



**HAL**  
open science

# Prime numbers and the Goldbach conjecture

Mohamed Sghiar

► **To cite this version:**

| Mohamed Sghiar. Prime numbers and the Goldbach conjecture. 2020. hal-02997473v1

**HAL Id: hal-02997473**

**<https://hal.science/hal-02997473v1>**

Preprint submitted on 10 Nov 2020 (v1), last revised 16 Dec 2020 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.


## Prime numbers and the Goldbach conjecture

M. SGHIAR

msghiar21@gmail.com

Presented to :

Université de Bourgogne Dijon, Faculté des sciences Mirande, Département  
de mathématiques . Laboratoire d'analyse, 9 av alain savary 21078 , Dijon  
cedex, France

Tel :  0033669753590

**Abstract** : Inspired by the article [2], the introduction of the function  $\hat{\mathbb{S}}$  whose integer zeros are the prime numbers, will allow me to demonstrate analytically the Goldbach conjecture [5].

**Résumé** : Inspiré de l'article [ 2], l'introduction de la fonction  $\hat{\mathbb{S}}$  dont les zéros entiers sont les nombres premiers, va me permettre de démontrer analytiquement la conjecture de Goldbach [5].

**Keywords** : Prime Number, the Goldbach conjectures, the Gamma function.

**Code MSC** : 11P32

## I- INTRODUCTION

In 1742, in a letter to Leonhard Euler, Christian Goldbach conjectured [5] : Any even integer greater than 3 can be written as the sum of two prime numbers.

Inspired by the function  $\mathbb{S}$  which I introduced in the article [2], I introduce the function  $\hat{\mathbb{S}}$  which, also like  $\mathbb{S}$ , allows to find the prime numbers .

The study of the function  $S_{2m}(z) = \hat{\mathbb{S}}^2(z) + \hat{\mathbb{S}}^2(2m - z)$  will allow me to demonstrate the Goldbach conjecture.

In this article I assume known the functions zeta  $\zeta$ , Gamma  $\Gamma : z \mapsto \int_0^{+\infty} t^{z-1} e^{-t} dt$  and their properties (see [3] and [4]).

## II- THE PROOF OF THE GOLDBACH CONJECTURE :

**Théorème** : [The Goldbach conjecture ]

Any even integer greater than 3 can be written as the sum of two prime numbers.

**Proposition 1.** [2] :

Let  $\mathbb{S}(z) = \zeta(-\frac{\Gamma(z)+1}{z/2})$ .

If  $z \in \mathbb{N}^*$  then  $\mathbb{S}(z) = 0 \iff z$  is a prime number

**Proof** :

It follows from Wilson's theorem [1] - which shows that  $p$  is prime if and only  $(p - 1)! \equiv -1 \pmod{p}$  - and the fact that the trivial zeros of  $\zeta$  are  $-2\mathbb{N}^*$ .

**Proposition 2 :**

Let  $\hat{\mathbb{S}}(z) = \sin\left(\frac{\Gamma(z)+1}{z}\pi\right)$ .

If  $z \in \mathbb{N}^*$  then  $\hat{\mathbb{S}} = 0 \iff z$  is a prime number

**Proof :** Inspired by **proposition 1**

**Proposition 3 :** If  $m \in \mathbb{N}^* \setminus \{1\}$ , then the equation :

$$\sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) = 0$$

admits at least one solution  $z$  in  $\mathbb{N}$  with  $z$  and  $2m - z$  are prime.

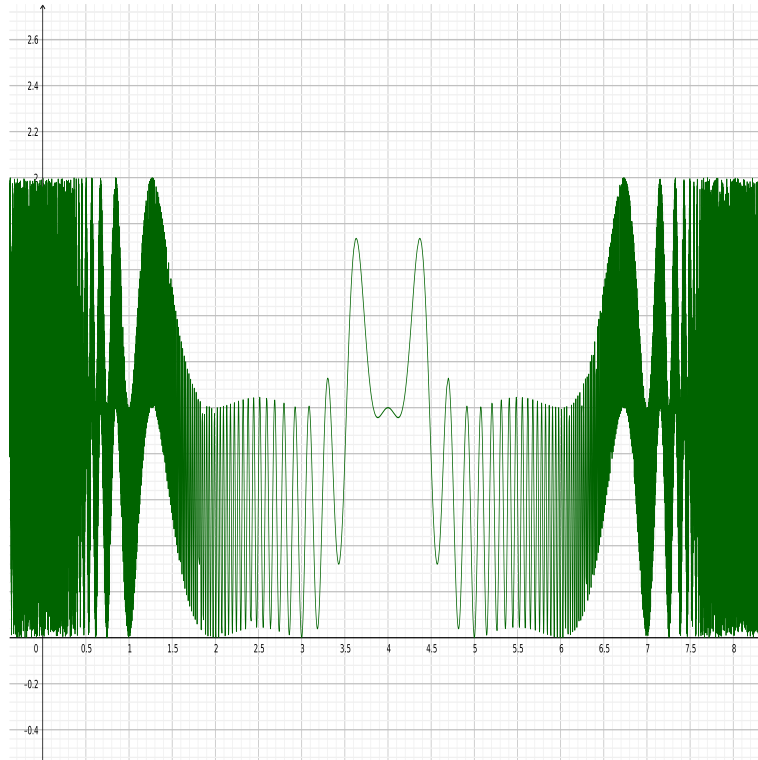


FIGURE 1 –  $S(z)$  for  $m=8$

**Proof of Proposition 3 :**

Consider the restriction to  $\mathbb{R}^{+*}$  of the function :

$$S(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$$

**1 step :** If the function  $S(z) = \sin^2\left(\frac{\Gamma(z)+1}{z}\pi\right) + \sin^2\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right)$  has a local

extremum at the point  $p \neq 2$  with  $p$  is prime :

In this case  $\frac{dS}{dz}(p) = 0$  and according to proposition 2 we have  $\hat{\mathbb{S}}(p) = \sin(\frac{\Gamma(p)+1}{p}\pi) = 0$

We have :

$$\frac{dS}{dz}(z) = 2\pi \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \cos\left(\frac{\Gamma(z)+1}{z}\pi\right) \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} + 2\pi \sin\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \cos\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2}$$

But as  $\frac{-(2m-p)\Gamma'(2m-p)+(\Gamma(2m-p)+1)}{(2m-p)^2} \neq 0$  (car  $-z\Gamma'(z) + \Gamma(z) + 1$ ) vanishes in  $z \sim 2.5 \notin \mathbb{N}$  , then  $2\sin\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right) \cos\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right) = 0$

And since we cannot have  $\cos\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right) = 0$ , because otherwise  $\frac{\Gamma(2m-p)+1}{2m-p} \in \mathbb{N} + \frac{1}{2}$ , and therefore  $p \in 2\mathbb{N}$ , which is not the case, then  $\sin\left(\frac{\Gamma(2m-p)+1}{2m-p}\pi\right) = 0$ , and from **proposition 2**, we conclude that  $2m - p$  is prime.

**2 step :** Let us show the existence of local minimums  $z$  in  $]1, m[$  as soon as  $m \geq 5$ , with  $S(z) = 0$  ,  $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$  and  $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$  :

First  $S$  admits a local extremum at a point  $z \in ]1, m[$  if  $m \geq 5$  : Indeed :

$$S(z) = 1 - \frac{1}{2}(\cos(2\frac{\Gamma(z)+1}{z}\pi) + \cos(2\frac{\Gamma(2m-z)+1}{2m-z}\pi))$$

If  $S$  does not admit a local extremum at a point  $z \in ]1, m[$  , then  $T(z) = (\cos(2\frac{\Gamma(z)+1}{z}\pi) + \cos(2\frac{\Gamma(2m-z)+1}{2m-z}\pi))$  does not admit a local extremum in  $z \in ]1, m[$ , and in this case, by symmetry of the curve of  $S$  with respect to the line  $x = m$ , we have  $\frac{dT}{dz}(z) \leq 0, \forall z \in ]1, m[$  or  $\frac{dT}{dz}(z) \leq 0, \forall z \in ]m, 2m-1[$  .

By symmetry suppose that  $\frac{dT}{dz}(z) \leq 0, \forall z \in ]1, m[$  :

$$\frac{dT}{dz}(z) = -2\pi \sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} - 2\pi \sin\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2}$$

By growth of the function  $\frac{z\Gamma'(z)-(\Gamma(z)+1)}{z^2}$  in  $\in [3, +\infty[$  we have :  $\frac{(2m-z)\Gamma'(2m-z)-(\Gamma(2m-z)+1)}{(2m-z)^2} \frac{z\Gamma'(z)-(\Gamma(z)+1)}{z^2} \not\leq 0$

1,  $\forall z \in [3, m[$ , so :

$$\sin\left(\frac{\Gamma(z)+1}{z}\pi\right) \geq \sin\left(\frac{\Gamma(2m-z)+1}{2m-z}\pi\right) \frac{(2m-z)\Gamma'(2m-z)-(\Gamma(2m-z)+1)}{(2m-z)^2} \frac{z\Gamma'(z)-(\Gamma(z)+1)}{z^2}; \forall z \in [3, m[$$

And for  $z$  such that  $\sin(\frac{\Gamma(2m-z)+1}{2m-z}\pi) = 1$  ( $z$  exists because the function  $L(x) = \frac{\Gamma(x+1)+1}{x+1} - \frac{\Gamma(x)+1}{x}$  check  $L(x) \geq 1, \forall x \geq 3.2$ ), we will have  $\sin(\frac{\Gamma(z)+1}{z}\pi) \not\leq 1$ , which is absurd. And therefore  $S(z) = \sin^2(\frac{\Gamma(z)+1}{z}\pi) + \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$  admits local extremum  $z_1$  in  $\in ]3, m[$

If  $z_1$  is not a local minimum, by taking the same reasoning by replacing 1 by  $z_1$ , we deduce the existence of a local extremum  $z_2$  in  $[3, z_1[$  or in  $]z_1, m[$  and hence there is a local minimum  $z$  between  $z_1$  et  $z_2$ .

Let  $z$  be a minimum of  $S$  in  $\in ]1, m[$ .

If  $S(z) \neq 0$ , then  $\frac{\Gamma(z)+1}{z}\pi \neq 0, \text{ mod}(\pi)$  ou  $\frac{\Gamma(2m-z)+1}{2m-z}\pi \neq 0, \text{ mod}(\pi)$

By symmetry of the curve of  $S$  with respect to the line  $x = m$ , we can assume that  $\frac{\Gamma(2m-z)+1}{2m-z}\pi \neq 0, \text{ mod}(\pi)$  and that  $\sin^2(\frac{\Gamma(z)+1}{z}\pi) \leq \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$ .

If  $\sin^2(\frac{\Gamma(z)+1}{z}\pi) < \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$  : If  $z'$  is too close to  $z$  with  $\sin^2(\frac{\Gamma(2m-z')+1}{2m-z'}\pi) < \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$ , then  $S(z') < S(z)$  which contradicts the minimality. It follows that  $\sin^2(\frac{\Gamma(z)+1}{z}\pi) = \sin^2(\frac{\Gamma(2m-z)+1}{2m-z}\pi)$

But from the equation :

$$0 = \frac{dS}{dz}(z) = 2\pi \sin(\frac{\Gamma(z)+1}{z}\pi) \cos(\frac{\Gamma(z)+1}{z}\pi) \frac{z\Gamma'(z) - (\Gamma(z)+1)}{z^2} + 2\pi \sin(\frac{\Gamma(2m-z)+1}{2m-z}\pi) \cos(\frac{\Gamma(2m-z)+1}{2m-z}\pi) \frac{-(2m-z)\Gamma'(2m-z) + (\Gamma(2m-z)+1)}{(2m-z)^2}$$

We deduce that :  $0 = \sin(\frac{\Gamma(z)+1}{z}\pi) \cos(\frac{\Gamma(z)+1}{z}\pi)$ , and therefore  $0 = \sin(\frac{\Gamma(z)+1}{z}\pi)$  by minimality of  $S(z)$ . Thus  $S(z) = 0$  with  $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$  and  $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$

**3 step** : Let us show the existence of the roots  $z \in \mathbb{N} \cap ]1, m[$  :

The case where  $m \in \{2, 3, 4\}$  is easy to see, since the Goldbach conjecture holds in these cases.

If  $m \geq 5$ , from step 2, we have  $S(z) = 0$  with :  $z \in ]1, m[$ ,  $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$  and  $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$

If  $z \notin \mathbb{N}$  :

First  $z \geq 3$  : because  $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$  and if  $z \not\leq 3$  then  $z = 1$  ou  $z = 2$  which is not the case because  $z \in \mathbb{R}^+ \setminus \mathbb{N}$

Let  $\frac{\Gamma(z)+1}{z} = k$ ,  $z = n + r$  with  $r \geq 2$ ,  $r \in \mathbb{R}^+ \setminus \mathbb{N}$  and  $n \geq 1$

We deduce :  $\prod_{i=1}^n (n - i + r) \Gamma(r) + 1 = kn + kr$

So  $\Gamma(r) = \frac{-1+kn+kr}{\prod_{i=1}^n (n-i+r)}$

As  $r \geq 2$ , then  $\Gamma'(r) \not\leq 0$ , and hence :

$$\prod_{i=1}^n (n - i + r) \not\leq (-1 + kn + kr) \sum_{j=1}^n \prod_{i \neq j; i=1}^n (n - i + r)$$

So  $\frac{1}{r} \geq \frac{1}{-1+kn+kr} \not\leq \sum_{i=1}^n \frac{1}{n-i+r}$ , which is not the case, and  $z \in \mathbb{N}$ .

**4 step : Proof of proposition 3**

The case where  $m \in \{2, 3, 4\}$  is easy to see, since the Goldbach conjecture holds in these cases.

If  $m \geq 5$ , in step 3 we have shown the existence of the roots  $z \in \mathbb{N} \cap ]1, m[$  such that  $\frac{\Gamma(z)+1}{z} \in \mathbb{N}$  and  $\frac{\Gamma(2m-z)+1}{2m-z} \in \mathbb{N}$ . And from proposition 2 we deduce that  $z$  and  $2m - z$  are prime with  $2m = z + (2m - z)$

**Proof of Theorem :**

It is deduced directly from **proposition 3**

**III- References**

- [1] Roshdi Rashed, Entre arithmétique et algèbre : Recherches sur l'histoire des mathématiques arabes, journal Paris, 1984,
- [2] M. SGHIAR, The Special Functions and the Proof of the Riemann's Hypothesis, IOSR Journal of Mathematics (IOSR-JM)e-ISSN : 2278-5728, p-ISSN : 2319-765X. Volume 16, Issue 3 Ser. II (May –June 2020), PP 10-12. [www.iosrjournals.org](http://www.iosrjournals.org)
- [3] [https://en.wikipedia.org/wiki/Gamma\\_function](https://en.wikipedia.org/wiki/Gamma_function).
- [4] [https://en.wikipedia.org/wiki/Riemann\\_zeta\\_function](https://en.wikipedia.org/wiki/Riemann_zeta_function).
- [5] [https://en.wikipedia.org/wiki/Goldbach%27s\\_conjecture](https://en.wikipedia.org/wiki/Goldbach%27s_conjecture).