

Homotopic Digital Rigid Motion: An Optimization Approach on Cellular Complexes

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Rigid motions in \mathbb{Z}^n : Problem

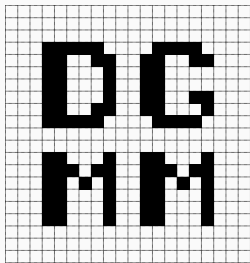
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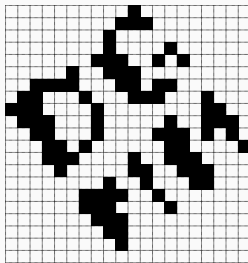
In \mathbb{R}^n , a rigid transformation, which is a combination of rotations and translations, is a geometric transformation that preserves the Euclidean distance between any pair of points.

Problem

In \mathbb{Z}^n , topology (and geometry) is often altered during such transformations due to the discontinuities induced by digitization.



original image

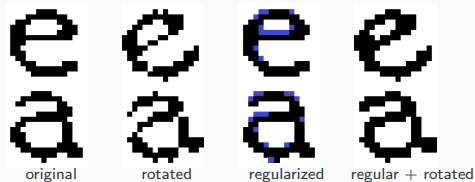


transformed image

Topological preservation: State of the art

State of the art

- A class of 2D images that preserve their topological properties during rigid motions, called **regular**, has been identified [1].

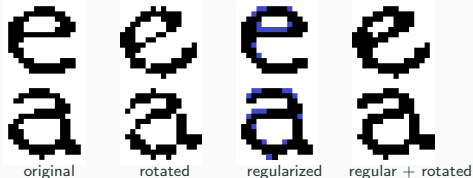


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Topological preservation: State of the art

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⇒ This notion can be **transposed to neither higher dimensions nor other transformations** [2].



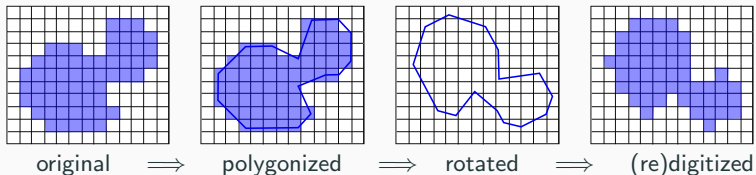
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[2] Passat, Kenmochi, Ngo, Pluta, "Rigid motions in the cubic grid: A discussion on topological issues", In Proc. DGCI 2019.

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- A class of 2D images that preserve their topological properties during rigid motions, called **regular**, has been identified [1].
- With a polygonal intermediate, a class of regions in \mathbb{R}^2 , which preserve their connectivity after the digitization, called **quasi-regular**, has been identified [3,4].



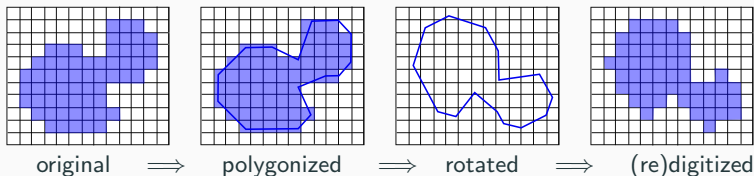
[3] Ngo, Passat, Kenmochi, Debled-Rennesson, "Geometric preservation of 2D digital objects under rigid motions", Journal of Mathematical Imaging and Vision, 2019.

[4] Ngo, Passat, Kenmochi, Debled-Rennesson, "Convexity invariance of voxel objects under rigid motions", in Proc. ICPR 2018.

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- With a polygonal intermediate, a class of regions in \mathbb{R}^2 , which preserve their connectivity after the digitization, called **quasi-regular**, has been identified [3,4].
⇒ Extension to higher dimensions **is not trivial** [2].

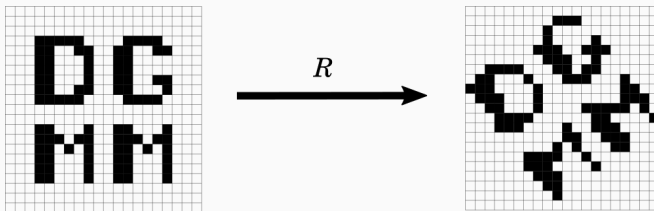


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Goal and contributions

Goal

Given a binary object X and a rigid transformation \mathcal{R} , construct a transformed binary object $X_{\mathcal{R}}$ **preserving the homotopy type**.

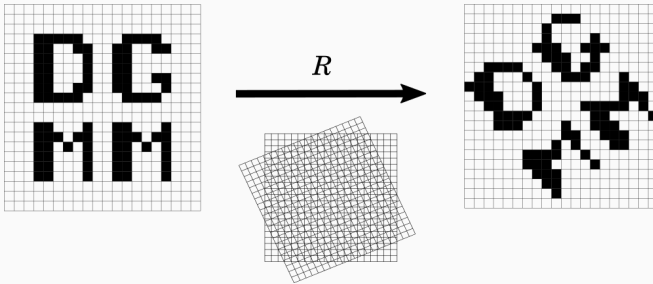


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Given a binary object X and a rigid transformation \mathcal{R} , construct a transformed binary object $X_{\mathcal{R}}$ **preserving the homotopy type**.

The problem is formulated as an **optimization problem in the space of cellular complexes** with the notion of collapse on complexes.

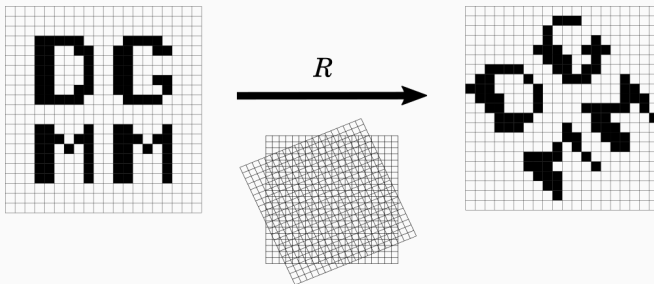


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Given a binary object X and a rigid transformation \mathcal{R} , construct a transformed binary object $X_{\mathcal{R}}$ **preserving the homotopy type**.

The problem is formulated as an **optimization problem in the space of cellular complexes** with the notion of collapse on complexes.



Advantages

The framework is **flexible** in higher dimensions as well as other geometric transformations.

Problem statement as a discrete optimization

Reaching the most similar $X_{\mathcal{R}}$ to X can be formalized as:

$$X_{\mathcal{R}} = \arg_{Y \in \mathbb{Z}^2} \min \mathcal{D}_{\mathcal{R},X}(Y)$$

where $\mathcal{D}_{\mathcal{R},X}(Y)$ is a dissimilarity measure between X and Y .

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Example of dissimilarity measure

- based on Gauss digitization:

$$\mathcal{D}_{\mathcal{R},X}^{\square}(Y) = |\square(\mathcal{R}(\square(X))) \setminus Y| + |Y \setminus \square(\mathcal{R}(\square(X)))|$$

- **Continuous analogue of X :** $\square(X) = X \oplus \square = X$
where \oplus is the dilation operator and \square is the structuring element $[\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$.
- **Gauss digitization of X :** $\square(X) = X \cap \mathbb{Z}^2$.

Problem statement as a discrete optimization

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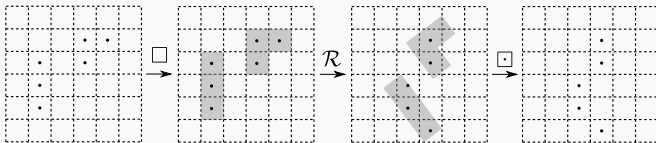
$$X_{\mathcal{R}} = \arg_{Y \in 2^{\mathbb{Z}^2}} \min \mathcal{D}_{\mathcal{R},X}(Y)$$

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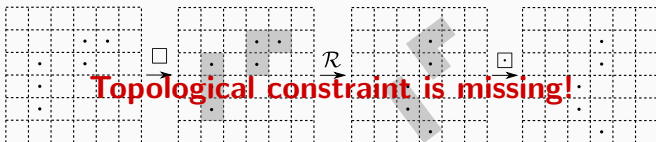
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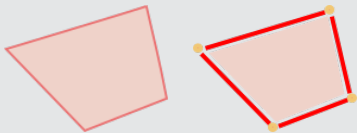
- based on Gauss digitization:

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Definition of cellular space

A closed convex polygon P and its partition $\mathcal{F}(P)$



$\mathcal{F}(P)$ contains:

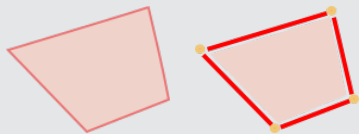
2-face (interior of P , $\overset{\circ}{P}$),

1-faces (edges of P), and

0-faces (vertices of P).

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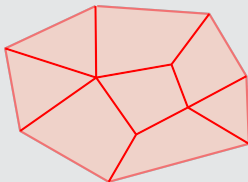
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A union of closed convex polygons Ω and its partition $\mathbb{K}(\Omega)$

Let $\Omega = \bigcup \mathcal{K}$ where \mathcal{K} is a set of closed, convex polygons such that for any pair $P_1, P_2 \in \mathcal{K}$, $\overset{\circ}{P}_1 \cap \overset{\circ}{P}_2 = \emptyset$. Then, $\mathbb{K}(\Omega) = \bigcup_{P \in \mathcal{K}} \mathcal{F}(P)$.



Definition of cellular complexes

Let \mathbb{K} be a cellular space and $f \in \mathbb{K}$ be a face.

Cell $C(f)$

The cell $C(f)$ induced by f is the subset of faces of \mathbb{K} such that $\bigcup C(f)$ is the smallest closed set that includes f .

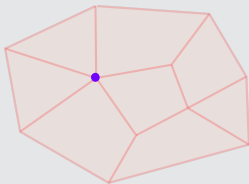


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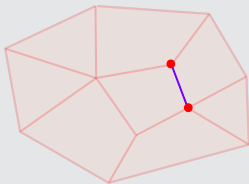


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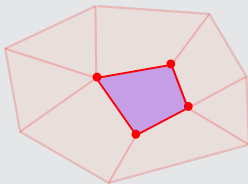


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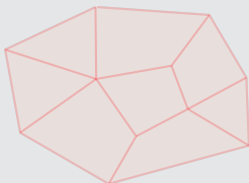
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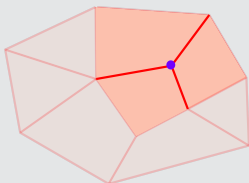
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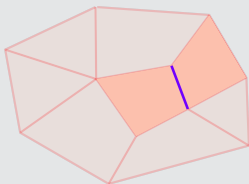
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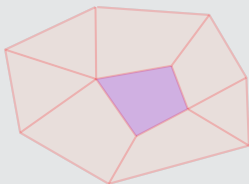
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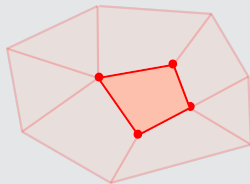
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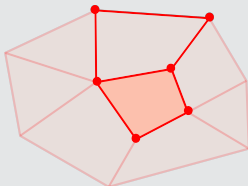
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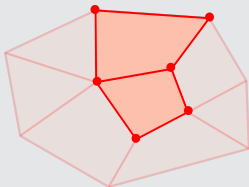
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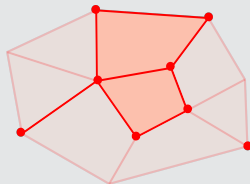
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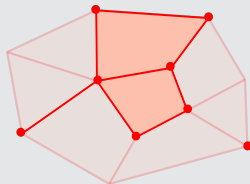
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The **embedding of K into \mathbb{R}^2** is defined by $\Pi_{\mathbb{R}^2}(K) = \bigcup K$.

If $X = \Pi_{\mathbb{R}^2}(K)$, K is the **embedding of X into \mathbb{K}** , $K = \Pi_{\mathbb{K}}(X)$.

Collapse on complexes

Let K be a complex defined in a cellular space \mathbb{K} .

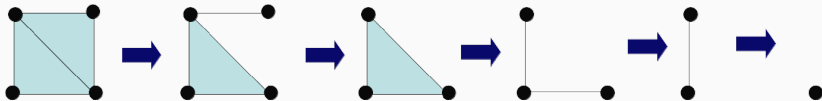
Elementary collapse

Suppose that τ and σ are two faces of K such that

- $\tau \subset \sigma$ with $\dim(\tau) = \dim(\sigma) - 1$ and
- σ is a maximal face of K and no other maximal face of K contains τ ,

then τ is called a **free face** and the removal of the faces, $K \setminus \{\tau, \sigma\}$, is called an **elementary collapse**.

If there is a sequence of elementary collapses from K to a complex K' , we say that K **collapses** to K' .



Simple cells

Let K be a complex defined in a cellular space \mathbb{K} on \mathbb{R}^2 .

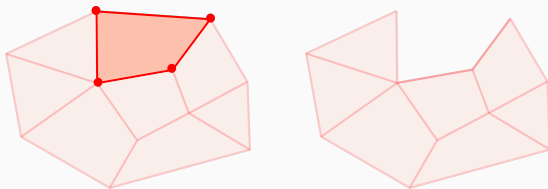
Let f_2 be a 2-face of K .

Let $D_d(f_2)$, $d = 0, 1$, be the subset of $C(f_2)$ composed by the d -faces f such that $S(f) \cap K = S(f) \cap C(f_2)$.

Simple cells

If $|D_1(f_2)| = |D_0(f_2)| + 1$, $C(f_2)$ is called a **simple 2-cell** for K .

- **Detachment** of a simple 2-cell $C(f_2)$ from K : collapse operation from K to $K \otimes C(f_2) = K \setminus (\{f_2\} \cup D_1(f_2) \cup D_0(f_2))$
- **Attachment** of a simple 2-cell $C(f_2)$ for $K \cup C(f_2)$ where $f \in \mathbb{K} \setminus K$: the inverse collapse operation from K into $K \cup C(f_2)$

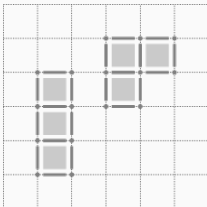


Cubical space \mathbb{F} induced by \mathbb{Z}^2

Let $\Delta = \mathbb{Z} + \frac{1}{2}$. The induced **cellular complex space** \mathbb{F} is composed of:

- set of 0-faces $\mathbb{F}_0 = \{\{\mathbf{d}\} \mid \mathbf{d} \in \Delta^2\}$
- set of 1-faces $\mathbb{F}_1 = \cup_{i=1,2} \{] \mathbf{d}, \mathbf{d} + \mathbf{e}_i [\mid \mathbf{d} \in \Delta^2\}$
- set of 2-faces $\mathbb{F}_2 = \{] \mathbf{d}, \mathbf{d} + \mathbf{e}_1 [\times] \mathbf{d}, \mathbf{d} + \mathbf{e}_2 [\mid \mathbf{d} \in \Delta^2\}$

where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

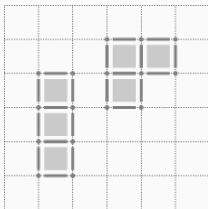


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- set of 1-faces $\mathbb{F}_1 = \bigcup_{i=1,2} \{]d, d + e_i[\mid d \in \Delta^2\}$
- set of 2-faces $\mathbb{F}_2 = \{]d, d + e_1[\times]d, d + e_2[\mid d \in \Delta^2\}$

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Given a digital object $X \subset \mathbb{Z}^2$, the **associated complex** $F = \Pi_{\mathbb{F}}(\square(X))$ is defined as:

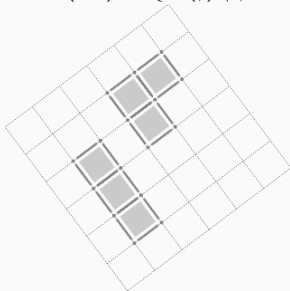
$$F = \bigcup_{x \in X} C(\blacksquare(x))$$

where $\blacksquare(p) = p \oplus] - \frac{1}{2}, \frac{1}{2}[^2$ for $p \in \mathbb{Z}^2$.

Cubical space \mathbb{G} induced by $\mathcal{R}(\mathbb{Z}^2)$

The **cubical space** \mathbb{G} induced by a rigid motion \mathcal{R} and \mathbb{Z}^2 is composed of the three sets of d -faces ($0 \leq d \leq 2$):

$$\mathbb{G}_d = \mathcal{R}(\mathbb{F}_d) = \{\mathcal{R}(f) \mid f \in \mathbb{F}_d\}$$



The continuous object $X_{\mathcal{R}} \subset \mathbb{Z}^2$ is modeled by the complex $G = \Pi_{\mathbb{G}}(X_{\mathcal{R}})$, which is defined by

$$G = \mathcal{R}(F) = \mathcal{R}(\Pi_{\mathbb{F}}(X)) = \{\mathcal{R}(f) \mid f \in \Pi_{\mathbb{F}}(X)\}$$

Rational rigid motions

Rigid motion $\mathcal{R}_{(\theta, \mathbf{t})} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

A rigid motion $\mathcal{R}_{(\theta, \mathbf{t})}$ is defined for any $\mathbf{p} \in \mathbb{R}^2$ as

$$\mathcal{R}_{(\theta, \mathbf{t})}(\mathbf{p}) = R(\theta) \cdot \mathbf{p} + \mathbf{t} \quad \text{where} \quad R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

is the rotation matrix of angle θ and \mathbf{t} is the translation vector.

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Here we consider that $R(\theta)$ is rational, i.e.,

$$\cos \theta = \frac{a}{c} \quad \text{and} \quad \sin \theta = \frac{b}{c}$$

where $(a, b, c) \in \mathbb{Z}^3$ is a Pythagorean triple, and $\mathbf{t} \in \mathbb{Q}^2$.

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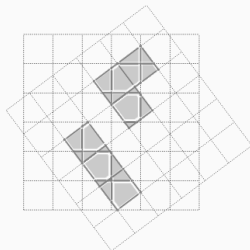
Remark

A family of rational rigid motions handled for application uses,

- are sufficiently dense, and
- leads to exact geometric calculations.

Cellular space \mathbb{H} refining \mathbb{F} and \mathbb{G}

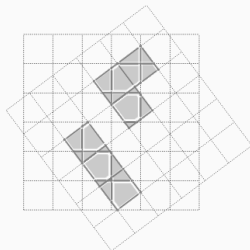
A new cellular space \mathbb{H} that refines both \mathbb{F} and \mathbb{G} is built.



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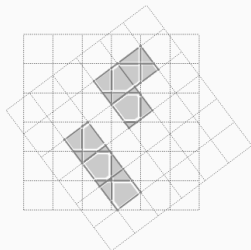


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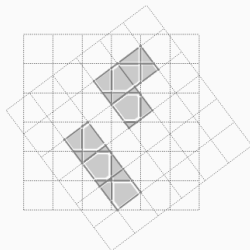
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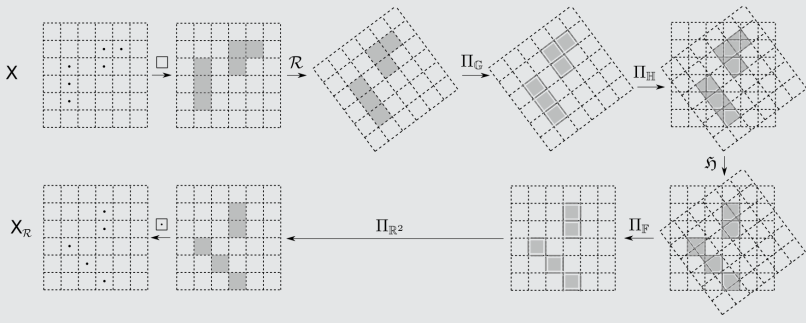
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The 2-faces of \mathbb{H} are convex polygons with 3 to 8 edges.

Optimization-based rigid motion with topological constraint

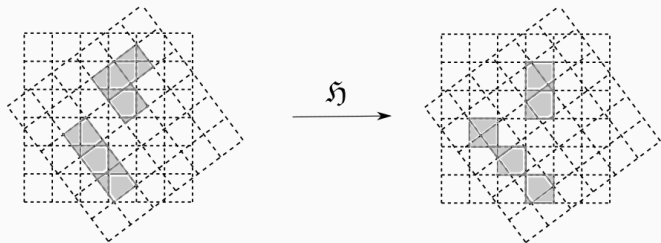
Proposed framework



Main steps:

1. Generate the cellular complex H on \mathbb{H} from $G = \Pi_{\mathbb{G}}(\mathcal{R}(\square(X)))$.
2. Apply a **homotopic transformation** \mathfrak{H} from H to \hat{H} .
3. Embed the cellular complex \hat{H} into \mathbb{F} , i.e. $\hat{F} = \Pi_{\mathbb{F}}(\hat{H})$.

Homotopic transformation \mathfrak{H} on \mathbb{H}



It is mandatory that:

- \mathfrak{H} is a homotopic transformation from H to \hat{H} represented by a sequence of additions / removals of simple 2-cells,
- \hat{H} can be embedded into \mathbb{F} , i.e. $\hat{F} = \Pi_{\mathbb{F}}(\hat{H})$ exists; and
- the digital analogue $X_{\mathcal{R}} = \square(\Pi_{\mathbb{R}^2}(\hat{H})) \subset \mathbb{Z}^2$ of \hat{H} be as close as possible to the solution of the optimization problem

$$X_{\mathcal{R}} = \arg_{Y \in 2^{\mathbb{Z}^2}} \min \mathcal{D}_{\mathcal{R}, X}(Y)$$

Heuristics for solving the optimization problem

A non-exhaustive list of ideas in order to reach the goal:

- Border processing
- Measure separability and gradient climbing
- Homotopic transformations in \mathbb{F}
- Termination issues
- Non-existence of solutions

General algorithm of homotopic digital rigid motion

Algorithm 1: Construction of \widehat{H} from H by \mathcal{H}

Input: $H \subset \mathbb{H}$, $\mathcal{D}_{\mathcal{R},X} : 2^{\mathbb{Z}^2} \rightarrow \mathbb{R}_+$

Output: $\widehat{H} \subset \mathbb{H}$

```
1  $\widehat{H} \leftarrow H$ 
2  $\mathbb{B}_2 \leftarrow \{f_2 \in \mathbb{F}_2 \mid \Phi(f_2) \not\subseteq \mathbb{H}_2(H) \wedge \Phi(f_2) \cap \mathbb{H}_2(H) \neq \emptyset\}$ 
3 while  $\mathbb{B}_2 \neq \emptyset$  do
4     choose  $f_2 \in \mathbb{B}_2$  wrt  $\mathcal{D}_{\mathcal{R},X}$ 
5      $\mathbb{B}_2 \leftarrow \mathbb{B}_2 \setminus \{f_2\}$ 
6      $(\mathbb{I}_2, \mathbb{O}_2) \leftarrow (\Phi(f_2) \cap \mathbb{H}_2(\widehat{H}), \Phi(f_2) \setminus \mathbb{H}_2(\widehat{H}))$ 
7     while  $\exists h_2 \in \mathbb{I}_2$  s.t.  $C(h_2)$  is simple for  $\widehat{H}$  do
8          $\widehat{H} \leftarrow \widehat{H} \otimes C(h_2)$ 
9          $(\mathbb{I}_2, \mathbb{O}_2) \leftarrow (\mathbb{I}_2 \setminus \{h_2\}, \mathbb{O}_2 \cup \{h_2\})$ 
10    if  $\mathbb{I}_2 \neq \emptyset$  then
11        while  $\exists h_2 \in \mathbb{O}_2$  s.t.  $C(h_2)$  is simple for  $\widehat{H}$  do
12             $\widehat{H} \leftarrow \widehat{H} \cup C(h_2)$ 
13             $(\mathbb{I}_2, \mathbb{O}_2) \leftarrow (\mathbb{I}_2 \cup \{h_2\}, \mathbb{O}_2 \setminus \{h_2\})$ 
14        if  $\mathbb{O}_2 \neq \emptyset$  then Failure of the process
```

Dissimilarity measures

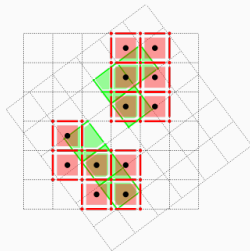
Examples of dissimilarity measure

- based on Gauss digitization:

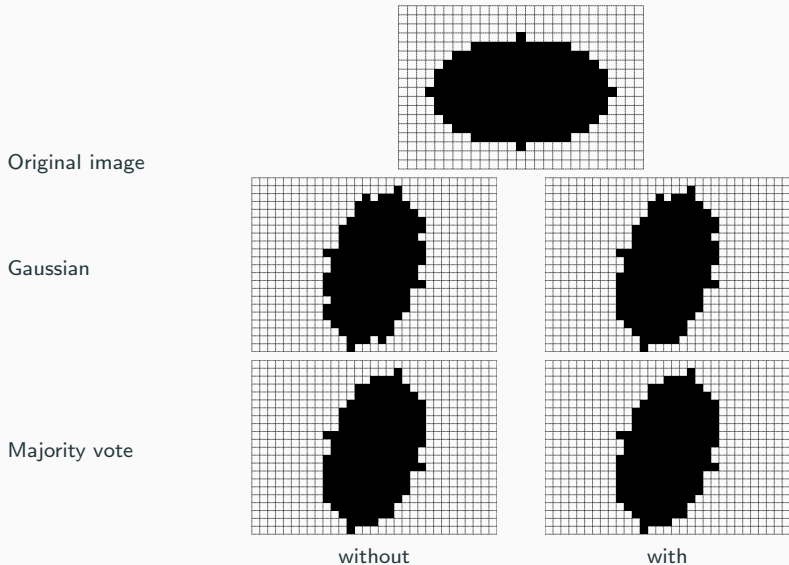
$$\mathcal{D}_{\mathcal{R},X}^{\square}(Y) = |\square(\mathcal{R}(\square(X))) \setminus Y| + |Y \setminus \square(\mathcal{R}(\square(X)))|$$

- based on majority vote digitization:

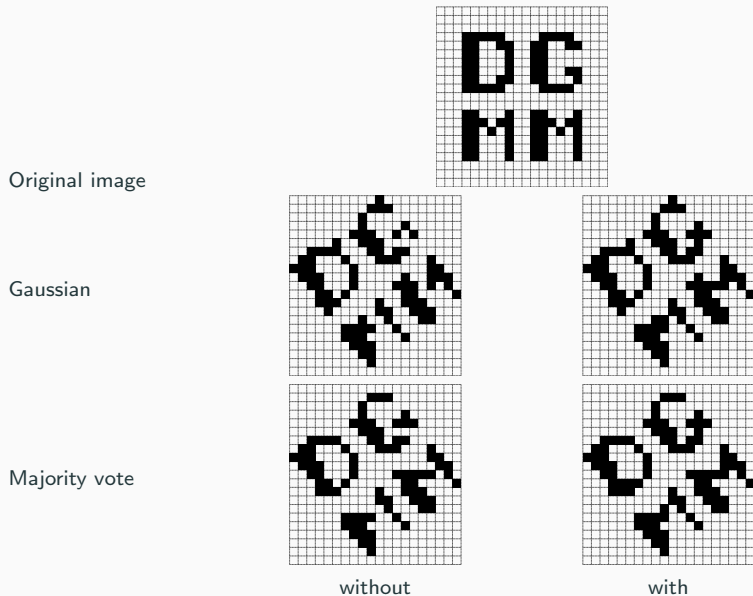
$$\mathcal{D}_{\mathcal{R},X}^{\square}(Y) = |\mathcal{R}(\square(X)) \setminus \square(Y)| + |\square(Y) \setminus \mathcal{R}(\square(X))|$$



Experiments with/without topological constraint

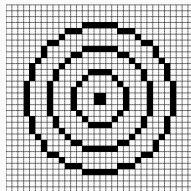


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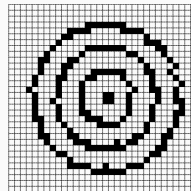
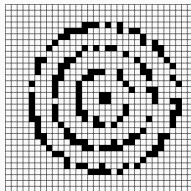


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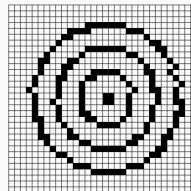
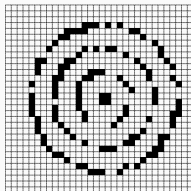
Original image



Gaussian



Majority vote



without

with

Conclusion and perspectives

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- The proposed approach of digital rigid motion ensures topological invariance between the initial object and its image.
- It relies on an optimization strategy under topological constraints on the cellular space.
- It may lead to non-existence of solutions.

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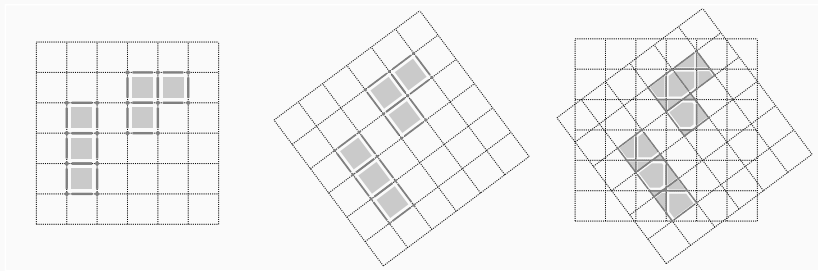
Perspectives

- Multigrid strategies may help us to handle solution non-existence.
- Other dissimilarity measures can be integrated in the framework.
- Adaptation of the proposed approach to other topological models and non-binary images will be investigated.
- Transformations in higher dimensions and other families of transformations will be considered.
- Other constraints, which are not only topological but also geometrical, can be combined.

Implementation issues

A first implementation of the homotopic digital rigid motion proceeds in three steps:

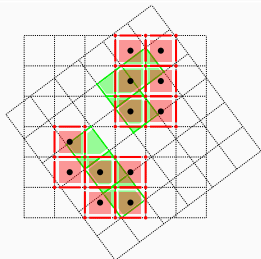
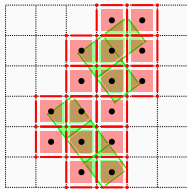
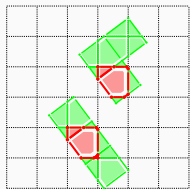
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3. update \hat{H} iteratively while the error measure is reduced and the solution is homotopic to H .

