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ALUTHGE OPERATOR FIELD AND ITS NUMERICAL RANGE AND SPECTRAL PROPERTIES

GILLES CASSIER^{1*} AND THOMAS PERRIN ²

ABSTRACT. For an arbitrary operator T acting on a Hilbert space we consider a field of operators $(\Delta_z(T))$ called the Aluthge operator field associated with T . After giving preliminary results, we establish that two fields (left and right), canonically linked to the Aluthge field $(\Delta_z(T))$ and a support subspace, are constant on each horizontal segment where they are defined. This result leads to a positive solution of a conjecture stated by Jung-Ko-Pearcy in 2000. Then we do a detailed spectral study of $(\Delta_z(T))$ and we give a Yamazaki type formula in this context.

1. INTRODUCTION

Let H be a Hilbert space and let $B(H)$ be the algebra of all bounded operators acting on H . We write $\mathcal{K}(H)$ for the ideal of compact operators, $\mathcal{C}(H)$ for the quotient algebra $B(H)/\mathcal{K}(H)$ (called the Calkin algebra) and π for the canonical surjection from $B(H)$ onto $\mathcal{C}(H)$. If $T \in B(H)$ we denote by $\sigma(T)$ the spectrum of T and by $r(T)$ its spectral radius. Among familiar sets associated with T , recall that the numerical range of T is defined as

$$W(T) = \{ \langle Tx \mid x \rangle : \|x\| = 1 \}.$$

As usual, we write $\sigma_p(T)$, $\sigma_{app}(T)$, $\sigma_{res}(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$ and $\sigma_{surj}(T)$ for the point spectrum, the approximate point spectrum, the residual spectrum, the essential spectrum, the left essential spectrum, the right essential spectrum and the surjective spectrum of T , respectively. Recall that an operator $T \in B(H)$ is said to be Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined by

$$\begin{aligned} \sigma_w(T) &= \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl} \} \\ \sigma_b(T) &= \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder} \}. \end{aligned}$$

Denote by \mathcal{S} the open strip of the complex plane defined by

$$\mathcal{S} = \{ z \in \mathbb{C}; 0 < \Re(z) < 1 \}.$$

Let $T \in B(H)$ and let $T = U|T|$ be its polar decomposition, we define the Aluthge field of operators associated with T by setting

$$\Delta_z(T) = |T|^z U |T|^{1-z}$$

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for any $z \in \overline{\mathcal{S}}$ (for u such that $\Re(u) \geq 0$, the complex power $|T|^u$ of $|T|$ is defined below in Section 2). The operator $\Delta(T) := \Delta_{1/2}(T)$ is called the Aluthge transform of T and was introduced in [1]. Since then, the Aluthge transformation has been studied by many authors in many areas of operator theory (see for example [2, 3, 4, 5, 7, 8, 9, 10, 12, 18, 20, 21, 22, 23, 24] and the references therein, the list is far to be exhaustive). In [1], A. Aluthge proved that if T is p -hyponormal with $p \in [1/2, 1[$ then $\Delta(T)$ is hyponormal, and if T is p -hyponormal with $p \in]0, 1/2[$ then $\Delta(T)$ is $(p+1/2)$ -hyponormal, thus showing the regularizing effect of the Aluthge transformation. For $z = 1$, the operator $\Delta_1(T)$ corresponds to the Duggal transform of T . For $\alpha \in]0, 1[$, it seems that the transformation $\Delta_\alpha : T \mapsto \Delta_\alpha(T)$ was firstly considered in [20].

In Section 2, we provide preliminary results which will be used throughout the paper. Some of them are of own interest (see for instance Theorem 2.5).

Section 3 is devoted to numerical range properties of the Aluthge field associated with an operator $T \in B(H)$. I. B. Jung, E. Ko and C. Pearcy (2000) have conjectured (conjecture 1.9 of [18]) that

$$W(\Delta(T)) \subseteq W(T) \quad (1.1)$$

for any $T \in B(H)$. In [22], P. Y. Wu has given the first result in this direction which is $\overline{W(\Delta(T))} \subseteq \overline{W(T)}$ (see also [23]). In the finite dimensional case, it gives (1.1). Later in [3], it is shown that $\overline{W(\Delta_\alpha(T))} \subseteq \overline{W(T)}$ for each $\alpha \in [0, 1]$. Notice that almost all results which are concerned with closure of numerical ranges are based on the following Hildebrandt's characterization of $\overline{W(T)}$ established in [16]

$$\overline{W(T)} = \bigcap_{\zeta \in \mathbb{C}} D(\zeta, \|\zeta I - T\|).$$

The main result of this paper is Theorem 3.1 asserting that the two fields of operators (left and right), canonically associated with the Aluthge field $(\Delta_z(T))$ and a support subspace (defined below in Subsection 2.6), are constant on each horizontal segment included in $\overline{\mathcal{S}}$. As a direct consequence, a positive answer to conjecture (1.1) is given. Even more, the inclusion

$$W(\Delta_z(T)) \subseteq W(T) \quad (1.2)$$

is obtained for each $z \in \overline{\mathcal{S}}$.

The last section deals with spectral properties of the Aluthge field $(\Delta_z(T))_{z \in \overline{\mathcal{S}}}$ associated with an operator $T \in B(H)$. We carefully study different spectra of $\Delta_z(T)$, showing that most of them are preserved by the transformation Δ_z when $z \in \mathcal{S}$. Using counter examples we prove that some of these spectra are not preserved when z belongs to the boundary of \mathcal{S} . Finally, by means of a short and direct proof, we show that a Yamazaki type formula holds for every $z \in \mathcal{S}$.

2. PRELIMINARIES

Let $T \in B(H)$, as usually we write $\mathcal{N}(T)$ for the nullspace of T , $\mathcal{R}(T)$ for its range, $\text{Lat}(T)$ for the set of invariant subspaces of T and $\gamma(T)$ for the reduced minimum modulus of T .

2.0.1. *Power operator field associated with a positive operator.* Let R be a positive operator, using the spectral measure E^R associated with R we can define the power operator field $z \longrightarrow R^z$ defined on $\mathcal{P}_+ = \{z \in \mathbb{C}; \Re(z) \geq 0\}$ by setting

$$R^z = \int_{]0, \|R\|]} t^z dE^R(t).$$

The next proposition is a direct consequence of functional calculus associated with a positive operator and Lebesgue theorem.

Proposition 2.1. *Let R be a positive operator acting on a Hilbert space H . Then the operator field*

$$z \longrightarrow R^z$$

is strongly continuous on the half plane \mathcal{P}_+ , holomorphic on $\{\Re(z) > 0\}$ and uniformly bounded on any closed strip of the form $\{z \in \mathbb{C}; a \leq \Re(z) \leq b\}$ where $0 \leq a \leq b$.

Let $R \in B(H)$ be a positive operator, the following proposition summarizes a number of useful properties of the power operator field associated with R .

Proposition 2.2. *Let R be a nonzero positive operator acting on a Hilbert space H . Then, we have:*

- (i) $R^0 = P_{\overline{\mathcal{R}(R)}} := P$ and $R^{z_1+z_2} = R^{z_1} \circ R^{z_2} = R^{z_2} \circ R^{z_1}$ for any $(z_1, z_2) \in \mathcal{P}_+^2$;
- (ii) $\mathcal{N}(R^z) = \mathcal{N}(R)$ for any $z \in \mathcal{P}_+$;
- (iii) $\overline{\mathcal{R}(R^z)} = \overline{\mathcal{R}(R)}$ for any $z \in \mathcal{P}_+$;
- (iv) $\gamma(R^{it}) = 1$ for any $t \in \mathbb{R}$ and $\gamma(R^z) = \gamma(R)^{2\Re(z)}$ for each $z \in \mathcal{P}_+ \setminus i\mathbb{R}$.
- (v) Let $z_0 \in \mathcal{P}_+$. Assume that R^{z_0} is semi-Fredholm, then R^{z_0} is Fredholm.

Moreover, if $z_0 \in \overset{\circ}{\mathcal{P}}_+$, then R^z is Fredholm for any $z \in \mathcal{P}_+$.

- (vi) Let Ω be a Borelian subset of \mathbb{R} with positive Lebesgue measure. Assume that $E \in \bigcap_{u \in i\Omega} \text{Lat}(R^u)$, then $E \in \text{Lat}(R^z)$ for any $z \in \overset{\circ}{\mathcal{P}}_+$.

Proof. (i) For any $z \in \mathcal{P}_+$ and any $t \in [0, \|R\|]$, set $\varphi_z(t) = \mathbf{1}_{]0, \|R\|]}(t)t^z$ ($\varphi_z(0) = 0$). Notice that $\varphi_0(t) = \mathbf{1}_{]0, \|R\|]}(t)$, hence $R^0 = P_{\overline{\mathcal{R}(R)}}$. Since $\varphi_{z_1+z_2} = \varphi_{z_1} \times \varphi_{z_2}$, the equality $R^{z_1+z_2} = R^{z_1} \circ R^{z_2} = R^{z_2} \circ R^{z_1}$ follows from the Borelian functional calculus associated with R .

- (ii) Let $x \in \mathcal{N}(R^z)$, then we have

$$0 = \int_{]0, \|R\|]} \exp(2\Re(z) \ln(t)) dE_{x,x}^R(t).$$

Therefore we have necessarily $E_{x,x}^R(]0, \|R\|]) = 0$ and then $E_{x,y}^R(]0, \|R\|]) = 0$ for any $y \in H$, so $Rx = \left[\int_0^{\|R\|} t dE^R(t) \right] x = 0$. Thus, $x \in \mathcal{N}(R)$. Conversely, if $x \in \mathcal{N}(R)$ we have $E_{x,y}^R = \langle x | y \rangle \delta_0$ for any $y \in H$. Then we get $\langle R^z x | y \rangle = \int_0^{\|R\|} \varphi_z(t) dE_{x,y}^R = 0$ for each $y \in H$ and hence $R^z x = 0$.

(iii) The equality $\overline{\mathcal{R}(R^z)} = \overline{\mathcal{R}(R)}$ follows from (ii).

(iv) Recall that the reduced modulus of an operator $T \in B(H)$ is defined by setting

$$\gamma(T) = \inf \{ \|Tx\|; x \in \mathcal{N}(T)^\perp \text{ and } \|x\| = 1 \}.$$

Let $z \in \mathcal{P}_+ \setminus i\mathbb{R}$. On the one hand, from (i) we infer that $|R^z|^2 = R^{\bar{z}}R^z = R^{2\Re(z)}$ and hence $|R^z| = R^{\Re(z)}$. On the other hand, since R is a positive operator, using Theorem 1.5 of [19], we see that $\gamma(R^{\Re(z)}) = \inf \{ t; t \in \sigma(R^{\Re(z)}) \setminus \{0\} \} = \inf \{ t^{\Re(z)}; t \in \sigma(R) \setminus \{0\} \} = \inf \{ t; t \in \sigma(R) \setminus \{0\} \}^{\Re(z)} = \gamma(R)^{\Re(z)}$. Let $z = it \in i\mathbb{R}$, then we have $|R^{it}| = R^0 = P$ ($P \neq 0$ because $R \neq 0$). It implies that $\gamma(R^{it}) = \gamma(|R^{it}|) = \gamma(P) = 1$.

(v) Assume that R^{z_0} is semi-Fredholm for some $z_0 \in \mathcal{P}_+$, then the range of R^{z_0} is closed and from (ii) we derive that $\dim(\mathcal{N}(R^{z_0})) = \dim(\mathcal{N}(R^{z_0})) < +\infty$, so R^{z_0} is Fredholm. Now suppose that $z_0 \in \overset{\circ}{\mathcal{P}}_+$ and let $z \in \mathcal{P}_+$. If $z \in i\mathbb{R}$, then the range of R^z is closed. Thus, we may supposed that $z \in \overset{\circ}{\mathcal{P}}_+$. By (iv) we have $\gamma(R^z) = \gamma(R^{z_0})^{\frac{\Re(z)}{\Re(z_0)}} > 0$, and hence the range of R^z is closed. By (ii), we also have $\dim(\mathcal{N}(R^z)) = \dim(\mathcal{N}(R^{\bar{z}})) = \dim(\mathcal{N}(R^{\bar{z}_0})) = \dim(\mathcal{N}(R^{z_0})) < +\infty$. The conclusion follows.

(vi) Let $E \in \bigcap_{u \in i\Omega} \text{Lat}(R^u)$ and x, y be two vectors of H . Without loss of generality, we may suppose that $\|R\| \leq 1$. We introduce the function f defined on the open unit disk \mathbb{D} by $f(w) = \langle R^{\frac{1-w}{1+w}} x \mid y \rangle$ ($\frac{1-w}{1+w} \in \overset{\circ}{\mathcal{P}}_+$). Using Proposition 2.1, we see that f is an analytic function. Besides, we have

$$|f(z)| \leq \int_{]0,1[} |r^{\frac{1-w}{1+w}}| |d|E_{x,y}^R|(r) \leq \int_{]0,1[} |d|E_{x,y}^R|(r) = |E_{x,y}^R|([0,1]) \leq \|x\| \|y\|.$$

Therefore, the function f belongs to the space H^∞ of bounded analytic functions on \mathbb{D} . Then, the function f admits nontangential limits almost everywhere (see for instance [15]). Using again Proposition 2.1, we see that for any $w \in \mathbb{T} \setminus \{-1\}$, we still have $\lim_{r \rightarrow 1^-} f(rw) = \langle R^{\frac{1-w}{1+w}} x \mid y \rangle$. Let $z_0 \in \overset{\circ}{\mathcal{P}}_+$ and w_0 be such that $z_0 = \frac{1-w_0}{1+w_0}$. Since $f \in H^\infty$ (see [15]), $f(w_0)$ is the Poisson integral of the boundary values of f . Then, we have

$$\langle R^{z_0} x \mid y \rangle = \int_{]-\pi, \pi[} \langle R^{\frac{1-e^{it}}{1+e^{it}}} x \mid y \rangle P_{w_0}(e^{it}) \frac{dt}{2\pi}. \quad (2.1)$$

Now, suppose that $x \in E$ and $y \in E^\perp$. Then $f(w) = 0$ for any $w \in \tilde{\Omega} = \{u \in \mathbb{T} \setminus \{-1\}; \frac{1-u}{1+u} \in \Omega\}$. Clearly $\tilde{\Omega}$ is of positive Lebesgue measure on the torus \mathbb{T} . As $f \in H^\infty$, we necessarily have $f(e^{it}) = 0$ for almost every $t \in]-\pi, \pi[$. From (2.1) we infer that $\langle R^{z_0} x \mid y \rangle = 0$, thus $R^{z_0} x \in E$. Consequently $E \in \text{Lat}(R^{z_0})$. \square

2.0.2. *Ultrapower of a Hilbert space.* Recall that a Banach limit \mathcal{L} is a state on $l^\infty(\mathbb{C})$ which is shift invariant (that is $\mathcal{L}((u_{n+1})_{n \geq 0}) = \mathcal{L}((u_n)_{n \geq 0})$). If the sequence $(u_n)_{n \geq 0}$ converges to l , we have necessarily

$$\mathcal{L}((u_n)_{n \geq 0}) = l. \quad (2.2)$$

Recall also that $0 \leq u_n \leq v_n$ implies $0 \leq \mathcal{L}((u_n)_{n \geq 0}) \leq \mathcal{L}((v_n)_{n \geq 0})$. The associated ultrapower of a Hilbert space H was firstly considered by S. K. Berberian in [6] and is constructed in the following way:

We consider the space $l^\infty(H)$ of bounded sequences $x = (x_n)$ of H , and define a positive sesquilinear form by setting

$$(x | y) = \mathcal{L}(\langle x_n | y_n \rangle)$$

and the corresponding Hilbertian seminorm

$$n(x) = \sqrt{(x | y)} = \sqrt{\mathcal{L}(\|x_n\|^2)}.$$

We set $\mathcal{N} = \{x \in l^\infty(H) : n(x) = 0\}$. The completion $\mathcal{H} := \mathcal{H}_\mathcal{L}$ of $l^\infty(H)/\mathcal{N}$ is called the ultrapower of H associated with \mathcal{L} . For any $x = (x_n) \in l^\infty(H)$, we denote by $\dot{x} = \widehat{(x_n)}$ the associated point in \mathcal{H} . The map

$$y \longmapsto \dot{y} = \widehat{(y_n)} \quad (\text{where } y_n = y \text{ for all } n)$$

is a canonical isometric embedding of H into \mathcal{H} . We consider the unital \mathbf{C}^* -morphism Φ , from the von Neumann algebra $l^\infty(B(H))$ ($l^\infty(B(H))$ can be identify with the algebra of diagonal operators acting on $l^2(H) = H \oplus \cdots \oplus H \oplus \cdots$) into $B(\mathcal{H})$, which is defined by setting

$$\Phi((T_n))(\dot{x}) = \widehat{(T_n(x_n))}.$$

Notice that $\|\Phi((T_n))\| \leq \sup_{n \geq 0} \|T_n\|$. We set $\hat{T} = \Phi((T_n))$ where $T_n = T$ for each $n \in \mathbb{N}$. The map $\varphi : T \longmapsto \hat{T}$ is a \mathbf{C}^* -embedding of $B(H)$ into $B(\mathcal{H})$. The next Proposition gives some useful properties.

Proposition 2.3. *Let $(T_n) \in l^\infty(B(H))$ and let $T \in B(H)$, then we have*

- (i) $\|\Phi((T_n))\| \in [\liminf \|T_n\|, \limsup \|T_n\|]$;
- (ii) $|\Phi((T_n))|^z = \Phi(|T_n|^z)$ for any $z \in \mathcal{P}_+$;
- (iii) $\Delta_z(\Phi((T_n))) = \Phi(\Delta_z(T_n))$ for any $z \in \mathcal{S}$;
- (iv) $\sigma(\hat{T}) = \sigma(T)$;
- (v) $\sigma_p(\hat{T}) = \sigma_{app}(\hat{T}) = \sigma_{app}(T)$;
- (vi) Assume that the sequence $(\|T_n\|)_{n \geq 0}$ is convergent, then there exists $x = (x_n) \in l^\infty(H)$ such that $|\Phi((T_n))|(\dot{x}) = \|\Phi((T_n))\|\dot{x}$ with $\|\dot{x}\| = 1$.

Proof. (i) For convenience we set $R = \Phi((T_n))$. Let $x = (x_n) \in l^\infty(H)$, then for any $p \in \mathbb{N}$ we have

$$\|T_{n+p}x_{n+p}\|^2 \leq \left[\sup_{n \geq p} \|T_n\|^2 \right] \|x_{n+p}\|^2,$$

which implies $\|R\dot{x}\|^2 \leq \sup_{n \geq p} \|T_n\|^2 \|\dot{x}\|^2$ and hence $\|R\| \leq \sup_{n \geq p} \|T_n\|$. Letting p goes to $+\infty$, we get $\|R\| \leq \limsup \|T_n\|$. Now, for each $n \in \mathbb{N}$ we choose a unit

vector x_n such that $\|T_n x_n\|^2 \geq \|T_n\|^2 - 1/(n+1)$ and consider $x = (x_n) \in l^\infty(H)$. Then, we have

$$\|R\|^2 \geq \|R\dot{x}\|^2 = \mathcal{L}(\|(T_{n+p}x_{n+p})\|^2) \geq \mathcal{L}\left(\|T_{n+p}\|^2 - \frac{1}{n+p+1}\right) \geq \inf_{n \geq p} \|T_n\|^2,$$

and hence $\|R\| \geq \liminf \|T_n\|$.

(ii) Using the continuous functional calculus associated with positive operators, we see that it is a direct consequence of the fact that Φ is a continuous morphism which preserves the composition and the adjonction.

(iii) Let $(T_n) \in l^\infty(B(H))$. For each $n \in \mathbb{N}$, consider the polar decomposition $T_n = U_n|T_n|$. Set $A = \Phi((T_n))$, $B = \Phi((\Delta_z(T_n)))$ and $U_\Phi = \Phi((U_n))$. Since $z \in \mathcal{S}$, we have $\Re(z) > 0$ and $\Re(1-z) > 0$ and hence $|A|^u = \Phi((|T_n|^u))$ for $u \in \{z, 1-z\}$ by (ii). It follows that $B = |A|^z U_\Phi |A|^{1-z}$. Let $A = U|A|$ be the polar decomposition of A . On the one hand, we have $\Delta_z(A)|A| = |A|^z U|A| |A|^{1-z} = |A|^z U_\Phi |A| |A|^{1-z} = B|A|$, so $\Delta_z(A)|_{\overline{\mathcal{R}(|A|)}} = B|_{\overline{\mathcal{R}(|A|)}}$. On the other hand, we have $0 = \Delta_z(A)|_{\mathcal{N}(|A|)} = B|_{\mathcal{N}(|A|)}$ by using Proposition 2.2 (ii). Therefore, we get the desired equality $\Delta_z(A) = B = \Phi((\Delta_z(T_n)))$ for any $z \in \mathcal{S}$.

(iv) The inclusion $\rho(T) \subseteq \rho(\hat{T})$ is clear. For the reverse inclusion, consider the canonical isometric embedding of H into \mathcal{H} . With this identification, we see that H is a reducing subspace for $\hat{T}|_H = T$ and hence $\rho(\hat{T}) \subseteq \rho(T)$.

(v) The equality $\sigma_{app}(\hat{T}) = \sigma_{app}(T)$ follows directly from the fact that a complex number λ does not belong to the approximate point spectrum $\sigma_{app}(A)$ of an operator A if and only if there exists $\rho > 0$ such that $(\bar{\lambda}I - A^*)(\lambda I - A) \geq \rho I$ and the fact that the application $T \mapsto \hat{T}$ is a \mathbf{C}^* -embedding of $B(H)$ into $B(\mathcal{H})$. Taking into account the construction of \mathcal{H} , we easily see that $\sigma_{app}(T) \subseteq \sigma_p(\hat{T})$. Then, the equality $\sigma_p(\hat{T}) = \sigma_{app}(\hat{T})$ comes from the previous equality $\sigma_{app}(\hat{T}) = \sigma_{app}(T)$ and usual properties.

(vi) In this case, observe that (i) implies that $\|\Phi((T_n))\| = \lim \|T_n\|$. For each $n \in \mathbb{N}$, let x_n be a unit vector such that $\|T_n x_n\|^2 \geq \|T_n\|^2 - 1/(n+1)$ and consider $x = (x_n) \in l^\infty(H)$. Then, we have $\|\dot{x}\| = 1$ and

$$\begin{aligned} \|\Phi((T_n))\|^2 &\geq \|\Phi((T_n))\dot{x}\|^2 = \|\Phi((|T_n|))\dot{x}\|^2 = \mathcal{L}(\|T_n x_n\|^2) \\ &\geq \mathcal{L}\left(\|T_n\|^2 - \frac{1}{n+1}\right) = \lim \|T_n\|^2 = \|\Phi((T_n))\|^2 = \|\Phi((T_n))\dot{x}\|^2. \end{aligned}$$

Thus, we get $\|\Phi((T_n))\dot{x}\| = \|\Phi((T_n))\|$ which forces $|\Phi((T_n))\dot{x}| = \|\Phi((T_n))\|\dot{x} = \|\Phi((T_n))\|\dot{x}$, since $|\Phi((T_n))|$ is a positive operator. \square

Remark 2.4. (1) In general, we do not have $\|\Phi((T_n))\| = \mathcal{L}(\|(T_n)\|)$. Indeed, set $T_n = I$ if n is even and $T_n = 0$ if n is odd, then we see that $\Phi((T_n))$ is a non-zero orthogonal projection and hence $\|\Phi((T_n))\| = 1$ while we have $\mathcal{L}(\|(T_n)\|) = 1/2$. (2) Consider a positive operator $A \in B(H)$ which is one-to-one but not invertible.

Let $z = it \in i\mathbb{R}$ and set $T_n = A$ for each $n \in \mathbb{N}$, then we easily check that $\Phi((T_n^{it}))$ is a unitary operator, but $\Phi((|T_n|))^{it}$ is not a unitary operator because $0 \in \sigma_p(|\hat{A}|)$ since $0 \in \sigma_p(\hat{A}) = \sigma_{app}(\hat{A}) = \sigma_{app}(A)$. Thus, Property (ii) cannot be extended to whole \mathcal{P}_+ .

(3) By keeping the notations used in proof of (iii), we can remark that U_Φ may be different from U . Indeed, let us consider any one-to-one operator $T \in B(H)$ whose range is dense in H but not equal to H , and set $T_n = T$ for each $n \in \mathbb{N}$. Then, the operators U_n are all unitary and hence U_Φ is a unitary operator. But U can not be a unitary operator since $0 \in \sigma_p(\hat{T})$. Thus, Φ does not preserve the polar decomposition.

The next result establishes some links between the iterated Aluthge fields associated with T and the powers of T .

Theorem 2.5. *Let $T \in B(H)$ be an operator of norm 1, then we have*

- (i) $\mathcal{N}(I - |T^{n+1}|) = \mathcal{N}(I - |\Delta_z^n(T)|)$ for each $z \in \mathcal{S}$;
- (ii) Let $z \in \mathcal{S}$, then $\|\Delta_z^n(T)\| = 1$ if and only if $\|T^{n+1}\| = 1$.

Proof. (i) We proceed by induction. Denote by $T = U|T|$ the polar decomposition of T . For $n = 0$, it is obvious. Assume that assertion (i) is true for some $n \in \mathbb{N}$. Let x be a unit vector in $\mathcal{N}(I - |T^{n+2}|)$, then we have $1 = \|T^{n+1}U|T|x\| \leq \| |T|x\| \leq 1$. Thus, we get $1 = \| |T|x\| = \| |T|\|$ which forces that $|T|x = x$, because $|T|$ is a positive operator. We infer that

$$\begin{aligned} 1 &= \|T^{n+2}x\| = \|T^{n+1}U|T|^{1-z}|T|^zx\| = \|U|T|^{1-z}\Delta_z^{n+1}(T)|T|^zx\| \\ &= \|U|T|^{1-z}\Delta_z^{n+1}(T)x\| \leq \|\Delta_z(T)^{n+1}x\| = \|\Delta_z(T)^{n+1}|x\| \leq \|\Delta_z(T)\| \leq 1. \end{aligned}$$

Set $A = \Delta_z(T)$, then we see that $\|A\| = 1$ and that $x \in \mathcal{N}(I - |A^{n+1}|)$. We can apply the induction assumption for A , this yields $x \in \mathcal{N}(I - |\Delta_z^n(A)|) = \mathcal{N}(I - |\Delta_z^{n+1}(T)|)$. Conversely, assume that

$$x \in \mathcal{N}(I - |\Delta_z^{n+1}(T)|) = \mathcal{N}(I - |\Delta_z^n(A)|).$$

By the induction assumption, it implies that $x \in \mathcal{N}(I - |A^{n+1}|)$, then we have $1 = \| |A^{n+1}|x\| = \|A^{n+1}x\| = \|\Delta_z(T)^{n+2}x\|$. We write $z = a + ib$ with $a \in]0, 1[$. We get $1 = \|\Delta_z(T)^{n+1}|T|^zU|T|^{1-z}x\| \leq \| |T|^{1-z}x\| \leq \| |T|^{1-a}x\| \leq 1$. Since $|T|^{1-a}$ is a positive operator, it leads to $|T|^{1-a}x = x$. As $1 - a > 0$, we see that $\mathbb{C}[t^{1-a}]$ is dense in the space $C[0, 1]$ of continuous functions on $[0, 1]$. Then, the continuous functional calculus associated with $|T|$ ensures that $|T|^u x = x$ for any $u \in \mathcal{S}$. Then,

$$\begin{aligned} 1 &= \|\Delta_z(T)^{n+1}|T|^zU|T|^{1-z}x\| = \| |T|^zT^{n+1}U|T|^{1-z}x\| \\ &= \| |T|^zT^{n+1}U|T|x\| = \| |T|^zT^{n+2}x\| \leq \|T^{n+2}x\| = \| |T|^{n+2}|x\| \leq 1, \end{aligned}$$

which implies that $x \in \mathcal{N}(I - |T^{n+2}|)$ and this completes the induction proof.

(ii) Assume that $\|\Delta_z^n(T)\| = 1$. Using Proposition 2.3 ((i), (iii) and (v)), we obtain

$$1 \in \sigma_{app}(|\Delta_z^n(T)|) = \sigma_p(\varphi(|\Delta_z^n(T)|)) = \sigma_p(|\varphi(\Delta_z^n(T))|) = \sigma_p(|\Delta_z^n(\hat{T})|).$$

Hence, there exists a unit vector $\hat{x} \in \mathcal{H}$ which belongs to $\mathcal{N}\left(I - |\Delta_z^n(\hat{T})|\right) = \mathcal{N}\left(I - |\hat{T}^{n+1}|\right)$ (by (i)). Therefore, we get $x = |\hat{T}^{n+1}|x$ and hence $1 = \|\hat{T}^{n+1}\| = \|T^{n+1}\|$. Using again (i) and Proposition 2.3, we prove in a similar way that the converse implication holds. \square

2.0.3. Support lines and support subspaces. Recall that a infinite straight line L in \mathbb{C} may be written as : $\Re(e^{i\theta}z) = r$ with $(r, \theta) \in \mathbb{R}^2$. Let C be a convex set in \mathbb{C} , we say that L is a support line of C if $\Re(e^{i\theta}z) \leq r$ for every $z \in C$ and $L \cap \partial C \neq \emptyset$. Let $T \in B(H)$, the well known theorem of Toeplitz and Hausdorff asserts that the numerical range $W(T)$ is convex. Therefore, we can consider support lines associated with $W(T)$ and observe that we have

Proposition 2.6. *Let $T \in B(H)$ and L a support line for $W(T)$ whose equation is $\Re(e^{i\theta}z) = r$, then the set of points $x \in H$ satisfying $\Re(e^{i\theta}\langle Tx | x \rangle) = r \|x\|^2$ is a closed vectorial subspace of H .*

Proof. Since L is a support line for $W(T)$, we see that the operator $rI - \Re(e^{i\theta}T)$ is a positive operator and then admits a square root. Thus, we have

$$\langle (rI - \Re(e^{i\theta}T))x | x \rangle = \|\sqrt{rI - \Re(e^{i\theta}T)}x\|^2.$$

It easily implies that the set of points $x \in H$ satisfying $\Re(e^{i\theta}\langle Tx | x \rangle) = r\|x\|^2$ is exactly $\mathcal{N}(rI - \Re(e^{i\theta}T))$, and the conclusion follows. \square

Such subspaces will be called *support subspaces* for T .

3. ALUTHGE FIELD OF OPERATORS AND NUMERICAL RANGE PROPERTIES

The following result is due to the first author. Let $T \in B(H)$, the most striking fact is that the two fields (left and right), canonically associated with the Aluthge field $(\Delta_z(T))_{z \in \bar{\mathcal{S}}}$ and a support subspace, are constant on each horizontal segment.

Theorem 3.1. *Let $T \in B(H)$, $z_0 \in \mathcal{S}$ and $\lambda \in W(\Delta_{z_0}(T)) \cap \partial W(T)$, then there exists a support subspace E for $\Delta_{z_0}(T)$ associated with a support line containing λ , not reduced to $\{0\}$, and such that the two operator fields $z \in \bar{\mathcal{S}} \rightarrow P_E \Delta_z(T)$ and $z \in \bar{\mathcal{S}} \rightarrow \Delta_z(T) P_E$ are constant on each horizontal segment contained in $\bar{\mathcal{S}}$. Moreover, if T is one-to-one and has a dense range, then the operator $A := P_E \Delta_z(T) P_E$ is normal. In particular, we always have*

$$W(\Delta_z(T)) \subseteq W(T)$$

for every $z \in \bar{\mathcal{S}}$ and any $T \in B(H)$.

Proof. Let $z = a + ib$ be in $\bar{\mathcal{S}}$ and λ be in $W(\Delta_z(T))$, then there exists a unit vector $x \in H$ such that $\lambda = \langle \Delta_z(T)x | x \rangle$. Using Proposition 2.1 and Proposition 2.2, we can write

$$\lambda = \langle \Delta_a(T)|T|^{-ib}x | |T|^{-ib}x \rangle = \langle \Delta_a(T) [|T|^{-ib}x + Qx] | [|T|^{-ib}x + Qx] \rangle$$

where $Q = P_{\mathcal{N}(T)} = I - P$. Observe that $\||T|^{-ib}x + Qx\|^2 = \|P|T|^{-ib}Px + Qx\|^2 = \|P|T|^{-ib}Px\|^2 + \|Qx\|^2 = \||T|^{-ib}Px\|^2 + \|Qx\|^2 = \|Px\|^2 + \|Qx\|^2 = \|x\|^2 = 1$. Therefore $\lambda \in W(\Delta_a(T))$. Conversely, assume that $\lambda = \langle \Delta_a(T)x | x \rangle$, in a

similar way we can see that $\lambda = \langle \Delta_z(T) [|T|^{ib}x + Qx] \mid [|T|^{ib}x + Qx] \rangle$ with $\| |T|^{ib}x + Qx \| = 1$, and hence $\lambda \in W(\Delta_z(T))$. Consequently, for each $z \in \overline{\mathcal{S}}$, we have

$$W(\Delta_z(T)) = W(\Delta_{\Re(z)}(T)). \quad (3.1)$$

From (3.1) and the fact that $\Delta_z(T) = |T|^{i\Im(z)} \Delta_{\Re(z)}(T) |T|^{-i\Im(z)}$, it suffices to prove Theorem 3.1 in the case where $z_0 = \alpha_0 \in]0, 1[$.

On the one hand, since $W(T)$ is convex and $\lambda \in \partial W(T)$, we know that there exists a support line L for $W(T)$ which contains λ . On the other hand, we have $\Delta_\beta(e^{i\theta}R) = e^{i\theta} \Delta_\beta(R)$ and $W(e^{i\theta}R) = e^{i\theta}W(R)$ for any triple $(R, \beta, \theta) \in B(H) \times [0, 1] \times \mathbb{R}$. Then, we are reduced to the case where there exists a unit vector $x_0 \in H$ and a real number r such that

$$\forall z \in W(T) \quad \Re(z) \leq r, \lambda = \langle \Delta_{\alpha_0}(T)x_0 \mid x_0 \rangle \text{ and } \Re(\lambda) = r. \quad (3.2)$$

Since $W(\Delta_\alpha(T)) \subseteq \overline{W(T)}$ for any $\alpha \in [0, 1]$, L is still a support line for $\Delta_{\alpha_0}(T)$ ($\lambda \in W(\Delta_{\alpha_0}(T))$). Let E be the support subspace for $\Delta_{\alpha_0}(T)$ associated with L (now L is determined by the equation $\Re(z) = r$). Observe that E is not reduced to $\{0\}$ because $x_0 \in E$. Let $T = U|T|$ be the polar decomposition of T and let x be a unit vector in H , we consider the function f defined on the closed strip $\overline{\mathcal{S}}$ by setting

$$f(z) = \exp(\langle |T|^z U |T|^{1-z} x \mid x \rangle). \quad (3.3)$$

Using Proposition 4.2 we see that the two operator fields $z \rightarrow |T|^z$ and $z \rightarrow |T|^{1-z}$ are strongly continuous on $\overline{\mathcal{S}}$. Since the strong convergence is preserved by products, it implies the strong continuity of the operator field $z \rightarrow |T|^z U |T|^{1-z}$ and hence the continuity of f on $\overline{\mathcal{S}}$. Besides, f is analytic in \mathcal{S} .

Let $s \in \mathbb{R}$. On the one hand, since $\Re(T) \leq rI$ we have

$$\begin{aligned} |f(is)| &= \exp(\langle \Re(|T|^{is} U |T|^{1-is}) x \mid x \rangle) \\ &= \exp(\langle \Re(T) |T|^{-is} x \mid |T|^{-is} x \rangle) \leq \exp(r \| |T|^{-is} x \|^2) \leq \exp(r). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} |f(1-is)| &= \exp(\langle \Re(|T|^{1-is} U |T|^{is}) x \mid x \rangle) \\ &= \exp(\langle \Re(|T|U) |T|^{is} x \mid |T|^{is} x \rangle) \\ &= \exp(\langle U^* \Re(T) U |T|^{is} x \mid |T|^{is} x \rangle) \leq \exp(r \| U |T|^{-is} x \|^2) \leq \exp(r). \end{aligned}$$

Then, the Hadamard three-lines theorem tells us that

$$|f(z)| \leq (\exp(r))^{\Re(z)} (\exp(r))^{1-\Re(z)} = \exp(r)$$

for any $z \in \overline{\mathcal{S}}$. This inequality leads to the next operatorial inequality

$$\Re(|T|^z U |T|^{1-z}) \leq rI, \quad (3.4)$$

valid for any $z \in \overline{\mathcal{S}}$. Let x be a unit vector in E and let g be the function on $\overline{\mathcal{S}}$ given by $g(z) = \langle \Re(|T|^z U |T|^{1-z}) x \mid x \rangle$. As x is a unit vector in E , we have $g(\alpha_0) = \langle \Re(\Delta_{\alpha_0}(T)) x \mid x \rangle = r$. Using (3.4), we derive that g is a harmonic function in the open set \mathcal{S} that is bounded above and attains its maximum at

$\alpha_0 \in \mathcal{S}$, therefore g is constant in \mathcal{S} , and by continuity, in fact in all $\overline{\mathcal{S}}$. Thus, we get

$$\Re(|T|^z U |T|^{1-z}) y = ry \quad (3.5)$$

for every $y \in E$ and every $z \in \overline{\mathcal{S}}$.

Let t be a real number and let $y \in E$, then (3.5) ensures that

$$|T|^{-it} \Re(\Delta_{\alpha_0}(T)) |T|^{it} y = \Re(\Delta_{\alpha_0-it}(T)) y = ry.$$

As $|T|^{it} |T|^{-it} = P_{\overline{\mathcal{R}(|T|)}}$ and $\mathcal{R}(\Re(\Delta_{\alpha_0}(T))) \subseteq \overline{\mathcal{R}(|T|)}$, we deduce that $|T|^{it} y \in E$ for any $t \in \mathbb{R}$. Applying Proposition 2.2 (vi), we derive that $|T|$ is necessarily of the form $|T| = L_1 \oplus L_2$ with respect the orthogonal sum $H = E \oplus E^\perp$, and where L_1 and L_2 are two positive operators. Let

$$U = \begin{pmatrix} U_1 & W \\ V & U_2 \end{pmatrix}$$

be the matrix representation relative to the direct sum $H = E \oplus E^\perp$ of the partial isometry U associated with the polar decomposition $T = U|T|$. Then, we get

$$\Re(\Delta_\beta(T)) = \frac{1}{2} \begin{pmatrix} L_1^\beta U_1 L_1^{1-\beta} + L_1^{1-\beta} U_1^* L_1^\beta & L_1^\beta W L_2^{1-\beta} + L_1^{1-\beta} V^* L_2^\beta \\ L_2^\beta V L_1^{1-\beta} + L_2^{1-\beta} W^* L_1^\beta & L_2^\beta U_2 L_2^{1-\beta} + L_2^{1-\beta} U_2^* L_2^\beta \end{pmatrix} \quad (3.6)$$

for any $\beta \in [0, 1]$. According to (3.5), we have $\Re(\Delta_\beta(T)) y = ry$ for any couple $(\beta, y) \in [0, 1] \times E$. It forces $\Re(\Delta_\beta(T))$ to be of the form

$$\Re(\Delta_\beta(T)) = \begin{pmatrix} rI_1 & 0 \\ 0 & * \end{pmatrix}. \quad (3.7)$$

Using (3.6) and (3.7), we obtain the following system of equations

$$\begin{cases} L_1^\beta U_1 L_1^{1-\beta} + L_1^{1-\beta} U_1^* L_1^\beta = 2rI_1 \\ L_2^\beta V L_1^{1-\beta} + L_2^{1-\beta} W^* L_1^\beta = 0 \end{cases} \quad (3.8)$$

for every $\beta \in [0, 1]$. Let $\alpha \in [0, 1]$, then by successively taking $\beta = \alpha$ and $\beta = 1 - \alpha \in [0, 1]$ in the first equation of the system (3.8), we get the system of equations

$$\begin{cases} L_1^\alpha U_1 L_1^{1-\alpha} + L_1^{1-\alpha} U_1^* L_1^\alpha = 2rI_1 \\ L_1^{1-\alpha} U_1 L_1^\alpha + L_1^\alpha U_1^* L_1^{1-\alpha} = 2rI_1 \end{cases} \quad (3.9)$$

Summing the two equations of the system (3.9) we get

$$L_1^\alpha \Re(U_1) L_1^{1-\alpha} + L_1^{1-\alpha} \Re(U_1) L_1^\alpha = 2rI_1. \quad (3.10)$$

Set $A_1 = \Re(U_1)$ and $P_1 = P_{\overline{\mathcal{R}(L_1)}}$, and let $\tilde{A}_1 = P_1 A_1 P_1$ and $\tilde{L}_1 = P_1 L_1 P_1$ be the respective compressions of A_1 and L_1 that act on the subspace $\overline{\mathcal{R}(L_1)}$. Since the operators P_1 and L_1 commute, we derive from (3.10) that

$$\begin{aligned} 2rP_1 &= P_1 L_1^\alpha P_1 A_1 P_1 L_1^{1-\alpha} P_1 + P_1 L_1^{1-\alpha} P_1 A_1 P_1 L_1^\alpha P_1 \\ &= \tilde{L}_1^\alpha \tilde{A}_1 \tilde{L}_1^{1-\alpha} + \tilde{L}_1^{1-\alpha} \tilde{A}_1 \tilde{L}_1^\alpha. \end{aligned}$$

Then, taking $\alpha = 0$ we obtain $2rP_1 = P_1 \tilde{A}_1 \tilde{L}_1 + \tilde{L}_1 \tilde{A}_1 P_1$. Multiplying this last equality by \tilde{L}_1 , once on the left and once on the right, we obtain $2r\tilde{L}_1 P_1 = 2r\tilde{L}_1 = \tilde{L}_1 P_1 \tilde{A}_1 \tilde{L}_1 + \tilde{L}_1^2 \tilde{A}_1 P_1 = \tilde{L}_1 \tilde{A}_1 \tilde{L}_1 + \tilde{L}_1^2 \tilde{A}_1$ and $2r\tilde{L}_1 = P_1 \tilde{A}_1 \tilde{L}_1^2 + \tilde{L}_1 \tilde{A}_1 P_1 \tilde{L}_1 =$

$P_1 \tilde{A}_1 \tilde{L}_1^2 + \tilde{L}_1 \tilde{A}_1 \tilde{L}_1$, from which we deduce that $\tilde{A}_1 \tilde{L}_1^2 = \tilde{L}_1^2 \tilde{A}_1$. Then, using the continuous functional calculus associated with the positive operator \tilde{L}_1 , we derive that

$$\tilde{A}_1 \tilde{L}_1 = \tilde{L}_1 \tilde{A}_1. \quad (3.11)$$

Subtracting the two equations of the system (3.9), we obtain $L_1^\alpha \mathfrak{S}(U_1) L_1^{1-\alpha} + L_1^{1-\alpha} (-\mathfrak{S}(U_1)) L_1^\alpha = 0$. Set $A_2 = \mathfrak{S}(U_1)$ and denote by \tilde{A}_2 the compression of A_2 to $\overline{\mathcal{R}(L_1)}$, we see that $\tilde{L}_1^\alpha \tilde{A}_2 \tilde{L}_1^{1-\alpha} = \tilde{L}_1^{1-\alpha} \tilde{A}_2 \tilde{L}_1^\alpha$. Then, making α to be 0 we have $P_1 \tilde{A}_2 \tilde{L}_1 = \tilde{L}_1 \tilde{A}_2 P_1$, and hence

$$\tilde{A}_2 \tilde{L}_1 = \tilde{L}_1 \tilde{A}_2. \quad (3.12)$$

Taking into account (3.11) and (3.12), we finally obtain

$$\tilde{U}_1 \tilde{L}_1 = \tilde{L}_1 \tilde{U}_1 \quad (3.13)$$

where \tilde{U}_1 is the compression of U_1 to $\overline{\mathcal{R}(L_1)}$.

Now, we exploit the second equation of (3.8). Taking successively $\beta = \alpha$ and $\beta = 1 - \alpha$ in this equation, we deduce

$$\begin{cases} \tilde{L}_2^\alpha \tilde{V} \tilde{L}_1^{1-\alpha} + \tilde{L}_2^{1-\alpha} \tilde{W}^* \tilde{L}_1^\alpha = 0 \\ \tilde{L}_2^{1-\alpha} \tilde{V} \tilde{L}_1^\alpha + \tilde{L}_2^\alpha \tilde{W}^* \tilde{L}_1^{1-\alpha} = 0 \end{cases} \quad (3.14)$$

where $P_2 = P_{\overline{\mathcal{R}(L_2)}}$, $\tilde{L}_2 = P_2 L_2 P_2$, $\tilde{V} = P_2 V P_1$ and $\tilde{W} = P_1 W P_2$. Set $C = \tilde{V} + \tilde{W}^*$, by summing the two equations of the system (3.14) we get $\tilde{L}_2^\alpha C \tilde{L}_1^{1-\alpha} + \tilde{L}_2^{1-\alpha} C \tilde{L}_1^\alpha = 0$. Thus we obtain

$$C \tilde{L}_1 = -\tilde{L}_2 C \quad (3.15)$$

by taking $\alpha = 0$. Similarly, by subtraction, we get from (3.14)

$$D \tilde{L}_1 = \tilde{L}_2 D \quad (3.16)$$

where $D = \tilde{V} - \tilde{W}^*$. Using (3.15) and (3.16), on the one hand we can see that $(D + C) \tilde{L}_1 = \tilde{L}_2 (D - C)$, and hence

$$\tilde{V} \tilde{L}_1 = -\tilde{L}_2 \tilde{W}^*. \quad (3.17)$$

On the other hand, we derive that $(D - C) \tilde{L}_1 = \tilde{L}_2 (D + C)$, and hence

$$\tilde{L}_2 \tilde{V} = -\tilde{W}^* \tilde{L}_1. \quad (3.18)$$

Combining (3.17) and (3.18), we see that $\tilde{V} \tilde{L}_1^2 = -\tilde{L}_2 \tilde{W}^* \tilde{L}_1 = \tilde{L}_2^2 \tilde{V}$ which implies $\tilde{V} p(\tilde{L}_1^2) = p(\tilde{L}_1^2) \tilde{V}$ for any complex polynomial p . Let $\beta \in]0, 1]$ and consider a sequence (p_n) of complex polynomials converging to the function φ in the space $C[0, \|T\|]$, where φ is defined by setting $\varphi(t) = t^\beta$. Using the functional calculus associated with the two positive operators \tilde{L}_1 and \tilde{L}_2 , we easily get

$$\tilde{V} \tilde{L}_1^\beta = \tilde{L}_2^\beta \tilde{V}. \quad (3.19)$$

We also observe that $\tilde{W}^* \tilde{L}_1^2 = -\tilde{L}_2 \tilde{V} \tilde{L}_1 = \tilde{L}_2^2 \tilde{W}^*$. By working in the same way, we come to the second intertwining relation

$$\tilde{W}^* \tilde{L}_1^\beta = \tilde{L}_2^\beta \tilde{W}^* \quad (3.20)$$

valid for any $\beta \in]0, 1[$. Let $\alpha \in]0, 1[$, by substituting the two intertwining relations (3.19) and (3.20) into the second equation of the system (3.14), we see that $0 = \tilde{L}_2^{1-\alpha} \tilde{V} \tilde{L}_1^\alpha + \tilde{L}_2^\alpha \tilde{W}^* \tilde{L}_1^{1-\alpha} = \tilde{L}_2^\alpha \tilde{L}_2^{1-\alpha} \tilde{V} + \tilde{L}_2^{1-\alpha} \tilde{L}_2^\alpha \tilde{W}^* = \tilde{L}_2 (\tilde{V} + \tilde{W}^*)$. Since by construction we have $\mathcal{R}(\tilde{V} + \tilde{W}^*) \subseteq \overline{\mathcal{R}(L_2)}$, we deduce that $(\tilde{V} + \tilde{W}^*)x \in \mathcal{N}(L_2) \cap \overline{\mathcal{R}(L_2)} = \{0\}$ for any $x \in E$, and hence

$$\tilde{W} = -\tilde{V}^*. \quad (3.21)$$

Combining (3.13), (3.19), (3.20) and (3.21), we successively obtain

$$\begin{aligned} L_1^\alpha U_1 L_1^{1-\alpha} &= \tilde{L}_1^\alpha \tilde{U}_1 \tilde{L}_1^{1-\alpha} = \tilde{U}_1 \tilde{L}_1 = P_1 U_1 L_1, \\ L_1^\alpha W L_2^{1-\alpha} &= \tilde{L}_1^\alpha \tilde{W} \tilde{L}_2^{1-\alpha} = \tilde{L}_1^\alpha \tilde{L}_1^{1-\alpha} \tilde{W} \\ &= \tilde{L}_1 \tilde{W} = -\tilde{L}_1 \tilde{V}^* = -L_1 P_1 V^* P_2 = -L_1 V^* P_2, \end{aligned}$$

From the above equalities, the strong continuity of $(\Delta_z(T))_{z \in \overline{\mathcal{S}}}$ and the fact that $\Delta_z(T) = |T|^{i\Im(z)} \Delta_{\Re(z)}(T) |T|^{-i\Im(z)}$, we obtain that the two operator fields $z \in \overline{\mathcal{S}} \rightarrow P_E \Delta_z(T)$ and $z \in \overline{\mathcal{S}} \rightarrow \Delta_z(T) P_E$ are constant on each horizontal segment contained in $\overline{\mathcal{S}}$.

Now, assume that T is one-to-one and has a dense range. Then, the operator U in the polar decomposition of T is unitary. Therefore, using (3.21) we derive that $I_1 = U_1 U_1^* + W W^* = U_1 U_1^* + V^* V$, and that $I_1 = U_1^* U_1 + V^* V$. Thus U_1 is normal (in a similar way, we can prove that U_2 is also normal). In this case $U_1 = \tilde{U}_1$ commutes with $L_1 = \tilde{L}_1$, and then $A = P_1 U_1 L_1 = U_1 L_1$ is normal.

Let $T \in B(H)$, the last step in the proof consists in showing that

$$W(\Delta_z(T)) \subseteq W(T)$$

for every $z \in \overline{\mathcal{S}}$. Taking into account (3.1), we are reduced to prove this inclusion for $z = \alpha_0 \in [0, 1]$. When $\alpha_0 \in \{0, 1\}$, we easily check this inclusion. Thus, we may assume that $\alpha_0 \in]0, 1[$. Since $W(\Delta_{\alpha_0}(T)) \subseteq \overline{W(T)}$, we can suppose that $\lambda \in W(\Delta_{\alpha_0}(T)) \cap \partial W(T)$. In this case we have shown that there exists a unit vector w such that $\lambda = \langle \Delta_{\alpha_0}(T)w \mid w \rangle$, and a support subspace E containing w such that the operator field $(\Delta_\alpha(T)P_E)_{\alpha \in [0,1]}$ is constant. We derive that

$$\lambda = \langle \Delta_{\alpha_0}(T)w \mid w \rangle = \langle \Delta_{\alpha_0}(T)P_E w \mid w \rangle = \langle \Delta_0(T)P_E w \mid w \rangle = \langle P_{\overline{\mathcal{R}(T)}} T w \mid w \rangle.$$

Two possibilities occurs:

- Firstly, if T is one-to-one, then $P_{\overline{\mathcal{R}(T)}} = I$ and there is nothing to do.
- Secondly, if T is not one-to-one, then $0 \in W(T)$. So, we may assume that $\lambda \neq 0$ and hence $P_{\overline{\mathcal{R}(T)}} w \neq 0$. We can write

$$\lambda = \left(1 - \|P_{\overline{\mathcal{R}(T)}} w\|^2\right) \times 0 + \|P_{\overline{\mathcal{R}(T)}} w\|^2 \left\langle T \left(\frac{P_{\overline{\mathcal{R}(T)}} w}{\|P_{\overline{\mathcal{R}(T)}} w\|} \right) \mid \frac{P_{\overline{\mathcal{R}(T)}} w}{\|P_{\overline{\mathcal{R}(T)}} w\|} \right\rangle,$$

and then the Toeplitz-Hausdorff theorem tells us that $\lambda \in W(T)$. This achieves the proof of Theorem 3.1. \square

Remark 3.2. (1) For $\alpha = \frac{1}{2}$, the inclusion $W(\Delta(T)) \subseteq W(T)$ given by Theorem 3.1 answers positively to Conjecture 1.9 of [18].

(2) Assume that T is one-to-one and has a dense range. Let E be a support subspace for $\Delta_z(T)$ ($z \in \mathcal{S}$) which is associated with a complex scalar λ and U be the partial isometry given by the polar decomposition of T . Then, we have seen in the proof of Theorem 3.1 that U takes the form

$$U = \begin{pmatrix} U_1 & * \\ * & U_2 \end{pmatrix},$$

with respect to the orthogonal decomposition $H = E \oplus E^\perp$, where the two operators U_1 and U_2 are normal.

4. SPECTRAL PROPERTIES FOR ALUTHGE OPERATOR FIELDS

4.0.1. *Different type of spectra and Aluthge operator fields.* Some spectral results are known for $\Delta_{\frac{1}{2}}(T) := \Delta(T)$ where $T \in B(H)$ (see for instance [17], and Theorem 1.3 and Theorem 1.5 of [18]). In the next theorem, we extend a number of useful spectral properties of $\Delta(T)$ to the Aluthge operator field $(\Delta_z(T))_{z \in \overline{\mathcal{S}}}$ and establish some new spectral properties in this context.

Theorem 4.1. *Let T be an operator acting on a Hilbert space H . Then the Aluthge operator field associated with T satisfies the following properties:*

- (i) *We have $\sigma_{surj}(\Delta_z(T)) \setminus \{0\} = \sigma_{surj}(T) \setminus \{0\}$ and $\sigma_{surj}(T) \subseteq \sigma_{surj}(\Delta_z(T))$ for any $z \in \overline{\mathcal{S}}$. The last inclusion may be strict.*
- (ii) *$\sigma(\Delta_z(T)) = \sigma(T)$ for any $z \in \overline{\mathcal{S}}$;*
- (iii) (a) *For any $z \in \overline{\mathcal{S}}$ we have $\sigma_e(\Delta_z(T)) = \sigma_e(T)$;*
 (b) *Let $z \in \overline{\mathcal{S}} \setminus [1 + i\mathbb{R}]$ then we have $\sigma_{le}(\Delta_z(T)) = \sigma_{le}(T)$. When $z \in 1 + i\mathbb{R}$, we have $\sigma_{le}(\Delta_z(T)) \subseteq \sigma_{le}(T)$ and the inclusion may be strict;*
 (c) *For any $z \in \overline{\mathcal{S}}$, we have $\sigma_{re}(T) \subseteq \sigma_{re}(\Delta_z(T))$. Moreover the inclusion may be strict for all $z \in \overline{\mathcal{S}}$;*
- (iv) *$\sigma_w(\Delta_z(T)) = \sigma_w(T)$ for any $z \in \overline{\mathcal{S}}$;*
- (v) *$\sigma_b(\Delta_z(T)) = \sigma_b(T)$ for any $z \in \overline{\mathcal{S}}$;*
- (vi) *$\sigma_p(\Delta_z(T)) = \sigma_p(T)$ for any $z \in \overline{\mathcal{S}}$;*
- (vii) *$\sigma_{app}(\Delta_z(T)) = \sigma_{app}(T)$ for any $z \in \overline{\mathcal{S}} \setminus [1 + i\mathbb{R}]$. If $z \in 1 + i\mathbb{R}$, we have $\sigma_{app}(\Delta_z(T)) \subseteq \sigma_{app}(T)$ and the inclusion may be strict;*
- (viii) *$\sigma_{res}(\Delta_z(T)) = \sigma_{res}(T)$ for each $z \in i\mathbb{R}$ and $\sigma_{res}(\Delta_z(T)) \subseteq \sigma_{res}(T)$ for any $z \in \overline{\mathcal{S}} \setminus [i\mathbb{R}]$. Moreover, in this latter case, the inclusion may be strict.*

Proof. (i) Let A, B be two operators acting on H and let λ be a nonzero complex number such that $\lambda I - AB$ is surjective. Then $\lambda I - AB$ admits a right inverse $X \in B(H)$. Set $Y = \frac{1}{\lambda} [I + BXA]$, then we have $(\lambda I - BA)Y = \frac{1}{\lambda} [\lambda I - BA + B(\lambda I - AB)XA] = \frac{1}{\lambda} [\lambda I - BA + BA] = I$, thus $\lambda I - AB$ admits a right inverse. As the operators A and B play a symmetric role, we deduce that $\sigma_{surj}(AB) \setminus \{0\} = \sigma_{surj}(BA) \setminus \{0\}$. Then, using Proposition 2.1 (i), we see

that $\sigma_{surj}(\Delta_z(T)) \setminus \{0\} = \sigma_{surj}(T) \setminus \{0\}$. Now, suppose that $\Delta_z(T)$ is onto, then it implies that $|T|^z$ is onto and that $|T|^{\bar{z}}$ is one-to-one. By Proposition 2.2 (ii), we have $\{0\} = \mathcal{N}(|T|^{\bar{z}}) = \mathcal{N}(|T|^z)$, we finally get that $|T|^z$ is invertible. Thus $T = (|T|^z)^{-1} \Delta_z(T) |T|^z$ is onto. Taking into account the two previous properties we get $\sigma_{surj}(T) \subseteq \sigma_{surj}(\Delta_z(T))$. Let T be a pure coisometry, then we have $0 \notin \sigma_{surj}(T)$. As in [7] (Proposition 4), we can see that $\Delta_z(T) = T^* T^2$ which is clearly not surjective and hence $0 \in \sigma_{surj}(\Delta_z(T))$. Therefore, the inclusion $\sigma_{surj}(T) \subseteq \sigma_{surj}(\Delta_z(T))$ may be strict.

(ii) Since $\sigma(|T|^z U |T|^{1-z}) \setminus \{0\} = \sigma(U |T|^{1-z} |T|^z) \setminus \{0\} = \sigma(T) \setminus \{0\}$, it remains to prove that T is invertible if and only if $\Delta_z(T)$ is invertible. Assume that T is invertible, then we successively see that $|T|$ and $|T|^z$ are invertible, hence $\Delta_z(T) = (|T|^z)^{-1} T |T|^z$ is invertible. Conversely, assume that $\Delta_z(T)$ is invertible, then there exists $X \in B(H)$ such that $|T|^z U |T|^{1-z} X = X |T|^z U |T|^{1-z} = I$, it implies that $|T|^z$ is onto and $\{0\} = \mathcal{N}(|T|^{\bar{z}}) = \mathcal{N}(|T|^z)$, thus $|T|^z$ is invertible and hence $T = |T|^z \Delta_z(T) (|T|^z)^{-1}$ is invertible.

(iii) Let $\lambda \in \mathbb{C}^*$ and suppose that there exists X such that $X(\lambda I - AB) = I + K_1$ (resp. $(\lambda I - AB)X = I + K_2$), where K_1, K_2 belong to the space $\mathcal{K}(H)$ of all compact operators on H . As in the proof of (i), set $Y = \frac{1}{\lambda} [I + BXA]$, then we easily check that $Y(\lambda I - BA) = I + \frac{1}{\lambda} BK_1 A$ (resp. $(\lambda I - BA)Y = I + \frac{1}{\lambda} BK_2 A$) with $\frac{1}{\lambda} BK_i A \in \mathcal{K}(H)$ ($i \in \{1, 2\}$). It follows that $\sigma_e(\Delta_z(T)) \setminus \{0\} = \sigma_e(T) \setminus \{0\}$, $\sigma_{le}(\Delta_z(T)) \setminus \{0\} = \sigma_{le}(T) \setminus \{0\}$ and $\sigma_{re}(\Delta_z(T)) \setminus \{0\} = \sigma_{re}(T) \setminus \{0\}$. Consequently, To prove (a) (b) and (c), it suffices to examine what happens at the point 0.

(a) In the sequel, we denote by \mathcal{F} (resp. \mathcal{SF}) the class of Fredholm (resp. semi-Fredholm) operators in $B(H)$. Assume that $0 \in \rho_e(T)$, then $T \in \mathcal{F}$ and hence $|T| \in \mathcal{F}$ which implies in turn that $U \in \mathcal{F}$. By Proposition 2.2 (v), we see that $|T|^u \in \mathcal{F}$ for any $u \in \bar{\mathcal{S}}$. Therefore $\pi(\Delta_z(T)) = \pi(|T|^z) \pi(U) \pi(|T|^{1-z})$ is invertible in $\mathcal{C}(H)$ as a product of elements of $GL(\mathcal{C}(H))$, equivalently $0 \in \rho_e(\Delta_z(T))$. Conversely, suppose that $0 \in \rho_e(\Delta_z(T))$, then we derive that $|T|^z$ and $|T|^{1-z}$ are both in \mathcal{SF} . Since $\Re(z)$ and $\Re(1-z)$ are not both zero, using Proposition 1 (v) we see that $|T|^u \in \mathcal{F}$ for any $u \in \bar{\mathcal{S}}$, and hence $\pi(U) = \pi(|T|^z)^{-1} \pi(\Delta_z(T)) \pi(|T|^{1-z})^{-1} \in GL(\mathcal{C}(H))$. Thus $\pi(T) = \pi(U) \pi(|T|) \in GL(\mathcal{C}(H))$ which gives that $0 \in \rho_e(T)$.

(b) Assume that $0 \in \rho_{le}(T)$, then $|T| \in \mathcal{SF}$ which implies, as we have seen before, that $|T|^u \in \mathcal{F}$ for any $u \in \bar{\mathcal{S}}$. Thus, $\pi(U)$ is left invertible in $\mathcal{C}(H)$ and hence $0 \in \rho_{le}(\Delta_z(T))$ for any $z \in \bar{\mathcal{S}}$. Now, let $z \in \bar{\mathcal{S}} \setminus [1 + i\mathbb{R}]$ and suppose that $0 \in \rho_{le}(\Delta_z(T))$, it implies that $|T|^{1-z}$ is in \mathcal{SF} . As $\Re(1-z) \neq 0$, we get that $|T|^u \in \mathcal{F}$ for any $u \in \bar{\mathcal{S}}$ and hence $U \in \mathcal{SF}$. We derive that $0 \in \rho_{le}(T)$. Summarizing, we obtain $\sigma_{le}(\Delta_z(T)) = \sigma_{le}(T)$ for each $z \in \bar{\mathcal{S}} \setminus [1 + i\mathbb{R}]$ and $\sigma_{le}(\Delta_z(T)) \subseteq \sigma_{le}(T)$ when $z \in 1 + i\mathbb{R}$.

Let $(e_n)_{n \geq 0}$ be the canonical orthonormal basis of the Hilbert space $H := l^2(\mathbb{N})$. One may then define operators A and B in $B(H)$ by setting $Ae_n = \frac{1}{2^n} e_{2n+1}$ and $Be_n = e_{2n}$. We consider the operator T acting on $\mathcal{H} = H \oplus H$ which is given by

the 2×2 matrix

$$T = \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix}.$$

We can check that $|A|e_n = \frac{1}{2^n}e_n$ and that $A^*B = 0$. Notice also that B is an isometry, then we get

$$|T| = \begin{pmatrix} |A| & 0 \\ 0 & I \end{pmatrix}.$$

Let $T = U|T|$ be the polar decomposition of T . Since $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and A is one-to-one, we derive that T is one-to-one, and hence U is an isometry. Taking into account that U is the unique isometry W such that $T = W|T|$, we see that U is necessarily given by

$$U = \begin{pmatrix} 0 & 0 \\ V & B \end{pmatrix}$$

where V is the partial isometry (here V is an isometry) associated with the polar decomposition of $A = V|A|$. Then, for any $z \in 1 + i\mathbb{R}$ we obtain

$$\Delta_z(T) = \begin{pmatrix} 0 & 0 \\ V|A|^{-i\Im(z)} & B \end{pmatrix}.$$

On the one hand, we see that $\Delta_z(T)$ is an isometry and hence $0 \notin \sigma_{le}(\Delta_z(T))$. On the other hand, let Q be the orthogonal projection on the subspace $H \oplus \{0\}$, then we get

$$TQ = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \in \mathcal{K}(\mathcal{H}).$$

Then, from Theorem 1.1 of [11] we deduce that $0 \in \sigma_{le}(T)$. Thus the inclusion $\sigma_{le}(\Delta_z(T)) \subseteq \sigma_{le}(T)$ is strict for each $z \in 1 + i\mathbb{R}$.

(c) Let $z \in \overline{\mathcal{S}}$ and suppose that $0 \in \rho_{re}(\Delta_z(T))$. We infer that $\pi(|T|^z)$ is right invertible in $\mathcal{C}(H)$ and hence invertible (Proposition 2.2 (v)). It successively implies that $\pi(U|T|^{1-z})$ and $\pi(T) = \pi(U|T|^{1-z})\pi(|T|^z)$ are right invertible in $\mathcal{C}(H)$, thus $0 \in \rho_{re}(T)$. Since $\sigma_{re}(\Delta_z(T)) \setminus \{0\} = \sigma_{re}(|T|^z U|T|^{1-z}) \setminus \{0\} = \sigma_{re}(U|T|^{1-z}|T|^z) \setminus \{0\} = \sigma_{re}(T) \setminus \{0\}$, we finally get $\sigma_{re}(T) \subseteq \sigma_{re}(\Delta_z(T))$. Let T be a pure coisometry of infinite dimensional null space. Using Proposition 4 of [7], we can see that $\Delta_z(T)^* = T^{*2}T$ has an infinite dimensional null space, and hence $0 \in \sigma_{re}(\Delta_z(T))$, but clearly $0 \notin \sigma_{re}(T)$ because T is right invertible. It follows that the inclusion may be strict for all $z \in \overline{\mathcal{S}}$.

(iv) We have seen before ((iii) (a)) that $\lambda I - \Delta_z(T) \in \mathcal{F}$ if and only if $\lambda I - T \in \mathcal{F}$. Besides that, we know that $\dim(\mathcal{N}(\lambda I - AB)) = \dim(\mathcal{N}(\lambda I - BA))$ for any non-zero λ , and for any pair $(A, B) \in B(H)$. It leads to $\sigma_w(\Delta_z(T)) \setminus \{0\} = \sigma_w(T) \setminus \{0\}$. Suppose now that T is a Weyl operator, then we have seen that $U \in \mathcal{F}$ and $|T|^u \in \mathcal{F}$ for any $u \in \overline{\mathcal{S}}$ (see (iii) (a)). Then $\Delta_z(T) \in \mathcal{F}$ and using the multiplicative property of the index we get $ind(\Delta_z(T)) = ind(|T|^z) + ind(U) + ind(|T|^{1-z}) = ind(T) = 0$, so $\Delta_z(T)$ is a Weyl operator. In the same manner, we prove that $\Delta_z(T)$ is a Weyl operator implies that T is a Weyl operator. Finally, we have obtained the desired equality $\sigma_w(\Delta_z(T)) = \sigma_w(T)$ for any $z \in \overline{\mathcal{S}}$.

(v) Recall that A is a Browder operator if and only if A is a Fredholm operator and $zI - A$ is invertible for sufficiently small $z \in \mathbb{C} \setminus \{0\}$ ([13] Theorem 7.9.3). Thus, the desired equality follows from (ii) and (iii) (a).

(vi) As $\sigma_p(|T|^z U |T|^{1-z}) \setminus \{0\} = \sigma_p(U |T|^{1-z} |T|^z) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$, it rests to prove that $0 \in \sigma_p(T)$ if and only if $0 \in \sigma_p(\Delta_z(T))$ for any $z \in \overline{\mathcal{S}}$. Assume there exists $x \neq 0$ such that $Tx = 0$. By Proposition 2.2 (ii), we know that $x \in \mathcal{N}(|T|^z) = \mathcal{N}(|T|^{1-z})$, thus $x \in \mathcal{N}(\Delta_z(T))$. Conversely, suppose that $0 \in \sigma_p(\Delta_z(T))$ and let $x \in [H \setminus \{0\}] \cap \mathcal{N}(\Delta_z(T))$, then $0 = U |T|^{1-z} \Delta_z(T)x = T U |T|^{1-z} x$. On the one hand, if $y = U |T|^{1-z} x \neq 0$, we have $\mathcal{N}(T) \neq \{0\}$. On the other hand, $y = 0$, using Proposition 2.2 we see that $|T|^{1-z} x \in \mathcal{N}(U) = \mathcal{N}(|T|) = \mathcal{N}(|T|^z)$ and hence $x \in \mathcal{N}(|T|) = \mathcal{N}(T)$. Consideration of both cases leads to $0 \in \sigma_p(T)$. This concludes the proof of this assertion.

(vii) Having in view that $\sigma_{app}(A) = \sigma_{le}(A) \cup \sigma_p(A)$ for each $A \in B(H)$ and using (iii) (b) and (vi), we get $\sigma_{app}(\Delta_z(T)) = \sigma_{app}(T)$ for any $z \in \overline{\mathcal{S}} \setminus [1 + i\mathbb{R}]$ and $\sigma_{app}(\Delta_z(T)) \subseteq \sigma_{app}(T)$ when $z \in 1 + i\mathbb{R}$. Moreover, let $z \in 1 + i\mathbb{R}$ and consider the operator T acting on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ and defined in (iii) (b). Since $\Delta_z(T)$ is an isometry, we clearly have $0 \notin \sigma_{app}(\Delta_z(T))$ but $\|T(e_n \oplus 0)\| \rightarrow 0$ which implies $0 \in \sigma_{app}(T)$. Therefore the inclusion $\sigma_{app}(\Delta_z(T)) \subseteq \sigma_{app}(T)$ may be strict when $z \in 1 + i\mathbb{R}$;

(viii) We have $\sigma_{res}(|T|^z U |T|^{1-z}) \setminus \{0\} = \sigma_{res}(U |T|^{1-z} |T|^z) \setminus \{0\} = \sigma_{res}(T) \setminus \{0\}$ for each $z \in \overline{\mathcal{S}}$. Let $z = ib \in i\mathbb{R}$, taking into account that T is one-to-one if and only if $\Delta_z(T)$ is also (see (vi)), we remark that $\Delta_z(T) = |T|^{ib} T |T|^{-ib}$ is unitarily similar to T as soon as $0 \in \sigma_{res}(T) \cup \sigma_{res}(\Delta_{ib}(T))$. We infer that $0 \in \sigma_{res}(\Delta_{ib}(T))$ if and only if $0 \in \sigma_{res}(T)$, and hence $\sigma_{res}(\Delta_{ib}(T)) = \sigma_{res}(T)$. From now on, we suppose that $z \in \overline{\mathcal{S}} \setminus [i\mathbb{R}]$. Assume that $0 \in \sigma_{res}(\Delta_z(T))$, then $\Delta_z(T)$ is one-to-one and there exists a non-zero $x \in H$ such that $\Delta_z(T)^* x = 0$. It implies that $|T|^u$ is one-to-one for any $u \in \mathcal{P}_+$, and hence that $U^* |T|^{\bar{z}} x = 0$. Obviously, we have $y = |T|^{\bar{z}} x \neq 0$ and $y \in \mathcal{N}(U^*) = \mathcal{N}(T^*)$. Finally, it leads to $0 \in \sigma_{res}(T)$, thus $\sigma_{res}(\Delta_z(T)) \subseteq \sigma_{res}(T)$ for any $z \in \overline{\mathcal{S}} \setminus [i\mathbb{R}]$.

Now, let $z \in \overline{\mathcal{S}} \setminus [i\mathbb{R}]$, H be $L^2([0, 1]dt)$, R be the positive operator defined by $R(f)(t) = t^{1/\Re(z)} f(t)$ for any $f \in H$. Consider an isometry U acting on H whose range is $\{\mathbf{1}\}^\perp$, and set $T = UR$. We easily check that T is one-to-one and that $T^*(\mathbf{1}) = 0$, so $0 \in \sigma_{res}(T)$. Observe that $R = |T|$ and that $\mathcal{N}(U) = \{0\} = \mathcal{N}(T)$, it follows that $T = UR$ is exactly the polar decomposition of T and then $\Delta_z(T) = R^z U R^{1-z}$. Suppose that $0 \in \sigma_{res}(\Delta_z(T))$, then there exists a non-zero $f \in H$ such that $0 = R^{1-\bar{z}} U^* R^{\bar{z}}(f)$, it implies that $R^{\bar{z}}(f) \in \mathcal{N}(U^*) = \mathbb{C}\mathbf{1}$. So $R^{\bar{z}}(f)$ is almost everywhere equal to a non-zero constant a . It forces the function $\psi : t \mapsto at^{-1}$ to be in $L^2([0, 1]dt)$, a fact which is absurd. Consequently, in this case, the inclusion $\sigma_{res}(\Delta_z(T)) \subseteq \sigma_{res}(T)$ is strict for any $z \in \overline{\mathcal{S}} \setminus [i\mathbb{R}]$. This completes the proof of Theorem 4.1.

□

Remark 4.2. Note that we have proved in passing (see (iv)) that if T is a Fredholm operator, then $\Delta_z(T) \in \mathcal{F}$ for any $z \in \overline{\mathcal{S}}$ and the index is constant on the Aluthge operator field $(\Delta_z(T))_{z \in \overline{\mathcal{S}}}$.

4.0.2. *Generalized Yamazaki formula.* Let $T \in B(H)$ and denote by $r(T)$ its spectral radius. Recall the famous Gelfand formula

$$r(T) = \lim \|T^n\|^{\frac{1}{n}} = \inf_{n \geq 0} \|T^n\|^{\frac{1}{n}}.$$

In [23], T. Yamazaki gives an alternative formula involving the Aluthge transformation $\Delta_{1/2}$:

$$r(T) = \lim \|\Delta_{\frac{1}{2}}^n(T)\| = \inf_{n \geq 0} \|\Delta_{\frac{1}{2}}^n(T)\|.$$

Notice that a simpler proof is given in [2]. See also [8] for other spectral radius formulas via Aluthge transformation. In [3], the authors proved that for any square matrix M and any $\alpha \in]0, 1[$ we have

$$r(M) = \lim \|\Delta_\alpha^n(M)\| = \inf_{n \geq 0} \|\Delta_\alpha^n(M)\|. \quad (4.1)$$

In [21], it is proved that Formula 4.1 holds if we replace M by any invertible operator acting on an infinite dimensional Hilbert space. In the next theorem, we show that we can drop the assumption that T is invertible and we can consider Δ_z for any $z \in \mathcal{S}$.

Theorem 4.3. *Let T be a operator acting on a Hilbert space H , then we have*

$$r(T) = \lim \|\Delta_z^n(T)\| = \inf_{n \geq 0} \|\Delta_z^n(T)\|$$

for any $z \in \mathcal{S}$.

Proof. Firstly, from Proposition 4.1 we know that $\sigma(\Delta_z^n(T)) = \sigma(T)$, so $r(T) = r(\Delta_z^n(T)) \leq \|\Delta_z^n(T)\|$ and hence $r(T) \leq \inf_{n \geq 0} \|\Delta_z^n(T)\|$. Thus from now on, by homogeneity we may assume that $\inf_{n \geq 0} \|\Delta_z^n(T)\| = 1$. We set $z = a + ib$ with $a \in]0, 1[$. Let A, B be two positive operators acting on H and $X \in B(H)$. Recall that Heinz inequality (see [14]) tells us that

$$\|A^r X B^{1-r}\| \leq \|AX\|^r \|XB\|^{1-r}$$

for each $r \in [0, 1]$. Using Heinz inequality, for any $m \geq 1$ we get

$$\begin{aligned} \|\Delta_z(T)^m\| &= \| |T|^z T^{m-1} U |T|^{1-z} \| \leq \| |T|^a T^{m-1} U |T|^{1-a} \| \\ &\leq \| |T| T^{m-1} U \| \| T^{m-1} U |T| \|^{1-a} \leq \| T^m \|^a \| T^m \|^{1-a} = \| T^m \|. \end{aligned} \quad (4.2)$$

By (4.2), we see that the sequence $(\|\Delta_z^n(T)^m\|)_{n \geq 0}$ is decreasing for any $m \in \mathbb{N}^*$. On the one hand, it implies that the operator $R = \Phi((\Delta_z^n(T))_{n \geq 0})$, acting on the ultrapower of \mathcal{H} associated with a Banach limit \mathcal{L} , is well defined. On the other hand, by Proposition 2.3 (i) and by (4.2) we get

$$\|R^m\| = \lim_{n \rightarrow +\infty} \|\Delta_z^n(T)^m\| \leq \|T^m\|,$$

a fact which leads to $r(R) \leq r(T)$. Let $m \geq 1$, observe that $\|\Delta_z^m(R)\| = \lim(\|\Delta_z^{m+n}(T)\|) = \inf_{n \geq 0} \|\Delta_z^n(T)\| = 1 = \|R\|$. Applying Theorem 2.5 (ii), we obtain that $\|R^{m+1}\| = 1$. It follows that $1 = r(R) \leq r(T)$ and finally that $r(T) = 1$. It ends the proof of Theorem 4.3. \square

Remark 4.4. Let $T \in B(H)$ be a one-to-one operator with dense range such that $r(T) < \|T\|$. In this case, we have

$$\|\Delta_{1+it}(T)\| = \| |T|U \| = \|T\| = \|T\| = \|\Delta_{it}(T)\|$$

for any $t \in \mathbb{R}$. If T is one-to-one with dense range, then $\Delta_z(T)$ is still one-to-one with dense range. So, by iteration we obtain $\|\Delta_{1+it}^n(T)\| = \|T\| = \|\Delta_{it}^n(T)\|$ for each $n \in \mathbb{N}$. Therefore, a Yamazaki type formula is not true in general when z belongs to the boundary of \mathcal{S} .

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