow me to demonstrate the De Polignac's conjecture.

By y the same techniques I will prove the Legendre's conjecture [9], and Landau's conjecture [10] .

In this article I assume known the functions zeta ζ , Gamma $\Gamma : z \mapsto \int_0^{+\infty} t^{z-1} e^{-t} dt$ and their properties (see [4] and [5]).

II- THE PROOF OF THE GOLDBACH CONJECTURE :

Theorem 1 : [The Goldbach conjecture]

Any even integer greater than 3 can be written as the sum of two prime numbers.

Proposition 1. [3] :

Let $\mathbb{S}(z) = \zeta(-\frac{\Gamma(z)+1}{z/2})$.

If $z \in \mathbb{N}^*$ then $\mathbb{S}(z) = 0 \iff z$ is a prime number

Proof :

It follows from Wilson's theorem [1] - which shows that p is prime if and only $(p - 1)! \equiv -1 \pmod{p}$ - and the fact that the trivial zeros of ζ are $-2\mathbb{N}^*$.

Proposition 2 :

Let $\hat{\mathbb{S}}(z) = \sin(\frac{\Gamma(z)+1}{z}\pi)$.

If $z \in \mathbb{N}^*$ then $\hat{\mathbb{S}} = 0 \iff z$ is a prime number

Proof : Inspired by **proposition 1**

Proposition 3 : If $m \in \mathbb{N}^* \setminus \{1\}$, then the equation :

$$\sin^2\left(\frac{\Gamma(z) + 1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m - z) + 1}{2m - z}\pi\right) = 0$$

admits at least one solution z in \mathbb{N} with z and $2m - z$ are primes.

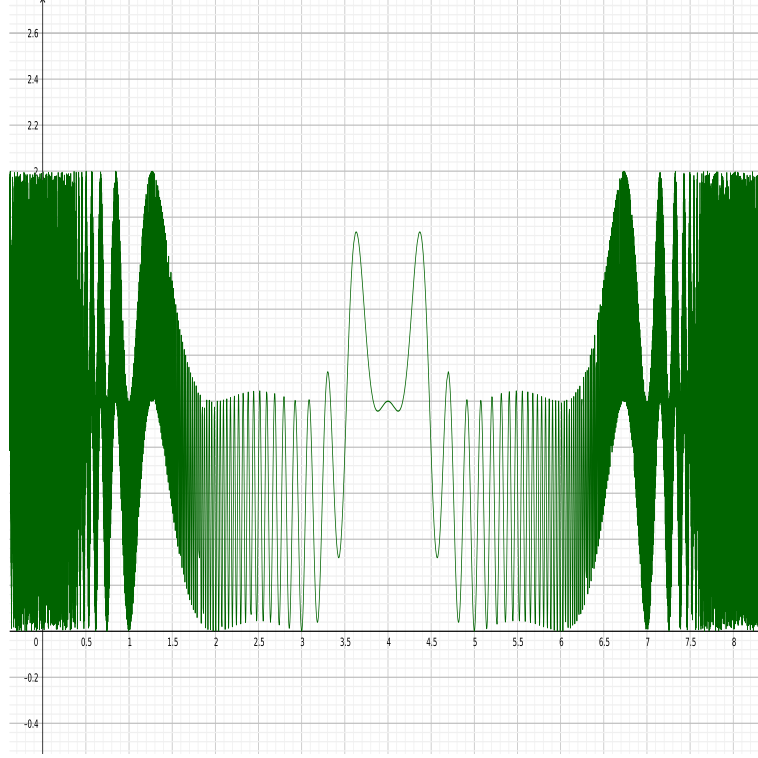


FIGURE 1 – $S(z)$ for $m=8$

Proof of Proposition 3 :

Consider the restriction to \mathbb{R}^{+*} of the function :

$$S(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$$

1 step : If the function $S(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$ has a local extremum at the point $p \neq 2$ with p is prime :

In this case $\frac{dS}{dz}(p) = 0$ and according to proposition 2 we have $\hat{\mathbb{S}}(p) = \sin\left(\frac{\Gamma(p)+1}{p}\pi\right) = 0$

We have :

$$\begin{aligned} \frac{dS}{dz}(z) = & 2\pi \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \cos\left(\frac{\Gamma(z)+1}{z}\pi\right) \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} + 2\pi \sin\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \\ & \cos\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2} \end{aligned}$$

But as $\frac{-(2m-p)\Gamma'(2m-p) + (\Gamma(2m-p)+1)}{(2m-p)^2} \neq 0$ (Because $-z\Gamma'(z) + \Gamma(z) + 1$) vanishes

in $z \sim 2.5 \notin \mathbb{N}$), then $2\sin\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right)\cos\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right) = 0$

And since we cannot have $\cos\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right) = 0$, because otherwise $\frac{\Gamma(2m-p)+1}{2m-p} \in \mathbb{N} + \frac{1}{2}$, and therefore $p \in 2\mathbb{N}$, which is not the case, then $\sin\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right) = 0$, and from **proposition 2**, we conclude that $2m - p$ is prime.

2 step : Let us show the existence of local minimums z in $]1, m[$ as soon as $m \geq 5$, with $S(z) = 0$, $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$:

First S admits a local extremum at a point $z \in]1, m[$ if $m \geq 5$: Indeed :

$$S(z) = 1 - \frac{1}{2}(\cos(2\frac{\Gamma(z)+1}{z}\pi) + \cos(2\frac{\Gamma(2m-z)+1}{2m-z}\pi))$$

If S does not admit a local extremum at a point $z \in]1, m[$, then $T(z) = (\cos(2\frac{\Gamma(z)+1}{z}\pi) + \cos(2\frac{\Gamma(2m-z)+1}{2m-z}\pi))$ does not admit a local extremum in $z \in]1, m[$, and in this case, by symmetry of the curve of S with respect to the line $x = m$, we have $\frac{dT}{dz}(z) \leq 0, \forall z \in]1, m[$ or $\frac{dT}{dz}(z) \leq 0, \forall z \in]m, 2m - 1[$.

By symmetry suppose that $\frac{dT}{dz}(z) \leq 0, \forall z \in]1, m[$:

$$\begin{aligned} \frac{dT}{dz}(z) = & -2\pi\sin\left(2\frac{\Gamma(z)+1}{z}\pi\right)\frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} - 2\pi\sin\left(2\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \\ & - \frac{(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2} \end{aligned}$$

By growth of the function $\frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2}$ in $\in]3, +\infty[$ we have : $\frac{(2m-z)\Gamma'(2m-z) - (\Gamma(2m-z)+1)}{(2m-z)^2} > \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2}$ $> 1, \forall z \in]3, m[$, so :

$$\sin\left(2\frac{\Gamma(z)+1}{z}\pi\right) \geq \sin\left(2\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{(2m-z)\Gamma'(2m-z) - (\Gamma(2m-z)+1)}{z\Gamma'(z) - (\Gamma(z)+1)}; \forall z \in]3, m[$$

And for z such that $\sin\left(2\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) = 1$ (z exists because the function $L(x) = \frac{\Gamma(x+1)+1}{x+1} - \frac{\Gamma(x)+1}{x}$ check $L(x) \geq 1, \forall x \geq 3.2$), we will have $\sin\left(2\frac{\Gamma(z)+1}{z}\pi\right) > 1$, which is absurd. And therefore $S(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$ admits local extremum z_1 in $\in]3, m[$

If z_1 is not a local minimum, by taking the same reasoning by replacing 1 by z_1 , we deduce the existence of a local extremum z_2 in $]3, z_1[$ or in $]z_1, m[$ and hence there is a local minimum z between z_1 et z_2 .

Let z be a minimum of S in $\in]1, m[$.

If $S(z) \neq 0$, then $\frac{\Gamma(z)+1}{z}\pi \neq 0, \text{ mod}(\pi)$ ou $\frac{\Gamma(2m-z)+1}{2m-z}\pi \neq 0, \text{ mod}(\pi)$

By symmetry of the curve of S with respect to the line $x = m$, we can assume that $\frac{\Gamma(2m-z)+1}{2m-z}\pi \neq 0, \text{ mod}(\pi)$ and that $\sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) \leq \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$.

If $\sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) < \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$: If z' is too close to z with $\sin^2\left(\frac{\Gamma(2m-z')+1}{2m-z'}\pi\right) < \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$, then $S(z') < S(z)$ which contradicts the minimality. It follows that $\sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) = \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$

But from the equation :

$$0 = \frac{dS}{dz}(z) = 2\pi \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \cos\left(\frac{\Gamma(z)+1}{z}\pi\right) \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} + 2\pi \sin\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \cos\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2}$$

We deduce that : $0 = \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \cos\left(\frac{\Gamma(z)+1}{z}\pi\right)$, and therefore $0 = \sin\left(\frac{\Gamma(z)+1}{z}\pi\right)$ by minimality of $S(z)$. Thus $S(z) = 0$ with $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$

3 step : Let us show the existence of the roots $z \in \mathbb{N} \cap]1, m[$:

The case where $m \in \{2, 3, 4\}$ is easy to see, since the Goldbach conjecture holds in these cases.

If $m \geq 5$, from step 2, we have $S(z) = 0$ with : $z \in]1, m[$, $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$

If $z \notin \mathbb{N}$:

- If $z = n + r$ with $r \geq 2$, $r \in \mathbb{R}^+ \setminus \mathbb{N}$ and $n \geq 1$

Let $\frac{\Gamma(z)+1}{z} = k$.

we deduce : $\prod_{i=1}^n (n - i + r)\Gamma(r) + 1 = kn + kr$

So $\Gamma(r) = \frac{-1+kn+kr}{\prod_{i=1}^n (n-i+r)}$

As $r \geq 2$, then $\Gamma'(r) > 0$, and therefore by deriving we will have :

$$k \prod_{i=1}^n (n - i + r) > (-1 + kn + kr) \sum_{j=1}^n \prod_{i \neq j; i=1}^n (n - i + r)$$

So $\frac{1}{r} \geq \frac{1}{-1+kn+kr} > \sum_{i=1}^n \frac{1}{n-i+r}$, which is not the case, and $z \in \mathbb{N}$.

- If $z = n + r$ with $r < 2$, $r \in \mathbb{R}^+ \setminus \mathbb{N}$ and $n \geq 1$

By setting $z = (n + 3) + (r - 3)$ we can suppose that $z = n + r$ with $r < -1$, $r \in \mathbb{R}^- \setminus \mathbb{N}$ and $n \geq 1$

Let $\frac{\Gamma(2m-z)+1}{2m-z} = l$, we deduce :

$$\prod_{i=1}^{2m-n-1} (2m - n - i - r)\Gamma(1 - r) + 1 = l(2m - n - 1) + l(1 - r)$$

So $\Gamma(1 - r) = \frac{-1+l(2m-n-1)+l(1-r)}{\prod_{i=1}^{2m-n-1} (2m-n-i-r)}$

But $-\Gamma'(1 - r) < 0$ if $r < -1$.

So :

$$-l \prod_{i=1}^{2m-n-1} (2m-n-i-r) < -(-1+l(2m-n-1)+l(1-r)) \sum_{j=1}^{2m-n-1} \prod_{i \neq j; i=1}^{2m-n-1} (2m-n-i-r)$$

Thus : $\frac{1}{1-r} \geq \frac{1}{-\frac{1}{l}+2m-n-1+(1-r)} > \sum_{i=1}^{2m-n-1} \frac{1}{2m-n-i-r}$, which is not the case, and $z \in \mathbb{N}$.

Note : *This result of step 3 can be noticed in the graphs of S_{2m} : If the curve of S_{2m} touches the segment $]1, 2m[$ at a point p , then p is a prime number !!*

4 step : Proof of proposition 3

The case where $m \in \{2, 3, 4\}$ is easy to see, since the Goldbach conjecture holds in these cases.

If $m \geq 5$, in step 3 we have shown the existence of the roots $z \in \mathbb{N} \cap]1, m[$ such that $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$. And from proposition 2 we deduce that z and $2m-z$ are primes with $2m = z + (2m-z)$

Proof of Theorem 1 :

It is deduced directly from **proposition 3**

III - The proof of Alphonse de Polignac's conjecture [7]

Lemma 1 :

Let f and g be two continuous functions from $[N, +\infty[$ over \mathbb{R}^+ with $\lim_{z \rightarrow +\infty} f(z) = +\infty$ and $\lim_{z \rightarrow +\infty} g(z) = +\infty$

If $M(z) = \sin^2(f(z)\pi) + \sin^2(g(z)\pi)$, then there exists $z \in [N, +\infty[$ such that $M(z)$ is a minimum of M .

Proof : Let :

$$\overline{M(f(z)\pi, g(z)\pi)} = \sin^2(\overline{f(z)\pi}) + \sin^2(\overline{g(z)\pi})$$

Where $\overline{Z} = Z \text{ mod}(2\pi)$

From hypotheses on f and g we deduce that $\{(\overline{f(z)\pi}, \overline{g(z)\pi}), z \in [N, +\infty[\} = [0, 2\pi] \times F$ where F is a part of $[0, 2\pi]$. If \tilde{F} is the closure of F , then $[0, 2\pi] \times \tilde{F}$ is a compact of $[0, 2\pi] \times [0, 2\pi]$ therefore (see [2]) \overline{M} has minima in points $(\overline{f(z)\pi}, b)$ of $[0, 2\pi] \times \tilde{F}$.

But if z_n is a sequence such that $z_n \rightarrow z$, then $\overline{g(z_n)\pi} \rightarrow b$. But as $\overline{g(z_n)\pi} \rightarrow \overline{g(z)\pi}$, then $b = \overline{g(z)\pi}$, and \overline{M} has minima in points $(\overline{f(z)\pi}, \overline{g(z)\pi})$. And therefore M has minima at points z .

The Polignac conjecture is a conjecture about the theory of numbers. It was stated by Alphonse de Polignac in 1849 [7].

Theorem 2 : The conjecture of Alphonse de Polignac [7]

Any non zero even number $2m$ is equal to the difference of two consecutive prime numbers in an infinite number of ways.

Proof :

It is demonstrated by the same technique used above to prove the Goldbach conjecture using the function S^+ instead of S :

$$S^+(z) = \sin^2\left(\frac{\Gamma(z) + 1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m + z) + 1}{2m + z}\pi\right)$$

- Using Lemma 1 above, we show the existence of an infinite number of minima z of S^+ .

And by showing as in steps 2, 3, above :

- The existence of an infinite number of local minimums z of S^+ in $]1, \infty[$, with $S(z) = 0$, $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m+z)+1}{2m+z} \in \mathbb{N}$

- The existence of an infinity of integer z roots of S^+ ($z \in \mathbb{N}$).

Then using proposition 2 - as in step 4 above - we deduce that z and $2m + z$ are prime with $2m = (2m + z) - z$

Corollary : [The conjecture of twin prime numbers] [8]

There exists an infinity of prime numbers p such that $p + 2$ is also prime.

IV - The proof of Legendre's conjecture [9]

The Legendre conjecture, proposed by Adrien-Marie Legendre, is stated as follows :

Theorem 3 : [Legendre's conjecture] : For all $n \geq 1$, there is a prime number between n^2 et $(n + 1)^2$

Proof :

We consider the function defined on $]1, 2n]$:

$$S_{le,n}(z) = \sin^2\left(\frac{\Gamma(z) + 1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(n^2 + z) + 1}{n^2 + z}\pi\right)$$

The result is true if $n < 3$.

- By the same techniques used above, we see that there are integer z roots of $S_{le,n}(z)$ ($z \in \mathbb{N} \cap]3, 2n]$ if $n \geq 3$). Then by using proposition 2 -as in step 4 above- we deduce that z and $n^2 + z$ are primes with $n^2 + z \in [n^2, (n + 1)^2]$

V - The proof of the Landau's conjecture [10]

Theorem 4 [Landau's conjecture] : There exists an infinity of primes p such that $p - 1$ is a perfect square. (or in other words : there is an infinity of prime numbers of the form $n^2 + 1$).

Proof :

We consider the function defined on $[1, +\infty]$:

$$S_{la}(z) = \sin^2\left(\frac{\Gamma(z) + 1}{z}\pi\right) + \sin^2(\pi\sqrt{z-1})$$

- Using Lemma 1 above, we show the existence of an infinity of minima z of S_{la}

- By the same techniques used above, we see that there exists an infinity of integer z roots of $S_{la}(z)$ ($z \in \mathbb{N} \cap [3, +\infty]$) with $z = n^2 + 1$. Then by using proposition 2 -as in step 4 above- we deduce that z is a prime with $z = n^2 + 1$

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