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► **To cite this version:**

Long Meng. Regularity of many-body Schrödinger evolution equation and its application to numerical analysis. 2020. hal-02994993v1

HAL Id: hal-02994993

<https://hal.science/hal-02994993v1>

Preprint submitted on 8 Nov 2020 (v1), last revised 20 Oct 2023 (v2)

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Regularity of many-body Schrödinger evolution equation and its application to numerical analysis

Long Meng  *

Abstract

A decade ago, the mixed regularity of stationary many-body Schrödinger equation has been studied by Harry Yserentant through the Pauli Principle and the Hardy inequality (Uncertainty Principle). In this article, we prove that the many-body evolution Schrödinger equation has a similar mixed regularity if the initial data u_0 satisfies the Pauli Principle. By generalization of the Strichartz estimates, our method also applies to the numerical approximation of this problem: based on these mixed derivatives, we design a new approximation which can hugely improve the computing capability especially in quantum chemistry.

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1 Introduction

In this article, we study the existence, mixed regularity and its application to numerical analysis of the following evolution Schrödinger equation:

$$\begin{cases} i\partial_t u = H(t)u, & t \in [-a, a] = I_a, x = (x_1, \dots, x_N) \in (\mathbb{R}^3)^N \\ u(0, x) = u_0(x) \end{cases} \quad (1)$$

with

$$H(t) = \sum_{j=1}^N -\Delta_j - \sum_{j=1}^N \sum_{\mu=1}^M V(x_j, t) + \sum_{k < j}^N W(x_j, x_k).$$

where

$$V(x_j, t) = \sum_{\mu=1}^M \frac{Z_\mu}{|x_j - a_\mu(t)|} \quad (2)$$

and

$$W(x_i, x_j) = \frac{1}{|x_i - x_j|}. \quad (3)$$

In physics and chemistry, this equation is used to describe the quantum mechanical many-body problem in which the electrons and nuclei interact by Coulomb attraction and repulsion forces. It acts on the functions with variables $x_1, \dots, x_N \in \mathbb{R}^3$, the coordinates of given N electrons. The atom μ is positioned at $a_\mu(t) \in \mathbb{R}^3$ dependently on time with the charge Z_μ .

1.1 The Existence of Solution

At the beginning, instead of studying these given potentials V and W , we consider a more general case:

Assumption 1.1. $V(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ satisfies

$$V \in L_{t,\text{loc}}^{\alpha_q}(L^{q/(q-2)}(\mathbb{R}^3)) + L_{t,\text{loc}}^{\beta_q}(L^\infty(\mathbb{R}^3))$$

and $W(x_j, x_k, t) = w(x_j - x_k, t)$ with $w \in \mathbb{R}^3 \times \mathbb{R}$ satisfies

$$w \in L_t^{\alpha_p}(L^{p/(p-2)}(\mathbb{R}^3)) + L_t^{\beta_p}(L^\infty(\mathbb{R}^3)).$$

for some p and q , such that

$$2 \leq p, q < 6$$

and

$$\theta_{\alpha,\beta} > 0$$

with

$$1/\theta_{\alpha,\beta} = \min\{3/p - 1/2 - 1/\alpha_p, 3/q - 1/2 - 1/\alpha_q, 1 - 1/\beta_p, 1 - 1/\beta_q\}. \quad (4)$$

Obviously, the case V and W in Equation (2) satisfies this Assumption, with $p = q = 4$ and $\alpha_p = \alpha_q = \beta_p = \beta_q = \infty$.

In the last century, for the one particle case which means $N = 1$ and $W = 0$, the evolution Schrödinger equation $i\partial_t u = H(t)u$ was well developed, see [16, 18]. In the

case when $H(t) = H_0$ is independent of t and selfadjoint, the Stone theorem guarantees the existence and uniqueness of the unitary group $U_0(t, s) = \exp(-i(t - s)H_0)$ such that $U_0 H^2(\mathbb{R}^N) \subset H^2(\mathbb{R}^N)$. In 1987, Yajima [20] proved the time-dependent case by Duhamel formula and Strichartz estimate, and then the Schrödinger equation with magnetic field [21]. And it is until this century that the existence of one kind of many-body Schrödinger equation was proved, also by Yajima, see [22]. Inspired by his works, we find out another way to prove the existence of the many-body Schrödinger evolution equation, which is in fact equivalent to the method of Yajima, but much easier to deal with the regularity of the Coulombic potential.

Let

$$r_{i,j} = x_i - x_j, \quad D_{i,j} = x_i + x_j,$$

and

$$\mathcal{R}_{i,j}u(r_{i,j}, D_{i,j}, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_N) = u(x_1, \dots, x_N). \quad (5)$$

Then, define the functional space

$$L_{x_i}^{p,2} = L^p(\mathbb{R}_{x_i}^3, L^2((\mathbb{R}^3)^{N-1}))$$

with the norm

$$\|u\|_{L_{x_i}^{p,2}}^p = \int_{\mathbb{R}_{x_i}^3} \left(\int_{(\mathbb{R}^3)^{N-1}} |u|^2 dx_1 \cdots \widehat{dx}_i \cdots dx_N \right)^{p/2} dx_i.$$

We shorten it by $\|u\|_{L_i^{p,2}}$, and define

$$L_{i,j}^{p,2} = L^p(\mathbb{R}_{r_{i,j}}^3, L^2((\mathbb{R}^3)^{N-1}))$$

with the norm

$$\|u\|_{L_{i,j}^{p,2}}^p = \int_{\mathbb{R}_{d_{i,j}}^3} \left(\int_{(\mathbb{R}^3)^{N-1}} |\mathcal{R}_{i,j}u|^2 dD_{i,j} dx_1 \cdots \widehat{dx}_i \cdots \widehat{dx}_j \cdots dx_N \right)^{p/2} dr_{i,j}.$$

The notation \widehat{dx}_j means that the integration over the j^{th} coordinate is omitted. Obviously, $\|u\|_{L_{i,j}^{p,2}} = \|\mathcal{R}_{i,j}u\|_{L_{d_{i,j}}^{p,2}}$.

Then we introduce the following functional space:

$$X(T) = L_t^\infty([0, T], L^2) \bigcap_{i < j} L_t^{\theta_p}([0, T], L_{i,j}^{p,2}) \bigcap_k L_t^{\theta_q}([0, T], L_k^{q,2})$$

with the norm

$$\|u\|_{X(T)} = \max_{\substack{1 \leq i < j \leq N \\ 1 \leq k \leq N}} \left\{ \|u\|_{L_t^\infty(L^2)}, \|u\|_{L_t^{\theta_p}(L_{i,j}^{p,2})}, \|u\|_{L_t^{\theta_q}(L_k^{q,2})} \right\},$$

where $2/\theta_p = 3(1/2 - 1/p)$ and $2/\theta_q = 3(1/2 - 1/q)$. And if $p, q = 2$, then $\theta_p, \theta_q = +\infty$. Herein we use the shorthand notation $X = X(T)$ without confusion.

And we use the notation

$$L_t^{\theta_q}(L_D^{q,2}) \quad (6)$$

to represent the separate functional spaces. If $q = 2$, then

$$L_t^{\theta_q}(L_D^{q,2}) := L_t^\infty(L^2).$$

If $D = \{k\}$, then

$$L_t^{\theta_q}(L_D^{q,2}) := L_t^{\theta_q}(L_k^{q,2}).$$

If $D = \{i, j\}$, then

$$L_t^{\theta_q}(L_D^{q,2}) := L_t^{\theta_q}(L_{i,j}^{q,2}).$$

Taking $U_0(t)$ the free propagator $\exp(it \sum_{j=1}^N \Delta_j)$, we have our first theorem:

Theorem 1.2. *Under the Assumption 1.1, the Equation (1) has a unique solution $u \in X(a)$, for every $u_0 \in L^2((\mathbb{R}^3)^N)$ and $s \in I_a$.*

And there is a constant C only dependent on p, q, V, W with $1/\theta_{\alpha,\beta} > 0$, if T small enough such that $CT^{1/\theta_{\alpha,\beta}}N(N+1) < 1/2$, we have

$$\|u\|_X \lesssim_{p,q} \|u_0\|_{L^2}$$

where $\theta_{\alpha,\beta}$ is defined by Equation (4).

Remark 1.3. *Indeed, the constant C satisfies the Inequality (16).*

Remark 1.4. *For some kinds of potentials V and W , for example the Coulombic potentials V and W which satisfy the Equation (2) and (3) respectively, the case $p, q = 6$ is also correct. We can use the strategy of proof of Theorem 1.5, regard the potentials V and W as $|V|^\alpha|V|^{1-\alpha}$ and $|W|^\alpha|W|^{1-\alpha}$ with $0 \leq \alpha \leq 1$ and introduce other factor \tilde{p} and \tilde{q} . Then we get the case $p, q = 6$.*

1.2 The Regularity under the Fixed Spin States

Nowadays, we return back to electronic evolution equation with V and W satisfying the Equation (2) and (3).

In physics, for electronic systems, or more general fermionic systems, the initial datum u_0 should satisfy the Pauli Exclusion Principle, which means it is of anti-symmetry under the change of electron coordinates for one spin state [13, 23]. If a particle has s spin states, then we label them by the integer

$$\sigma \in \{1, 2, \dots, s\}.$$

Suppose there are N particles and the i^{th} particle has s_i spin states. Then a wave function for these N particles can then be written as

$$u(x_1, \sigma_1, \dots, x_N, \sigma_N)$$

where $1 \leq \sigma_i \leq s_i$. For the fixed spin σ systems, u is only a function of x_1, \dots, x_N , then it can be regarded as

$$u(x_1, \dots, x_N).$$

Let

$$\mathcal{I}_s = \{i | \sigma_i = s\}, \quad s = 1, \dots, N,$$

and $P_{i,j}$ is one permutation that exchange the position of variable x_i, x_j and the spin σ_i, σ_j simultaneously. By the Pauli Principle, we know

$$u(P_{i,j}x) = -u(x), \quad \text{if } \exists 1 \leq l \leq s, \text{ s.t. } i, j \in \mathcal{I}_l. \quad (7)$$

In fact, in many-body quantum mechanics, fruitful results derive from the anti-symmetry. In the past three decades, the stability of Coulomb systems has been studied extensively (see [13] for a textbook presentation). For all normalized, anti-symmetric wave function ψ with s spin state,

$$(\psi, H(0)\psi) \geq -0.231s^{2/3}N(1 + 2.16 \max_j Z_j(M/N)^{1/3})^2,$$

through the Lieb-Thirring inequalities which are one of the most important consequence of Pauli Exclusion Principle. And recently, new methods for the Lieb-Thirring inequality has been developed by lots of mathematicians, for example R. Frank and D. Lundholm, [5, 14, 15].

For one smooth function u with s spin states, for the fixed σ , the Equation (7) holds, thus we know $|u(x)| \sim |x_i - x_j|^\alpha$ for some $\alpha \geq 1$ when $|x_i - x_j| \rightarrow 0$. Because of this observation, Yserentant [23, 25] found out the new mixed regularity and applied it to the numerical analysis.

Denote

$$\mathcal{L}_{\mathcal{I}_l} = \bigotimes_{i \in \mathcal{I}_l} \nabla_i$$

with ∇_i is the gradient to the i th electron, and \otimes is the tensor product.

Provided that $\Omega > C(N + \sum_\mu Z_\mu)N^{1/2} + \max\{\lambda, 0\}$, Yserentant [23, 25] tells us that if λ is the eigenvalue of the operator H , then for the eigenvalue equation

$$Hu = \lambda u,$$

there exists one anti-symmetric solution u , and

$$\|\mathcal{L}_{\mathcal{I}_l} u_H\|_{L^2} \leq \|\mathcal{L}_{\mathcal{I}_l} u_L\|_{L^2}, \quad \|\nabla \mathcal{L}_{\mathcal{I}_l} u_H\|_{L^2} \leq \Omega \|\nabla \mathcal{L}_{\mathcal{I}_l} u_L\|_{L^2} \quad (8)$$

with

$$\hat{u}_L = \hat{u} \mathbb{1}_{|\omega| < \Omega}, \quad u_H = u - u_L.$$

with \hat{u} is the Fourier Transform of u .

If $H(t) = H$ is independent of t , obviously it is selfadjoint with the domain $H^1((\mathbb{R}^3)^N)$. Hence there is for each Borel set $A \subset \mathbb{R}$, a projection, $E_A(H)$, so that $H = \int \lambda dE_\lambda$ and $\exp(itH) = \int \exp(it\lambda) dE_\lambda$. It is natural to consider the similar question: if u_0 anti-symmetric, and $\|\mathcal{L}_{\mathcal{I}_l} u_0\|_{L^2} < \infty$, does $\|\mathcal{L}_{\mathcal{I}_l} u\|_{L^2} < \infty$ hold?

There are fruitful works about the regularity of the eigenfunctions of the Hamiltonian operator H . Beginning from the work of Kato [11], in which he derived the famous cusp conditions that establish a connection between the function values and certain first order directional derivatives at the points where two particles meet and the corresponding interaction potential becomes singular, Fournais and others directed attention primarily to the local behaviour of the eigenfunctions near the singular points of the interaction potentials, rather than like Yserentant showing that the eigenfunctions possess global, square-integrable weak derivatives of partly very high order, see [2-4, 8, 9]. Now, we do

the similar work of Yserentant, showing that the solutions of the electronic evolution Schrödinger equation has similar mixed high order derivative regularity.

To simplify the notation, we denote

$$1/\theta = \min\{3/(2p) + 3/(2\tilde{p}) - 1/2, 3/(2q) + 3/(2\tilde{q}) - 1/2\}. \quad (9)$$

Our main result is Theorem 1.5, 1.8 and 1.10:

Theorem 1.5. *If u_0 has the fixed spin states σ , $\mathcal{L}_{\mathcal{I}_l}u_0 \in L^2((\mathbb{R}^3)^N)$, $l = 1, \dots, s$, and $0 < \alpha < 1/2$, $\frac{6}{3-2\alpha} < p, q \leq 6$, the solution of Equation (1) has a unique solution u with the same spin states σ , and $\mathcal{L}_{\mathcal{I}_l}u \in X(a)$ for $s \in I_a$.*

And there is a constant C_1 only dependent on α , \tilde{p} , p , \tilde{q} and q with $\frac{6}{1+2\alpha} < \tilde{p}, \tilde{q} \leq 6$ and $1/\theta > 0$, if T small enough such that $C_1(\sum_{\mu} Z_{\mu} + N)NT^{1/\theta} < 1/2$, we have

$$\|\mathcal{L}_{\mathcal{I}_l}u\|_{L_t^{\infty}(L^2)} \leq \|\mathcal{L}_{\mathcal{I}_l}u\|_X \lesssim_{p,q} \|\mathcal{L}_{\mathcal{I}_l}u_0\|_{L^2},$$

where θ satisfies the Equation (9).

Remark 1.6. *Indeed, the constant C_1 satisfies the Inequality (20).*

If u_0 has N spins states, and for every $1 \leq l \leq N$, $|\mathcal{I}_l| = 1$, then it can be regarded as the case without spin states. So $\mathcal{L}_{\mathcal{I}_l} = \nabla_l$. Thus we have the following corollary:

Corollary 1.7. *If $\nabla_l u_0 \in L^2((\mathbb{R}^3)^N)$, $l = 1, \dots, N$, and $0 < \alpha < 1/2$, $\frac{6}{3-2\alpha} < p, q \leq 6$, the solution of Equation (1) has a unique solution u , and $\mathcal{L}_{\mathcal{I}_l}u \in X(a)$ for $s \in I_a$.*

And if $C_1(\sum_{\mu} Z_{\mu} + N)NT^{1/\theta} < 1/2$, we have

$$\|\nabla_l u\|_{L_t^{\infty}(L^2)} \leq \|\nabla_l u\|_X \lesssim_{p,q} \|\nabla_l u_0\|_{L^2},$$

where θ satisfies the Equation (9).

In Yserentant's works, the author also introduced another type of operator

$$\mathcal{K}_{\mathcal{I}_l} = \prod_{j \in \mathcal{I}_l} (1 - \Delta_j)^{1/2}$$

which is equivalent to $\mathcal{L}_{\mathcal{I}_l}$ in the L^2 functional space. However, it is not so evident for the X functional space, not only because of the $L^p - L^q$ type functional space, but also the change of variable in the integration. Luckily, after generalization of Calderón-Zygmund inequality and observation of the special property of our functional space, we found out some useful inequalities in Section 2.3. Then, we have the following Theorem:

Theorem 1.8. *If u_0 has the fixed spin states σ , $\mathcal{K}_{\mathcal{I}_l}u_0 \in L^2((\mathbb{R}^3)^N)$, $l = 1, \dots, s$, and $0 < \alpha < 1/2$, $\frac{6}{3-2\alpha} < p, q \leq 6$, the solution of Equation (1) has a unique solution u with the same spin states σ , and $\mathcal{K}_{\mathcal{I}_l}u \in X(a)$ for $s \in I_a$.*

And there is a constant C_2 ($C_2 > C_1$) only dependent on α , \tilde{p} , p , \tilde{q} and q with $\frac{6}{1+2\alpha} < \tilde{p}, \tilde{q} \leq 6$ and $1/\theta > 0$, if T small enough such that $C_2(\sum_{\mu} Z_{\mu} + N)NT^{1/\theta} < 1/2$, we have

$$\|\mathcal{K}_{\mathcal{I}_l}u\|_{L_t^{\infty}(L^2)} \leq \|\mathcal{K}_{\mathcal{I}_l}u\|_X \lesssim_{p,q} \|\mathcal{K}_{\mathcal{I}_l}u_0\|_{L^2}.$$

Remark 1.9. *Indeed, the constant C_2 satisfies the Inequality (21).*

1.3 The Numerical Analysis

Similar to [24], it is interesting to consider the numerical approximation of Equation (1).

We construct the projection firstly.

Define by $\Omega(R)$ the following hyperbolic cross space

$$\Omega(R) = \left\{ (\omega_1, \dots, \omega_N) \in (\mathbb{R}^3)^N \mid \sum_{1 \leq l \leq s} \prod_{i \in \mathcal{I}_l} (1 + |\omega_i|^2)^{1/2} \leq R \right\}.$$

And let $\chi : (\mathbb{R}^3)^N \rightarrow [0, 1]$ now be a symmetric function with the values $\chi_R(\omega) = 1$ for $\omega \in \Omega(R)$. Then, we have the following operator:

$$(P_{R,\chi}u)(x) = \left(\frac{1}{\sqrt{2\pi}} \right)^{3N} \int_{\omega \in (\mathbb{R}^3)^N} \chi_R(\omega) \hat{u}(\omega) \exp(i\omega \cdot x) d\omega.$$

For example, let $\chi_R(\omega) = \mathbb{1}_{\Omega(R)}(\omega)$, then the operator $P_{R,\chi}$ is the projection on the Fourier space.

As the choice of χ_R has few influences to our result, we shorten $P_{R,\chi}$ by P_R without confusion. And then we have the following approximation of Equation (1):

$$\begin{cases} i\partial_t u_R = H_R(u_R), & t \in [-a, a] = I_a, \quad x = (x_1, \dots, x_N) \in (\mathbb{R}^3)^N \\ u_R(0, x) = P_R(u_0)(x) \end{cases} \quad (10)$$

with

$$H_R(u) = \sum_{j=1}^N -\Delta_j u - \sum_{j=1}^N \sum_{\mu=1}^M P_R(V(x_j, t)u) + \sum_{k < j}^N P_R(W(x_j, x_k)u).$$

As a consequence of our main Theorem 1.5 and Theorem 1.8, we have:

Theorem 1.10. *If u_0 has the fixed spin states σ , $\mathcal{K}_{\mathcal{I}_l} u_0 \in L^2((\mathbb{R}^3)^N)$, $l = 1, \dots, s$, and $0 < \alpha < 1/2$, $\frac{6}{3-2\alpha} < p, q < 6$, then the solution of Equation (10) has unique solution u_R .*

And there is a constant C_3 ($C_3 > C_2$) only dependent on α , \tilde{p} , p , \tilde{q} and q with $\frac{6}{1+2\alpha} < \tilde{p}, \tilde{q} < 6$ and $1/\theta > 0$, if T small enough such that $C_3(\sum_{\mu} Z_{\mu} + N)NT^{1/\theta} < 1/2$, we have

$$\|u - u_R\|_{L_t^{\infty}(L^2)} \leq \|u - u_R\|_X \lesssim_{p,q} 1/R \sum_{l=1}^s \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}, \quad (11)$$

where u is the solution of Equation (1).

Remark 1.11. *Indeed, the constant C_3 satisfies the Inequality (22).*

Indeed, it provides us several numerical methods, [6, 24]. For the numerical analysis, normally, we split $\Omega(R)$ into finitely many subdomains by means of a C^{∞} -partition of unity $\sum_{l=1}^L \psi_l = 1$ on $\Omega(R)$ with $l \in (\mathbb{N}^3)^N$, i.e. each $\psi_l(\omega) \in C^{\infty}$ has compact support. It forms the basis of many possible approximation procedures that differ mainly by the way how the partition of unity is actually chosen and how the parts are finally approximated by functions in finite dimensional spaces. Let

$$u_l(x) = \left(\frac{1}{\sqrt{2\pi}} \right)^{3N} \int \psi_l \hat{u}(\omega) \exp(i\omega \cdot x) d\omega,$$

and

$$\psi_l(\omega) = (\widehat{\phi}_l)^2.$$

[24] tells us that the part u_l of u can be approximated arbitrarily well by the functions in the space

$$\mathcal{V}_l = \text{span}\{\phi_l(\cdot - D_l^{-1}k) | k \in \mathbb{Z}^{3N}\}$$

with

$$D_l \omega = \frac{4}{\pi} (2^{l_1} \omega_1, \dots, 2^{l_N} \omega_N).$$

Hence, taking the symmetric $\sum_{l=1}^L \psi_l$ and let $\chi_R = \sum_{l=1}^L \psi_l$. En consequence, we get the P_R and then u_R which satisfies the Inequality (11).

Outline of the paper. Before giving the proofs of the main results, we pause to outline the structure of this paper.

- In Section 2 we introduce the tools that we need: the Hardy-type inequalities, the generalization of Strichartz estimates, and the Sobolev inequalities in $L^p - L^2$ functional spaces.
- In Section 3 we prove the existence of the general many-body Schrödinger equation, namely the Theorem 1.2.
- In Section 4, we return back to Coulombic potentials, and study its regularity. Under the assumption of the initial datum that u_0 has the fixed spin states σ , we get our main results: Theorem 1.5 and 1.8. The Sobolev inequalities play one central role in the proof of the Theorem 1.8.
- In Section 5, we design one new hyperbolic cross space approximation and derive the numerical analysis by using the Theorem 2.8.

2 Preliminary

2.1 Hardy Type Inequality

For the mixed regularity, we need to study the Hardy type inequalities. By a similar methods to [23, lemma 1], we generalize the Hardy inequality:

Lemma 2.1. *If $u \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, then*

$$\int_{\mathbb{R}^3} \frac{1}{|x|^{k-2}} |\nabla u(x)|^2 dx \geq \frac{(k-3)^2}{4} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^k} dx$$

for $k \in [2, 3) \cup (3, 5)$.

Proof. Let $d(x) = |x|$. We have the relationship:

$$(k-1) \frac{1}{d^k} = -\nabla \frac{1}{d^{k-1}} \cdot \nabla d,$$

and because of $\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^k} dx < \infty$, hence by the integration by part we obtain

$$(k-1) \int_{\mathbb{R}^3} \frac{1}{d^k} u^2 = \int_{\mathbb{R}^3} \frac{1}{d^{k-1}} \nabla \cdot (u^2 \nabla d) dx.$$

Using $\Delta d = \frac{2}{d}$ on the right hand, then

$$(k-1) \int_{\mathbb{R}^3} \frac{1}{d^k} u^2 dx = 2 \int_{\mathbb{R}^3} \frac{1}{d^{k-1}} u \nabla u \cdot \nabla d dx + 2 \int_{\mathbb{R}^3} \frac{1}{d^k} u^2 dx,$$

by the Cauchy-Schwarz inequality, we yield

$$\left| \frac{k-3}{2} \right| \int_{\mathbb{R}^3} \frac{1}{d^k} u^2 \leq \left(\int_{\mathbb{R}^3} \frac{1}{d^k} u^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^3} \frac{1}{d^{k-2}} |\nabla d \cdot \nabla u|^2 dx \right)^{1/2}$$

and as $|\nabla d| = 1$, finally get the estimate. \square

Using Lemma 2.1 twice and the Fubini's Theorem, we have the following corollary.

Corollary 2.2. *If $u \in C_0^\infty((\mathbb{R}^3)^2)$ with $u(x, y) = -u(y, x)$ for $x, y \in \mathbb{R}^3$. Then we have the following inequality:*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|^{k-4}} |\nabla_x \nabla_y u(x, y)|^2 dx dy \geq \frac{(k-5)^2 (k-3)^2}{16} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x, y)|^2}{|x-y|^k} dx dy$$

for $k \in [4, 5)$.

When $k = 3$, from Lemma 2.1, we can only know

$$\int_{\mathbb{R}^3} \frac{|\nabla u(x)|^2}{|x|} dx \geq 0,$$

which means that $\int_{\mathbb{R}^3} \frac{|\nabla u(x)|^2}{|x|} dx$ has no relation with $\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^3} dx$.

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Nowadays, we consider the cylindrical coordinates in \mathbb{R}^3 , let

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta$$

then we have the following unit vectors

$$\vec{r} = (\cos \theta, \sin \theta, 0), \quad \vec{\theta} = (-\sin \theta, \cos \theta, 0).$$

Let $A = \vec{\theta}/r$. Indeed it is the Aharonov-Bohm magnetic vector potential. So we have the following covariant derivatives:

$$D_\alpha = -i\nabla + \alpha A.$$

Then, we have the magnetic Hardy-type inequality:

Lemma 2.3. *If $u \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, then*

$$\int_{\mathbb{R}^3} \frac{|D_\alpha u(x)|^2}{|x|} dx \geq \min_{k \in \mathbb{Z}} (k - \alpha)^2 \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^3} dx.$$

Proof. Indeed, using the cylindrical coordinates (r, θ, x_3) we have $u(x_1, x_2, x_3) = (1/\sqrt{2\pi}) \sum_k u_k(r, x_3) e^{ik\theta}$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|D_\alpha u(x)|^2}{|x|} dx &= \int_{\mathbb{R}_z} \int_0^\infty \int_0^{2\pi} \left(|u'_r|^2 + |u'_{x_3}|^2 + \left| \frac{i u'_\theta + \alpha u}{r} \right|^2 \right) d\theta dr dx_3 \\ &\geq \frac{1}{2\pi} \int_{\mathbb{R}_z} \int_0^\infty \int_0^{2\pi} \left| \sum_k \frac{\alpha - k}{r} u_k e^{ik\theta} \right|^2 d\theta dr dx_3 \\ &= \int_{\mathbb{R}_z} \int_0^\infty \sum_k \left| \frac{\alpha - k}{r} u_k \right|^2 dr dx_3 \\ &\geq \min_{k \in \mathbb{Z}} (k - \alpha)^2 \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^3} dx. \end{aligned}$$

□

Remark 2.4. This kind of magnetic Hardy inequalities has been well developed for the 2d case, which can be used to study the many-body Hardy inequalities, see [7, 15].

2.2 Strichartz Estimate

At the beginning, we recall the free propagator $U_0 = \exp(it \sum_{j=1}^N \Delta)$.

Denoting the integral operator

$$(Su)(t) = \int_0^t U_0(t - \tau) u(\tau) d\tau.$$

and

$$Qu(t) = \sum_{j=1}^N (SV(x_j, \cdot)u)(t) - i \sum_{j < k} (SW(x_j, x_k)u(\cdot))(t),$$

we consider the integral equation:

$$u(t) = U_0(t)u_0 + iQu(t). \quad (12)$$

Before the discussion about U_0 , we need the following properties.

Lemma 2.5.

$$\mathcal{R}_{i,j} \nabla_i = (\nabla_{d_{i,j}} + \nabla_{D_{i,j}}) \mathcal{R}_{i,j}, \quad \mathcal{R}_{i,j} \nabla_j = (\nabla_{D_{i,j}} - \nabla_{d_{i,j}}) \mathcal{R}_{i,j}$$

and

$$-\mathcal{R}_{i,j} \Delta_i = |\nabla_{d_{i,j}} + \nabla_{D_{i,j}}|^2 \mathcal{R}_{i,j}, \quad -\mathcal{R}_{i,j} \Delta_j = |\nabla_{d_{i,j}} - \nabla_{D_{i,j}}|^2 \mathcal{R}_{i,j}$$

Proof. We know $x_i = (d_{i,j} + D_{i,j})/2$ and $x_j = (D_{i,j} - d_{i,j})/2$, then

$$\nabla_{d_{i,j}} \mathcal{R}_{i,j} u = 1/2 \mathcal{R}_{i,j} (\nabla_i - \nabla_j) u,$$

and

$$\nabla_{D_{i,j}} \mathcal{R}_{i,j} u = 1/2 \mathcal{R}_{i,j} (\nabla_i + \nabla_j) u.$$

Then the first equation holds.

For the second, we just use the fact

$$-\Delta_i = \nabla_i^* \cdot \nabla_i = |\nabla_i|^2.$$

Together with the first equation, we yield the results. □

Then the next integrability property of the free propagator $U_0(t)$ is fundamental in the following discussions.

Lemma 2.6 (Kato). *Let $2 \leq p \leq \infty$, then*

$$\|U_0(t)u\|_{L_D^{p,2}} \lesssim_p |t|^{-3(1/2-1/p)} \|u\|_{L_D^{p',2}}, \quad D \subset \{1, \dots, N\}, \quad 1 \leq |D| \leq 2.$$

Proof. For the case $|D| = 1$, it is just the normal Kato inequality. For the another case, let $D = \{i, j\}$. Notice that by Lemma 2.5

$$-\mathcal{R}_{i,j}\Delta_x - \mathcal{R}_{i,j}\Delta_y = -2\Delta_{d_{i,j}}\mathcal{R}_{i,j} - 2\Delta_{D_{i,j}}\mathcal{R}_{i,j}.$$

Then, we know

$$\mathcal{R}_{i,j}U_0(t)u = \tilde{U}_0(t)\mathcal{R}_{i,j}u.$$

with $\tilde{U}_0(t) = \exp(-i(\sum_{k \neq i,j} \Delta_k + 2\Delta_{d_{i,j}} + 2\Delta_{D_{i,j}}))$.

Therefore,

$$\begin{aligned} \|U_0(t)u\|_{L_{i,j}^{p,2}} &= \|\mathcal{R}_{i,j}U_0(t)u\|_{L_{d_{i,j}}^{p,2}} \\ &= \|\tilde{U}_0(t)\mathcal{R}_{i,j}u\|_{L_{d_{i,j}}^{p,2}} \\ &\lesssim_p |t|^{-3(1/2-1/p)} \|\mathcal{R}_{i,j}u\|_{L_{d_{i,j}}^{p,2}} \\ &\lesssim_p |t|^{-3(1/2-1/p)} \|u\|_{L_{i,j}^{p,2}}. \end{aligned}$$

Get conclusion. □

Then, we have the following Strichartz estimates:

Lemma 2.7 (Strichartz estimate). *[22] [12] For $D, D' \subset \{1, \dots, N\}$, $1 \leq |D|, |D'| \leq 2$ and $2 \leq p, q \leq 6$, we have*

$$\|U_0(t)f\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_p \|f\|_{L^2}, \quad (13a)$$

$$\left\| \int U(s)^* u(s) ds \right\|_{L^2} \lesssim_p \|u\|_{L_t^{\theta'_p}(L_D^{p',2})}, \quad (13b)$$

$$\|Su\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_{p,q} \|u\|_{L_t^{\theta'_q}(L_{D'}^{q',2})}. \quad (13c)$$

Normally, the operator bounded in L^2 functional space is not bounded in the $L_D^{p,2}$ functional space. But the following theorem tells us that after applying the operator S , the bounded operator in L^2 is also bounded in $L_D^{p,2}$.

Theorem 2.8. *If $2 \leq p, q < 6$, for one operator P acting on $L^2(\mathbb{R}^{3N})$, if $[P, U_0] = 0$ and $\|Pf_0\|_{L^2} \leq \|f\|_{L^2}$, then*

$$\|PSf(\cdot, x)\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_{p,q} \|f\|_{L_t^{\theta'_q}(L_{D'}^{q',2})}.$$

And this inequality has the same optimal constant with Inequality (13c).

Proof. It is to prove

$$\left\| P \int_0^t U_0(t-s)f(s, x)ds \right\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_{p,q} \|f\|_{L_t^{\theta'_q}(L_{D'}^{q',2})}.$$

Then instead of proving this inequality, we prove the following one,

$$\left\| P \int_0^T U_0(t-s)f(s, x)ds \right\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_{p,q} \|f\|_{L_t^{\theta'_q}(L_{D'}^{q',2})}$$

then by Christ-Kiselev lemma, get conclusion.

Since P and U_0 commute, we have

$$\left\| P \int_0^T U_0(t-s)f(s, x)ds \right\|_{L_t^{\theta_p}(L_D^{p,2})} = \left\| U_0(t)P \int_0^T U(s)^* f(s, x)ds \right\|_{L_t^{\theta_p}(L_D^{p,2})}.$$

By Inequality (13a), we have

$$\left\| P \int_0^T U_0(t-s)f(s, x)ds \right\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_p \left\| P \int_0^T U(s)^* f(s, x)ds \right\|_{L^2}.$$

Then, by $\|Pf_0\|_{L^2} \leq \|f\|_{L^2}$, we have

$$\left\| P \int_0^T U_0(t-s)f(s, x)ds \right\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_p \left\| \int_0^T U(s)^* f(s, x)ds \right\|_{L^2}.$$

By Inequality (13b), we have

$$\left\| P \int_0^T U_0(t-s)f(s, x)ds \right\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_{p,q} \|f\|_{L_t^{\theta'_q}(L_{D'}^{q',2})}.$$

□

Corollary 2.9. For $D, D' \subset \{1, \dots, N\}$, $1 \leq |D|, |D'| \leq 2$ and $2 \leq p, q < 6$, we have

$$\|SP_R u\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_{p,q} \|u\|_{L_t^{\theta'_q}(L_{D'}^{q',2})}, \quad (14a)$$

$$\|S(1 - P_R)u\|_{L_t^{\theta_p}(L_D^{p,2})} \lesssim_{p,q} 1/R \left\| \sum_{1 \leq l \leq s} \mathcal{K}_{\mathcal{I}_l} u \right\|_{L_t^{\theta'_q}(L_{D'}^{q',2})}. \quad (14b)$$

And these inequalities have the same optimal constant with Inequality (13c).

Proof. Obviously, by the definition of P_R , we have

$$[P_R, U_0] = 0$$

and

$$\|P_R u\|_{L^2} \leq \|u\|_{L^2}.$$

Let $P = P_R$, then we get the Inequality (14a). Besides,

$$\|(1 - P_R)u\|_{L^2} \leq \|\mathbb{1}_{\Omega(R)^c} \widehat{u}\|_{L^2}$$

For all wave vector ω outside the domain $\Omega(R)$, we have

$$1 \leq 1/R \sum_{1 \leq l \leq s} \prod_{i \in \mathcal{I}_l} (1 + |\omega_i|^2)^{1/2}.$$

By the definition of norm, we know

$$\|(1 - P_R)u\|_{L^2} \leq 1/R \left\| \sum_{1 \leq l \leq s} \prod_{i \in \mathcal{I}_l} (1 + |\omega_i|^2)^{1/2} \hat{u} \right\|_{L^2} = 1/R \left\| \sum_{1 \leq l \leq s} \mathcal{K}_{\mathcal{I}_l} u \right\|_{L^2}.$$

Given $[\mathcal{K}_{\mathcal{I}_l}, U_0] = 0$, then take $P = R(1 - P_R) \left(\sum_{1 \leq l \leq s} \mathcal{K}_{\mathcal{I}_l} \right)^{-1}$, we get conclusion. \square

2.3 Sobolev Inequalities

Because of the unusuality of our functional space, we need to reconstruct some Sobolev inequalities which will be useful for the regularity. We generalized the Calderón-Zygmund inequality to satisfy the new functional space $L_i^{p,2}$ in Appendix. The following inequalities are the application of the new Calderón-Zygmund inequality and then we make it compatible for the functional space $L_{i,j}^{p,2}$.

Theorem 2.10. *For $1 < p < \infty$, the following inequalities hold:*

$$\|\nabla_i u\|_{L_i^{p,2}} \lesssim_p \|(1 - \Delta_i)^{1/2} u\|_{L_i^{p,2}}, \quad i = 1, \dots, N \quad (15a)$$

$$\|u\|_{L_i^{p,2}} \lesssim_p \|(1 - \Delta_i)^{1/2} u\|_{L_i^{p,2}}, \quad i = 1, \dots, N \quad (15b)$$

$$\|(1 - \nabla_i)u\|_{L_i^{p,2}} \lesssim_p \|(1 - \Delta_i)^{1/2} u\|_{L_i^{p,2}}, \quad i = 1, \dots, N \quad (15c)$$

$$\|\nabla_i u\|_{L_{i,j}^{p,2}} \lesssim_p \|(1 - \Delta_i)^{1/2} u\|_{L_{i,j}^{p,2}}, \quad i, j = 1, \dots, N \quad (15d)$$

$$\|u\|_{L_{i,j}^{p,2}} \lesssim_p \|(1 - \Delta_i)^{1/2} u\|_{L_{i,j}^{p,2}}, \quad i, j = 1, \dots, N \quad (15e)$$

$$\|(1 - \nabla_i)u\|_{L_{i,j}^{p,2}} \lesssim_p \|(1 - \Delta_i)^{1/2} u\|_{L_{i,j}^{p,2}}, \quad i = 1, \dots, N. \quad (15f)$$

Proof. For the first inequality, we only need to study equivalently the following inequality

$$\|\nabla_i (1 - \Delta_i)^{-1/2} u\|_{L_i^{p,2}} \lesssim_p \|u\|_{L_i^{p,2}}.$$

Obviously,

$$a(\xi) = \frac{\xi}{(1 + |\xi|^2)^{1/2}} \quad \text{for } \xi \in \mathbb{R}^3.$$

Using Theorem A.3, get conclusion. And since $\xi \in \mathbb{R}^3$, we know the optimal constant of this inequality is independent on N .

The second and third inequalities are similar.

For the fourth inequality, by Lemma 2.3, we know

$$\|\nabla_i u\|_{L_{i,j}^{p,2}} = \|\mathcal{R}_{i,j} \nabla_i u\|_{L_{d_{i,j}}^{p,2}} = \|(\nabla_{d_{i,j}} + \nabla_{D_{i,j}}) \mathcal{R}_{i,j} u\|_{L_{d_{i,j}}^{p,2}}.$$

Define the Fourier transform just for the variable $D_{i,j}$ by \mathcal{F}_D , and by Parseval's Theorem, then

$$\begin{aligned} \|\nabla_i u\|_{L_{i,j}^{p,2}} &= \|(\nabla_{d_{i,j}} - i\xi_{D_{i,j}}) \mathcal{F}_D \mathcal{R}_{i,j} u\|_{L_{d_{i,j}}^{p,2}} \\ &= \|\nabla_{d_{i,j}} \exp(-id_{i,j} \cdot \xi_{D_{i,j}}) \mathcal{F}_D \mathcal{R}_{i,j} u\|_{L_{d_{i,j}}^{p,2}} \\ &\lesssim_p \|(1 - \Delta_{d_{i,j}})^{1/2} \exp(-id_{i,j} \cdot \xi_{D_{i,j}}) \mathcal{F}_D \mathcal{R}_{i,j} u\|_{L_{d_{i,j}}^{p,2}}. \end{aligned}$$

So in order to get the result, we only need to prove for every u ,

$$(1 - \Delta_{d_{i,j}})^{1/2} \exp(-id_{i,j} \cdot \xi_{D_{i,j}})u = \exp(-id_{i,j} \cdot \xi_{D_{i,j}})(1 + |\nabla_{d_{i,j}} - i\xi_{D_{i,j}}|)^{1/2}u.$$

It is correct by the fact

$$(1 - \Delta)^{1/2} = 2/\pi \int_0^\infty \frac{1 - \Delta}{1 - \Delta + t^2} dt$$

and

$$(1 - \Delta + t^2)^{-1} = \int_0^\infty \exp(-(1 - \Delta + t^2)s) ds.$$

Finally, repeating the strategy of the fourth inequality, we get the fifth and sixth inequalities. \square

Remark 2.11. *The inequalities we get in Theorem 2.10 work not only on i , but also on j .*

3 Existence of Solution

Proof of Theorem 1.2. In fact, we just need to analyze the term $SW(x_i, x_j)u$. Since

$$W \in L_t^{\alpha p}(L^{p/(p-2)}) + L_t^{\beta p}(L^\infty),$$

we have

$$W = W_1 + W_2, \quad W_1 \in L_t^{\alpha p}(L^{p/(p-2)}), \quad W_2 \in L_t^{\beta p}(L^\infty)$$

then,

$$\begin{aligned} \| |W|^{1/2}u \|_{L^2}^2(t) &= \int |W|(t)|u|^2(t) dx \\ &\leq \int |W_1|(t)|u|^2(t) dx + \int |W_2|(t)|u|^2(t) dx \\ &\leq \|W_1\|_{L^{p/(p-2)}}(t) \|u\|_{L_{i,j}^{p,2}}^2(t) + \|W_2\|_{L^\infty}(t) \|u\|_{L^2}^2(t). \end{aligned}$$

Therefore, for every $v \in L_t^{\theta'_q}(L_D^{q',2})$ with $1 \leq |D| \leq 2$ and $2 \leq q < 6$.

$$\begin{aligned}
& \int_0^T \langle SW(x_i, x_j)u(t), v \rangle dt \\
&= \int_0^T \langle W(x_i, x_j)^{1/2}u, W(x_i, x_j)^{1/2}S^*v(s) \rangle ds \\
&\leq \int_0^T \| |W(x_i, x_j)|^{1/2}u \|_{L^2} \| |W(x_i, x_j)|^{1/2}S^*v \|_{L^2} ds \\
&\leq \int_0^T \left(\|W_2\|_{L^\infty}^{1/2} \|u\|_{L^2} + \|W_1\|_{L^{p/(p-2)}}^{1/2} \|u\|_{L_{i,j}^{p,2}} \right) \\
&\quad \left(\|W_2\|_{L^\infty}^{1/2} \|S^*v\|_{L^2} + \|W_1\|_{L^{p/(p-2)}}^{1/2} \|S^*v\|_{L_{i,j}^{p,2}} \right) ds \\
&\leq \int_0^T \left(\|W_2\|_{L^\infty} \|u\|_{L^2} + \|W_1\|_{L^{p/(p-2)}}^{1/2} \|W_2\|_{L^\infty}^{1/2} \|u\|_{L_{i,j}^{p,2}} \right) \|S^*v\|_{L^2} ds \\
&\quad + \int_0^T \left(\|W_1\|_{L^{p/(p-2)}}^{1/2} \|W_2\|_{L^\infty}^{1/2} \|u\|_{L^2} + \|W_1\|_{L^{p/(p-2)}} \|u\|_{L_{i,j}^{p,2}} \right) \|S^*v\|_{L_{i,j}^{p,2}} ds \\
&\lesssim_{p,q,W} T^{1/\theta_{\alpha,\beta}} \left(\|u\|_{L_t^\infty(L^2)} + \|u\|_{L_t^{\theta_p}(L_{i,j}^{p,2})} \right) \|v\|_{L_t^{\theta'_q}(L_D^{q',2})} \\
&\lesssim_{p,q,W} T^{1/\theta_{\alpha,\beta}} \|u\|_X \|v\|_{L_t^{\theta'_q}(L_D^{q',2})}.
\end{aligned}$$

Choosing one sequence $v_n \in L_t^{\theta'_q}(L_D^{q',2})$ with $\|v_n\|_{L_t^{\theta'_q}(L_D^{q',2})} = 1$, such that

$$\|SWu\|_{L_t^{\theta_q}(L_D^{q,2})} = \lim_{n \rightarrow \infty} \left| \langle SWu, v_n \rangle_{L_t^{\theta_q}(L_D^{q,2}), L_t^{\theta'_q}(L_D^{q',2})} \right| \lesssim_{p,q,W} T^{1/\theta_{\alpha,\beta}} \|u\|_X.$$

Let $L_t^{\theta_q}(L_D^{q,2}) = L_t^\infty(L^2)$ or $L_t^{\theta_p}(L_{i,j}^{p,2})$ or $L_t^{\theta_q}(L_k^{q,2})$. Then obviously,

$$\|SWu\|_X \lesssim_{p,q,W} T^{1/\theta_{\alpha,\beta}} \|u\|_X.$$

Similarly, we have

$$\|SVu\|_X \lesssim_{p,q,V} T^{1/\theta_{\alpha,\beta}} \|u\|_X$$

Hence, there is a constant C only dependent on p, q, V, W , such that

$$\|Qu\|_X \leq CT^{1/\theta_{\alpha,\beta}} N(N+1) \|u\|_X. \tag{16}$$

Let T small enough, such that $CT^{1/\theta_{\alpha,\beta}} N(N+1) < 1/2$, the operator Q is a contraction on X . Since, by Lemma 2.7, $u_0(t) = U_0(t)u_0 \in X$, for any $u_0 \in L^2$, it follows that the integral equation

$$u(t) = u_0(t) + iQu(t)$$

has a unique solution $u(t) = (\mathbb{1} - iQ)^{-1}u_0(t) \in X$. And

$$\|u\|_X \leq 2\|U_0(t)u_0\|_X \lesssim_{p,q} \|u_0\|_{L^2}.$$

Besides the standard continuation procedure for the solution of linear integral equations yields a global unique solution $u \in X(a)$. \square

4 Regularity of the Equation

Before analyzing this section, we study the following equation firstly:

$$\begin{cases} i\partial_t u_\epsilon = H_\epsilon u_\epsilon, & t \in [-a, a] = I_a, x = (x_1, \dots, x_N) \in (\mathbb{R}^3)^N \\ u_\epsilon(0, x) = u_0(x) \end{cases}$$

with

$$H_\epsilon = \sum_{j=1}^N -\Delta_j - \sum_{j=1}^N \sum_{\mu=1}^M V_\epsilon(x_j) + \sum_{k < j}^N W_\epsilon(x_j, x_k),$$

where

$$V_\epsilon(x_j) = \sum_{\mu=1}^M \frac{Z_\mu}{|x_j - a_\mu(t)| + \epsilon}$$

and

$$W_\epsilon(x_j, x_k) = \frac{1}{|x_k - x_j| + \epsilon}.$$

Let

$$Q_\epsilon u_\epsilon(t) = \sum_{j=1}^N (S V_\epsilon(x_j, \cdot) u_\epsilon)(t) - i \sum_{j < k} (S W_\epsilon(x_j, x_k) u_\epsilon(\cdot))(t), \quad (17)$$

Lemma 4.1. *For $\epsilon > 0$, if u_0 has the fixed spin states σ , then the above equation has a unique solution with the same spin states and the solution $u_\epsilon \in C_0^\infty((\mathbb{R}^3)^N)$.*

Proof. Taking the all kinds of derivatives, the potential V_ϵ and W_ϵ are still smooth, hence in $L_t^\infty(L^{q/(q-2)}) + L_t^\infty(L^\infty)$ and $L_t^\infty(L^{p/(p-2)}) + L_t^\infty(L^\infty)$ respectively. From Theorem 1.2, we know the equation has a unique solution.

And by the smoothness of V_ϵ and W_ϵ , we know the solution $u_\epsilon \in C_0^\infty((\mathbb{R}^3)^N)$.

Let $P_{\mathcal{I}_l}$ is the permutation operator, denote \mathcal{A} by

$$\mathcal{A}u(x) = \frac{1}{|\mathcal{I}_l|!} \sum_{P_{\mathcal{I}_l}} \text{Sign}(P_{\mathcal{I}_l}) u(P_{\mathcal{I}_l}x).$$

If u_ϵ is a solution, then $\mathcal{A}u_\epsilon(x)$ is another solution too. By the uniqueness of solution, we know u_ϵ has the same spin states. \square

Therefore, we can use Corollary 2.2 for u_ϵ .

Proof of Theorem 1.5. Taking the operator \mathcal{L} to the integral equation, we have

$$\mathcal{L}_{\mathcal{I}_l} u_\epsilon(t) = U_0(t) \mathcal{L}_{\mathcal{I}_l} u_0 + i \sum_{j=1}^N (S \mathcal{L}_{\mathcal{I}_l} V_\epsilon(x_j, \cdot) u_\epsilon)(t) - i \sum_{j < k} (S \mathcal{L}_{\mathcal{I}_l} W_\epsilon(x_j, x_k) u_\epsilon(\cdot))(t). \quad (18)$$

The key point is to study the term $S \mathcal{L}_{\mathcal{I}_l} W_\epsilon(x_j, x_k) u_\epsilon(\cdot)$ and $S \mathcal{L}_{\mathcal{I}_l} V_\epsilon(x_j, \cdot) u_\epsilon(\cdot)$, herein we use the Strichartz estimate. And in fact, we just need to deal with $S \mathcal{L}_{\mathcal{I}_l} W_\epsilon(x_j, x_k) u_\epsilon(\cdot)$, for the term $S \mathcal{L}_{\mathcal{I}_l} V_\epsilon(x, \cdot) u_\epsilon(\cdot)$ is same.

Similar to operator \mathcal{L} , we define the following operators:

$$\mathcal{L}_{\mathcal{I}_l, j} = \bigotimes_{i \in \mathcal{I}_l, i \neq j} \nabla_i, \quad \mathcal{L}_{\mathcal{I}_l, j, k} = \bigotimes_{i \in \mathcal{I}_l, i \neq j, k} \nabla_i.$$

For every $v \in L_t^{\theta'_q}(L_D^{q',2})$ with $1 \leq |D| \leq 2$, we consider the following inner product:

$$\int_0^T \langle [S\mathcal{L}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon](t), v(t) \rangle dt.$$

If $j, k \notin \mathcal{I}_l$, we have

$$\mathcal{L}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon = W_\epsilon(x_j, x_k)\mathcal{L}_{\mathcal{I}_l}u_\epsilon.$$

And if $j \in \mathcal{I}_l$, and $k \notin \mathcal{I}_l$,

$$\mathcal{L}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon = W_\epsilon(x_j, x_k)\mathcal{L}_{\mathcal{I}_l}u_\epsilon + (\nabla_j W_\epsilon(x_j, x_k))\mathcal{L}_{\mathcal{I}_l, j}u_\epsilon.$$

Analogously for $k \in \mathcal{I}_l$, and $j \notin \mathcal{I}_l$. Finally if $j, k \in \mathcal{I}_l$,

$$\begin{aligned} & \mathcal{L}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon \\ &= W_\epsilon(x_j, x_k)\mathcal{L}_{\mathcal{I}_l}u_\epsilon + (\nabla_j W_\epsilon(x_j, x_k))\mathcal{L}_{\mathcal{I}_l, j}u_\epsilon \\ & \quad + (\nabla_k W_\epsilon(x_j, x_k))\mathcal{L}_{\mathcal{I}_l, k}u_\epsilon + (\nabla_j \nabla_k W_\epsilon(x_j, x_k))\mathcal{L}_{\mathcal{I}_l, j, k}u_\epsilon. \end{aligned}$$

Then, we have

$$\begin{aligned} & \int_0^T \langle [S\mathcal{L}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon](t), v(t) \rangle dt \\ & \lesssim \left| \int_0^T \langle [SW_\epsilon\mathcal{L}_{\mathcal{I}_l}u_\epsilon](t), v(t) \rangle dt \right| + \left| \int_0^T \langle [S(\nabla_j W_\epsilon)\mathcal{L}_{\mathcal{I}_l, j}u_\epsilon](t), v(t) \rangle dt \right| \\ & \quad + \left| \int_0^T \langle [S(\nabla_k W_\epsilon)\mathcal{L}_{\mathcal{I}_l, k}u_\epsilon](t), v(t) \rangle dt \right| + \left| \int_0^T \langle [S(\nabla_j \nabla_k W_\epsilon)\mathcal{L}_{\mathcal{I}_l, j, k}u_\epsilon](t), v(t) \rangle dt \right| \end{aligned}$$

For $\alpha \in (0, 1/2)$, by

$$\begin{aligned} |W_\epsilon(x_j, x_k)| & \leq \frac{1}{|x_j - x_k|}, \\ |\nabla_j W_\epsilon(x_j, x_k)| & \lesssim \frac{1}{|x_j - x_k|^2}, \\ |\nabla_j \nabla_k W_\epsilon(x_j, x_k)| & \lesssim \frac{1}{|x_j - x_k|^3}, \end{aligned}$$

we yield

$$\begin{aligned} & \int_0^T \langle [SW_\epsilon\mathcal{L}_{\mathcal{I}_l}u_\epsilon](t), v(t) \rangle dt \\ &= \int_0^T \left\langle \frac{1}{|x_j - x_k|^\alpha} \mathcal{L}_{\mathcal{I}_l}u_\epsilon(s), \frac{1}{|x_j - x_k|^{1-\alpha}} [S^*v](s) \right\rangle ds \\ & \lesssim \int_0^T \left\| \frac{1}{|x_j - x_k|^\alpha} \mathcal{L}_{\mathcal{I}_l}u_\epsilon \right\|_{L^2} \left\| \frac{1}{|x_j - x_k|^{1-\alpha}} [S^*v] \right\|_{L^2} ds; \end{aligned}$$

for the second and third term,

$$\begin{aligned} & \int_0^T \langle [S(\nabla_j W_\epsilon)\mathcal{L}_{\mathcal{I}_l, j}u_\epsilon](t), v(t) \rangle dt \\ & \lesssim \int_0^T \left\| \frac{1}{|x_j - x_k|^{1+\alpha}} \mathcal{L}_{\mathcal{I}_l, j}u_\epsilon \right\|_{L^2} \left\| \frac{1}{|x_j - x_k|^{1-\alpha}} [S^*v] \right\|_{L^2} ds \\ & \lesssim 1/(1-2\alpha) \int_0^T \left\| \frac{1}{|x_j - x_k|^\alpha} \mathcal{L}_{\mathcal{I}_l}u_\epsilon \right\|_{L^2} \left\| \frac{1}{|x - y|^{1-\alpha}} [S^*v] \right\|_{L^2} ds; \end{aligned}$$

and the fourth term

$$\begin{aligned}
& \int_0^T \langle [S(\nabla_j \nabla_k W_\epsilon) \mathcal{L}_{\mathcal{I}_l, j, k} u_\epsilon](t), v(t) \rangle dt \\
& \lesssim \int_0^T \left\| \frac{1}{|x_j - x_k|^{2+\alpha}} \mathcal{L}_{\mathcal{I}_l, j, k} u_\epsilon \right\|_{L^2} \left\| \frac{1}{|x_j - x_k|^{1-\alpha}} [S^* v] \right\|_{L^2} ds \\
& \lesssim 1/(1-2\alpha)^2 \int_0^T \left\| \frac{1}{|x_j - x_k|^\alpha} \mathcal{L}_{\mathcal{I}_l} u_\epsilon \right\|_{L^2} \left\| \frac{1}{|x_j - x_k|^{1-\alpha}} [S^* v] \right\|_{L^2} ds.
\end{aligned}$$

By the Hölder inequality, we have

$$\left\| \frac{v}{|x_j - x_k|^{1-\alpha}} \right\|_2 \leq \left\| \frac{1}{|x|^{1-\alpha}} \right\|_{L^{\tilde{r}}(\mathbb{B}(0,1))} \|v\|_{L^{\tilde{p},2}_{j,k}} + \|v\|_{L^2} \lesssim_{\alpha, \tilde{p}} \|v\|_{L^{\tilde{p},2}_{j,k}} + \|v\|_{L^2}, \quad (19a)$$

$$\left\| \frac{u_\epsilon}{|x_j - x_k|^\alpha} \right\|_2 \leq \left\| \frac{1}{|x|^\alpha} \right\|_{L^r(\mathbb{B}(0,1))} \|u_\epsilon\|_{L^{\tilde{p},2}_{j,k}} + \|u_\epsilon\|_{L^2} \lesssim_{\alpha, p} \|u_\epsilon\|_{L^{\tilde{p},2}_{j,k}} + \|u_\epsilon\|_{L^2}, \quad (19b)$$

for $1/2 = 1/p + 1/r$, $1/2 = 1/\tilde{p} + 1/\tilde{r}$, $\tilde{p}, p \leq 6$, $\alpha r < 3$, $(1-\alpha)\tilde{r} < 3$, namely

$$\frac{6}{1+2\alpha} < \tilde{p} \leq 6, \quad \frac{6}{3-2\alpha} < p \leq 6.$$

Then

$$\begin{aligned}
& \int_0^T \langle [S \mathcal{L}_{\mathcal{I}_l} W_\epsilon u_\epsilon](t), v(t) \rangle dt \\
& \lesssim_{\alpha, p, \tilde{p}} \int_0^T \left(\|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L^{\tilde{p},2}_{j,k}} + \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L^2} \right) \left(\|[S^* v]\|_{L^{\tilde{p},2}_{j,k}} + \|[S^* v]\|_{L^2} \right) ds \\
& \lesssim_{\alpha, p, \tilde{p}} \left(\|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^{\theta'_p}(L^{\tilde{p},2}_{j,k})} + \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^{\theta'_p}(L^2)} \right) \|[S^* v]\|_{L_t^{\theta_{\tilde{p}}}(L^{\tilde{p},2}_{j,k})} \\
& \quad + \left(\|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^1(L^{\tilde{p},2}_{j,k})} + \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^1(L^2)} \right) \|[S^* v]\|_{L_t^\infty(L^2)} \\
& \lesssim_{\alpha, p, \tilde{p}} T^{1/\theta'_p - 1/\theta_p} \left(\|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^{\theta_p}(L^{\tilde{p},2}_{j,k})} + \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^{\theta_p}(L^2)} \right) \|[S^* v]\|_{L_t^{\theta_{\tilde{p}}}(L^{\tilde{p},2}_{j,k})} \\
& \quad + T^{1-1/\theta_p} \left(\|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^1(L^{\tilde{p},2}_{j,k})} + \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_{L_t^1(L^2)} \right) \|[S^* v]\|_{L_t^\infty(L^2)} \\
& \lesssim_{\alpha, p, \tilde{p}} T^{1/\theta} \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_X \|v\|_{L_t^{\theta'_q}(L_D^{q',2})}.
\end{aligned}$$

Choosing a sequence $\|v_n\|_{L_t^{\theta'_q}(L_D^{q',2})} = 1$, for $q = 2$ or $\frac{6}{3-2\alpha} < q \leq 6$ such that

$$\|S \mathcal{L}_{\mathcal{I}_l} W(x_j, x_k) u_\epsilon\|_{L_t^{\theta_q}(L_D^{q,2})} = \lim_{n \rightarrow \infty} \left| \langle S \mathcal{L}_{\mathcal{I}_l} W(x_j, x_k) u_\epsilon, v_n \rangle_{L_t^{\theta_q}(L_D^{q,2}), L_t^{\theta'_q}(L_D^{q',2})} \right|.$$

Let $L_t^{\theta_q}(L_D^{q,2}) = L_t^\infty(L^2)$ or $L_t^{\theta_p}(L_{i,j}^{p,2})$ or $L_t^{\theta_q}(L_k^{q,2})$. Then,

$$\|S \mathcal{L}_{\mathcal{I}_l} W(x_j, x_k) u_\epsilon\|_X \lesssim_{\alpha, p, \tilde{p}, q} T^{1/\theta} \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_X.$$

Similarly, there is a $\frac{6}{1+2\alpha} < \tilde{q} \leq 6$, such that

$$\begin{aligned}
& \int_0^T \langle [S \mathcal{L}_{\mathcal{I}_l} V(x_j, \cdot) u_\epsilon](t), v(t) \rangle dt \\
& \lesssim_{\alpha, p, \tilde{q}, q} \sum_{\mu} Z_\mu T^{1/\theta} \|\mathcal{L}_{\mathcal{I}_l} u_\epsilon\|_X \|v\|_{L_t^{\theta'_p}(L_D^{p',2})},
\end{aligned}$$

and

$$\|S\mathcal{L}_{\mathcal{I}_l}V(x_j, \cdot)u_\epsilon\|_X \lesssim_{\alpha, \tilde{p}, p, \tilde{q}, q} \sum_{\mu} Z_{\mu} T^{1/\theta} \|\mathcal{L}_{\mathcal{I}_l}u_\epsilon\|_X.$$

Hence, there is a constant C_1 only dependent on $\alpha, \tilde{p}, p, \tilde{q}$ and q , such that

$$\|\mathcal{L}_{\mathcal{I}_l}Q_\epsilon u_\epsilon\|_X \leq C_1 \left(\sum_{\mu} Z_{\mu} + N \right) NT^{1/\theta} \|\mathcal{L}_{\mathcal{I}_l}u_\epsilon\|_X. \quad (20)$$

Let $C_1(\sum_{\mu} Z_{\mu} + N)NT^{1/\theta} < 1/2$, by Equation (18) we have

$$\|\mathcal{L}_{\mathcal{I}_l}u_\epsilon\|_X \leq \|U_0\mathcal{L}_{\mathcal{I}_l}u_0\|_X + 1/2\|\mathcal{L}_{\mathcal{I}_l}u_\epsilon\|_X,$$

thus,

$$\|\mathcal{L}_{\mathcal{I}_l}u_\epsilon\|_X \leq 2\|U_0\mathcal{L}_{\mathcal{I}_l}u_0\|_X \lesssim_{p,q} \|\mathcal{L}_{\mathcal{I}_l}u_0\|_{L^2}.$$

Let $\epsilon \rightarrow 0$, we know

$$\|\mathcal{L}_{\mathcal{I}_l}u\|_X \lesssim_{p,q} \|\mathcal{L}_{\mathcal{I}_l}u_0\|_{L^2},$$

which implies

$$u_\epsilon \xrightarrow{*} u \text{ in } X.$$

We also have these other convergences:

$$V_\epsilon \rightarrow V \text{ in } L_t^\infty(L_x^\infty) + L_t^\infty(L_x^r),$$

and

$$W_\epsilon \rightarrow W \text{ in } L_t^\infty(L_x^\infty) + L_t^\infty(L_x^r).$$

with $0 < r < 3$.

Thus u is the solution in the sense of distributions and satisfies $\mathcal{L}_{\mathcal{I}_l}u \in X$. \square

Combining the Theorem 2.10, we can prove the Theorem 1.8.

Proof of Theorem 1.8. Denote

$$\mathcal{K}_{\mathcal{I}_l, j} = \prod_{i \neq j} (1 - \Delta_i)^{1/2}$$

and

$$\mathcal{K}_{\mathcal{I}_l, j, k} = \prod_{i \neq j, k} (1 - \Delta_i)^{1/2}.$$

Analogously, we study the term $SKW_\epsilon(x_j, x_k)u_\epsilon(\cdot)$ firstly. If $j, k \in \mathcal{I}_l$,

$$\mathcal{K}_{\mathcal{I}_l} \cdot \mathcal{K}_{\mathcal{I}_l} = \mathcal{K}_{\mathcal{I}_l, j, k} \cdot \mathcal{K}_{\mathcal{I}_l, j, k} (1 - \nabla_j \cdot \nabla_j - \nabla_k \cdot \nabla_k + (-\nabla_j \cdot \nabla_j)(-\nabla_k \cdot \nabla_k)),$$

then

$$\begin{aligned} & \int_0^T \langle SK_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon, \mathcal{K}_{\mathcal{I}_l}v \rangle \\ &= \langle SW_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon, \mathcal{K}_{\mathcal{I}_l, j, k}v \rangle \\ & \quad + \langle S\nabla_j W_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon, \nabla_j \mathcal{K}_{\mathcal{I}_l, j, k}v \rangle \\ & \quad + \langle S\nabla_k W_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon, \nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}v \rangle \\ & \quad + \langle S\nabla_j \nabla_k W_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon, \nabla_j \nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}v \rangle. \end{aligned}$$

After calculation, we have

$$\begin{aligned}
& \int_0^T \langle S\mathcal{K}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon, \mathcal{K}_{\mathcal{I}_l}\mathcal{I}_l v \rangle \\
& \lesssim_{\alpha, p, \tilde{p}} \int_0^T \left(\|\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L_{\mathcal{I}_l, j, k}^{p, 2}} + \|\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L^2} \right) \\
& \quad \times \left(\| [S^*\mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L_{j, k}^{\tilde{p}, 2}} + \| [S^*\mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L^2} \right) ds \\
& + \int_0^T \left(\|\nabla_j \mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L_{\mathcal{I}_l, j, k}^{p, 2}} + \|\nabla_j \mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L^2} \right) \\
& \quad \times \left(\| [S^*\nabla_j \mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L_{j, k}^{\tilde{p}, 2}} + \| [S^*\nabla_j \mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L^2} \right) ds \\
& + \int_0^T \left(\|\nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L_{j, k}^{p, 2}} + \|\nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L^2} \right) \\
& \quad \times \left(\| [S^*\nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L_{j, k}^{\tilde{p}, 2}} + \| [S^*\nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L^2} \right) ds \\
& + \int_0^T \left(\|\nabla_j \nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L_{j, k}^{p, 2}} + \|\nabla_j \nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon\|_{L^2} \right) \\
& \quad \times \left(\| [S^*\nabla_j \nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L_{\mathcal{I}_l, j, k}^{\tilde{p}, 2}} + \| [S^*\nabla_j \nabla_k \mathcal{K}_{\mathcal{I}_l, j, k}v] \|_{L^2} \right) ds
\end{aligned}$$

By the Theorem 2.10, we have for $p, \tilde{p} = 2$ or $\frac{6}{3-2\alpha} < p, \tilde{p} \leq 6$

$$\|\nabla_j^{l_1} \nabla_k^{l_2} \mathcal{K}_{j, k}v\|_{L_{\mathcal{I}_l, j, k}^{\tilde{p}, 2}} \lesssim_{\tilde{p}} \|\mathcal{K}_{\mathcal{I}_l}v\|_{L_{j, k}^{\tilde{p}, 2}}, \quad l_1, l_2 = 0, 1.$$

and same for u_ϵ . Then, we yield

$$\begin{aligned}
& \int_0^T \langle S\mathcal{K}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon, \mathcal{K}_{\mathcal{I}_l}v \rangle \\
& \lesssim_{\alpha, p, \tilde{p}} \int_0^T \left(\|\mathcal{K}_{\mathcal{I}_l}u_\epsilon\|_{L_{j, k}^{p, 2}} + \|\mathcal{K}_{\mathcal{I}_l}u_\epsilon\|_{L^2} \right) \\
& \quad \times \left(\| [S^*\mathcal{K}_{\mathcal{I}_l}v] \|_{L_{j, k}^{p, 2}} + \| [S^*\mathcal{K}_{\mathcal{I}_l}v] \|_{L^2} \right) ds.
\end{aligned}$$

If $j \in \mathcal{I}_l$ and $k \notin \mathcal{I}_l$,

$$\mathcal{K}_{\mathcal{I}_l} \cdot \mathcal{K}_{\mathcal{I}_l} = \mathcal{K}_{\mathcal{I}_l, j} \cdot \mathcal{K}_{\mathcal{I}_l, j}(1 - \nabla_j \cdot \nabla_j)$$

then

$$\begin{aligned}
& \int_0^T \langle S\mathcal{K}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon, \mathcal{K}_{\mathcal{I}_l}v \rangle \\
& = \langle SW_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j}u_\epsilon, \mathcal{K}_{\mathcal{I}_l, j}v \rangle \\
& \quad + \langle S\nabla_j W_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j}u_\epsilon, \nabla_j \mathcal{K}_{\mathcal{I}_l, j}v \rangle.
\end{aligned}$$

Repeating the above calculation and by Theorem 2.10, we have

$$\begin{aligned}
& \int_0^T \langle S\mathcal{K}_{\mathcal{I}_l}W_\epsilon(x_j, x_k)u_\epsilon, \mathcal{K}_{\mathcal{I}_l}v \rangle \\
& \lesssim_{\alpha, p, \tilde{p}} \int_0^T \left(\|\mathcal{K}_{\mathcal{I}_l}u_\epsilon\|_{L_{j, k}^{p, 2}} + \|\mathcal{K}_{\mathcal{I}_l}u_\epsilon\|_{L^2} \right) \\
& \quad \times \left(\| [S^*\mathcal{K}_{\mathcal{I}_l}v] \|_{L_{j, k}^{\tilde{p}, 2}} + \| [S^*\mathcal{K}_{\mathcal{I}_l}v] \|_{L^2} \right) ds.
\end{aligned}$$

Analogously for $j \notin \mathcal{I}_l$ and $k \in \mathcal{I}_l$. Finally if $i, k \notin \mathcal{I}_l$, obviously

$$\int_0^T \langle SK_{\mathcal{I}_l} W_\epsilon(x_j, x_k) u_\epsilon, \mathcal{K}_{\mathcal{I}_l} v \rangle = \langle SW_\epsilon(x_j, x_k) \mathcal{K}_{\mathcal{I}_l} u_\epsilon, \mathcal{K}_{\mathcal{I}_l} v \rangle.$$

after calculation, we have

$$\begin{aligned} & \int_0^T \langle SK_{\mathcal{I}_l} W_\epsilon(x_j, x_k) u_\epsilon, \mathcal{K}_{\mathcal{I}_l} v \rangle \\ & \lesssim_{p, \tilde{p}} \int_0^T \left(\|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_{L_{j,k}^{p,2}} + \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_{L^2} \right) \\ & \quad \times \left(\|[S^* \mathcal{K}_{\mathcal{I}_l} v]\|_{L_{j,k}^{\tilde{p},2}} + \|[S^* \mathcal{K}_{\mathcal{I}_l} v]\|_{L^2} \right) ds. \end{aligned}$$

Thus, for any $1 \leq j < k \leq N$, we have

$$\begin{aligned} & \int_0^T \langle SK_{\mathcal{I}_l} W_\epsilon(x_j, x_k) u_\epsilon, \mathcal{K}_{\mathcal{I}_l} v \rangle \\ & \lesssim_{\alpha, \tilde{p}, p, \tilde{q}, q} \int_0^T \left(\|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_{L_{j,k}^{p,2}} + \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_{L^2} \right) \\ & \quad \times \left(\|[S^* \mathcal{K}_{\mathcal{I}_l} v]\|_{L_{j,k}^{p,2}} + \|[S^* \mathcal{K}_{\mathcal{I}_l} v]\|_{L^2} \right) ds. \end{aligned}$$

And similarly for the term $SK_{\mathcal{I}_l} V_\epsilon u_\epsilon$.

Repeating the same procedure of Theorem 1.5, there is a constant C_2 only dependent on α , \tilde{p} , p , \tilde{q} and q such that

$$\|\mathcal{K}_{\mathcal{I}_l} Q_\epsilon u_\epsilon\|_X \leq C_1 \left(\sum_\mu Z_\mu + N \right) N T^{1/\theta} \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X. \quad (21)$$

And if $C_2 \left(\sum_\mu Z_\mu + N \right) N T^{1/\theta} < 1/2$, we can get

$$\|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X \lesssim_{p,q} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

Taking $\epsilon \rightarrow 0$, we have

$$\|\mathcal{K}_{\mathcal{I}_l} u\|_X \lesssim_{p,q} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

□

5 Numerical Analysis

Lemma 5.1. *Under the assumption of Theorem 1.10, we have*

$$\|u - P_R u\|_X \lesssim_{p,q} 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}$$

Proof. By the Equation (12), we know

$$(P_R u)(t) = (P_R U_0(t) u_0) + i(P_R Q u(t)).$$

Thus,

$$\|u - P_R u\|_X \leq \|(1 - P_R) U_0 u\|_X + \|(1 - P_R) Q u\|_X.$$

By the definition of P_R and Lemma 2.7, we know

$$\|(1 - P_R)U_0u\|_X \lesssim_{p,q} \|(1 - P_R)u_0\|_{L^2} \lesssim_{p,q} 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l}u_0\|_{L^2}.$$

Instead of studying $(1 - P_R)Qu$ directly, we study $(1 - P_R)Q_\epsilon u_\epsilon$, and then take the convergence. So just need to analyze $(1 - P_R)S(W_\epsilon(j, \cdot)u_\epsilon)$ and $(1 - P_R)S(W_\epsilon(x_j, x_k)u_\epsilon)$. They are similar, so we just deal with the latter.

We consider the following inner product

$$\begin{aligned} & \int_0^T \left\langle (1 - P_R)S(W_\epsilon(x_j, x_k)u_\epsilon), \sum_{1 \leq l \leq s} \mathcal{K}_{\mathcal{I}_l}v \right\rangle dt \\ &= \sum_{1 \leq l \leq s} \int_0^T \langle \mathcal{K}_{\mathcal{I}_l}S(W_\epsilon(x_j, x_k)u_\epsilon), (1 - P_R)v \rangle dt. \end{aligned}$$

Let

$$\begin{aligned} \tilde{\mathcal{K}}_{\mathcal{I}_l, j} &= (1 + \nabla_j) \prod_{l \neq j} (1 - \Delta_l)^{1/2}, \\ \tilde{\mathcal{K}}_{\mathcal{I}_l, j, k} &= (1 + \nabla_j) \otimes (1 + \nabla_k) \prod_{l \neq j, k} (1 - \Delta_l)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{K}}^j &= \frac{(1 - \nabla_j)}{(1 - \Delta_j)^{1/2}}, \\ \tilde{\mathcal{K}}^{j, k} &= \frac{(1 - \nabla_j) \otimes (1 - \nabla_k)}{(1 - \Delta_j)^{1/2} (1 - \Delta_k)^{1/2}}. \end{aligned}$$

Then,

$$\mathcal{K}_{\mathcal{I}_l} = \tilde{\mathcal{K}}^j \cdot \tilde{\mathcal{K}}_{\mathcal{I}_l, j} = \tilde{\mathcal{K}}^{j, k} \cdot \tilde{\mathcal{K}}_{\mathcal{I}_l, j, k}.$$

If $j, k \in \mathcal{I}_l$, we consider the following inner product

$$\begin{aligned} & \int_0^T \langle \mathcal{K}_{\mathcal{I}_l}S(W_\epsilon(x_j, x_k)u_\epsilon), (1 - P_R)v \rangle dt \\ &= \int_0^T \left\langle \tilde{\mathcal{K}}_{\mathcal{I}_l, j, k}^* (W_\epsilon(x_j, x_k)u_\epsilon), S^*(1 - P_R)\tilde{\mathcal{K}}^{j, k}v \right\rangle ds \\ &= \int_0^T \left\langle (1 - \nabla_j)(1 - \nabla_k)(W_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon), S^*(1 - P_R)\tilde{\mathcal{K}}^{j, k}v \right\rangle ds. \end{aligned}$$

Or if $j \in \mathcal{I}_l$ and $k \notin \mathcal{I}_l$, we consider the following inner product

$$\begin{aligned} & \int_0^T \langle \mathcal{K}_{\mathcal{I}_l}S(W_\epsilon(x_j, x_k)u_\epsilon), (1 - P_R)v \rangle dt \\ &= \int_0^T \left\langle \tilde{\mathcal{K}}_{\mathcal{I}_l, j}^* (W_\epsilon(x_j, x_k)u_\epsilon), S^*(1 - P_R)\tilde{\mathcal{K}}^jv \right\rangle ds \\ &= \int_0^T \left\langle (1 - \nabla_j)(W_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l, j, k}u_\epsilon), S^*(1 - P_R)\tilde{\mathcal{K}}^jv \right\rangle ds. \end{aligned}$$

Analogously for $j \notin \mathcal{I}_l$ and $k \in \mathcal{I}_l$. And finally, if $j, k \notin \mathcal{I}_l$,

$$\int_0^T \langle \mathcal{K}_{\mathcal{I}_l} S(W_\epsilon(x_j, x_k)u_\epsilon), (1 - P_R)v \rangle dt = \int_0^T \langle (W_\epsilon(x_j, x_k)\mathcal{K}_{\mathcal{I}_l}u_\epsilon), S^*(1 - P_R)v \rangle dt.$$

Before repeating the proof of Theorem 1.8, we only need to deal with

$$\|S^*(1 - P_R)\tilde{\mathcal{K}}^j v\|_{L_t^{\theta_{\tilde{p}}}(L_{j,k}^{\tilde{p},2})}$$

and

$$\|S^*(1 - P_R)\tilde{\mathcal{K}}^{j,k} v\|_{L_t^{\theta_{\tilde{p}}}(L_{j,k}^{\tilde{p},2})}.$$

By Theorem 2.10 and Corollary 2.9, we know

$$\begin{aligned} \|S^*(1 - P_R)\tilde{\mathcal{K}}^j v\|_{L_t^{\theta_{\tilde{p}}}(L_{j,k}^{\tilde{p},2})} &\lesssim_{\tilde{p},p,q} \|S^*(1 - P_R)v\|_{L_t^{\theta_{\tilde{p}}}(L_{j,k}^{\tilde{p},2})} \\ &\lesssim_{\tilde{p},p,q} 1/R \left\| \sum_l \mathcal{K}_{\mathcal{I}_l} v \right\|_{L_t^{\theta'_q}(L_D^{q',2})} \end{aligned}$$

and

$$\begin{aligned} \|S^*(1 - P_R)\tilde{\mathcal{K}}^{j,k} v\|_{L_t^{\theta_{\tilde{p}}}(L_{j,k}^{\tilde{p},2})} &\lesssim_{\tilde{p},p,q} \|S^*(1 - P_R)v\|_{L_t^{\theta_{\tilde{p}}}(L_{j,k}^{\tilde{p},2})} \\ &\lesssim_{\tilde{p},p,q} 1/R \left\| \sum_l \mathcal{K}_{\mathcal{I}_l} v \right\|_{L_t^{\theta'_q}(L_D^{q',2})}. \end{aligned}$$

Now, we know

$$\begin{aligned} &\int_0^T \langle \mathcal{K}_{\mathcal{I}_l} S(W_\epsilon(x_j, x_k)u_\epsilon), (1 - P_R)v \rangle dt \\ &\lesssim_{\alpha, \tilde{p}, p, \tilde{q}, q} T^{1/\theta} / R \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X \left\| \sum_l \mathcal{K}_{\mathcal{I}_l} v \right\|_{L_t^{\theta'_q}(L_D^{q',2})}. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^T \left\langle (1 - P_R)S(W_\epsilon(x_j, x_k)u_\epsilon), \sum_{1 \leq l \leq s} \mathcal{K}_{\mathcal{I}_l} v \right\rangle dt \\ &\lesssim_{\alpha, \tilde{p}, p, \tilde{q}, q} T^{1/\theta} / R \left(\sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X \right) \left\| \sum_l \mathcal{K}_{\mathcal{I}_l} v \right\|_{L_t^{\theta'_q}(L_D^{q',2})}. \end{aligned}$$

Therefore,

$$\|(1 - P_R)S(W_\epsilon(x_j, x_k)u_\epsilon)\|_X \lesssim_{\alpha, \tilde{p}, p, \tilde{q}, q} T^{1/\theta} / R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X.$$

Similarly for the term $(1 - P_R)S(V_\epsilon(x_j, \cdot)u_\epsilon)$

Hence, there is a constant $C_3 > C_2$ only dependent on $\alpha, \tilde{p}, p, \tilde{q}$ and q , such that

$$\|(1 - P_R)Q_\epsilon u_\epsilon\|_X \leq C_3 \left(\sum_\mu Z_\mu + N \right) NT^{1/\theta} / R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X. \quad (22)$$

If $C_3(\sum_\mu Z_\mu + N)NT^{1/\theta} < 1/2$, then $C_2(\sum_\mu Z_\mu + N)NT^{1/\theta} < 1/2$. By Theorem 1.8 we have

$$\|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X \lesssim_{p,q} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

Thus

$$\|(1 - P_R)Q_\epsilon u_\epsilon\|_X \leq 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_\epsilon\|_X \lesssim_{p,q} 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

Therefore,

$$\|u_\epsilon - P_R u_\epsilon\|_X \lesssim_{p,q} 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

Finally, taking $\epsilon \rightarrow 0$, we have

$$\|u - P_R u\|_X \lesssim_{p,q} 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

□

Proof of Theorem 1.10. The proof of existence is similar to Theorem 1.2 except the following modification:

$$\int_0^T \langle P_R S W(x_i, x_j) u(t), v \rangle dt = \int_0^T \langle S W(x_i, x_j) u(t), P_R v \rangle ds$$

Given the symmetry of the projector P_R , we know if the Equation (10) has a solution, the solution u_R keeps the spin states. For the existence, we only need study the term $\|P_R S^* v\|_{L_t^{\theta'_p}(\tilde{L}_D^{\tilde{p}', 2})}$.

By the Corollary 2.9, we have

$$\|P_R S^* v\|_{L_t^{\theta'_q}(\tilde{L}_D^{q, 2})} \lesssim_{\tilde{p}, p} \|v\|_{L_t^{\theta'_p}(\tilde{L}_D^{p', 2})}.$$

Thus by Theorem 1.2 and under the assumption of Theorem 1.10, for the Equation (10), there exists a unique solution u_R , such that

$$\|u\|_X \lesssim_{p,q} \|u_0\|_{L^2}.$$

Instead of studying $\|u - u_R\|_X$ directly, we analyze $\|P_R - u_R\|_X$ at the beginning. By the Formula 12, we know

$$u_R - P_R u = iP_R Q(u - u_R)$$

Repeating the above process, and under the assumption of Theorem 1.8, we know

$$\|u_R - P_R u\|_X \leq 1/2 \|u - u_R\|_X \leq 1/2 \|u_R - P_R u\|_X + 1/2 \|u - P_R u\|_X$$

Then,

$$\|u_R - P_R u\|_X \leq \|u - P_R u\|_X \lesssim_{p,q} 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

Finally, we know

$$\|u - u_R\|_{X_{\text{anti}}} \leq \|u - P_R u\|_{X_{\text{anti}}} + \|u_R - P_R u\|_{X_{\text{anti}}} \lesssim_{p,q} 1/R \sum_{1 \leq l \leq s} \|\mathcal{K}_{\mathcal{I}_l} u_0\|_{L^2}.$$

□

A The Calderón-Zygmund Inequality

Unlike the usual Calderón-Zygmund inequality, we need to prove a new one which is compatible for our special functional space. But the proof is similar, so we just give the sketch of proof, for the details, see [17, p.27-38].

Definition A.1. Let $n \in \mathbb{N}$ and let $\Delta_n := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | x = y\}$ be the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. Fix two constants $C > 0$ and $0 < \sigma \leq 1$. A Calderón-Zygmund pair on \mathbb{R}^n with constants C and σ is a pair (T_x, K) , consisting of a bounded linear operator $T_x : L^2(\mathbb{R}^n, \mathbb{C}) \rightarrow L^2(\mathbb{R}^n, \mathbb{C})$ working on variable $x \in \mathbb{R}^n$ and a continuous function $K : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_n \rightarrow \mathbb{C}$, satisfying the following axioms.

- $\|T_x f\|_{L^2} \leq C \|f\|_{L^2}$ for all $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{C})$.
- For $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, if $f(x, z) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{C}$ is a continuous function with compact support then

$$(T_x f)(x, z) = \int_{\mathbb{R}^n} K(x, y) f(y, z) dy.$$

- Let $x, y \in \mathbb{R}^n$ such that $x \neq y$. Then

$$|K(x, y)| \leq \frac{C}{|x - y|^n}.$$

- Let $x, x', y, y' \in \mathbb{R}^n$ such that $x \neq y$, $x' \neq y'$, and $x' \neq y$. Then

$$|y - y'| < \frac{1}{2}|x - y| \implies |K(x, y) - K(x, y')| \leq \frac{C|y - y'|^\sigma}{|x - y|^{n+\sigma}},$$

$$|x - x'| < \frac{1}{2}|x - y| \implies |K(x, y) - K(x', y)| \leq \frac{C|x - x'|^\sigma}{|x - y|^{n+\sigma}}.$$

Theorem A.2. Fix an integer $m, n \in \mathbb{N}$, a real number $1 < p < \infty$, and two constants $C > 0$ and $0 < \sigma \leq 1$. Then there exists a constant $c = c(n, p, \sigma, C)$ such that every Calderón-Zygmund pair (T_x, K) working only on the variable $x \in \mathbb{R}^n$ with constant C and σ satisfies the inequality

$$\|T_x f(x, y)\|_{L^p(\mathbb{R}^n, L^2(\mathbb{R}^m))} \leq c \|f(x, y)\|_{L^p(\mathbb{R}^n, L^2(\mathbb{R}^m))}$$

for all $f \in L^2(\mathbb{R}^{n+m}, \mathbb{C}) \cap L^p(\mathbb{R}^n, L^2(\mathbb{R}^m))$.

Sketch of proof. Let $b_f(x) = \left(\int_{\mathbb{R}^m} |f(x, y)|^2 dy\right)^{1/2}$, and μ the Lebesgue measure on \mathbb{R}^n . Then define the function $\kappa_f : (0, \infty) \rightarrow [0, \infty]$ by

$$\kappa_f(t) := \mu(\{t \geq 0 | |b_f(x)| > t\}) \text{ for } t > 0.$$

We shorten the operator T_x by T without confusion.

Step 1. (Calderón Zygmund Decomposition).

Decompose $b_f(x)$ in place of $f(x, y)$ directly. Then for $t > 0$, there exists a countable collection of closed cubes $Q_i \subset \mathbb{R}^n$ with pairwise disjoint interiors such that

$$\mu(Q_i) < \frac{1}{t} \int_{Q_i} |b_f(x)| dx \leq 2^n \mu(Q_i) \text{ for all } i \in \mathbb{N}$$

and

$$|b_f(x)| \leq t \text{ for almost all } x \in \mathbb{R}^n \setminus B$$

where $B := \cup_{i=1}^{\infty} Q_i$.

Step 2. (Construction of function).

Define $g, h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ by

$$g(x, y) := f(x, y) \mathbb{1}_{\mathbb{R}^n \setminus B} + \sum_i \frac{\int_{Q_i} f(x, y) dx}{\mu(Q_i)} \mathbb{1}_{Q_i}, \quad h := f - g.$$

Then,

$$b_g(x) = b_f(x) \leq t \text{ for almost all } x \in \mathbb{R}^n \setminus B, \quad (23)$$

and by Minkowski's inequality,

$$\begin{aligned} b_g(x) &= \frac{1}{\mu(Q_i)} \left(\int_{\mathbb{R}^m} \left| \int_{Q_i} f dx \right|^2 dy \right)^{1/2} \\ &\leq \frac{1}{\mu(Q_i)} \int_{Q_i} |b_f(x)| dx \quad \text{for } x \in B. \\ &\leq 2^n t \end{aligned} \quad (24)$$

Combining Equation (23) and Equation (24) together, we know

$$\|g\|_{L^1(L^2)} = \|b_g\|_{L^1} \leq \|b_f\|_{L^1} = \|f\|_{L^1(L^2)}, \quad \|h\|_{L^1(L^2)} \leq 2\|f\|_{L^1(L^2)}. \quad (25)$$

Hence, we have

$$\kappa_{Tg} \leq \frac{1}{t^2} \int_{\mathbb{R}^n} |b_g(x)|^2 dx \leq \frac{2^n}{t} \int_{\mathbb{R}^n} |b_g(x)| dx \leq \frac{2^n}{t} \int_{\mathbb{R}^n} |b_f(x)| dx \leq \frac{2^n}{t} \|f\|_{L^1(L^2)}.$$

Step 3. (Estimate for κ_{Th}). Define $h_i(x)$ by

$$h_i(x, y) = h(x, y) \mathbb{1}_{Q_i}.$$

Denote by $q_i \in Q_i$ the center of the cube Q_i and by $2r_i > 0$ its length. Then $|x - q_i| \leq \sqrt{n}r_i$ for all Q_i . Then we have

$$(Th_i)(x, y) = \int_{Q_i} K(x, z) h_i(z, y) dz = \int_{Q_i} (K(x, z) - K(x, q_i)) h_i(z, y) dz.$$

Hence, by Minkowski's inequality

$$b_{Th_i}(x) \leq \int_{Q_i} |K(x, z) - K(x, q_i)| |b_{h_i}(z)| dz.$$

Then, by the standard proof of Calderón zygmond inequality, we know, there is a constant c dependent on n such that

$$\kappa_{Th}(t) \leq c \left(\mu(B) + \frac{1}{t} \|b_h\|_{L^1} \right) \text{ for all } t > 0.$$

Besides,

$$\mu(B) = \sum_i \mu(Q_i) \leq \frac{1}{t} \sum_i \int_{Q_i} |b_f(x)| dx \leq \frac{1}{t} \|f\|_{L^1(L^2)}.$$

Together with Equation (25),

$$\kappa_{Th}(t) \leq \frac{3c}{t} \|f\|_{L^1(L^2)}.$$

By the triangle inequality, we know

$$b_{Tf}(x) \leq b_{Tg} + b_{Th},$$

therefore,

$$\kappa_{Tf}(2t) \leq \kappa_{Tg}(t) + \kappa_{Th}(t) \leq \frac{2^{n+1} + 6c}{2t} \|f\|_{L^1(L^2)}.$$

Finally, using the standard method, we get conclusion. \square

If $a : \mathbb{R}^n \rightarrow \mathbb{C}$ is a bounded measurable function, it determines a bounded linear operator

$$T_a : L^2(\mathbb{R}^n, \mathbb{C}) \rightarrow L^2(\mathbb{R}^n, \mathbb{C})$$

given by

$$T_a u := \widetilde{a\tilde{u}}$$

for $u \in L^2(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{C})$, and \tilde{u} is the inverse Fourier Transform.

Theorem A.3. *For every integer $m, n \in \mathbb{N}$, every constant $C > 0$, and every real number $1 < p < \infty$, there exists a constant $c = c(n, p, C)$ with the following significance. Let $a : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$ be a C^{n+2} function that satisfies the inequality*

$$|\partial^\alpha a(\xi)| \leq \frac{C}{|\xi|^\alpha}$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| \leq n + 2$. Then

$$\|T_a f\|_{L^p(\mathbb{R}^n, L^2(\mathbb{R}^m))} \leq c \|f\|_{L^p(\mathbb{R}^n, L^2(\mathbb{R}^m))}$$

for all $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m), \mathbb{C} \cap L^p(\mathbb{R}^n, L^2(\mathbb{R}^m))$.

The proof is same with the normal Mihklin Multiplier Theorem except using the Theorem A.2 instead of the normal one.

Acknowledgement

The author thanks Éric SÉRÉ for his fruitful discussions on the full work of the paper. This work was supported by grants from Région Ile-de-France.

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