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Prime numbers, Goldbach's conjecture, and De Polignac's conjecture

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Abstract : Inspired by the article [2], the introduction of the function $\hat{\mathbb{S}}$ whose integer zeros are the prime numbers, will allow me to demonstrate analytically the Goldbach's conjecture and the De Polignac's Conjecture [5].

Résumé : Inspiré de l'article [2], l'introduction de la fonction $\hat{\mathbb{S}}$ dont les zéros entiers sont les nombres premiers, va me permettre de démontrer analytiquement la conjecture de Goldbach et la conjecture de De Polignac [5].

Keywords : Prime Number, the Goldbach conjectures, the Gamma function, the De Polignac's Conjecture .

Code MSC : 11P32, 11N05, 11N36

In this version I give an improvement of the proof of step 3, and I give an important note on the curve of the function S_{2m} :

***Note : This result of step 3 can be noticed in the graphs of S_{2m} :
If the curve of S_{2m} touches the segment $]1, 2m[$ at a point p , then p is a prime number !!***

I- INTRODUCTION

In 1742, in a letter to Leonhard Euler, Christian Goldbach conjectured [5] : Any even integer greater than 3 can be written as the sum of two prime numbers.

De Polignac's conjecture, stated by Alphonse De Polignac in 1849 [6], is as follows : Any non-zero even number $2m$ is equal to the difference of two consecutive prime numbers in an infinite number of ways.

Inspired by the function \mathbb{S} which I introduced in the article [2], I introduce the function $\hat{\mathbb{S}}$ which, also like \mathbb{S} , allows to find the prime numbers .

The study of the function $S_{2m}(z) = \hat{\mathbb{S}}^2(z) + \hat{\mathbb{S}}^2(2m - z)$ will allow me to demonstrate the Goldbach conjecture. And the study of the function : $S_{2m}(z) = \hat{\mathbb{S}}^2(z) + \hat{\mathbb{S}}^2(2m + z)$ will allow me to demonstrate the De Polignac's conjecture.

In this article I assume known the functions zeta ζ , Gamma $\Gamma : z \mapsto \int_0^{+\infty} t^{z-1} e^{-t} dt$ and their properties (see [3] and [4]).

II- THE PROOF OF THE GOLDBACH CONJECTURE :

Théorème : [The Goldbach conjecture]

Any even integer greater than 3 can be written as the sum of two prime numbers.

Proposition 1. [2] :

Let $\mathbb{S}(z) = \zeta(-\frac{\Gamma(z)+1}{z/2})$.

If $z \in \mathbb{N}^*$ then $\mathbb{S}(z) = 0 \iff z$ is a prime number

Proof :

It follows from Wilson's theorem [1] - which shows that p is prime if and only $(p - 1)! \equiv -1 \pmod{p}$ - and the fact that the trivial zeros of ζ are $-2\mathbb{N}^*$.

Proposition 2 :

Let $\hat{\mathbb{S}}(z) = \sin(\frac{\Gamma(z)+1}{z}\pi)$.

If $z \in \mathbb{N}^*$ then $\hat{\mathbb{S}}(z) = 0 \iff z$ is a prime number

Proof : Inspired by **proposition 1**

Proposition 3 : If $m \in \mathbb{N}^* \setminus \{1\}$, then the equation :

$$\sin^2\left(\frac{\Gamma(z) + 1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m - z) + 1}{2m - z}\pi\right) = 0$$

admits at least one solution z in \mathbb{N} with z and $2m - z$ are primes.

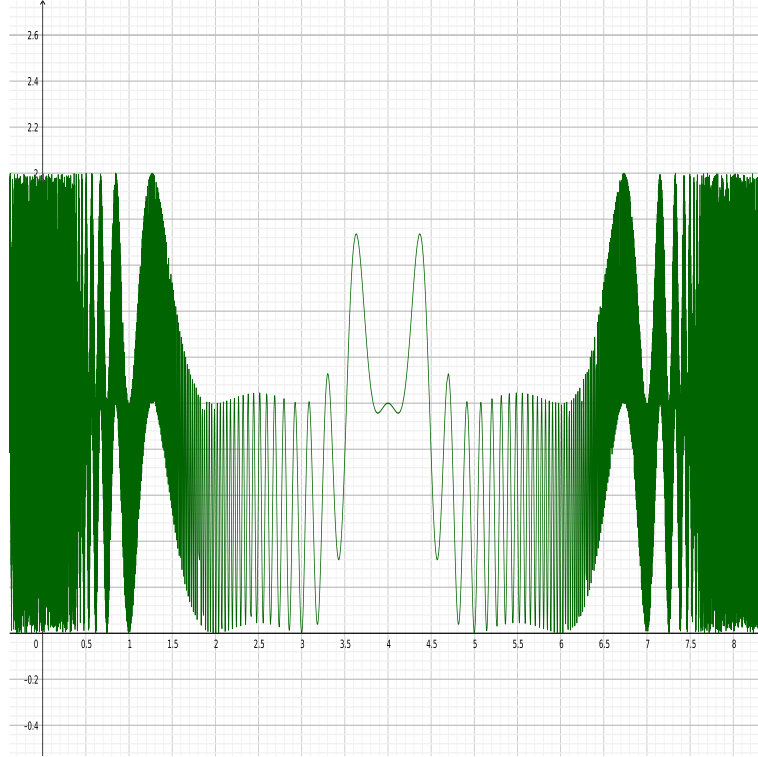


FIGURE 1 – $S(z)$ for $m=8$

Proof of Proposition 3 :

Consider the restriction to \mathbb{R}^{+*} of the function :

$$S(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$$

1 step : If the function $S(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$ has a local extremum at the point $p \neq 2$ with p is prime :

In this case $\frac{dS}{dz}(p) = 0$ and according to proposition 2 we have $\hat{\mathbb{S}}(p) = \sin\left(\frac{\Gamma(p)+1}{p}\pi\right) = 0$

We have :

$$\begin{aligned} \frac{dS}{dz}(z) = & 2\pi \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \cos\left(\frac{\Gamma(z)+1}{z}\pi\right) \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} + 2\pi \sin\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \\ & \cos\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2} \end{aligned}$$

But as $\frac{-(2m-p)\Gamma'(2m-p) + (\Gamma(2m-p)+1)}{(2m-p)^2} \neq 0$ (car $-z\Gamma'(z) + \Gamma(z) + 1$) vanishes in

$z \sim 2.5 \notin \mathbb{N}$), then $2\sin(\frac{\Gamma(2m-p)+1}{2m-p}\pi)\cos(\frac{\Gamma(2m-p)+1}{2m-p}\pi) = 0$

And since we cannot have $\cos(\frac{\Gamma(2m-p)+1}{2m-p}\pi) = 0$, because otherwise $\frac{\Gamma(2m-p)+1}{2m-p} \in \mathbb{N} + \frac{1}{2}$, and therefore $p \in 2\mathbb{N}$, which is not the case, then $\sin(\frac{\Gamma(2m-p)+1}{2m-p}\pi) = 0$, and from **proposition 2**, we conclude that $2m - p$ is prime.

2 step : Let us show the existence of local minimums z in $]1, m[$ as soon as $m \geq 5$, with $S(z) = 0$, $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$:

First S admits a local extremum at a point $z \in]1, m[$ if $m \geq 5$: Indeed :

$$S(z) = 1 - \frac{1}{2}(\cos(2\frac{\Gamma(z)+1}{z}\pi) + \cos(2\frac{\Gamma(2m-z)+1}{2m-z}\pi))$$

If S does not admit a local extremum at a point $z \in]1, m[$, then $T(z) = (\cos(2\frac{\Gamma(z)+1}{z}\pi) + \cos(2\frac{\Gamma(2m-z)+1}{2m-z}\pi))$ does not admit a local extremum in $z \in]1, m[$, and in this case, by symmetry of the curve of S with respect to the line $x = m$, we have $\frac{dT}{dz}(z) \leq 0, \forall z \in]1, m[$ or $\frac{dT}{dz}(z) \leq 0, \forall z \in]m, 2m-1[$.

By symmetry suppose that $\frac{dT}{dz}(z) \leq 0, \forall z \in]1, m[$:

$$\begin{aligned} \frac{dT}{dz}(z) = & -2\pi\sin(\frac{\Gamma(z)+1}{z}\pi)\frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} - 2\pi\sin(\frac{\Gamma(2m-z)+1}{2m-z}\pi) \\ & \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2} \end{aligned}$$

By growth of the function $\frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2}$ in $\in [3, +\infty[$ we have : $\frac{(2m-z)\Gamma'(2m-z) - (\Gamma(2m-z)+1)}{(2m-z)^2} > \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} > 1, \forall z \in [3, m[$, so :

$$\sin(\frac{\Gamma(z)+1}{z}\pi) \geq \sin(\frac{\Gamma(2m-z)+1}{2m-z}\pi) \frac{(2m-z)\Gamma'(2m-z) - (\Gamma(2m-z)+1)}{(2m-z)^2} \frac{z^2}{z\Gamma'(z) - (\Gamma(z)+1)}; \forall z \in [3, m[$$

And for z such that $\sin(\frac{\Gamma(2m-z)+1}{2m-z}\pi) = 1$ (z exists because the function $L(x) = \frac{\Gamma(x+1)+1}{x+1} - \frac{\Gamma(x)+1}{x}$ check $L(x) \geq 1, \forall x \geq 3.2$), we will have $\sin(\frac{\Gamma(z)+1}{z}\pi) > 1$, which is absurd. And therefore $S(z) = \sin^2(\frac{\Gamma(z)+1}{z}\pi) + \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$ admits local extremum z_1 in $\in [3, m[$

If z_1 is not a local minimum, by taking the same reasoning by replacing 1 by z_1 , we deduce the existence of a local extremum z_2 in $[3, z_1[$ or in $]z_1, m[$ and hence there is a local minimum z between z_1 et z_2 .

Let z be a minimum of S in $\in]1, m[$.

If $S(z) \neq 0$, then $\frac{\Gamma(z)+1}{z}\pi \neq 0, \text{ mod}(\pi)$ ou $\frac{\Gamma(2m-z)+1}{2m-z}\pi \neq 0, \text{ mod}(\pi)$

By symmetry of the curve of S with respect to the line $x = m$, we can assume that $\frac{\Gamma(2m-z)+1}{2m-z}\pi \neq 0, \text{ mod}(\pi)$ and that $\sin^2(\frac{\Gamma(z)+1}{z}\pi) \leq \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$.

If $\sin^2(\frac{\Gamma(z)+1}{z}\pi) < \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$: If z' is too close to z with $\sin^2(\frac{\Gamma(2m-z')+1}{2m-z'}\pi) < \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$, then $S(z') < S(z)$ which contradicts the minimality. It follows that $\sin^2(\frac{\Gamma(z)+1}{z}\pi) = \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$

But from the equation :

$$0 = \frac{dS}{dz}(z) = 2\pi \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \cos\left(\frac{\Gamma(z)+1}{z}\pi\right) \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} + 2\pi \sin\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \cos\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2}$$

We deduce that : $0 = \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \cos\left(\frac{\Gamma(z)+1}{z}\pi\right)$, and therefore $0 = \sin\left(\frac{\Gamma(z)+1}{z}\pi\right)$ by minimality of $S(z)$. Thus $S(z) = 0$ with $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$

3 step : Let us show the existence of the roots $z \in \mathbb{N} \cap]1, m[$:

The case where $m \in \{2, 3, 4\}$ is easy to see, since the Goldbach conjecture holds in these cases.

If $m \geq 5$, from step 2, we have $S(z) = 0$ with : $z \in]1, m[$, $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$

If $z \notin \mathbb{N}$:

- If $z = n + r$ with $r \geq 2$, $r \in \mathbb{R}^+ \setminus \mathbb{N}$ and $n \geq 1$

Let $\frac{\Gamma(z)+1}{z} = k$.

we deduce : $\prod_{i=1}^n (n - i + r)\Gamma(r) + 1 = kn + kr$

So $\Gamma(r) = \frac{-1+kn+kr}{\prod_{i=1}^n (n-i+r)}$

As $r \geq 2$, then $\Gamma'(r) > 0$, and therefore by deriving we will have :

$$k \prod_{i=1}^n (n - i + r) > (-1 + kn + kr) \sum_{j=1}^n \prod_{i \neq j; i=1}^n (n - i + r)$$

So $\frac{1}{r} \geq \frac{1}{-1+kn+kr} > \sum_{i=1}^n \frac{1}{n-i+r}$, which is not the case, and $z \in \mathbb{N}$.

- If $z = n + r$ with $r < 2$, $r \in \mathbb{R}^+ \setminus \mathbb{N}$ and $n \geq 1$

By setting $z = (n + 3) + (r - 3)$ we can suppose that $z = n + r$ with $r < -1$, $r \in \mathbb{R}^- \setminus \mathbb{N}$ and $n \geq 1$

Let $\frac{\Gamma(2m-z)+1}{2m-z} = l$, we deduce :

$$\prod_{i=1}^{2m-n-1} (2m - n - i - r)\Gamma(1 - r) + 1 = l(2m - n - 1) + l(1 - r)$$

So $\Gamma(1 - r) = \frac{-1+l(2m-n-1)+l(1-r)}{\prod_{i=1}^{2m-n-1} (2m-n-i-r)}$

But $-\Gamma'(1 - r) < 0$ if $r < -1$.

So :

$$-l \prod_{i=1}^{2m-n-1} (2m-n-i-r) < -(-1+l(2m-n-1)+l(1-r)) \sum_{j=1}^{2m-n-1} \prod_{i \neq j; i=1}^{2m-n-1} (2m-n-i-r)$$

Thus : $\frac{1}{1-r} \geq \frac{1}{-\frac{1}{l}+2m-n-1+(1-r)} > \sum_{i=1}^{2m-n-1} \frac{1}{2m-n-i-r}$, which is not the case, and $z \in \mathbb{N}$.

Note : *This result of step 3 can be noticed in the graphs of S_{2m} : If the curve of S_{2m} touches the segment $]1, 2m[$ at a point p , then p is a prime number !!*

4 step : Proof of proposition 3

The case where $m \in \{2, 3, 4\}$ is easy to see, since the Goldbach conjecture holds in these cases.

If $m \geq 5$, in step 3 we have shown the existence of the roots $z \in \mathbb{N} \cap]1, m[$ such that $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$. And from proposition 2 we deduce that z and $2m-z$ are primes with $2m = z + (2m-z)$

Proof of Theorem 1 :

It is deduced directly from **proposition 3**

III - The proof of Alphonse de Polignac's conjecture [6]

Theorem 2 : The conjecture of Alphonse de Polignac [6]

Any non zero even number $2m$ is equal to the difference of two consecutive prime numbers in an infinite number of ways.

Proof :

It is demonstrated by the same technique used above to prove the Goldbach conjecture using the function S^+ instead of S :

$$S^+(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m+z)+1}{2m+z}\pi\right)$$

And by showing as in steps 2, 3, above :

- The existence of an infinite number of local minimums z of S^+ in $]1, \infty[$, with $S(z) = 0$, $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$ and $\frac{\Gamma(2m+z)+1}{2m+z} \in \mathbb{N}$
- The existence of an infinity of integer z roots of S^+ ($z \in \mathbb{N}$).

Then using proposition 2 - as in step 4 above - we deduce that z and $2m+z$ are prime with $2m = (2m+z) - z$

Corollary : [The conjecture of twin prime numbers] [7]

There exists an infinity of prime numbers p such that $p+2$ is also prime.

IV- References

- [1] Roshdi Rashed, *Entre arithmétique et algèbre : Recherches sur l'histoire des mathématiques arabes*, journal Paris, 1984,
- [2] M. SGHIAR, *The Special Functions and the Proof of the Riemann's Hypothesis*, IOSR Journal of Mathematics (IOSR-JM)e-ISSN : 2278-5728, p-ISSN : 2319-765X. Volume 16, Issue 3 Ser. II (May –June 2020), PP 10-12. www.iosrjournals.org
- [3] https://en.wikipedia.org/wiki/Gamma_function.
- [4] https://en.wikipedia.org/wiki/Riemann_zeta_function.
- [5] https://en.wikipedia.org/wiki/Goldbach%27s_conjecture.
- [6] https://fr.wikipedia.org/wiki/Conjecture_de_Polignac
- [7] https://fr.wikipedia.org/wiki/Nombres_premiers_jumeaux