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Abstract. We show that the Brydges-Fröhlich-Spencer-Dynkin and the Le Jan's isomorphisms between the Gaussian free fields and the occupation times of symmetric Markov processes generalizes to β-Dyson's Brownian motion. For $\beta \in \{1,2,4\}$ this is a consequence of the Gaussian case, however the relation holds for general β. We further raise the question whether there is an analogue of β-Dyson's Brownian motion on general electrical networks, interpolating and extrapolating the fields of eigenvalues in matrix valued Gaussian free fields. In the case $n = 2$ we give a simple construction.

1. Introduction

There is a class of results, known as *isomorphism theorems*, relating the squares of Gaussian free fields (GFFs) to occupation times of symmetric Markov processes. They originate from the works in mathematical physics [Sym69, BFS82]. For a review, see [MR06, Szn12]. Here in particular we will be interested in the Brydges-Fröhlich-Spencer-Dynkin isomorphism [BFS82, Dyn84a, Dyn84b] and in the Le Jan's isomorphism [LJ10, LJ11]. The BFS-Dynkin isomorphism involves Markovian paths with fixed ends. Le Jan's isomorphism involves a Poisson point process of Markovian loops, with an intensity parameter $\alpha = 1/2$ in the case of real scalar GFFs. For vector valued GFFs with d components, the intensity parameter is $\alpha = d/2$. We show that both Le Jan's and BFS-Dynkin isomorphisms have a generalization to β-Dyson's Brownian motion.

For $\beta \in \{1,2,4\}$, a β-Dyson's Brownian motion is the diffusion of eigenvalues in a Brownian motion on the space of real symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$), respectively quaternionic Hermitian ($\beta = 4$) matrices. Yet, the β-Dyson's Brownian motion is defined for every $\beta \geqslant 0$. The one-dimensional marginals of β-Dyson's Brownian motion are Gaussian beta ensembles GβE. The generalization of Le Jan's and BFS-Dynkin isomorphisms works for every $\beta \geqslant 0$, and for $\beta \in \{1,2,4\}$ is follows from the Gaussian case. The intensity parameter α appearing in the Le Jan's type isomorphism is given by

$$2\alpha = d(\beta, n) = n + n(n-1)\frac{\beta}{2}.,$$

where n is the number of "eigenvalues". In particular, α takes not only half-integer values, as in the Gaussian case, but a whole half-line of values. The BFS-Dynkin type isomorphism involves polynomials defined by a recurrence with a structure similar to that of the Schwinger-Dyson equation for GβE. These polynomials also give the symmetric moments of the β-Dyson's Brownian motion.

We further ask the question whether an analogue of GβE and β-Dyson's Brownian motion could exist on electrical networks and interpolate and extrapolate the distributions of the eigenvalues in matrix valued GFFs. Our motivation for this is that such analogues could be related to Poisson point process of random walk loops, in particular to those of non half-integer intensity parameter. If the underlying graph is a tree, the construction of such analogues is straightforward, by taking β-Dyson's Brownian motions along each branch of the tree. However, if the graph contains cycles, this is not immediate, and one does not expect a Markov property for the obtained fields. However, in the simplest case $n = 2$, we provide a construction working on any graph.

Key words and phrases. Dyson's Brownian motion, Gaussian beta ensembles, Gaussian free field, isomorphism theorems, local time, permanental fields, topological expansion .

Our article is organized as follows. In Section 2 we recall the BFS-Dynkin and the Le Jan's isomorphisms in the particular case of 1D Brownian motion. In Section 3 we recall the definition of Gaussian beta ensembles and the corresponding Schwinger-Dyson equation. In Section 4 we give the recurrence on polynomials that will be used for the BFS-Dynkin type isomorphism. The Section 5 deals with β-Dyson's Brownian motion and the corresponding isomoprhims. Section 6 deals with general electrical networks. We give our construction for $n = 2$ and ask our questions for $n \geqslant 3$.

2. Isomorphism theorems for 1D Brownian motion

Let $(B_t)_{t\geqslant 0}$ be the standard Brownian motion on $\mathbb{R}$. L^x will denote the Brownian local times:

$$L^x((B_s)_{0\leqslant s\leqslant t}) = \lim_{\varepsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbf{1}_{|B_s - x|<\varepsilon} ds.$$

$p(t, x, y)$ will denote the heat kernel on $\mathbb{R}$, and $p_{\mathbb{R}_+}(t, x, y)$ the heat kernel on $\mathbb{R}_+$ with condition 0 in 0:

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}, \qquad p_{\mathbb{R}_+}(t,x,y) = p(t,x,y) - p(t,x,-y).$$

$\mathbb{P}^{t,x,y}(\cdot)$ will denote the Brownian bridge probability from x to y in time t, and $\mathbb{P}^{t,x,y}_{\mathbb{R}_+}(\cdot)$ (for $x, y > 0$) the probability measures where one conditions $\mathbb{P}^{t,x,y}(\cdot)$ on that the bridge does not hit 0. $(G_{\mathbb{R}_+}(x, y))_{x,y\geqslant 0}$ be the Green's function of $\frac{1}{2}\frac{d^2}{dx^2}$ on $\mathbb{R}_+$ with 0 condition in 0, and for $K > 0$, $(G_K(x, y))_{x,y\geqslant 0}$ the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - K$ on $\mathbb{R}$:

$$\begin{aligned} G_{\mathbb{R}_+}(x,y) &= 2x\wedge y = \int_0^{+\infty} p_{\mathbb{R}_+}(t,x,y)dt, \\ G_K(x,y) &= \frac{1}{\sqrt{2K}} e^{-\sqrt{2K}|y-x|} = \int_0^{+\infty} p(t,x,y) e^{-Kt} dt. \end{aligned}$$

Let $(\mu^{x,y}_{\mathbb{R}_+})_{x,y>0}$, resp. $(\mu^{x,y}_K)_{x,y\in\mathbb{R}}$ be the following measures on finite duration paths:

$$\mu^{x,y}_{\mathbb{R}_+}(\cdot) := \int_0^{+\infty} \mathbb{P}^{t,x,y}_{\mathbb{R}_+}(\cdot) p_{\mathbb{R}_+}(t,x,y)dt, \qquad \mu^{x,y}_K(\cdot) := \int_0^{+\infty} \mathbb{P}^{t,x,y}(\cdot) p(t,x,y) e^{-Kt} dt. \tag{2.1}$$

The total mass of $\mu^{x,y}_{\mathbb{R}_+}$, resp. $\mu^{x,y}_K$, is $G_{\mathbb{R}_+}(x, y)$, resp. $G_K(x, y)$. The image of $\mu^{x,y}_{\mathbb{R}_+}$, resp. $\mu^{x,y}_K$, by time reversal is $\mu^{y,x}_{\mathbb{R}_+}$, resp. $\mu^{y,x}_K$.

Let T_x denote the first hitting time of a level x by the Brownian motion $(B_t)_{t\geqslant 0}$. γ will denote a generic path of $\mathbb{R}$. Let $(\check{\mu}^{x,y}(\cdot))_{x<y\in\mathbb{R}}$, resp. $(\check{\mu}^{x,y}_K(\cdot))_{x<y\in\mathbb{R}}$ be the following measures on paths from x to y:

$$\check{\mu}^{x,y}(F(\gamma)) = \mathbb{E}_{B_0=y}[F((B_{T_x - t})_{0\leqslant t\leqslant T_x})], \qquad \check{\mu}^{x,y}_K(F(\gamma)) = \mathbb{E}_{B_0=y}\big[e^{-KT_x} F((B_{T_x-t})_{0\leqslant t\leqslant T_x})\big].$$

$\check{\mu}^{x,y}$ has total mass 1 (probability measure), whereas the total mass of $\check{\mu}^{x,y}_K$ is

$$\mathbb{E}_{B_0=y}\big[e^{-KT_x}\big] = e^{-\sqrt{2K}|y-x|} = \frac{G_K(x,y)}{G_K(x,x)}.$$

For $0 < x \leqslant y < z$, the measure $\mu^{x,z}_{\mathbb{R}_+}$ can be obtained as the image of the product measure $\mu^{x,y}_{\mathbb{R}_+} \otimes \check{\mu}^{y,z}$ under the concatenation of two paths. Similarly, for $x \leqslant y < z \in \mathbb{R}$, the measure $\mu^{x,z}_K$ is the image of $\mu^{x,y}_K \otimes \check{\mu}^{y,z}_K$ under the concatenation of two paths.

Let $(W(x))_{x\in R}$ denote a two-sided Brownian motion, i.e. $(W(x))_{x\geqslant 0}$ and $(W(-x))_{x\geqslant 0}$ being two independent standard Brownian motion started from 0 ($W(0) = 0$). Note that here x is rather a one-dimensional space variable then a time variable. $dW(x)$ is a white noise on $\mathbb{R}$. Let $(\phi_{\mathbb{R}_+}(x))_{x\geqslant 0}$ denote the process $(\sqrt{2}W(x))_{x\geqslant 0}$. The covariance function of $\phi_{\mathbb{R}_+}$ is $G_{\mathbb{R}_+}$. Let $(\phi_K(x))_{x\in\mathbb{R}}$ be the stationary Ornstein–Uhlenbeck process with invariant measure $\mathcal{N}(0, 1/\sqrt{2K})$. It is a solution to the SDE

$$d\phi_K(x) = \sqrt{2}dW(x) - \sqrt{2K}\phi_K(x)dx.$$

The covariance function of ϕ_K is G_K.

What follows is the BFS-Dynkin isomorphism (Theorem 2.2 in [BFS82], Theorems 6.1 and 6.2 in [Dyn84a], Theorem 1 in [Dyn84b]) in the particular case of a 1D Brownian motion. In general, the BFS-Dynkin isomorphism relates the squares of Gaussian free fields to local times of symmetric Markov processes.

Theorem 2.1 (Brydges-Fröhlich-Spencer [BFS82], Dynkin [Dyn84a, Dyn84b])**.** *Let F be a bounded measurable functional on $\mathcal{C}(\mathbb{R}_+)$, resp. on $\mathcal{C}(\mathbb{R})$. Let $k \geqslant 1$ and $x_1, x_2, \dots, x_{2k}$ in $(0,+\infty)$, resp. in $\mathbb{R}$. Then*

$$\mathbb{E}\Big[\prod_{i=1}^{2k}\phi_{\mathbb{R}_+}(x_i)F(\phi_{\mathbb{R}_+}^2/2)\Big] = \sum_{\substack{(\{a_i,b_i\})_{1\leqslant i\leqslant k}\\ \text{partition in pairs}\\ \text{of } [\![1,2k]\!]}} \int_{\gamma_1,\dots,\gamma_k} \mathbb{E}\Big[F(\phi_{\mathbb{R}_+}^2/2+L(\gamma_1)+\dots+L(\gamma_k))\Big]\prod_{i=1}^{k}\mu_{\mathbb{R}_+}^{x_{a_i},x_{b_i}}(d\gamma_i),$$

resp.

$$\mathbb{E}\Big[\prod_{i=1}^{2k}\phi_{K}(x_i)F(\phi_{K}^2/2)\Big] = \sum_{\substack{(\{a_i,b_i\})_{1\leqslant i\leqslant k}\\ \text{partition in pairs}\\ \text{of } [\![1,2k]\!]}} \int_{\gamma_1,\dots,\gamma_k} \mathbb{E}\Big[F(\phi_{K}^2/2+L(\gamma_1)+\dots+L(\gamma_k))\Big]\prod_{i=1}^{k}\mu_{K}^{x_{a_i},x_{b_i}}(d\gamma_i),$$

where the sum runs over the $(2k)!/(2^k k!)$ partitions in pairs, γ_i-s are Brownian paths and $L(\gamma_i)$ are the corresponding occupation fields $x \mapsto L^x(\gamma_i)$.

Remark 2.2. *Since for $x<y$, the measure $\mu_{\mathbb{R}_+}^{x,y}$, resp. $\mu_K^{x,y}$, can be decomposed as $\mu_{\mathbb{R}_+}^{x,x}\otimes\check{\mu}^{x,y}$, resp. $\mu_K^{x,x}\otimes\check{\mu}_K^{x,y}$, Theorem 2.1 can be rewritten using only the measures of type $\mu_{\mathbb{R}_+}^{x,x}$ and $\check{\mu}^{x,y}$, resp. $\mu_K^{x,x}$ and $\check{\mu}_K^{x,y}$.*

To a wide class of symmetric Markov processes one can associate in a natural way an infinite, σ-finite measure on loops [LW04, LTF07, LL10, LJ10, LJ11, LJMR15, FR14]. It originated from the works in mathematical physics [Sym65, Sym66, Sym69, BFS82]. Here we recall it in the setting of a 1D Brownian motion, which has been studied in [Lup18]. Of course, the range on a loop will be just a segment on the line, but it will carry a non-trivial Brownian local time process which will be of interest for us.

Given a Brownian loop γ, $T(\gamma)$ will denote its duration. The measures on (rooted) loops are

$$\mu_{\mathbb{R}_+}^{\text{loop}}(d\gamma) := \frac{1}{T(\gamma)}\int_{\mathbb{R}_+}\mu_{\mathbb{R}_+}^{x,x}(d\gamma)dx, \qquad \mu_K^{\text{loop}}(d\gamma) = \frac{1}{T(\gamma)}\int_{\mathbb{R}}\mu_K^{x,x}(d\gamma)dx. \tag{2.2}$$

Usually one considers unrooted loops, but this will not be important here. The 1D Brownian *loop-soups* are the Poisson point processes, denoted $\mathcal{L}_{\mathbb{R}_+}^{\alpha}$, resp. $\mathcal{L}_K^{\alpha}$, of intensity $\alpha\mu_{\mathbb{R}_+}^{\text{loop}}$, resp. $\alpha\mu_K^{\text{loop}}$, where $\alpha>0$ is an intensity parameter. $L(\mathcal{L}_{\mathbb{R}_+}^{\alpha})$, resp. $L(\mathcal{L}_K^{\alpha})$, will denote the occupation field of $\mathcal{L}_{\mathbb{R}_+}^{\alpha}$, resp. $\mathcal{L}_K^{\alpha}$:

$$L^x(\mathcal{L}_{\mathbb{R}_+}^{\alpha}) := \sum_{\gamma\in\mathcal{L}_{\mathbb{R}_+}^{\alpha}} L^x(\gamma), \qquad L^x(\mathcal{L}_K^{\alpha}) := \sum_{\gamma\in\mathcal{L}_K^{\alpha}} L^x(\gamma).$$

The following statement deals with the law of $L(\mathcal{L}_{\mathbb{R}_+}^{\alpha})$, resp. $L(\mathcal{L}_K^{\alpha})$. See Proposition 4.6, Property 4.11 and Corollary 5.5 in [Lup18]. For the analogous statements in discrete space setting, see Corollary 5, Proposition 6, Theorem 13 in [LJ10] and Corollary 1, Section 4.1, Proposition 16, Section 4.2, Theorem 2, Section 5.1 in [LJ11] In general, one gets α-permanental fields (see also [LJMR15, FR14]). For $\alpha=\frac{1}{2}$ in particular, one gets square Gaussians. We recall that given a matrix $M=(M_{ij})_{1\leqslant i,j\leqslant k}$, its α-permanent is

$$\operatorname{Perm}_\alpha(M) := \sum_{\substack{\sigma \text{ permutation}\\ \text{of } \{1,2,\dots,k\}}} \alpha^{\#\text{ cycles of }\sigma}\prod_{i=1}^{k} M_{i\sigma(i)}.$$

Theorem 2.3 (Le Jan [LJ10, LJ11], Lupu [Lup18])**.** *For every $\alpha > 0$ and $x \in \mathbb{R}_+$, resp. $x \in \mathbb{R}$, the r.v. $L^x(\mathcal{L}^\alpha_{\mathbb{R}_+})$, resp. $L^x(\mathcal{L}^\alpha_K)$, follows the distribution* $\mathrm{Gamma}(\alpha, G_{\mathbb{R}_+}(x,x)^{-1})$*, resp.* $\mathrm{Gamma}(\alpha, G_K(x,x)^{-1})$*. Moreover, the process $\alpha \mapsto L^x(\mathcal{L}^\alpha_{\mathbb{R}_+})$, resp. $L^x(\mathcal{L}^\alpha_K)$ is a pure jump Gamma subordinator with Lévy measure*

$$\mathbf{1}_{l>0}\frac{e^{-l/G_{\mathbb{R}_+}(x,x)}}{l}dl, \qquad \text{resp. } \mathbf{1}_{l>0}\frac{e^{-l/G_K(x,x)}}{l}dl.$$

Let $x_1, x_2, \dots, x_k \in \mathbb{R}_+$, resp. $\mathbb{R}$. Then

$$\mathbb{E}\Big[\prod_{i=1}^k L^{x_i}(\mathcal{L}^\alpha_{\mathbb{R}_+})\Big] = \mathrm{Perm}_\alpha\big(G_{\mathbb{R}_+}(x_i,x_j)_{1\leqslant i,j\leqslant k}\big), \ \mathbb{E}\Big[\prod_{i=1}^k L^{x_i}(\mathcal{L}^\alpha_K)\Big] = \mathrm{Perm}_\alpha\big(G_K(x_i,x_j)_{1\leqslant i,j\leqslant k}\big).$$

For $x \geqslant 0$, $x \mapsto L^x(\mathcal{L}^\alpha_{\mathbb{R}_+})$ is a solution to the SDE

$$dL^x(\mathcal{L}^\alpha_{\mathbb{R}_+}) = 2\big(L^x(\mathcal{L}^\alpha_{\mathbb{R}_+})\big)^{\frac{1}{2}}dW(x) + 2\alpha dx,$$

with initial condition $L^0(\mathcal{L}^\alpha_{\mathbb{R}_+}) = 0$. That is to say it is a square Bessel process of dimension 2α. For $x \in \mathbb{R}$ $x \mapsto L^x(\mathcal{L}^\alpha_K)$ is a stationary solution to the SDE

$$dL^x(\mathcal{L}^\alpha_K) = 2\big(L^x(\mathcal{L}^\alpha_K)\big)^{\frac{1}{2}}dW(x) - 2\sqrt{2K}L^x(\mathcal{L}^\alpha_K) + 2\alpha dx.$$

In particular, for $\alpha = \frac{1}{2}$, one has the following identities in law between stochastic processes:

$$L(\mathcal{L}^\alpha_{\mathbb{R}_+}) \overset{(\text{ law })}{=} \frac{1}{2}\phi^2_{\mathbb{R}_+}, \qquad L(\mathcal{L}^\alpha_K) \overset{(\text{ law })}{=} \frac{1}{2}\phi^2_K. \tag{2.3}$$

3. Gaussian beta ensembles

For references on Gaussian beta ensembles, see [DE02, For15], Section 1.2.2 in [EKR18], and Section 4.5 in [AGZ09]. Fix $n \geqslant 2$. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$, $D(\lambda)$ will denote the Vandermonde determinant

$$D(\lambda) := \prod_{1\leqslant j<j'\leqslant n}(\lambda_{j'} - \lambda_j).$$

For $q \geqslant 1$, $p_q(\lambda)$ will denote the q-th power sum polynomial

$$p_q(\lambda) := \sum_{j=1}^n \lambda_j^q.$$

By convention,

$$p_0(\lambda) = n.$$

A Gaussian beta ensemble GβE follows the distribution

$$\frac{1}{Z_{\beta,n}}|D(\lambda)|^\beta e^{-\frac{1}{2}p_2(\lambda)}\prod_{j=1}^n d\lambda_j, \tag{3.1}$$

where $Z_{\beta,n}$ is given by (Formula (17.6.7) in [Meh04] and Formula (1.2.23) in [EKR18])

$$Z_{\beta,n} = (2\pi)^{\frac{n}{2}}\prod_{j=1}^n \frac{\Gamma\big(1 + j\frac{\beta}{2}\big)}{\Gamma\big(1 + \frac{\beta}{2}\big)}.$$

The brackets $\langle\cdot\rangle_{\beta,n}$ will denote the expectation with respect to (3.1). For $\beta = 0$ one gets n i.i.d. $\mathcal{N}(0,1)$ Gaussians. For β equal to 1, 2, resp. 4, one gets the eigenvalue distribution of GOE, GUE, resp. GSE random matrices [Meh04, EKR18]. Usually the GβE are studied for $\beta > 0$ [DE02], but the distribution (3.1) is well defined for all $\beta > -\frac{2}{n}$. For $\beta \in (-\frac{2}{n}, 0)$ there is an attraction between the λ_j-s instead of a repulsion as for $\beta > 0$. Moreover, as $\beta \to -\frac{2}{n}$, λ under (3.1) converges in law to

$$\Big(\frac{1}{\sqrt{n}}\xi, \frac{1}{\sqrt{n}}\xi, \dots, \frac{1}{\sqrt{n}}\xi\Big),$$

where ξ follows $\mathcal{N}(0,1)$.

Let $d(\beta, n)$ denote

$$d(\beta, n) = n + n(n-1)\frac{\beta}{2}.$$

For $\beta \in \{1, 2, 4\}$, $d(\beta, n)$ is the dimension of the corresponding space of matrices.

Let be $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$, where $m \geqslant 1$, and for all $k \in \{1, 2, \ldots, m\}$, $\nu_k \in \mathbb{N}\backslash\{0\}$. We will denote

$$m(\nu) = m, \qquad |\nu| = \sum_{k=1}^{m(\nu)} \nu_k.$$

$p_\nu(\lambda)$ will denote

$$p_\nu(\lambda) := \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda).$$

By convention, we set $p_\varnothing(\lambda) = 1$ and $|\varnothing| = 0$. Note that $p_\varnothing(\lambda) \neq p_0(\lambda)$. We are interested in the expression of the moments $\langle p_\nu(\lambda)\rangle_{\beta,n}$. These are 0 if $|\nu|$ is not even. For $|\nu|$ even, these moments are given by a recurrence known as *loop equation or Schwinger-Dyson equation* (Lemma 4.13 in [LC09], slide 3/15 in [LC13] and Section 4.1.1 in [EKR18]). For the combinatorial interpretation of the solutions in terms of ribbon graphs or surfaces, see [LC13, LC09]. See the Appendix for the expression of some moments.

Proposition 3.1 (Schwinger-Dyson equation [LC09, LC13, EKR18])**.** *For every $\beta > -2/n$ and every ν as above with $|\nu|$ even,*

$$\begin{aligned}\langle p_\nu(\lambda)\rangle_{\beta,n} &= \frac{\beta}{2}\sum_{i=1}^{\nu_{m(\nu)}-1} \langle p_{(\nu_r)_{r\neq m(\nu)}}(\lambda) p_{i-1}(\lambda) p_{\nu_{m(\nu)}-1-i}(\lambda)\rangle_{\beta,n} \\ &\quad + \Big(1 - \frac{\beta}{2}\Big)(\nu_{m(\nu)} - 1)\langle p_{(\nu_r)_{r\neq m(\nu)}}(\lambda) p_{\nu_{m(\nu)}-2}(\lambda)\rangle_{\beta,n} \\ &\quad + \sum_{k=1}^{m(\nu)-1} \nu_k \langle p_{(\nu_r)_{r\neq k, m(\nu)}}(\lambda) p_{\nu_k+\nu_{m(\nu)}-2}(\lambda)\rangle_{\beta,n},\end{aligned} \tag{3.2}$$

where $p_0(\lambda) = n$. In particular, for q even,

$$\langle p_q(\lambda)\rangle_{\beta,n} = \frac{\beta}{2}\sum_{i=1}^{q-1}\langle p_{i-1}(\lambda)p_{q-1-i}(\lambda)\rangle_{\beta,n} + \Big(1 - \frac{\beta}{2}\Big)(q-1)\langle p_{q-2}(\lambda)\rangle_{\beta,n},$$

and for ν with $\nu_{m(\nu)} = 1$,

$$\langle p_\nu(\lambda)\rangle_{\beta,n} = \sum_{k=1}^{m(\nu)-1} \nu_k\langle p_{(\nu_r)_{r\neq k,m(\nu)}}(\lambda)p_{\nu_k-1}(\lambda)\rangle_{\beta,n}.$$

The recurrence (3.2) *and the initial condition $p_0(\lambda) = n$ determine all the moments $\langle p_\nu(\lambda)\rangle_{\beta,n}$.*

Proof. Note that (3.2) determines the moments $\langle p_\nu(\lambda)\rangle_{\beta,n}$ because on the left-hand side one has a degree $|\nu|$, and on the right-hand side all the terms have a degree $|\nu| - 2$. For a proof of (3.2) for $\beta > 0$, see Lemma 4.13 in [LC09] and Section 4.1.1 in [EKR18]. For $\beta \in (-2/n, 0)$, the proof works the same, with some care about the divergences in the density. Alternatively, on can use the analiticity in β to extend to $\beta \in (-\frac{2}{n}, 0)$. □

Next are some elementary properties of GβE, which follow from the form of the density (3.1).

Proposition 3.2. *The following holds.*

1. *For every $\beta > -2/n$, $\frac{1}{\sqrt{n}}p_1(\lambda)$ under GβE has for distribution $\mathcal{N}(0,1)$.*
2. *For every $\beta > -2/n$, $p_2(\lambda)/2$ under GβE has for distribution* $\mathrm{Gamma}(d(\beta,n)/2, 1)$.
3. *$p_1(\lambda)$ and $\lambda - \frac{1}{n}p_1(\lambda)$ under GβE are independent.*
4. *$\frac{1}{2}\big(p_2(\lambda) - \frac{1}{n}p_1(\lambda)^2\big) = \frac{1}{2}p_2\big(\lambda - \frac{1}{n}p_1(\lambda)\big)$ under GβE has for distribution* $\mathrm{Gamma}((d(\beta,n) - 1)/2, 1)$.

Next is an embryonic version of the BFS-Dynkin isomorphism (Theorem (2.1)) for the GβE. One should imagine that the state space is reduced to one vertex, and a particle on it gets killed at an exponential time.

Proposition 3.3. *Let $\beta > -2/n$. The following holds.*

(1) *Let $a \geqslant 0$. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $\langle |h(\lambda)| \rangle_{\beta,n} < +\infty$. Assume that h is a-homogeneous, that is to say $h(s\lambda) = s^a h(\lambda)$ for every $s > 0$. Let $F : [0, +\infty) \to \mathbb{R}$ be a bounded measurable function. Let θ be a r.v. with distribution* $\mathrm{Gamma}((d(\beta, n) + a)/2, 1)$. *Then*

$$\langle h(\lambda) F(p_2(\lambda)/2) \rangle_{\beta,n} = \langle h(\lambda) \rangle_{\beta,n} \mathbb{E}[F(\theta)]. \tag{3.3}$$

(2) *In particular, let ν be a finite family of positive integers such that $|\nu|$ is even. Let $\mathcal{T}_1, \dots, \mathcal{T}_{|\nu|/2}$ be an i.i.d. family of exponential times of mean 1, independent of the GβE. Then*

$$\langle p_\nu(\lambda) F(p_2(\lambda)/2) \rangle_{\beta,n} = \langle p_\nu \rangle_{\beta,n} \mathbb{E}\big[\langle F(p_2(\lambda)/2 + \mathcal{T}_1 + \dots + \mathcal{T}_{|\nu|/2}) \rangle_{\beta,n}\big].$$

Proof. (1) clearly implies (2). It is enough to check (3.3) for F of form $F(t) = e^{-Kt}$, with $K > 0$. Then

$$\begin{aligned}
\langle h(\lambda) e^{-\frac{1}{2}K p_2(\lambda)} \rangle_{\beta,n} &= \frac{1}{Z_{\beta,n}} \int_{\mathbb{R}^n} h(\lambda) |D(\lambda)|^\beta e^{-\frac{1}{2}(K+1)p_2(\lambda)} \prod_{j=1}^n d\lambda_j \\
&= \frac{(K+1)^{-\frac{n}{2}}}{Z_{\beta,n}} \int_{\mathbb{R}^n} h((K+1)^{-\frac{1}{2}}\tilde{\lambda}) |D((K+1)^{-\frac{1}{2}}\tilde{\lambda})|^\beta e^{-\frac{1}{2}p_2(\tilde{\lambda})} \prod_{j=1}^n d\tilde{\lambda}_j \\
&= (K+1)^{-\frac{1}{2}\left(n + n(n-1)\frac{\beta}{2} + a\right)} \langle h(\tilde{\lambda}) \rangle_{\beta,n},
\end{aligned}$$

where on the second line we used the change of variables $\tilde{\lambda} = (K+1)^{\frac{1}{2}}\lambda$, and on the third line the homogeneity. Further,

$$(K+1)^{-\frac{1}{2}\left(n + n(n-1)\frac{\beta}{2} + a\right)} = \mathbb{E}[e^{-K\theta}]. \qquad \square$$

4. A recurrence on formal polynomials

We consider a family of formal commuting polynomial variables $(\mathsf{Y}_{kk}, \check{\mathsf{Y}}_{k\,k+1})_{k \geqslant 1}$. We will consider finite families of positive integers $\nu = (\nu_1, \nu_2, \dots, \nu_{m(\nu)})$ with $|\nu|$ even. The order of the ν_k will matter. We want to construct a family of polynomials $P_{\nu,\beta,n}$ with parameters ν,β and n, where $P_{\nu,\beta,n}$ has for variables $(\mathsf{Y}_{kk})_{1 \leqslant k \leqslant m(\nu)}$ and $(\check{\mathsf{Y}}_{k-1\,k})_{2 \leqslant k \leqslant m(\nu)}$. To simplify the notations, we will drop the subscripts β, n and just write P_ν. The polynomials P_ν will appear in the expression of the symmetric moments of β-Dyson's Brownian motion and the corresponding BFS-Dynkin type isomorphism. We will give a recursive definition of the P_ν-s.

The solutions to the recurrence (3.2), which for $\beta \in (-2/n, +\infty)$ are the moments $\langle p_\nu(\lambda) \rangle_{\beta,n}$, will be denoted $c(\nu, \beta, n)$. By convention, $c((0), \beta, n) = n$ and $c(\varnothing, \beta, n) = 1$. For $k \geqslant k' \in \mathbb{N}$, $[\![k, k']\!]$ will denote the interval of integers

$$[\![k, k']\!] = \{k, k+1, \dots, k'\}.$$

For $k \geqslant 1$ and P a polynomial, $P^{k\leftarrow}$ will denote the polynomial in the variables $(\mathsf{Y}_{k'k'})_{1 \leqslant k' \leqslant k}$ and $(\check{\mathsf{Y}}_{k'-1\,k'})_{2 \leqslant k' \leqslant k}$, obtained from P by replacing each variable $\mathsf{Y}_{k'k'}$ with $k' \geqslant k+1$ by the variable Y_{kk}, and each variable $\check{\mathsf{Y}}_{k'-1\,k'}$ with $k' \geqslant k+1$ by the constant 1. Note that $P_\nu^{m(\nu)\leftarrow} = P_\nu$ and that $P_\nu^{1\leftarrow}$ is an univariate polynomial in Y_{11}. For Y a formal polynomial variable, $\deg_{\mathsf{Y}}$ will denote the partial degree in Y.

Definition 4.1. *The family of polynomials $(P_\nu)_{|\nu| \text{ even}}$ is defined by the following.*

(1) $P_\nu^{1\leftarrow} = c(\nu, \beta, n) \mathsf{Y}_{11}^{|\nu|/2}$.

(2) *If $m(\nu) \geqslant 2$, then for every $k \in [\![2, m(\nu)]\!]$,*

$$
\begin{aligned}
(4.1) \qquad \frac{\partial}{\partial \mathsf{Y}_{kk}} P_{\nu}^{k\leftarrow} \quad = \quad & \frac{\beta}{2} \sum_{\substack{k \leqslant k' \leqslant m(\nu) \\ \nu_{k'} > 2}} \frac{\nu(k')}{2} \sum_{i=2}^{\nu_{k'}-2} P^{k\leftarrow}_{((\nu_r)_{r \neq k'}, i-1, \nu_{k'}-1-i)} \\
& + \frac{\beta}{2} n \sum_{\substack{k \leqslant k' \leqslant m(\nu) \\ \nu_{k'} > 2}} \nu(k') P^{k\leftarrow}_{((\nu_r)_{r \neq k'}, \nu_{k'}-2)} \\
& + \frac{\beta}{2} n^2 \sum_{\substack{k \leqslant k' \leqslant m(\nu) \\ \nu_{k'} = 2}} P^{k\leftarrow}_{(\nu_r)_{r \neq k'}} \\
& + \Big(1 - \frac{\beta}{2}\Big) \sum_{\substack{k \leqslant k' \leqslant m(\nu) \\ \nu_{k'} > 2}} \frac{\nu_{k'}(\nu_{k'}-1)}{2} P^{k\leftarrow}_{((\nu_r)_{r \neq k'}, \nu_{k'}-2)} \\
& + \Big(1 - \frac{\beta}{2}\Big) n \sum_{\substack{k \leqslant k' \leqslant m(\nu) \\ \nu_{k'} = 2}} P^{k\leftarrow}_{(\nu_r)_{r \neq k'}} \\
& + \sum_{\substack{k \leqslant k' < k'' \leqslant m(\nu) \\ \nu_{k'} + \nu_{k''} > 2}} \nu_{k'} \nu_{k''} P^{k\leftarrow}_{((\nu_r)_{r \neq k', k''}, \nu_{k'} + \nu_{k''} - 2)} \\
& + n \sum_{\substack{k \leqslant k' < k'' \leqslant m(\nu) \\ \nu_{k'} = \nu_{k''} = 1}} P^{k\leftarrow}_{(\nu_r)_{r \neq k', k''}}.
\end{aligned}
$$

If $k = m(\nu)$, then the last two lines of (4.1) are zero.

(3) *If $m(\nu) \geqslant 2$, then for every $k \in [\![2, m(\nu)]\!]$, the polynomial $P_{\nu}^{k\leftarrow}(\mathsf{Y}_{kk} = 0)$ (i.e. the part of $P_{\nu}^{k\leftarrow}$ that does not contain terms in Y_{kk}) is such that each of its monomials Q satisfies*

$$\deg_{\check{\mathsf{Y}}_{k-1\,k}} Q = \sum_{k \leqslant r \leqslant m(\nu)} \nu_r.$$

Proposition 4.2. *Definition 4.1 uniquely defines a family of polynomials $(P_\nu)_{|\nu| \text{ even}}$. Moreover, the following properties hold.*

(1) *For every Q monomial of P_ν and every $k \in [\![2, m(\nu)]\!]$,*

$$\deg_{\check{\mathsf{Y}}_{k-1\,k}} Q + 2 \sum_{k \leqslant k' \leqslant m(\nu)} \deg_{\mathsf{Y}_{k'k'}} Q = \sum_{k \leqslant k' \leqslant m(\nu)} \nu_{k'},$$

and

$$2 \sum_{1 \leqslant k' \leqslant m(\nu)} \deg_{\mathsf{Y}_{k'k'}} Q = |\nu|.$$

(2) *For every $k \in [\![1, m(\nu)]\!]$ and every permutation σ of $[\![k, m(\nu)]\!]$,*

$$P^{k\leftarrow}_{(\nu_r)_{1 \leqslant r \leqslant k-1}, (\nu_{\sigma(r)})_{k \leqslant r \leqslant m(\nu)}} = P_{\nu}^{k\leftarrow}.$$

Proof. The fact that the polynomials P_ν are well defined can be proved by induction on $|\nu|/2$.

For $|\nu|/2 = 1$, there are only two polynomials, $P_{(2)}$ and $P_{(1,1)}$. According to the condition (1),

$$P_{(2)} = c((2), \beta, n)\mathsf{Y}_{11} = d(\beta, n)\mathsf{Y}_{11} = \Big(\frac{\beta}{2} n^2 + \Big(1 - \frac{\beta}{2}\Big) n\Big) \mathsf{Y}_{11}.$$

The conditions (2) and (3) do not apply for $P_{(2)}$. For $P_{(1,1)}$, according to the condition (2),

$$\frac{\partial}{\partial \mathsf{Y}_{22}} P_{(1,1)} = 0.$$

Thus, $P_{(1,1)}$ contains no terms in Y_{22}. According to the condition (3), $P_{(1,1)} = \widetilde{P}(\mathsf{Y}_{11})\check{\mathsf{Y}}_{12}$. From the condition (1) we further get

$$P_{(1,1)} = c((1,1),\beta,n)\mathsf{Y}_{11}\check{\mathsf{Y}}_{12} = n\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}.$$

The induction step works as follows. Assume $|\nu|/2 \geqslant 2$. The right hand side of (4.1) involves only families of integers $\tilde{\nu}$ with $|\tilde{\nu}| = |\nu|-2$. According to the induction hypotheses, $\dfrac{\partial}{\partial \mathsf{Y}_{kk}} P_\nu^{k\leftarrow}$ is uniquely determined for every $k \in [\![2, m(\nu)]\!]$. Thus, for every $k \in [\![2, m(\nu)]\!]$, $P_\nu^{k\leftarrow} - P_\nu^{k\leftarrow}(\mathsf{Y}_{kk} = 0)$ is uniquely determined. By the condition (1), $P_\nu^{1\leftarrow}$ is also uniquely determined. By the condition (3), for every $k \in [\![2, m(\nu)]\!]$,

$$P_\nu^{k\leftarrow}(\mathsf{Y}_{kk} = 0) = \big(P_\nu^{k-1\leftarrow} - \big(P_\nu^{k\leftarrow} - P_\nu^{k\leftarrow}(\mathsf{Y}_{kk} = 0)\big)^{k-1\leftarrow}\big)\check{\mathsf{Y}}_{k-1\,k}^{|(\nu_r)_{k\leqslant r\leqslant m(\nu)}|}.$$

Thus, all the polynomials $(P_\nu^{k\leftarrow})_{1\leqslant k\leqslant m(\nu)}$ are uniquely determined, with consistency by the $P \mapsto P^{k\leftarrow}$ operations. Finally, $P_\nu = P_\nu^{m(\nu)\leftarrow}$.

The properties (1) and (2) again follow easily by induction on $|\nu|/2$. $\square$

Next are the expressions for $P_{(1,1,\dots,1)}$ and $P_{(2,2,\dots,2)}$ that can be proved by induction.

Proposition 4.3. *Let $m \in \mathbb{N}\backslash\{0\}$. Let $\mathsf{M} = (\mathsf{M}_{kk'})_{1\leqslant k,k'\leqslant m}$ be the formal symmetric matrix with entries given by*

$$\mathsf{M}_{kk} = \mathsf{Y}_{kk}, \qquad \text{for } k < k',\ \mathsf{M}_{kk'} = \mathsf{M}_{k'k} = \mathsf{Y}_{kk} \prod_{k+1\leqslant r\leqslant k'} \check{\mathsf{Y}}_{r-1\,r}. \tag{4.2}$$

The following holds.

(1) *Assume m is even, and let $\nu = (1,1,\dots,1)$, where 1 appears m times. Then $P_{(1,1,\dots,1)}$ satisfies the Wick's rule for Gaussians:*

$$P_{(1,1,\dots,1)} = n^{\frac{m}{2}} \sum_{\substack{(\{a_i,b_i\})_{1\leqslant i\leqslant m/2} \\ \text{partition in pairs} \\ \text{of } [\![1,m]\!]}} \mathsf{M}_{a_i b_i},$$

where the sum runs over the $m!/(2^{\frac{m}{2}}(m/2)!)$ partitions in pairs.

(2) *Let $\nu = (2,2,\dots,2)$, where 2 appears m times. Then*

$$P_{(2,2,\dots,2)} = 2^m \operatorname{Perm}_{d(\beta,n)/2}(\mathsf{M}).$$

For other examples of P_ν, see the Appendix.

Next we observe that for $\beta = -\frac{2}{n}$, the polynomials P_ν give the moments of the stochastic processes $(\phi_{\mathbb{R}_+}(x))_{x\geqslant 0}$ and $(\phi_K(x))_{x\in\mathbb{R}}$ introduced in Section 2, which are Gaussian.

Proposition 4.4. *Let $n \geqslant 1$. Let $K > 0$. Let ν be a finite family of positive integers with $|\nu|$ even. Let $x_1 \leqslant \dots \leqslant x_{m(\nu)}$ be $m(\nu)$ points in $(0,+\infty)$, resp. in $\mathbb{R}$. Then*

$$P_{\nu,\beta=-\frac{2}{n},n}((\mathsf{Y}_{kk} = 2x_k)_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = 1)_{2\leqslant k\leqslant m(\nu)}) = n^{m(\nu)-|\nu|/2}\mathbb{E}\Big[\prod_{k=1}^{m(\nu)} \phi_{\mathbb{R}_+}(x_k)^{\nu_k}\Big],$$

resp.

$$P_{\nu,\beta=-\frac{2}{n},n}((\mathsf{Y}_{kk} = 1/\sqrt{2K})_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = e^{-\sqrt{2K}(x_k-x_{k-1})})_{2\leqslant k\leqslant m(\nu)})$$
$$= n^{m(\nu)-|\nu|/2}\mathbb{E}\Big[\prod_{k=1}^{m(\nu)} \phi_K(x_k)^{\nu_k}\Big].$$

That is to say, the variables Y_{kk} are replaced by $G_{\mathbb{R}_+}(x_k,x_k)$, resp. $G_K(x_k,x_k)$, and the variables $\check{\mathsf{Y}}_{k-1\,k}$ by $G_{\mathbb{R}_+}(x_{k-1},x_k)/G_{\mathbb{R}_+}(x_{k-1},x_{k-1})$, resp. $G_K(x_{k-1},x_k)/G_K(x_{k-1},x_{k-1})$.

Proof. First, one can check that

$$c\Big(\nu, \beta = -\frac{2}{n}, n\Big) = n^{m(\nu)-|\nu|/2}\frac{|\nu|!}{2^{|\nu|/2}(|\nu|/2)!}. \tag{4.3}$$

This follows from Proposition (3.2). The key point is that

$$d\Big(\beta = -\frac{2}{n}, n\Big) = 1.$$

Given ν a finite family of positive integers, let $\mathbf{k}_\nu : [\![1, |\nu|]\!] \mapsto [\![1, m(\nu)]\!]$ be the function such that

$$\mathbf{k}_\nu^{-1}(1) = [\![1, \nu_1]\!], \qquad \text{for } k' \in [\![2, m(\nu)]\!],\ \mathbf{k}_\nu^{-1}(k') = [\![\nu_1 + \cdots + \nu_{k'-1} + 1, \nu_1 + \cdots + \nu_{k'}]\!]. \tag{4.4}$$

Further, let $(\widetilde{P}_\nu)_{|\nu| \text{ even}}$ be the following formal polynomials:

$$\widetilde{P}_\nu = n^{m(\nu)-|\nu|/2} \sum_{\substack{(\{a_i,b_i\})_{1\leqslant i\leqslant |\nu|/2} \\ \text{partition in pairs} \\ \text{of } [\![1,|\nu|]\!]}} \mathsf{M}_{\mathbf{k}_\nu(a_i)\mathbf{k}_\nu(b_i)},$$

where the $\mathsf{M}_{kk'}$ are given by (4.2). To conclude, we need only to check that $\widetilde{P}_\nu = P_{\nu,\beta=-\frac{2}{n},n}$ for all ν with $|\nu|$ even. From (4.3) follows that the $\widetilde{P}_\nu$ satisfy the condition (1) in Definition 4.1. The condition (3) is immediate since $\widetilde{P}_\nu^{k\leftarrow}(\mathsf{Y}_{kk} = 0)$ corresponds to the partitions in pairs where each element of $\mathbf{k}_\nu^{-1}([\![k, m(\nu)]\!])$ is paired with an element of $\mathbf{k}_\nu^{-1}([\![1, k-1]\!])$. One can further check the recurrence (4.1), and this amounts to counting the pairs in $\mathbf{k}_\nu^{-1}([\![k, m(\nu)]\!])$. $\square$

5. Isomorphisms for β-Dyson's Brownian motion

5.1. β-Dyson's Brownian motions and the occupation fields of 1D Brownian loop-soups.

For references on β-Dyson's Brownian motion, see [Dys62, Cha92, RS93, CL97, CL07], Chapter 9 in [Meh04] and Section 4.3 in [AGZ09]. Let $\beta \geqslant 0$ and $n \geqslant 2$. The β-Dyson's Brownian motion is the process $(\lambda(x) = (\lambda_1(x), \dots, \lambda_n(x)))_{x\geqslant 0}$ with $\lambda_1(x) \geqslant \cdots \geqslant \lambda_n(x)$, satisfying the SDE

$$d\lambda_j(x) = \sqrt{2}dW_j(x) + \beta \sum_{j'\neq j} \frac{dx}{\lambda_j(x) - \lambda_{j'}(x)}, \tag{5.1}$$

with initial condition $\lambda(0) = 0$. $(dW_j(x))_{1\leqslant j\leqslant n}$ are independent white noises. Since we will be interested in isomorphisms with Brownian local times, the variable x corresponds here to a one-dimensional spatial variable rather then a time variable. For every $x > 0$, $\lambda(x)/\sqrt{G_{\mathbb{R}_+}(x,x)} = \lambda(x)/\sqrt{2x}$, is distributed, up to a reordering of the $\lambda_j(x)$-s, as a GβE (3.1). For β equal to $1, 2$ resp. 4, $(\lambda(x))_{x\geqslant 0}$ is the diffusion of eigenvalues in a Brownian motion on the space of real symmetric, complex Hermitian, resp. quaternionic Hermitian matrices. For $\beta \geqslant 1$, there is no collision between the $\lambda_j(x)$-s, and for $\beta \in [0, 1)$ two consecutive $\lambda_j(x)$-s can collide, but there is no collision of three or more particles [CL07]. Note that for $\beta > 0$ and $j \in [\![2, n]\!]$, $(\lambda_j(x) - \lambda_{j-1}(x))/2$ behaves near level 0 like a Bessel process of dimension $\beta + 1$ reflected at level 0, and since $\beta + 1 > 1$, the complication with the principal value and the local time at zero does not occur; see Chapter 10 in [Yor97]. In particular, each $(\lambda_j(x))_{x\geqslant 0}$ is a semimartingale. For $\beta = 0$, $(\lambda(x)/\sqrt{2})_{x\geqslant 0}$ is just a reordered family of n i.i.d. standard Brownian motions.

Remark 5.1. *We restrict to $\beta \geqslant 0$ because the case $\beta < 0$ has not been considered in the literature. The problem is the extension of the process after a collision of $\lambda_j(x)$-s. The collision of three or more particles, including all the n together for $\beta < -\frac{2(n-3)}{n(n-1)}$, is no longer excluded. However, we believe that the β-Dyson's Brownian motion can be defined for all $\beta > -\frac{2}{n}$. This* *is indeed the case if $n = 2$. One can use the reflected Bessel processes for that. Let $(\rho(x))_{x\geqslant 0}$ be the Bessel process of dimension $\beta + 1$, reflected at level 0, satisfying away from 0 the SDE*

$$d\rho(x) = dW(x) + \frac{\beta}{2\rho(x)}dx,$$

with $\rho(0) = 0$. The reflected version is precisely defined for $\beta > -1 = \frac{-2}{2}$; see Section XI.1 in [RY99] *and Section 3 in* [Law19]*. Let $(\widetilde{W}(x))_{x\geqslant 0}$ be a standard Brownian motion started from 0, independent from $(W(x))_{x\geqslant 0}$ Then, for $n = 2$ one can construct the β-Dyson's Brownian motion as*

$$\lambda_1(x) = \widetilde{W}(x) + \rho(x), \qquad \lambda_2(x) = \widetilde{W}(x) - \rho(x). \tag{5.2}$$

Next are some simple properties of the β-Dyson's Brownian motion.

Proposition 5.2. *The following holds.*

1. *The process $\big(\frac{1}{\sqrt{n}}p_1(\lambda(x))\big)_{x\geqslant 0}$ has same law as $\phi_{\mathbb{R}_+}$.*
2. *The process $(\frac{1}{2}p_2(\lambda(x)))_{x\geqslant 0}$ is a square Bessel process of dimension $d(\beta, n)$ started from 0.*
3. *The processes $(p_1(\lambda(x)))_{x\geqslant 0}$ and $\big(\lambda(x) - \frac{1}{n}p_1(\lambda(x))\big)_{x\geqslant 0}$ are independent.*
4. *The process $\big(\frac{1}{2}\big(p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2\big)\big)_{x\geqslant 0}$ is a square Bessel process of dimension $d(\beta, n) - 1$ started from 0.*

Proof. With Itô's formula, we get

$$dp_1(\lambda(x)) = \sqrt{2}\sum_{j=1}^{n} dW_j(x),$$

$$d\frac{1}{2}p_2(\lambda(x)) = 2\sum_{j=1}^{n}\frac{\lambda_j(x)}{\sqrt{2}}dW_j(x) + d(\beta, n)dx,$$

$$d\frac{1}{2}\Big(p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2\Big) = 2\sum_{j=1}^{n}\frac{\lambda_j(x) - \frac{1}{n}p_1(\lambda(x))}{\sqrt{2}}dW_j(x) + (d(\beta, n) - 1)dx, \tag{5.3}$$

where the points $x \in \mathbb{R}_+$ for which $\lambda_j(x) = \lambda_{j-1}(x)$ for some $j \in [\![2, n]\!]$ can be neglected. This gives (1), (2) and (4) since the processes

$$d\widetilde{W}(x) = \sum_{j=1}^{n}\frac{\lambda_j(x)}{\sqrt{p_2(\lambda(x))}}dW_j(x), \quad \widetilde{W}(0) = 0,$$

and

$$d\breve{W}(x) = \sum_{j=1}^{n}\frac{\lambda_j(x) - \frac{1}{n}p_1(\lambda(x))}{\sqrt{p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2}}dW_j(x), \quad \breve{W}(0) = 0,$$

are both standard Brownian motions. Again, one can neglect the points $x \in \mathbb{R}_+$ where $p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2 = 0$, which only occur for $n = 2$.

For (3), we have that

$$\begin{aligned} d\Big(\lambda_j(x) - \frac{1}{n}p_1(\lambda(x))\Big) &= \sqrt{2}d\Big(W_j(x) - \frac{1}{n}p_1(W(x))\Big) \\ &\quad + \beta\sum_{j'\neq j}\frac{dx}{\big(\lambda_j(x) - \frac{1}{n}p_1(\lambda(x))\big) - \big(\lambda_{j'}(x) - \frac{1}{n}p_1(\lambda(x))\big)}, \end{aligned}$$

where

$$p_1(W(x)) = \sum_{j'=1}^{n} W_{j'}(x).$$

The Brownian motion $p_1(W) = \frac{1}{\sqrt{2}}p_1(\lambda)$ is independent from the family of Brownian motions $\big(W_j - \frac{1}{n}p_1(W)\big)_{1\leqslant j\leqslant n}$. Further, the measurability of $\big(\lambda_j - \frac{1}{n}p_1(\lambda)\big)_{1\leqslant j\leqslant n}$ with respect to $\big(W_j - \frac{1}{n}p_1(W)\big)_{1\leqslant j\leqslant n}$ follows from the pathwise uniqueness of the solution to (5.1); see Theorem 3.1 in [CL97]. □

By combining Proposition 5.2 with Theorem 2.3, we get a first relation between the β-Dyson's Brownian motion and 1D Brownian local time. Compare it with Le Jan's isomorphism (2.3).

Corollary 5.3. *The process $\big(\frac{1}{2}p_2(\lambda(x))\big)_{x\geqslant 0}$ has same law as the occupation field $(L^x(\mathcal{L}^{\alpha}_{\mathbb{R}_+}))_{x\geqslant 0}$ of a 1D Brownian loop-soup $\mathcal{L}^{\alpha}_{\mathbb{R}_+}$, with the correspondence*

$$2\alpha = d(\beta, n) = n + n(n-1)\frac{\beta}{2}. \tag{5.4}$$

Further, let $\mathcal{L}^{\alpha-\frac{1}{2}}_{\mathbb{R}_+}$ and $\widetilde{\mathcal{L}}^{\frac{1}{2}}_{\mathbb{R}_+}$ be two independent 1D Brownian loop-soups, α still given by (5.4). Then, one has the following identity in law between pairs of processes:

$$\Big(\frac{1}{2}\Big(p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2\Big), \frac{1}{2n}p_1(\lambda(x))^2\Big)_{x\geqslant 0} \overset{(law)}{=} (L^x(\mathcal{L}^{\alpha-\frac{1}{2}}_{\mathbb{R}_+}), L^x(\widetilde{\mathcal{L}}^{\frac{1}{2}}_{\mathbb{R}_+}))_{x\geqslant 0}.$$

5.2. **Symmetric moments of β-Dyson's Brownian motion.** We will denote by $\langle\cdot\rangle^{\mathbb{R}_+}_{\beta,n}$ the expectation with respect the β-Dyson's Brownian motion (5.1).

Proposition 5.4. *Let ν be a finite family of positive integers, with $|\nu|$ even. Let $P_\nu = P_{\nu,\beta,n}$ be the polynomial given by Definition 4.1. Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{m(\nu)} \in \mathbb{R}_+$. Then,*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k))\Big\rangle^{\mathbb{R}_+}_{\beta,n} = P_\nu((\Upsilon_{kk} = 2x_k)_{1\leqslant k\leqslant m(\nu)}, (\check{\Upsilon}_{k-1\,k} = 1)_{2\leqslant k\leqslant m(\nu)}).$$

We start by some lemmas.

Lemma 5.5. *Let $q \geqslant 3$. Then*

$$\begin{aligned} dp_q(\lambda(x)) &= q\sqrt{2}\sum_{j=1}^{n}\lambda_j(x)^{q-1}dW_j(x) + \frac{\beta}{2}q\sum_{i=2}^{q-2}p_{i-1}(\lambda(x))p_{q-1-i}(\lambda(x))dx \\ &\quad + 2\frac{\beta}{2}nqp_{q-2}(\lambda(x))dx + \Big(1 - \frac{\beta}{2}\Big)q(q-1)p_{q-2}(\lambda(x))dx. \end{aligned}$$

Proof. By Itô's formula,

$$dp_q(\lambda(x)) = q\sqrt{2}\sum_{j=1}^{n}\lambda_j(x)^{q-1}dW_j(x) + q(q-1)p_{q-2}(\lambda(x))dx + \beta q\sum_{1\leqslant j<j'\leqslant n}\frac{\lambda_j(x)^{q-1} - \lambda_{j'}(x)^{q-1}}{\lambda_j(x) - \lambda_{j'}(x)}dx.$$

But

$$\sum_{1\leqslant j<j'\leqslant n}\frac{\lambda_j(x)^{q-1} - \lambda_{j'}(x)^{q-1}}{\lambda_j(x) - \lambda_{j'}(x)} = \sum_{1\leqslant j<j'\leqslant n}\sum_{r=0}^{q-2}\lambda_j(x)^r\lambda_{j'}(x)^{q-2-r} = \Big(n - \frac{q-1}{2}\Big)p_{q-2}(\lambda(x)) + \frac{1}{2}\sum_{i=2}^{q-2}p_{i-1}(\lambda(x))p_{q-1-i}(\lambda(x)).$$ □

Lemma 5.6. *Let $q, q' \geqslant 1$ with $q + q' > 2$. Then*

$$d\langle p_q(\lambda(x)), p_{q'}(\lambda(x))\rangle = 2qq'p_{q+q'-2}(\lambda(x))dx.$$

Moreover,

$$d\langle p_1(\lambda(x)), p_1(\lambda(x))\rangle = 2ndx.$$

Proof. This is a straightforward computation. $\square$

Lemma 5.7. *Let ν be a finite family of positive integers and let $q \geqslant 0$. Then the process*

$$\int_0^x p_\nu(\lambda(y)) \sum_{j=1}^n \lambda_j(y)^q dW_j(y) \tag{5.5}$$

is a martingale.

Proof. (5.5) is a local martingale. It actually has a locally $\mathbb{L}^1$ bounded quadratic variation:

$$\Big\langle \int_0^x p_\nu(\lambda(y))^2 p_{2q}(\lambda(y)) dy \Big\rangle_{\beta,n}^{\mathbb{R}_+} = \langle p_\nu(\lambda)^2 p_{2q}(\lambda)\rangle_{\beta,n} \int_0^x (2y)^{|\nu|+q} dy < +\infty$$

So it is a true martingale. $\square$

Proof of Proposition 5.4. The proof is done by induction on $|\nu|/2$.

The case $|\nu|/2 = 1$ corresponds to $\nu = (1,1)$ or $\nu = (2)$. These are treated by Proposition 4.3 and Proposition 5.2, and taking into account that the square Bessel processes are permanental fields.

Now consider the induction step. Assume $|\nu|/2 \geqslant 2$. Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{m(\nu)} \in \mathbb{R}_+$. For $k \in [\![1, m(\nu)]\!]$, $f_k(x)$ will be the function

$$f_k(x) := \Big\langle \prod_{k'=1}^{k-1} p_{\nu_{k'}}(\lambda(x_{k'})) \prod_{k'=k}^{m(\nu)} p_{\nu_{k'}}(\lambda(x)) \Big\rangle_{\beta,n}^{\mathbb{R}_+}.$$

We have that

$$f_1(x_1) = c(\nu,\beta,n)(2x_1)^{|\nu|/2} = P_\nu^{1\leftarrow}(\mathsf{Y}_{11} = 2x_1), \tag{5.6}$$

where for the second equality we applied the condition (1) in Definition 4.1. If $m(\nu) = 1$, there is nothing more to check. In the case $m(\nu) \geqslant 2$, we need only to check that for every $k \in [\![2, m(\nu)]\!]$ and every $x > x_{k-1}$,

$$\begin{aligned} \frac{d}{dx} f_k(x) &= \frac{\partial}{\partial x} P_\nu^{k\leftarrow}((\mathsf{Y}_{k'k'} = 2x_{k'})_{1\leqslant k'\leqslant k-1}, \mathsf{Y}_{kk} = 2x, (\check{\mathsf{Y}}_{k'-1\,k'} = 1)_{2\leqslant k'\leqslant k}) \\ &= 2\Big(\frac{\partial}{\partial \mathsf{Y}_{kk}} P_\nu^{k\leftarrow}\Big)((\mathsf{Y}_{k'k'} = 2x_{k'})_{1\leqslant k'\leqslant k-1}, \mathsf{Y}_{kk} = 2x, (\check{\mathsf{Y}}_{k'-1\,k'} = 1)_{2\leqslant k'\leqslant k}). \end{aligned} \tag{5.7}$$

Indeed, given (5.6), by applying (5.7) to $k = 2$, we further get

$$f_2(x_2) = P_\nu^{2\leftarrow}(\mathsf{Y}_{11} = 2x_1, \mathsf{Y}_{22} = 2x_2, \check{\mathsf{Y}}_{12} = 1),$$

and by successively applying (5.7) to $k = 3, \ldots, k = m(\nu)$, we at the end get

$$f_{m(\nu)}(x_{m(\nu)}) = P_\nu^{m(\nu)\leftarrow}((\mathsf{Y}_{k'k'} = 2x_{k'})_{1\leqslant k'\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k'-1\,k'} = 1)_{2\leqslant k'\leqslant m(\nu)}),$$

which is exactly what we want. To show (5.7), we proceed as follows. Let $(\mathcal{F}_x)_{x\geqslant 0}$ be the filtration of the Brownian motions $((W_j(x))_{1\leqslant j\leqslant n})_{x\geqslant 0}$. Then, for $x > x_{k-1}$,

$$f_k(x) = \Big\langle \prod_{k'=1}^{k-1} p_{\nu_{k'}}(\lambda(x_{k'})) \Big\langle \prod_{k'=k}^{m(\nu)} p_{\nu_{k'}}(\lambda(x)) \Big| \mathcal{F}_{x_{k-1}} \Big\rangle_{\beta,n}^{\mathbb{R}_+} \Big\rangle_{\beta,n}^{\mathbb{R}_+},$$

where $\langle \cdot | \mathcal{F}_{x_{k-1}} \rangle_{\beta,n}^{\mathbb{R}_+}$ denotes the conditional expectation. To express

$$\Big\langle \prod_{k'=k}^{m(\nu)} p_{\nu_{k'}}(\lambda(x)) \Big| \mathcal{F}_{x_{k-1}} \Big\rangle_{\beta,n}^{\mathbb{R}_+},$$

we apply Itô's formula to

$$\prod_{k'=k}^{m(\nu)} p_{\nu_{k'}}(\lambda(x)) - \Big\langle \prod_{k'=k}^{m(\nu)} p_{\nu_{k'}}(\lambda(x_{k-1})) \Big\rangle_{\beta,n}^{\mathbb{R}_+}.$$

The local martingale part is, according to Lemma 5.7 a true martingale, and thus gives a 0 conditional expectation. The bounded variation part is a linear combination of terms of form $p_{\tilde{\nu}}(\lambda(x))dx$, with

$$|\tilde{\nu}| = \Big(\sum_{k'=k}^{m(\nu)} \nu_{k'}\Big) - 2,$$

the exact expressions being given by Lemma 5.5 and Lemma 5.6. By comparing these expressions with the recurrence (4.1), and using the induction hypothesis at the step $|\nu|/2-1$, we get (5.7). $\square$

Note that in the proof above we did not use the condition (3) in Definition 4.1. It will be needed only later.

5.3. **BFS-Dynkin isomorphism for β-Dyson's Brownian motion.** We will denote by Υ a generic finite family of continuous paths on $\mathbb{R}$, $\Upsilon = (\gamma_1, \dots, \gamma_J)$, and $J(\Upsilon)$ will denote the size J of the family. We will consider finite Brownian measures on Υ where $J(\Upsilon)$ is not fixed but may take several values under the measure. Given $x \in \mathbb{R}$, $L^x(\Upsilon)$ will denote the sum of Brownian local times in x:

$$L^x(\Upsilon) = \sum_{i=1}^{J(\Upsilon)} L^x(\gamma_i).$$

$L(\Upsilon)$ will denote the occupation field $x \mapsto L^x(\Upsilon)$.

Given ν a finite family of positive integers with $|\nu|$ even and $0 < x_1 < x_2 < \cdots < x_{m(\nu)}$, $\mu_{\mathbb{R}_+}^{\nu,x_1,\dots,x_{m(\nu)}}(d\Upsilon)$ (also depending on β and n) will be the measure on finite families of continuous paths obtained by substituting in the polynomial $P_\nu = P_{\nu,\beta,n}$ for each variable Y_{kk} the measure $\mu_{\mathbb{R}_+}^{x_k,x_k}$, and for each variable $\check{\mathsf{Y}}_{k-1\,k}$ the measure $\check{\mu}_{\mathbb{R}_+}^{x_{k-1},x_k}$; see Section 2. Since we will deal with the functional $L(\Upsilon)$ under $\mu_{\mathbb{R}_+}^{\nu,x_1,\dots,x_{m(\nu)}}(d\Upsilon)$, the order of the Brownian measures in a product will not matter. For instance, for $\nu = (2,1,1)$ (see Appendix),

$$P_{(2,1,1)} = \Big(\frac{\beta}{2}n^3 + \Big(1-\frac{\beta}{2}\Big)n^2\Big)\mathsf{Y}_{11}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23} + 2n\mathsf{Y}_{11}^2\check{\mathsf{Y}}_{12}^2\check{\mathsf{Y}}_{23},$$

and

$$\begin{aligned}\mu_{\mathbb{R}_+}^{(2,1,1),x_1,x_2,x_3,x_4} &= \Big(\frac{\beta}{2}n^3 + \Big(1-\frac{\beta}{2}\Big)n^2\Big)\mu_{\mathbb{R}_+}^{x_1,x_1} \otimes \mu_{\mathbb{R}_+}^{x_2,x_2} \otimes \check{\mu}_{\mathbb{R}_+}^{x_2,x_3} \\ &\quad + 2n\mu_{\mathbb{R}_+}^{x_1,x_1} \otimes \mu_{\mathbb{R}_+}^{x_1,x_1} \otimes \check{\mu}_{\mathbb{R}_+}^{x_1,x_2} \otimes \check{\mu}_{\mathbb{R}_+}^{x_1,x_2} \otimes \check{\mu}_{\mathbb{R}_+}^{x_2,x_3}.\end{aligned}$$

Next is a version of BFS-Dynkin isomorphism (Theorem (2.1)) for β-Dyson's Brownian motion.

Proposition 5.8. *Let ν be a finite family of positive integers, with $|\nu|$ even and let $0 < x_1 < x_2 < \cdots < x_{m(\nu)}$. Let F be a bounded measurable functional on $\mathcal{C}(\mathbb{R}_+)$. Then*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) F\Big(\frac{1}{2}p_2(\lambda)\Big)\Big\rangle_{\beta,n}^{\mathbb{R}_+} = \int_{\Upsilon} \Big\langle F\Big(\frac{1}{2}p_2(\lambda) + L(\Upsilon)\Big)\Big\rangle_{\beta,n}^{\mathbb{R}_+} \mu_{\mathbb{R}_+}^{\nu,x_1,\dots,x_{m(\nu)}}(d\Upsilon). \tag{5.8}$$

Remark 5.9. *In the limiting case when $x_k = x_{k-1}$ for some $k \in [\![2, m(\nu)]\!]$, $\check{\mathsf{Y}}_{k-1\,k}$ in P_ν has to be replaced by the constant 1 instead of a measure on Brownian paths.*

Remark 5.10. *For $\beta \in \{0,1,2,4\}$, (5.8) reduces to the Gaussian case of Theorem (2.1).*

We start by some intermediate lemmas. Recall that $(\mathcal{F}_x)_{x\geqslant 0}$ denotes the filtration of the Brownian motions $((W_j(x))_{1\leqslant j\leqslant n})_{x\geqslant 0}$ in (5.1). $\chi(x)$ will be a continuous non-negative function with compact support in $(0,+\infty)$. $u_{\chi\downarrow}$ will denote the unique solution to

$$\frac{1}{2}\frac{d^2}{dx}u = \chi u$$

which is positive non-increasing on $\mathbb{R}_+$, with $u_{\chi\downarrow}(0) = 1$. See Section 2.1 in [Lup18] for details. Then

$$u_{\chi\downarrow}(+\infty) = \lim_{x\to+\infty} u_{\chi\downarrow}(x) > 0.$$

Lemma 5.11. *Let $\mathcal{D}_\chi(+\infty)$ be the positive r.v.*

$$\mathcal{D}_\chi(+\infty) := u_{\chi\downarrow}(+\infty)^{-\frac{1}{2}d(\beta,n)} \exp\Big(-\frac{1}{2}\int_0^{+\infty} p_2(\lambda(y))\chi(y)dy\Big). \tag{5.9}$$

Then $\langle \mathcal{D}_\chi(+\infty)\rangle_{\beta,n}^{\mathbb{R}_+} = 1$. Moreover,

$$\begin{aligned}\mathcal{D}_\chi(x) &:= \langle \mathcal{D}_\chi(+\infty)|\mathcal{F}_x\rangle_{\beta,n}^{\mathbb{R}_+} \\ &= u_{\chi\downarrow}(x)^{-\frac{1}{2}d(\beta,n)} \exp\Big(-\frac{1}{2}\int_0^{x} p_2(\lambda(y))\chi(y)dy\Big)\exp\Big(\frac{1}{4}p_2(\lambda(x))\frac{u_{\chi\downarrow}'(x)}{u_{\chi\downarrow}(x)}\Big).\end{aligned} \tag{5.10}$$

Let

$$\mathcal{M}_\chi(x) := \frac{1}{\sqrt{2}}\int_0^x \frac{u_{\chi\downarrow}'(y)}{u_{\chi\downarrow}(y)}\sum_{j=1}^n \lambda_j(y)dW_j(y).$$

Then $(\mathcal{M}_\chi(x))_{x\geqslant 0}$ is a martingale with respect the filtration $(\mathcal{F}_x)_{x\geqslant 0}$ and for all $x \geqslant 0$,

$$\mathcal{D}_\chi(x) = \exp\Big(\mathcal{M}_\chi(x) - \frac{1}{2}\langle \mathcal{M}_\chi, \mathcal{M}_\chi\rangle(x)\Big).$$

Proof. (5.9) and (5.10) follow from the properties of square Bessel processes. See Theorem (1.7), Section XI.1 in [RY99]. $(\mathcal{M}_\chi(x))_{x\geqslant 0}$ is obviously a (true) martingale, as can be seen with the quadratic variation. Further,

$$d\Big(\frac{1}{4}p_2(\lambda(x))\frac{u_{\chi\downarrow}'(x)}{u_{\chi\downarrow}(x)}\Big) = d\mathcal{M}_\chi(x) + \frac{1}{2}p_2(\lambda(x))\chi(x)dx - \frac{1}{4}p_2(\lambda(x))\frac{u_{\chi\downarrow}'(x)^2}{u_{\chi\downarrow}(x)^2}dx + \frac{1}{2}d(\beta,n)\frac{u_{\chi\downarrow}'(x)}{u_{\chi\downarrow}(x)}dx,$$

and

$$d\frac{1}{2}\langle \mathcal{M}_\chi, \mathcal{M}_\chi\rangle(x) = \frac{1}{4}p_2(\lambda(x))\frac{u_{\chi\downarrow}'(x)^2}{u_{\chi\downarrow}(x)^2}dx.$$

Thus

$$d\Big(\mathcal{M}_\chi(x) - \frac{1}{2}\langle \mathcal{M}_\chi, \mathcal{M}_\chi\rangle(x)\Big) = d\log(\mathcal{D}_\chi(x)).$$

□

Lemma 5.12. *Let be $(\tilde\lambda(x) = (\tilde\lambda_1(x),\dots,\tilde\lambda_n(x)))_{x\geqslant 0}$ with $\tilde\lambda_1(x) \geqslant \cdots \geqslant \tilde\lambda_n(x)$, satisfying the SDE*

$$d\tilde\lambda_j(x) = \sqrt{2}dW_j(x) + \frac{u_{\chi\downarrow}'(x)}{u_{\chi\downarrow}(x)}\tilde\lambda_j(x)dx + \beta\sum_{j'\neq j}\frac{dx}{\tilde\lambda_j(x) - \tilde\lambda_{j'}(x)}, \tag{5.11}$$

with initial condition $\tilde\lambda(0) = 0$. Further consider a change of measure with density $\mathcal{D}_\chi(+\infty)$ (5.9) on the filtered probability space with filtration $(\mathcal{F}_x)_{x\geqslant 0}$. Then λ after the change of measure and $\tilde\lambda$ before the change of measure have same law.

Proof. The existence and uniqueness of strong solutions to (5.11) is given by Theorem 3.1 in [CL97]. The rest is a consequence of Girsanov's theorem; see Theorems (1.7) and (1.12), Section VIII.1, in [RY99]. Indeed,

$$d\langle W_j(x), \mathcal{M}_\chi(x)\rangle = \frac{1}{\sqrt{2}}\frac{u_{\chi\downarrow}'(x)}{u_{\chi\downarrow}(x)}\lambda_j(x)dx.$$

Thus, after the change of measure, the

$$W_j(x) - \frac{1}{\sqrt{2}}\int_0^x \frac{u_{\chi\downarrow}'(y)}{u_{\chi\downarrow}(y)}\lambda_j(y)dy$$

for $j \in [\![1, n]\!]$ are n i.i.d. standard Brownian motions. □

Let ψ_χ denote the following diffeomorphism of $\mathbb{R}_+$:

$$\psi_\chi(x) = \int_0^x \frac{dy}{u_{\chi\downarrow}(y)^2}.$$

ψ_χ^{-1} will denote the inverse diffeomorphism.

Lemma 5.13. *If $\tilde{\lambda}$ is a solution to the SDE* (5.11)*, then the process*

$$\Big(\frac{1}{u_{\chi\downarrow}(\psi_\chi^{-1}(x))}\tilde{\lambda}(\psi_\chi^{-1}(x))\Big)_{x\geqslant 0}$$

satisfies the SDE (5.1).

Proof. The process $\Big(\frac{1}{u_{\chi\downarrow}(x)}\tilde{\lambda}(x)\Big)_{x\geqslant 0}$ satisfies

$$d\Big(\frac{1}{u_{\chi\downarrow}(x)}\tilde{\lambda}_j(x)\Big) = \frac{\sqrt{2}}{u_{\chi\downarrow}(x)}dW_j(x) + \beta\sum_{j'\neq j}\frac{1}{u_{\chi\downarrow}(x)^{-1}\tilde{\lambda}_j(x) - u_{\chi\downarrow}(x)^{-1}\tilde{\lambda}_{j'}(x)}\frac{dx}{u_{\chi\downarrow}(x)^2}.$$

By further performing the change of variable given by ψ_χ, one gets (5.1). $\square$

In the sequel $(G_{\mathbb{R}_+,\chi}(x,y))_{x,y\geqslant 0}$ will denote the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \chi$ on $\mathbb{R}_+$ with condition 0 in 0. Then for $0 \leqslant x \leqslant y$,

$$G_{\mathbb{R}_+,\chi}(x,y) = 2u_{\chi\downarrow}(x)\psi_\chi(x)u_{\chi\downarrow}(y). \tag{5.12}$$

Indeed,

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\Big(2u_{\chi\downarrow}(x)\psi_\chi(x)u_{\chi\downarrow}(y)\Big) = \chi(y)\Big(2u_{\chi\downarrow}(x)\psi_\chi(x)u_{\chi\downarrow}(y)\Big),$$

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(2u_{\chi\downarrow}(x)\psi_\chi(x)u_{\chi\downarrow}(y)\Big) &= \frac{1}{2}\frac{\partial}{\partial x}\Big(2u'_{\chi\downarrow}(x)\psi_\chi(x)u_{\chi\downarrow}(y) + 2\frac{u_{\chi\downarrow}(y)}{u_{\chi\downarrow}(x)}\Big)\\ &= \chi(x)\Big(2u_{\chi\downarrow}(x)\psi_\chi(x)u_{\chi\downarrow}(y)\Big) + 0,\end{aligned}$$

and

$$\frac{1}{2}\Big(\frac{\partial}{\partial x}\Big|_{x=y} - \frac{\partial}{\partial y}\Big|_{y=x}\Big)\Big(2u_{\chi\downarrow}(x)\psi_\chi(x)u_{\chi\downarrow}(y)\Big) = 1.$$

Lemma 5.14. *Let $(\tilde{\lambda}(x))_{x\geqslant 0}$ be the solution to* (5.11) *with $\tilde{\lambda}(0) = 0$. Let ν be a finite family of positive integers, with $|\nu|$ even. Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{m(\nu)} \in \mathbb{R}_+$. Then,*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\tilde{\lambda}(x_k))\Big\rangle_{\beta,n}^{\mathbb{R}_+} = P_\nu((\mathsf{Y}_{kk} = G_{\mathbb{R}_+,\chi}(x_k,x_k))_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = u_{\chi\downarrow}(x_k)/u_{\chi\downarrow}(x_{k-1}))_{2\leqslant k\leqslant m(\nu)}).$$

Proof. From Lemma 5.13 and Proposition 5.4 it follows that

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\tilde{\lambda}(x_k))\Big\rangle_{\beta,n}^{\mathbb{R}_+} = \Big(\prod_{k=1}^{m(\nu)} u_{\chi\downarrow}(x_k)^{\nu_k}\Big)P_\nu((\mathsf{Y}_{kk} = 2\psi_\chi(x_k))_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = 1)_{2\leqslant k\leqslant m(\nu)}).$$

Further, let Q be a monomial of P_ν. One has to check that

$$\begin{aligned}&\Big(\prod_{k=1}^{m(\nu)} u_{\chi\downarrow}(x_k)^{\nu_k}\Big)Q((\mathsf{Y}_{kk} = 2\psi_\chi(x_k))_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = 1)_{2\leqslant k\leqslant m(\nu)})\\ &\qquad = Q((\mathsf{Y}_{kk} = G_{\mathbb{R}_+,\chi}(x_k,x_k))_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = u_{\chi\downarrow}(x_k)/u_{\chi\downarrow}(x_{k-1}))_{2\leqslant k\leqslant m(\nu)}).\end{aligned}$$

This follows from (5.12) and the point (1) in Proposition 4.2. $\square$

Proof of Proposition 5.8. It is enough to show (5.8) for functionals F of form

$$F((\ell(x))_{x\geqslant 0}) = \exp\Big(-\int_{\mathbb{R}_+} \ell(x)\chi(x)dx\Big),$$

where χ is a continuous non-negative function with compact support in $(0,+\infty)$. For such a χ,

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \exp\Big(-\frac{1}{2}\int_{\mathbb{R}_+} p_2(\lambda(x))\chi(x)dx\Big)\Big\rangle_{\beta,n}^{\mathbb{R}_+} =$$

$$\Big\langle \exp\Big(-\frac{1}{2}\int_{\mathbb{R}_+} p_2(\lambda(x))\chi(x)dx\Big)\Big\rangle_{\beta,n}^{\mathbb{R}_+} \Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\tilde{\lambda}(x_k))\Big\rangle_{\beta,n}^{\mathbb{R}_+},$$

where $\tilde{\lambda}$ is given by (5.11), with $\tilde{\lambda}(0) = 0$. The symmetric moments of $\tilde{\lambda}$ are given by Lemma 5.14. To conclude, we use that

$$\int_\gamma \exp\Big(-\int_{\mathbb{R}_+} L^z(\gamma)\chi(z)dz\Big)\mu_{\mathbb{R}_+}^{x,x}(d\gamma) = G_{\mathbb{R}_+,\chi}(x,x),$$

and for $0 < x < y$,

$$\int_\gamma \exp\Big(-\int_{\mathbb{R}_+} L^z(\gamma)\chi(z)dz\Big)\check{\mu}^{x,y}(d\gamma) = \frac{G_{\mathbb{R}_+,\chi}(x,y)}{G_{\mathbb{R}_+,\chi}(x,x)} = \frac{u_{\chi\downarrow}(y)}{u_{\chi\downarrow}(x)};$$

see Section 3.2 in [Lup18]. $\square$

5.4. **The stationary case.** In this section we consider the stationary β-Dyson's Brownian motion on the whole line and state the analogues of Propositions 5.2, 5.4 and 5.8 for it. The proofs are omitted, as they are similar to the previous ones. As previously, $n \geqslant 2$ and $\beta \geqslant 0$. Let $K > 0$. We consider the process the process $(\lambda(x) = (\lambda_1(x), \dots, \lambda_n(x)))_{x\in\mathbb{R}}$ with $\lambda_1(x) \geqslant \dots \geqslant \lambda_n(x)$, satisfying the SDE

$$d\lambda_j(x) = \sqrt{2}dW_j(x) - \sqrt{2K}\,\lambda_j(x) + \beta\sqrt{2K}\sum_{j'\neq j} \frac{dx}{\lambda_j(x) - \lambda_{j'}(x)}, \tag{5.13}$$

the dW_j, $1 \leqslant j \leqslant n$ being n i.i.d. white noises on $\mathbb{R}$, and λ being stationary, with $(2K)^{\frac{1}{4}}\lambda(x)$ being distributed according to (3.1) (up to reordering of the $\lambda_j(x)$).

Proposition 5.15. *The following holds.*

1. *The process* $\big(\frac{1}{\sqrt{n}}p_1(\lambda(x))\big)_{x\in\mathbb{R}}$ *has same law as* ϕ_K.
2. *Consider a 1D Brownian loop-soup* $\mathcal{L}_K^\alpha$, *with* α *given by* (5.4). *The process* $(\frac{1}{2}p_2(\lambda(x)))_{x\in\mathbb{R}}$ *has same law as the occupation field* $(L^x(\mathcal{L}_K^\alpha))_{x\in\mathbb{R}}$.
3. *The processes* $(p_1(\lambda(x)))_{x\in\mathbb{R}}$ *and* $\big(\lambda(x) - \frac{1}{n}p_1(\lambda(x))\big)_{x\in\mathbb{R}}$ *are independent.*
4. *Let* $\mathcal{L}_K^{\alpha-\frac{1}{2}}$ *and* $\tilde{\mathcal{L}}_K^{\frac{1}{2}}$ *be two independent 1D Brownian loop-soups,* α *given by* (5.4). *Then, one has the following identity in law between pairs of processes:*

$$\Big(\frac{1}{2}\Big(p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2\Big), \frac{1}{2n}p_1(\lambda(x))^2\Big)_{x\in\mathbb{R}} \overset{(law)}{=} (L^x(\mathcal{L}_K^{\alpha-\frac{1}{2}}), L^x(\tilde{\mathcal{L}}_K^{\frac{1}{2}}))_{x\in\mathbb{R}}.$$

$\langle\cdot\rangle_{\beta,n}^K$ will denote the expectation with respect the stationary β-Dyson's Brownian motion. Given ν a finite family of positive integers with $|\nu|$ even and $x_1 < x_2 < \dots < x_{m(\nu)} \in \mathbb{R}$, $\mu_K^{\nu,x_1,\dots,x_{m(\nu)}}(d\Upsilon)$ (also depending on β and n) will be the measure on finite families of continuous paths obtained by substituting in the polynomial $P_\nu = P_{\nu,\beta,n}$ for each variable Y_{kk} the measure $\mu_K^{x_k,x_k}$, and for each variable $\check{\mathsf{Y}}_{k-1\,k}$ the measure $\check{\mu}_K^{x_{k-1},x_k}$.

Proposition 5.16. *Let ν a finite family of positive integers with $|\nu|$ even. Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{m(\nu)} \in \mathbb{R}$. Then,*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \Big\rangle_{\beta,n}^{K} = P_\nu((\mathsf{Y}_{kk} = 1/\sqrt{2K})_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = e^{-\sqrt{2K}(x_k - x_{k-1})})_{2\leqslant k\leqslant m(\nu)}).$$

Further, let F be a bounded measurable functional on $\mathcal{C}(\mathbb{R})$. For $x_1 < x_2 < \cdots < x_{m(\nu)} \in \mathbb{R}$,

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) F\Big(\frac{1}{2}p_2(\lambda)\Big) \Big\rangle_{\beta,n}^{K} = \int_{\Upsilon} \Big\langle F\Big(\frac{1}{2}p_2(\lambda) + L(\Upsilon)\Big) \Big\rangle_{\beta,n}^{K} \mu_K^{\nu,x_1,\dots,x_{m(\nu)}}(d\Upsilon).$$

6. The case of general electrical networks: a construction for $n = 2$ and further questions

6.1. **Formal polynomials for $n = 2$.** In this section $n = 2$, and β is arbitrary, considered as a formal parameter. Note that $d(\beta, n = 2) = \beta + 2$. In Section 4 we introduced the formal commuting polynomial variables $(\mathsf{Y}_{kk})_{k\geqslant 1}$. Here we further consider the commuting variables $(\mathsf{Y}_{kk'})_{1\leqslant k<k'}$, and by convention set $\mathsf{Y}_{kk'} = \mathsf{Y}_{k'k}$ for $k' < k$. Given $\tilde{\nu} = (\tilde{\nu}_1, \dots, \tilde{\nu}_m)$ with $\tilde{\nu}_k \in \mathbb{N}$ (value 0 allowed), $\mathfrak{P}_{\tilde{\nu},\beta}$ will be the following multivariate polynomial in the variables $(\mathsf{Y}_{kk'})_{1\leqslant k\leqslant k'\leqslant m}$:

$$\mathfrak{P}_{\tilde{\nu},\beta} := \mathrm{Perm}_{\frac{\beta+1}{2}}((\mathsf{Y}_{f(i)f(j)})_{1\leqslant i,j\leqslant \tilde{\nu}_1+\cdots+\tilde{\nu}_m}),$$

where f is a map $f : [\![1, \tilde{\nu}_1 + \cdots + \tilde{\nu}_m]\!] \to [\![1, m]\!]$, such that for every $k \in [\![1, m]\!]$, $|f^{-1}(k)| = \tilde{\nu}_k$. It is clear that $\mathfrak{P}_{\tilde{\nu},\beta}$ does not depend on the particular choice of f. In case $\tilde{\nu}_1 = \cdots = \tilde{\nu}_m = 0$, by convention we set $\mathfrak{P}_{\tilde{\nu},\beta} = 1$. Given ν a finite family of positive integers with $|\nu|$ even, let $\mathbf{k}_\nu : [\![1, |\nu|]\!] \mapsto [\![1, m(\nu)]\!]$ be the map given by (4.4). Let $\mathcal{I}_\nu$ be the following set of subsets of $[\![1, |\nu|]\!]$:

$$\mathcal{I}_\nu := \{I \subseteq [\![1, |\nu|]\!] |\, \forall k \in [\![1, m(\nu)]\!], |\mathbf{k}_\nu^{-1}(k)\backslash I| \text{ is even }\},$$

where $|\cdot|$ denotes the cardinal. Note that necessarily, for every $I \in \mathcal{I}_\nu$, the cardinal $|I|$ is even. Let $\widehat{P}_{\nu,\beta}$ be the following multivariate polynomial in the variables $(\mathsf{Y}_{kk'})_{1\leqslant k\leqslant k'\leqslant m(\nu)}$:

$$\widehat{P}_{\nu,\beta} := \sum_{I\in\mathcal{I}_\nu} 2^{m(\nu)-|I|/2} \Big(\sum_{\substack{(\{a_i,b_i\})_{1\leqslant i\leqslant |I|/2} \\ \text{partition in pairs} \\ \text{of } I}} \prod_{i=1}^{|I|/2} \mathsf{Y}_{\mathbf{k}_\nu(a_i)\mathbf{k}_\nu(b_i)} \Big) \mathfrak{P}_{(\frac{1}{2}|\mathbf{k}_\nu^{-1}(k)\backslash I|)_{1\leqslant k\leqslant m(\nu)},\beta}.$$

By construction, for every Q monomial of $\widehat{P}_{\nu,\beta}$ and every $k \in [\![1, m(\nu)]\!]$,

$$2 \deg_{\mathsf{Y}_{kk}} Q + \sum_{\substack{1\leqslant k'\leqslant m(\nu) \\ k'\neq k}} \deg_{\mathsf{Y}_{kk'}} Q = \nu_k. \tag{6.1}$$

Proposition 6.1. *Let ν be finite family of positive integers with $|\nu|$ even. $P_{\nu,\beta,n=2}$ is obtained from $\widehat{P}_{\nu,\beta}$ by replacing each variable $\mathsf{Y}_{kk'}$ with $1 \leqslant k < k' \leqslant m(\nu)$ by $\mathsf{Y}_{kk} \prod_{k+1\leqslant r\leqslant k'} \check{\mathsf{Y}}_{r-1\,r}$:*

$$P_{\nu,\beta,n=2} = \widehat{P}_{\nu,\beta}\Big(\big(\mathsf{Y}_{kk'} = \mathsf{Y}_{kk} \prod_{k+1\leqslant r\leqslant k'} \check{\mathsf{Y}}_{r-1\,r}\big)_{1\leqslant k<k'\leqslant m(\nu)}\Big).$$

Proof. Let be

$$\widetilde{P}_{\nu,\beta} := \widehat{P}_{\nu,\beta}\Big(\big(\mathsf{Y}_{kk'} = \mathsf{Y}_{kk} \prod_{k+1\leqslant r\leqslant k'} \check{\mathsf{Y}}_{r-1\,r}\big)_{1\leqslant k<k'\leqslant m(\nu)}\Big).$$

We want to show the equality $\widetilde{P}_{\nu,\beta} = P_{\nu,\beta,n=2}$. Since a direct combinatorial proof would be a bit lengthy, we proceed differently. Let $\beta \geqslant 0$ and let $(\lambda(x) = (\lambda_1(x), \lambda_2(x)))_{x\geqslant 0}$ be the β-Dyson's

Brownian motion (5.1) in the case $n = 2$. We use its construction through (5.2). We claim that for $x_1, x_2, \dots, x_{m(\nu)} \in \mathbb{R}_+$,

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \Big\rangle_{\beta,n=2}^{\mathbb{R}_+} = \widehat{P}_{\nu,\beta}\big((\mathsf{Y}_{kk'} = G_{\mathbb{R}_+}(x_{k-1}, x_k))_{1\leqslant k\leqslant k'\leqslant m(\nu)}\big).$$

Indeed, in the expansion of

$$\Big(\widetilde{W}(x_k) + \rho(x_k)\Big)^{\nu_k} + \Big(\widetilde{W}(x_k) - \rho(x_k)\Big)^{\nu_k}$$

only enter the even powers of $\rho(x_k)$, which is how $\mathcal{I}_\nu$ appears. Then one uses that the square Bessel process $(\rho(x))_{x\geqslant 0}$ is a $(\beta+1)/2$-permanental field with kernel $(G_{\mathbb{R}_+}(x, y))_{x,y\in\mathbb{R}_+}$. Because of the particular form of $G_{\mathbb{R}_+}$, we have that for $x_1 \leqslant x_2 \leqslant \dots \leqslant x_{m(\nu)} \in \mathbb{R}_+$,

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \Big\rangle_{\beta,n=2}^{\mathbb{R}_+} = \widetilde{P}_{\nu,\beta}((\mathsf{Y}_{kk} = 2x_k)_{1\leqslant k\leqslant m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = 1)_{2\leqslant k\leqslant m(\nu)}).$$

By combining with Proposition 5.4, we get that the following multivariate polynomials in the variables $(\mathsf{Y}_{kk})_{1\leqslant k\leqslant m(\nu)}$ are equal for $\beta \geqslant 0$:

$$\widetilde{P}_{\nu,\beta}((\check{\mathsf{Y}}_{k-1\,k} = 1)_{2\leqslant k\leqslant m(\nu)}) = P_{\nu,\beta,n=2}((\check{\mathsf{Y}}_{k-1\,k} = 1)_{2\leqslant k\leqslant m(\nu)}).$$

Since the coefficients of both are polynomials in β, the equality above holds for general β. To conclude the equality $\widetilde{P}_{\nu,\beta} = P_{\nu,\beta,n=2}$, we have to deal with the variables $(\check{\mathsf{Y}}_{k-1\,k})_{2\leqslant k\leqslant m(\nu)}$. For this we use that both $P_{\nu,\beta,n=2}$ and $\widetilde{P}_{\nu,\beta}$ satisfy the point (1) of Proposition 4.2. For $\widetilde{P}_{\nu,\beta}$ this follows from (6.1). □

6.2. **A construction on discrete electrical networks for $n = 2$.** Let $\mathcal{G} = (V, E)$ be an undirected connected graph, with V finite. We do not allow multiple edges or self-loops. The edges $\{x, y\} \in E$ are endowed with conductances $C(x, y) = C(y, x) > 0$. There is also a not uniformly zero killing measure $(K(x))_{x\in V}$, with $K(x) \geqslant 0$. We see $\mathcal{G}$ as an electrical network. Let $\Delta_{\mathcal{G}}$ denote the discrete Laplacian

$$(\Delta_{\mathcal{G}} f)(x) = \sum_{y\sim x} C(x, y)(f(y) - f(x)).$$

Let $(G_{\mathcal{G},K}(x, y))_{x,y\in V}$ be the massive Green's function $G_{\mathcal{G},K} = (-\Delta_{\mathcal{G}} + K)^{-1}$. The (massive) real scalar Gaussian free field (GFF) is the centered random Gaussian field on V with covariance $G_{\mathcal{G},K}$, or equivalently with density

$$\frac{1}{((2\pi)^{|V|}\det G)^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}\sum_{x\in V} K(x)\varphi(x)^2 - \frac{1}{2}\sum_{\{x,y\}\in E} C(x, y)(\varphi(y) - \varphi(x))^2\Big). \tag{6.2}$$

Let X_t be the continuous time Markov jump process to nearest neighbors with jump rates given by the conductances. X_t is also killed by K. Let $\zeta \in (0, +\infty]$ be the first time X_t gets killed by K. Let $p_{\mathcal{G},K}(t, x, y)$ be the transition probabilities of $(X_t)_{0\leqslant t<\zeta}$. Then $p_{\mathcal{G},K}(t, x, y) = p_{\mathcal{G},K}(t, y, x)$ and

$$G_{\mathcal{G},K}(x, y) = \int_0^{+\infty} p_{\mathcal{G},K}(t, x, y)dt.$$

Let $\mathbb{P}^{t,x,y}_{\mathcal{G},K}$ be the bridge probability measure from x to y, where one conditions by $t < \zeta$. For $x, y \in V$, let $\mu^{x,y}_{\mathcal{G}}$ be the following measure on paths:

$$\mu^{x,y}_{\mathcal{G},K}(\cdot) := \int_0^{+\infty} \mathbb{P}^{t,x,y}_{\mathcal{G},K}(\cdot) p_{\mathcal{G},K}(t, x, y)dt.$$

It is the analogue of (2.1). The total mass of $\mu_{\mathcal{G}}^{x,y}$ is $G_{\mathcal{G},K}(x,y)$, and the image of $\mu_{\mathcal{G}}^{x,y}$ by time reversal is $\mu_{\mathcal{G}}^{y,x}$. Similarly, one defines the measure on (rooted) loops by

$$\mu_{\mathcal{G},K}^{\text{loop}}(d\gamma) := \frac{1}{T(\gamma)} \sum_{x \in V} \mu_{\mathcal{G},K}^{x,x}(d\gamma),$$

where $T(\gamma)$ denotes the duration of the loop γ. It is the analogue of (2.2). $\mu_{\mathcal{G},K}^{\text{loop}}$ has an infinite total mass because it puts an infinite mass on trivial "loops" that stay in one vertex. For $\alpha > 0$, one considers Poisson point processes $\mathcal{L}_{\mathcal{G},K}^{\alpha}$ of intensity $\alpha\mu_{\mathcal{G},K}^{\text{loop}}$. These are (continuous time) *random walk loop-soups*. For details, see [LTF07, LL10, LJ10, LJ11].

For a continuous time path γ on $\mathcal{G}$ of duration $T(\gamma)$ and $x \in V$, we denote

$$L^x(\gamma) := \int_0^{T(\gamma)} \mathbf{1}_{\gamma(s)=x} ds.$$

Further,

$$L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha}) := \sum_{\gamma \in \mathcal{L}_{\mathcal{G},K}^{\alpha}} L^x(\gamma).$$

One has equality in law between $(L^x(\mathcal{L}_{\mathcal{G},K}^{\frac{1}{2}}))_{x \in V}$ and $(\frac{1}{2}\phi_{\mathcal{G},K}(x)^2)_{x \in V}$, where $\phi_{\mathcal{G},K}$ is the GFF distributed according to (6.2) [LJ10, LJ11]. This is the analogue of (2.3). For general $\alpha > 0$, the occupation field $(L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha}))_{x \in V}$ the α-permanental field with kernel $G_{\mathcal{G},K}$ [LJ10, LJ11, LJMR15]. In this sense it is analogous to squared Bessel processes. If $(\chi(x))_{x \in V} \in \mathbb{R}^V$ is such that $-\Delta_{\mathcal{G}} + K - \chi$ is positive definite, then

$$\mathbb{E}\Big[\exp\Big(\sum_{x \in V} \chi(x) L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha})\Big)\Big] = \left(\frac{\det(-\Delta_{\mathcal{G}} + K)}{\det(-\Delta_{\mathcal{G}} + K - \chi)}\right)^{\alpha}. \tag{6.3}$$

Now we proceed with our construction. Fix $\beta > -1$. Let $\alpha = \frac{1}{2}d(\beta, n = 2) = \frac{\beta+2}{2} > \frac{1}{2}$. Let $\phi_{\mathcal{G},K}$ be a GFF distributed according to (6.2), and $\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}}$ an independent random walk loop-soup. For $x \in V$ we set

$$\lambda_1(x) := \frac{1}{\sqrt{2}}\phi_{\mathcal{G},K}(x) + \sqrt{L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})}, \qquad \lambda_2(x) := \frac{1}{\sqrt{2}}\phi_{\mathcal{G},K}(x) - \sqrt{L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})},$$

and $\lambda := (\lambda_1(x), \lambda_2(x))_{x \in V}$. $\langle\cdot\rangle_{\beta,n=2}^{\mathcal{G},K}$ will denote the expectation with respect to λ. As in Section 5.3, $\Upsilon = (\gamma_1, \dots, \gamma_{J(\Upsilon)})$ will denote a generic family of continuous time paths, this time on the graph $\mathcal{G}$. For $x \in V$,

$$L^x(\Upsilon) := \sum_{i=1}^{J(\Upsilon)} L^x(\gamma_i),$$

and $L(\Upsilon)$ will denote the occupation field of Υ, $x \mapsto L^x(\Upsilon)$. Given ν a finite family of positive integers with $|\nu|$ even, and $x_1, x_2, \dots, x_{m(\nu)} \in V$, $\hat{\mu}_{\mathcal{G},K}^{\nu,\beta,x_1,\dots,x_{m(\nu)}}$ will denote the measure on families of $|\nu|/2$ paths on $\mathcal{G}$ obtained by substituting in the polynomial $\widehat{P}_{\nu,\beta}$ for each variable $\mathsf{Y}_{kk'}$, $1 \leqslant k \leqslant k' \leqslant m(\nu)$, the measure $\mu_{\mathcal{G},K}^{x_k,x_{k'}}$. The order of the paths will not matter.

Proposition 6.2. *The following holds.*

1. *For every $x \in V$, $(\lambda_1(x)/\sqrt{G_{\mathcal{G},K}(x,x)}, \lambda_2(x)/\sqrt{G_{\mathcal{G},K}(x,x)})$ is distributed, up to reordering, according to* (3.1) *for $n = 2$.*
2. *Let $x, y \in V$. Let*

$$\eta = \frac{G_{\mathcal{G},K}(x,x)G_{\mathcal{G},K}(y,y)}{G_{\mathcal{G},K}(x,y)^2} \geqslant 1. \tag{6.4}$$

Then the couple $(\sqrt{2}\lambda(x)/\sqrt{G_{\mathcal{G},K}(x,x)}, \sqrt{2\eta}\lambda(y)/\sqrt{G_{\mathcal{G},K}(y,y)})$ is distributed like the β-Dyson's Brownian motion (5.1) *at points* 1 *and η, for $n = 2$.*

(3) *Let ν be finite family of positive integers with $|\nu|$ even and $x_1, x_2, \dots, x_{m(\nu)} \in V$. Then*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \Big\rangle_{\beta,n=2}^{\mathcal{G},K} = \widehat{P}_{\nu,\beta}((\Upsilon_{kk'} = G_{\mathcal{G},K}(x_k, x_{k'}))_{1\leqslant k\leqslant k'\leqslant m(\nu)}).$$

(4) *(BFS-Dynkin's isomorphism) Moreover, given F a measurable bounded function on $\mathbb{R}^V$,*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) F\Big(\frac{1}{2}p_2(\lambda)\Big) \Big\rangle_{\beta,n=2}^{\mathcal{G},K} = \int_{\Upsilon} \Big\langle F\Big(\frac{1}{2}p_2(\lambda) + L(\Upsilon)\Big) \Big\rangle_{\beta,n=2}^{\mathcal{G},K} \hat{\mu}_{\mathcal{G},K}^{\nu,\beta,x_1,\dots,x_{m(\nu)}}(d\Upsilon). \tag{6.5}$$

(5) *For $\beta \in \{1, 2, 4\}$, $(\lambda_1(x), \lambda_2(x))_{x\in V}$ is distributed like the ordered family of eigenvalues in a GFF with values in 2×2 real symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$), resp. quaternionic Hermitian ($\beta = 4$) matrices, with density proportional to*

$$\exp\Big(- \frac{1}{2} \sum_{x\in V} K(x) \operatorname{Tr}(M(x)^2) - \frac{1}{2} \sum_{\{x,y\}\in E} C(x, y) \operatorname{Tr}((M(y) - M(x))^2)\Big). \tag{6.6}$$

(6) *Assume that $\beta > 0$. Let ϕ_1 and ϕ_2 be two independent scalar GFFs distributed according to (6.2). $\mathcal{L}_{\mathcal{G},K}^{\alpha-1}$ be a random walk loop-soup independent from (ϕ_1, ϕ_2), with still $\alpha = \frac{\beta+2}{2}$. Then $(\lambda_1(x), \lambda_2(x))_{x\in V}$ is the ordered family of eigenvalues in the matrix valued field*

$$\begin{pmatrix} \phi_1(x) & \sqrt{L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-1})} \\ \sqrt{L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-1})} & \phi_2(x) \end{pmatrix}, \quad x \in V. \tag{6.7}$$

(7) *Given another killing measure $\widetilde{K} \in \mathbb{R}_+^V$, non uniformly zero, and $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2)$ the field obtained by using $\widetilde{K}$ instead of K, the density of the law of $\tilde{\lambda}$ with respect to that of λ is*

$$\Big(\frac{\det(-\Delta_{\mathcal{G}} + \widetilde{K})}{\det(-\Delta_{\mathcal{G}} + K)}\Big)^{\frac{\beta+2}{2}} \exp\Big(- \frac{1}{2} \sum_{x\in V} (\widetilde{K}(x) - K(x)) p_2(\lambda(x))\Big).$$

Proof. (1) This follows from Proposition 3.2 and the fact that $\phi_{\mathcal{G},K}(x)/\sqrt{G_{\mathcal{G},K}(x,x)}$ is distributed according to $\mathcal{N}(0, 1)$, and $L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})/\sqrt{G_{\mathcal{G},K}(x,x)}$ according to Gamma $\big(\alpha - \frac{1}{2}, 1\big)$.

(2) One uses (5.2). Indeed, $(\sqrt{2}\phi_{\mathcal{G},K}(x)/\sqrt{G_{\mathcal{G},K}(x,x)}, \sqrt{2\eta}\phi_{\mathcal{G},K}(y)/\sqrt{G_{\mathcal{G},K}(y,y)})$ is distributed as $(\phi_{\mathbb{R}_+}(1), \phi_{\mathbb{R}_+}(\eta))$, and $(\sqrt{2}L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})/\sqrt{G_{\mathcal{G},K}(x,x)}, \sqrt{2\eta}L^y(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})/\sqrt{G_{\mathcal{G},K}(y,y)})$ is distributed as $(\rho(1), \rho(\eta))$. The latter can be seen using the moments, that characterize the finite-dimensional marginals of the Bessel process ρ.

(3) This follows by expanding

$$\Big(\frac{1}{\sqrt{2}}\phi_{\mathcal{G},K}(x_k) + \sqrt{L^{x_k}(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})}\Big)^{\nu_k} + \Big(\frac{1}{\sqrt{2}}\phi_{\mathcal{G},K}(x_k) - \sqrt{L^{x_k}(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})}\Big)^{\nu_k}.$$

(4) The GFF $\phi_{\mathcal{G},K}$ satisfies the BFS-Dynkin isomorphism; see Theorem 2.2 in [BFS82], Theorems 6.1 and 6.2 in [Dyn84a], Theorem 1 in [Dyn84b]. Moreover, there is a version of BFS-Dynkin isomorphism for the occupation field $L(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})$ obtained by applying Palm's identity to Poisson point processes; see Theorem 1.3 in [LJMR15] and Sections 3.4 and 4.3 in [Lup18]. Together, this implies (6.5).

(5) Recall that for all three matrix spaces considered, $\beta+2$ is the dimension. Given $(M(x))_{x\in V}$ an matrix field distributed according to (6.6), $M_0(x)$ will denoted $M(x) - \frac{1}{2}\operatorname{Tr}(M(x))\mathbf{I}_2$, where $\mathbf{I}_2$ is the 2×2 identity matrix, so that $\operatorname{Tr}(M_0(x)) = 0$. Since the hyperplane of zero trace matrices is orthogonal to $\mathbf{I}_2$ for the inner product $(A, B) \mapsto \operatorname{Re}(\operatorname{Tr}(AB))$, we get that $(M_0(x))_{x\in V}$ and $(\operatorname{Tr}(M(x)))_{x\in V}$ are independent. Moreover, $(\frac{1}{\sqrt{2}}\operatorname{Tr}(M(x)))_{x\in V}$ is distributed as the scalar GFF

(6.2). As for $(\mathrm{Tr}(M(x)^2))_{x\in V}$, on one hand it is the sum of $\beta+2$ i.i.d. squares of scalar GFFs (6.2) corresponding to the entries of the matrices. On the other hand,

$$\mathrm{Tr}(M(x)^2) = \mathrm{Tr}(M_0(x)^2) + \frac{1}{2}\,\mathrm{Tr}(M(x))^2.$$

So $(\mathrm{Tr}(M_0(x)^2))_{x\in V}$ is distributed as the sum of $\beta+1$ i.i.d. squares of scalar GFFs (6.2). So in particular, this is the same distributions as for $(2L^x(\mathcal{L}_{\mathcal{G},K}^{\frac{\beta+1}{2}}))_{x\in V}$. Finally, the eigenvalues of $M(x)$ are

$$\frac{1}{2}\,\mathrm{Tr}(M(x)) \pm \frac{1}{\sqrt{2}}\sqrt{\mathrm{Tr}(M_0(x)^2)}.$$

(6) The eigenvalues of the matrix (6.7) are

$$\frac{\phi_1(x)+\phi_2(x)}{2} \pm \sqrt{L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-1}) + (\phi_2(x)-\phi_1(x))^2/4}.$$

$(\phi_1+\phi_2)/\sqrt{2}$ and $(\phi_2-\phi_1)/\sqrt{2}$ are two independent scalar GFFs. Moreover,

$$L(\mathcal{L}_{\mathcal{G},K}^{\alpha-1}) + \frac{1}{4}(\phi_2-\phi_1)^2$$

has same distribution as $L(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})$.

(7) The density of the GFF $\phi_{\mathcal{G},\widetilde{K}}$ with respect to $\phi_{\mathcal{G},K}$ is

$$\Big(\frac{\det(-\Delta_{\mathcal{G}}+\widetilde{K})}{\det(-\Delta_{\mathcal{G}}+K)}\Big)^{\frac{1}{2}} \exp\Big(-\frac{1}{2}\sum_{x\in V}(\widetilde{K}(x)-K(x))\varphi(x)^2\Big).$$

The density of $L(\mathcal{L}_{\mathcal{G},\widetilde{K}}^{\alpha-\frac{1}{2}})$ with respect to $L(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})$ is

$$\Big(\frac{\det(-\Delta_{\mathcal{G}}+\widetilde{K})}{\det(-\Delta_{\mathcal{G}}+K)}\Big)^{\alpha-\frac{1}{2}} \exp\Big(-\sum_{x\in V}(\widetilde{K}(x)-K(x))L^x(\mathcal{L}_{\mathcal{G},K}^{\alpha-\frac{1}{2}})\Big),$$

as can be seen from the Laplace transform (6.3). □

6.3. **Further questions.** Here we present our questions that motivated this paper. The first question is combinatorial. We would like to have the polynomials $P_{\nu,\beta,n}$ given by Definition 4.1 under a more explicit form. The recurrence on polynomials (4.1) is closely related to the Schwinger-Dyson equation (3.2). Its very form suggests that the polynomials $P_{\nu,\beta,n}$ might be expressible as weighted sums over maps drawn on 2D compact surfaces (not necessarily connected), where the maps associated to ν have $m(\nu)$ vertices with degrees given by $\nu_1,\nu_2,\dots,\nu_{m(\nu)}$, with powers of n corresponding to the number of faces. This is indeed the case for $\beta\in\{1,2,4\}$, and this corresponds to the topological expansion of matrix integrals [BIPZ78, IZ80, MW03, Lup19].

Question 6.3. *Is there a more explicit expression for the polynomials $P_{\nu,\beta,n}$? Can they be expressed as weighted sums over the maps on 2D surfaces (topological expansion)?*

The second question is whether there is a natural generalization of Gaussian beta ensembles and β-Dyson's Brownian motion to electrical networks. For $n=2$, such a generalization was given in Section 6.2.

Question 6.4. *We are in the setting on an electrical network $\mathcal{G}=(V,E)$ endowed with a killing measure K, as in Section 6.2. Given $n\geqslant 3$ and $\beta>-\frac{2}{n}$, is there a distribution on the fields $(\lambda(x)=(\lambda_1(x),\lambda_2(x),\dots,\lambda_n(x)))_{x\in V}$, with $\lambda_1(x)>\lambda_2(x)>\dots>\lambda_n(x)$, satisfying the following properties?*

1. *For $\beta\in\{1,2,4\}$, λ is distributed as the fields of ordered eigenvalues in a GFF with values into $n\times n$ matrices, real symmetric ($\beta=1$), complex Hermitian ($\beta=2$), resp. quaternionic Hermitian ($\beta=4$).*
2. *For $\beta=0$, λ is obtained by reordering n i.i.d. scalar GFFs* (6.2).

(3) *As $\beta \to -\frac{2}{n}$, λ converges in law to*

$$\Big(\frac{1}{\sqrt{n}}\phi_{\mathcal{G},K}, \frac{1}{\sqrt{n}}\phi_{\mathcal{G},K}, \dots, \frac{1}{\sqrt{n}}\phi_{\mathcal{G},K}\Big),$$

where $\phi_{\mathcal{G},K}$ is a scalar GFF (6.2).

(4) *For every $x \in V$, $\lambda(x)/\sqrt{G_{\mathcal{G},K}(x,x)}$ is distributed, up to reordering, as the $G\beta E$ (3.1).*

(5) *For every $x, y \in V$, the couple $(\sqrt{2}\lambda(x)/\sqrt{G_{\mathcal{G},K}(x,x)}, \sqrt{2\eta}\lambda(y)/\sqrt{G_{\mathcal{G},K}(y,y)})$, with η given by (6.4), is distributed as the β-Dyson's Brownian motion (5.1) at points 1 and η.*

(6) *The fields $p_1(\lambda)$ and $\lambda - \frac{1}{n}p_1(\lambda)$ are independent.*

(7) *The field $\frac{1}{\sqrt{n}}p_1(\lambda)$ is distributed as a scalar GFF (6.2).*

(8) *The field $\frac{1}{2}\big(p_2(\lambda) - \frac{1}{n}p_1(\lambda)^2\big)$ is the $\alpha - \frac{1}{2}$-permanental field with kernel $G_{\mathcal{G},K}$, where $\alpha = \frac{1}{2}d(\beta,n)$, and in particular is distributed as the occupation field of the continuous-time random walk loop-soup $\mathcal{L}^{\alpha-\frac{1}{2}}_{\mathcal{G},K}$.*

(9) *The field $\frac{1}{2}p_2(\lambda)$ is the α-permanental field with kernel $G_{\mathcal{G},K}$, where $\alpha = \frac{1}{2}d(\beta,n)$, and in particular is distributed as the occupation field of the continuous-time random walk loop-soup $\mathcal{L}^{\alpha}_{\mathcal{G},K}$ (already implied by (6)+(7)+(8)).*

(10) *The symmetric moments*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \Big\rangle_{\beta,n}^{\mathcal{G},K}$$

are linear combination of products

$$\prod_{1 \leqslant k \leqslant k' \leqslant m(\nu)} G_{\mathcal{G},K}(x_k, x_{k'})^{a_{kk'}},$$

with $a_{kk'} \in \mathbb{N}$ and for every $k \in [\![1, m(\nu)]\!]$,

$$2a_{kk} + \sum_{\substack{1 \leqslant k' \leqslant m(\nu) \\ k' \neq k}} a_{kk'} = \nu_k,$$

the coefficients of the linear combination being universal polynomials in β and n, not depending on the electrical network and its parameters; see also Question 6.3.

(11) *Given $\widetilde{K} \in \mathbb{R}_+^V$, non uniformly zero, and $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n)$ the field associated to the killing measure $\widetilde{K}$ instead of K, the law of $\tilde{\lambda}$ has the following density with respect to that of λ:*

$$\Big(\frac{\det(-\Delta_{\mathcal{G}} + \widetilde{K})}{\det(-\Delta_{\mathcal{G}} + K)}\Big)^{\frac{1}{2}d(\beta,n)} \exp\Big(-\frac{1}{2}\sum_{x \in V}(\widetilde{K}(x) - K(x))p_2(\lambda(x))\Big).$$

(12) *λ satisfies a BFS-Dynkin type isomorphism with continuous time random walks (already implied by (10)+(11)).*

If the graph $\mathcal{G}$ is a tree, the answer for the properties (1),(2),(4),(5),(6),(7),(8),(9),(11),(12) is yes, at least for $\beta \geqslant 0$. In absence of cycles, λ satisfies a Markov property, and along each branch of the tree one has the values of a β-Dyson's Brownian motion at different positions. On the random walk loop-soup side, (8) and (9) is ensured by the covariance of the loop-soups under the rewiring of graphs; see Chapter 7 in [LJ11]. Constructing λ on a tree for $\beta \in \big(-\frac{2}{n}, 0\big)$ is a matter of constructing the corresponding β-Dyson's Brownian motion. However, if the graph $\mathcal{G}$ contains cycles, constructing λ is not immediate, and we have not encountered such a construction in the literature. One does not expect a Markov property, since already for $\beta \in \{1, 2, 4\}$ one has to take into account the angular part of the matrices.

Appendix: A list of moments for GβE and the corresponding formal polynomials

$$\begin{aligned}\langle p_1(\lambda)^2\rangle_{\beta,n} &= n,\\ P_{(1,1)} &= n\mathsf{Y}_{11}\check{\mathsf{Y}}_{12},\end{aligned}$$

$$\begin{aligned}\langle p_2(\lambda)\rangle_{\beta,n} &= \frac{\beta}{2}n^2+\Big(1-\frac{\beta}{2}\Big)n = d(\beta,n),\\ P_{(2)} &= \Big(\frac{\beta}{2}n^2+\Big(1-\frac{\beta}{2}\Big)n\Big)\mathsf{Y}_{11} = d(\beta,n)\mathsf{Y}_{11},\end{aligned}$$

$$\begin{aligned}\langle p_1(\lambda)^4\rangle_{\beta,n} &= 3n^2,\\ P_{(1,1,1,1)} &= n^2\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{33}\check{\mathsf{Y}}_{34}+2n^2\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23}^2\check{\mathsf{Y}}_{34},\end{aligned}$$

$$\begin{aligned}\langle p_2(\lambda)p_1(\lambda)^2\rangle_{\beta,n} &= \frac{\beta}{2}n^3+\Big(1-\frac{\beta}{2}\Big)n^2+2n,\\ P_{(2,1,1)} &= \Big(\frac{\beta}{2}n^3+\Big(1-\frac{\beta}{2}\Big)n^2\Big)\mathsf{Y}_{11}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23}+2n\mathsf{Y}_{11}^2\check{\mathsf{Y}}_{12}^2\check{\mathsf{Y}}_{23},\\ P_{(1,2,1)} &= \Big(\frac{\beta}{2}n^3+\Big(1-\frac{\beta}{2}\Big)n^2+2n\Big)\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23},\\ P_{(1,1,2)} &= \Big(\frac{\beta}{2}n^3+\Big(1-\frac{\beta}{2}\Big)n^2\Big)\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{33}+2n\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23}^2,\end{aligned}$$

$$\begin{aligned}\langle p_2(\lambda)^2\rangle_{\beta,n} &= \frac{\beta^2}{4}n^4+2\frac{\beta}{2}\Big(1-\frac{\beta}{2}\Big)n^3\\ &\quad+\Big(\Big(1-\frac{\beta}{2}\Big)^2+2\frac{\beta}{2}\Big)n^2+2\Big(1-\frac{\beta}{2}\Big)n\\ &= d(\beta,n)(d(\beta,n)+2),\\ P_{(2,2)} &= \Big(\frac{\beta^2}{4}n^4+2\frac{\beta}{2}\Big(1-\frac{\beta}{2}\Big)n^3+\Big(1-\frac{\beta}{2}\Big)^2n^2\Big)\mathsf{Y}_{11}\mathsf{Y}_{22}\\ &\quad+\Big(2\frac{\beta}{2}n^2+2\Big(1-\frac{\beta}{2}\Big)n\Big)\mathsf{Y}_{11}^2\check{\mathsf{Y}}_{12}^2,\end{aligned}$$

$$\begin{aligned}\langle p_3(\lambda)p_1(\lambda)\rangle_{\beta,n} &= 3\frac{\beta}{2}n^2+3\Big(1-\frac{\beta}{2}\Big)n,\\ P_{(3,1)} &= \Big(3\frac{\beta}{2}n^2+3\Big(1-\frac{\beta}{2}\Big)n\Big)\mathsf{Y}_{11}^2\check{\mathsf{Y}}_{12},\\ P_{(1,3)} &= \Big(3\frac{\beta}{2}n^2+3\Big(1-\frac{\beta}{2}\Big)n\Big)\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{22},\end{aligned}$$

$$\begin{aligned}\langle p_4(\lambda)\rangle_{\beta,n} &= 2\frac{\beta^2}{4}n^3+5\frac{\beta}{2}\Big(1-\frac{\beta}{2}\Big)n^2+\Big(\frac{\beta}{2}+3\Big(1-\frac{\beta}{2}\Big)^2\Big)n,\\ P_{(4)} &= \Big(2\frac{\beta^2}{4}n^3+5\frac{\beta}{2}\Big(1-\frac{\beta}{2}\Big)n^2+\Big(\frac{\beta}{2}+3\Big(1-\frac{\beta}{2}\Big)^2\Big)n\Big)\mathsf{Y}_{11}^2,\end{aligned}$$

$$\begin{aligned}\langle p_3(\lambda)^2\rangle_{\beta,n} &= 12\frac{\beta^2}{4}n^3+27\frac{\beta}{2}\Big(1-\frac{\beta}{2}\Big)n^2+\Big(3\frac{\beta}{2}+15\Big(1-\frac{\beta}{2}\Big)^2\Big)n,\\ P_{(3,3)} &= 9\Big(\frac{\beta^2}{4}n^3+2\frac{\beta}{2}\Big(1-\frac{\beta}{2}\Big)n^2+\Big(1-\frac{\beta}{2}\Big)^2n\Big)\mathsf{Y}_{11}^2\check{\mathsf{Y}}_{12}\mathsf{Y}_{22}\\ &\quad+3\Big(\frac{\beta^2}{4}n^3+3\frac{\beta}{2}\Big(1-\frac{\beta}{2}\Big)n^2+\Big(\frac{\beta}{2}+2\Big(1-\frac{\beta}{2}\Big)^2\Big)n\Big)\mathsf{Y}_{11}^3\check{\mathsf{Y}}_{12}^3.\end{aligned}$$

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