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On linearised and elliptic versions of the Kashiwara-Vergne Lie algebra

Elise Raphael and Leila Schneps

Abstract

The goal of this article is to define a linearized or depth-graded version $\mathfrak{lkv}$, and a closely related elliptic version $\mathfrak{krv}_{ell}$, of the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$ originally constructed by Alekseev and Torossian as the space of solutions to the linearized Kashiwara-Vergne problem. We show how the elliptic Lie algebra $\mathfrak{krv}_{ell}$ is related to earlier constructions of elliptic versions $\mathfrak{grt}_{ell}$ and $\mathfrak{ds}_{ell}$ of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ and the double shuffle Lie algebra $\mathfrak{ds}$. In particular we show that there is an injective Lie morphism $\mathfrak{ds}_{ell} \hookrightarrow \mathfrak{krv}_{ell}$, and an injective Lie algebra morphism $\mathfrak{krv} \to \mathfrak{krv}_{ell}$ extending the known morphisms $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ (Enriquez section) and $\mathfrak{ds} \to \mathfrak{ds}_{ell}$ (Écalle map).

1. Introduction

This article studies two Lie algebras closely related to the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$ defined in [AT]: firstly, a linearized (or depth-graded) version $\mathfrak{lkv}$, and secondly, an elliptic version $\mathfrak{krv}_{ell}$ whose construction is closely related to that of $\mathfrak{lkv}$. The results are motivated by the comparison of $\mathfrak{krv}$ with two other Lie algebras familiar from the theory of multiple zeta values: the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ and the double shuffle Lie algebra $\mathfrak{ds}$. Our definition of $\mathfrak{lkv}$ is an analog of the definition of the bigraded linearized double shuffle Lie algebra $\mathfrak{ls}$, whose structure has given rise to many results and conjectures, in particular the famous Broadhurst-Kreimer conjecture. Our definition of $\mathfrak{krv}_{ell}$ is an analog of the definition of the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$, which itself is related on the one hand to $\mathfrak{ls}$ and on the other to the elliptic Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_{ell}$. We explore all the relations between these different objects.

Like $\mathfrak{grt}$ and $\mathfrak{ds}$, the Lie algebra $\mathfrak{krv}$ is equipped with a depth filtration; we write gr for the associated gradeds. We show that in analogy with the known injective map $gr\,\mathfrak{ds} \to \mathfrak{ls}$, there is an injective map $gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkv}$ (Proposition 2). We also show that the injective map $gr\,\mathfrak{ds} \hookrightarrow gr\,\mathfrak{krv}$ arising from the injective Lie morphism $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ extends to an injective Lie morphism $\mathfrak{ls} \hookrightarrow \mathfrak{lkv}$, and that the parts of these spaces of depths $d = 1, 2, 3$ are isomorphic for all weights n (Theorem 3), which yields the dimensions of the bigraded parts of $\mathfrak{lkv}$ and $gr\,\mathfrak{krv}$ of depths $1, 2, 3$ in all weights, since these dimensions are well-known for $\mathfrak{ls}$. Finally, we define the elliptic version $\mathfrak{krv}_{ell}$ as a subspace of derivations of the free Lie algebra on two generators, and prove that it is closed under the Lie bracket of derivations (Theorem 5). We also define an injective Lie morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ (Theorem 6) in analogy with the section map $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ ([E]) and the mould-theoretic double shuffle map $\mathfrak{ds} \to \mathfrak{ds}_{ell}$ ([S3]). Finally, although we were not able to prove the existence of an injection $\mathfrak{grt}_{ell} \hookrightarrow \mathfrak{krv}_{ell}$, we define a Lie subalgebra $\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{grt}_{ell}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} & \hookrightarrow & \mathfrak{krv} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathfrak{grt}}_{ell} & \hookrightarrow & \mathfrak{ds}_{ell} & \hookrightarrow & \mathfrak{krv}_{ell} \end{array}$$

The main technique used for the constructions in this article is the mould theory developed by J. Écalle, to which we provide a brief introduction in §3, with complements in §4.

1.1. **Special types of derivations of $\mathfrak{lie}_2$.** Let $\mathfrak{lie}_2$ denote the degree completion of the free Lie algebra over $\mathbb{Q}$ on non-commutative variables x and y. The Lie algebra $\mathfrak{lie}_2$ has a weight grading by the degree (=weight) of the polynomials, and a depth grading by the y-degree (=depth) of the polynomials. We write $(\mathfrak{lie}_2)_n$ for the graded part of weight n, $(\mathfrak{lie}_2)^r$ for the graded part of depth r, and $(\mathfrak{lie}_2)^r_n$ for the intersection, which is finite-dimensional.

All the Lie algebras we will study in this article (the well-known ones $\mathfrak{krv}$, $\mathfrak{grt}$ and $\mathfrak{ds}$ as well as the linearized $\mathfrak{ls}$, and the spaces $\mathfrak{lkv}$ and $\mathfrak{krv}_{ell}$ that we introduce) can be viewed either as Lie subalgebras of particular subalgebras of the derivations of $\mathfrak{lie}_2$, equipped with the bracket of derivations, or as subspaces of $\mathfrak{lie}_2$ equipped with particular Lie brackets coming from the Lie bracket of derivations. Both ways of considering our spaces are natural and useful, and we go back and forth between them as convenient for our proofs.

Let $\mathfrak{der}_2$ denote the algebra of derivations on $\mathfrak{lie}_2$. It is a Lie algebra under the Lie bracket given by the commutator of derivations. For $a, b \in \mathfrak{lie}_2$, we write $D_{b,a}$ for the derivation defined by $x \mapsto b$ and $y \mapsto a$. The bracket is explicitly given by

$$[D_{b,a}, D_{b',a'}] = D_{\tilde{b},\tilde{a}} \tag{1}$$

with

$$\tilde{b} = D_{b,a}(b') - D_{b',a'}(b), \qquad \tilde{a} = D_{b,a}(a') - D_{b',a'}(a). \tag{2}$$

• Let $\mathfrak{oder}_2$ denote the Lie subalgebra of $\mathfrak{der}_2$ of derivations $D = D_{b,a}$ that annihilate the bracket $[x, y]$ and such that neither $D(x)$ nor $D(y)$ have a linear term in x. The map $\mathfrak{oder}_2 \to \mathfrak{lie}_2$ given by $D \mapsto D(x)$ is injective (see Corollary 18).

• Let $\mathfrak{tder}_2$ denote the Lie subalgebra of $\mathfrak{der}_2$ of *tangential derivations*, which are the derivations $E_{a,b}$ for elements $a, b \in \mathfrak{lie}_2$ such that a has no linear term in x and b has no linear term in y, such that there exists $c \in \mathfrak{lie}_2$ such that setting $z = -x - y$,

$$E_{a,b}(x) = [x, a], \quad E_{a,b}(y) = [y, b] \quad \text{and} \quad E_{a,b}(z) = [z, c].$$

The Lie bracket is explicitly given by

$$[E_{a,b}, E_{a',b]}] = E_{\tilde{a},\tilde{b}} \tag{3}$$

where

$$\tilde{a} = [a, a'] + E_{a,b}(a') - E_{a',b'}(a), \quad \tilde{b} = [b, b'] + E_{a,b}(b') - E_{a',b'}(b). \tag{4}$$

• Let $\mathfrak{sder}_2$ denote the Lie subalgebra of $\mathfrak{tder}_2$ of *special tangential derivations*, i.e. derivations such that $E_{a,b}(z) = [x, a] + [y, b] = 0$.

• Let $\mathfrak{ider}_2$ be the Lie subalgebra of $\mathfrak{tder}_2$ of *Ihara derivations*, which are those that annihilate x, i.e. those of the form $d_b = E_{0,b}$. The derivation d_b is defined by its values on x and y

$$d_b(x) = 0, \qquad d_b(y) = [y, b]. \tag{5}$$

The Lie bracket on $\mathfrak{ider}_2$ is given by $[d_b, d_{b'}] = d_{\{b,b'\}}$, where $\{b, b'\}$ is the *Poisson* (or Ihara) bracket given by

$$\{b, b'\} = [b, b'] + d_b(b') - d_{b'}(b), \tag{6}$$

i.e. the second term of (4).

We have the following diagram showing the connections between these subspaces:

$$\begin{array}{ccc} \mathfrak{oder}_2 & \hookrightarrow & \mathfrak{der}_2 \\ & & \uparrow \\ & & \mathfrak{tder}_2 \\ & & / \quad \backslash \\ \mathfrak{sder}_2 & \xrightarrow{\sim} & \mathfrak{ider}_2 \end{array} \tag{7}$$

The isomorphism between $\mathfrak{sder}_2$ and $\mathfrak{ider}_2$ is given in Lemma 25.

1.2. **Definition of the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$.** The universal enveloping algebra of $\mathfrak{lie}_2$ is isomorphic to the degree completion $\mathrm{Ass}_2 = \mathbb{Q}\langle\langle x, y\rangle\rangle$ of the free associative algebra with non-commutative generators x, y, i.e. to the ring of power series in x and y.

Definition 1. The trace vector space $\mathfrak{tr}_2$ (cf. [AT]) is defined to be the quotient of Ass_2 by the equivalence relation given between words in x and y by $w \sim w'$ if w' can be obtained from w by a cyclic permutation of the letters of the word w, and extended linearly to polynomials. The natural projection is denoted

$$\mathrm{tr} : \mathrm{Ass}_2 \to \mathfrak{tr}_2.$$

For any polynomial $f \in \mathrm{Ass}_2$ with constant term c, we can decompose f in two ways as

$$f = c + f_x x + f_y y = c + x f^x + y f^y \tag{8}$$

for uniquely determined polynomials f_x, f_y, f^x, f^y in Ass_2.

Definition 2. The *divergence* map is given by

$$\mathrm{div} : \begin{array}{ccc} \mathfrak{tder}_2 & \longrightarrow & \mathfrak{tr}_2 \\ u = E_{a,b} & \longmapsto & \mathrm{tr}(a_x x + b_y y). \end{array}$$

Definition 3. The *Kashiwara-Vergne Lie algebra* $\mathfrak{krv}_2$ is defined to be the subspace of $\mathfrak{sder}_2$ of derivations $E_{a,b}$ such that there exists a one-variable power series $h(x) \in \mathbb{Q}[x]$ of degree ≥ 2 such that

$$\mathrm{div}(E_{a,b}) = tr\big(h(x+y) - h(x) - h(y)\big). \tag{9}$$

This definition comes from [AT], where it was shown that $\mathfrak{krv}_2$ is actually a Lie subalgebra of $\mathfrak{sder}_2$. This Lie algebra inherits a weight-grading from that of $\mathfrak{lie}_2$, for which $E_{a,b}$ is of weight n if b (and thus also a) is a Lie polynomial of homogeneous degree n. In particular, the weight 1 part of $\mathfrak{krv}_2$ is spanned by the single element $u = E_{y,x}$, and the weight 2 part is zero. In this article, we do not consider the weight 1 part of $\mathfrak{krv}_2$. For convenience, we set $\mathfrak{krv} = \oplus_{n\geq 3}(\mathfrak{krv}_2)_n$, where $(\mathfrak{krv}_2)_n$ denotes the weight graded part of $\mathfrak{krv}_2$ of weight n. We have

$$\mathfrak{krv}_2 = (\mathfrak{krv}_2)_1 \oplus \mathfrak{krv} = \mathbb{Q}[E_{y,x}] \oplus \mathfrak{krv}.$$

Because the other Lie algebras in the literature that are most often compared with the Kashiwara-Vergne Lie algebra have no weight 1 or weight 2 parts, it makes most sense to compare them with $\mathfrak{krv}$. Thus it is $\mathfrak{krv}$ that we study for the remainder of this article.

The Lie algebra $\mathfrak{krv}$ also inherits a depth filtration from the depth grading on $\mathfrak{lie}_2$, for which $E_{a,b}$ is of depth r if r is the smallest number of y's occurring in any monomial of b. We write $gr\,\mathfrak{krv}$ for the associated graded for this depth filtration, so that $gr\,\mathfrak{krv}$ is a Lie algebra that is bigraded for the weight and the depth; we write $gr^r_n\,\mathfrak{krv}$ for the part of weight n and depth r. Essentially, an element of $gr\,\mathfrak{krv}$ is a derivation $E_{\bar{a},\bar{b}} \in \mathfrak{sder}_2$ where $\bar{a}, \bar{b}$ are the lowest-depth parts (i.e. the parts of lowest y-degree) of elements $a, b \in \mathfrak{lie}_2$ such that $E_{a,b} \in \mathfrak{krv}$. If $\bar{b}$ is of homogeneous y-degree r, then $\bar{a}$ is of homogeneous y-degree $r+1$.

Example. The smallest element of $\mathfrak{krv}$ is in weight 3 and is given by $E_{a,b}$ with

$$a = [[x,y],y], \quad b = [x,[x,y]].$$

Since $\bar{a} = a$ and $\bar{b} = b$, this is also equal to $E_{\bar{a},\bar{b}} \in gr\,\mathfrak{krv}$. The next smallest element of $\mathfrak{krv}$ is in weight 5, and the depth-graded part $E_{\bar{a},\bar{b}}$ is given by

$$\bar{a} = [x,[x,[[x,y],y]]] - 2[[x,[x,y]],[x,y]], \qquad \bar{b} = [x,[x,[x,[x,y]]]].$$

1.3. **The Grothendieck-Teichmüller and double shuffle Lie algebras.** Recall that the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ is the space of polynomials $b \in \mathfrak{lie}_2$ satisfying the famous *pentagon relation*, equipped with the Poisson bracket (6). This algebra was first introduced by Y. Ihara in [I], with three defining relations, as a particular derivation algebra of $\mathfrak{lie}_2$ (via the association $b \mapsto d_b$ as in (5)); H. Furusho subsequently showed that the pentagonal relation implies the other two (cf. [F1]).

Recall also that the double shuffle Lie algebra $\mathfrak{ds}$ is the space of polynomials $b \in \mathfrak{lie}_2$ satisfying a particular set of conditions on the coefficients called the *stuffle relations*, studied in the first place by Racinet (cf. [R]), who gave a quite difficult proof that $\mathfrak{ds}$ is also a Lie algebra under the Poisson bracket (6). This proof was later somewhat streamlined by Furusho (cf. [F2], Appendix), and a recent preprint [EF] gives another proof with a different approach, identifying the space as a stabilizer. Putting together basic elements from Écalle's mould theory also yields a completely different and very simple proof of this result ([SS]).

There is a commutative triangle of injective Lie morphisms

$$\begin{array}{ccc} \mathfrak{grt} & \to & \mathfrak{ds} \\ & \searrow & \swarrow \\ & \mathfrak{krv}. & \end{array} \tag{10}$$

The existence of the injection $\mathfrak{grt} \to \mathfrak{ds}$ was proven in [F1]; it is given by $b(x,y) \mapsto b(x,-y)$. The existence of the injection $\mathfrak{grt} \to \mathfrak{krv}$ was proven in [AT]; it is given by $b(x,y) \mapsto b(z,y)$ where $z = -x-y$. Finally, the existence of the injection $\mathfrak{ds} \to \mathfrak{krv}$ was proven in [S1] (using results from Écalle's mould theory), and is given, of course, by $b(x,y) \mapsto b(z,-y)$. In particular, these morphisms respect the weight gradings and depth filtrations on all three spaces.

1.4. **The linearized Kashiwara-Vergne Lie algebra: main results.** For $i \geq 1$, set $C_i = ad(x)^{i-1}(y)$ for $i \geq 1$, and let $\mathfrak{lie}_C$ denote the degree completion of the Lie algebra $L[[C_1, C_2, \ldots]]$ on the C_i. By Lazard elimination, $\mathfrak{lie}_C$ is free on the C_i and

$$\mathfrak{lie}_2 \simeq \mathbb{Q}[x] \oplus \mathfrak{lie}_C. \tag{11}$$

Thus, Lazard elimination shows that every polynomial $b \in \mathfrak{lie}_2$ having no linear term in x can be written uniquely as a Lie polynomial in the C_i.

Definition 4. Let the push-operator be defined on monomials in x, y by

$$push(x^{a_0} y x^{a_1} y \cdots y x^{a_r}) = x^{a_r} y x^{a_0} y \cdots y x^{a_{r-1}}. \tag{12}$$

The push is considered to act trivially on constants and powers of x^n, so we can extend it to all of Ass_2 by linearity. A polynomial b in x, y is said to be

- *push-invariant* if $push(b) = b$, and
- *push-neutral* if $b^r + push(b^r) + \cdots + push^r(b^r) = 0$ for all $r \geq 1$, where b^r denotes the depth r part of b. Finally, we say that b is
- *circ-neutral* if b^y is push-neutral in depths $r > 1$.

Definition 5. The *linearized Kashiwara-Vergne Lie algebra* $\mathfrak{lkv}$ is the space of elements $b \in \mathfrak{lie}_C$ of degree ≥ 3 such that

(i) b is push-invariant, and

(ii) b is circ-neutral.

Our first result on $\mathfrak{lkv}$ is that is a bigraded Lie algebra.

Proposition 1. *The space $\mathfrak{lkv}$ is bigraded by weight and depth, and forms a Lie algebra under the Poisson bracket defined in (6).*

In §1.4 below, we define a larger space, the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$, and show in Theorem 5 that it is a Lie algebra. Although it might be possible (albeit laborious) to prove Proposition 1 directly, it turns out to follow immediately from Theorem 5, due to the fact that there is a simple injection of $\mathfrak{lkv}$ into the larger space $\mathfrak{krv}_{ell}$ (see Proposition 6 following Theorem 5) whose image is easily identifiable as the intersection of two Lie subalgebras. For this reason, the proof of Proposition 1 can be found in Corollary 22 at the end of §4.1, following the proof of Theorem 5.

In §2, we show how we derive the definition of $\mathfrak{lkv}$ via a reformulation of the defining properties of $\mathfrak{krv}$, in the sense that the defining properties of $\mathfrak{lkv}$ are merely truncations of the two reformulated defining properties of $\mathfrak{krv}$ to their lowest-depth parts. This construction automatically gives the following result on $\mathfrak{lkv}$, whose proof is in §2.3.

Proposition 2. *There is an injective Lie algebra morphism*

$$gr\, \mathfrak{krv} \hookrightarrow \mathfrak{lkv}.$$

We conjecture that these two spaces are in fact isomorphic.

In using this type of definition for $\mathfrak{lkv}$, we are following the analogous situation of the well-known double shuffle Lie algebra $\mathfrak{ds}$ and the associated linearized double shuffle space $\mathfrak{ls}$ studied in many articles (cf. for example [Br]). The bigraded

linearized space $\mathfrak{ls}$ is defined as the set of Lie polynomials $f \in \mathfrak{lie}_2$ of weight $n \geq 3$ such that the polynomial $f_y y$, rewritten in the variables $y_i = x^{n-1}y$ for $n \geq 1$, is an element of the free Lie algebra on the y_i. One also adds the extra assumption that if f is of depth 1, then it is of odd weight, an assumption which is not needed for $\mathfrak{lkv}$ as it follows from the push-invariance condition in the definition. By its very construction, there is an injective Lie algebra homomorphism

$$gr\,\mathfrak{ds} \hookrightarrow \mathfrak{ls}, \tag{13}$$

and it is conjectured that these two spaces are isomorphic, but like for $\mathfrak{lkv}$, this is still an open question.

The injective Lie algebra morphism (10) from $\mathfrak{ds}$ to $\mathfrak{krv}$ yields a corresponding bigraded injective map:

$$gr\,\mathfrak{ds} \hookrightarrow gr\,\mathfrak{krv}. \tag{14}$$

Our next result extends this map to the more general linearized spaces $\mathfrak{ls}$ and $\mathfrak{lkv}$.

Theorem 3. *The Lie injection (14) extends to a bigraded Lie injection on the associated linearized spaces, giving the following commutative diagram:*

$$\begin{array}{ccc} gr\,\mathfrak{ds} & \hookrightarrow & gr\,\mathfrak{krv}. \\ \downarrow & & \downarrow \\ \mathfrak{ls} & \hookrightarrow & \mathfrak{lkv}. \end{array}$$

For all $n \geq 3$ and $r = 1, 2, 3$, the map is an isomorphism of the bigraded parts

$$\mathfrak{ls}_n^r \simeq \mathfrak{lkv}_n^r.$$

This theorem will be proved in §3, using mould theory, to which we give a brief and elementary introduction. Mould theory is also essential for all the proofs concerning the elliptic Kashiwara-Vergne Lie algebra defined in the next subsection.

Adding a variety of known results in the depth 2 and depth 3 situations to this result, we obtain the following corollary.

Corollary 4. *The following spaces are isomorphic for $n \geq 3$ and $r = 1, 2, 3$:*

$$gr_n^r\mathfrak{grt} \simeq gr_n^r\mathfrak{ds} \simeq gr_n^r\mathfrak{krv} \simeq \mathfrak{ls}_n^r \simeq \mathfrak{lkv}_n^r.$$

In particular, all of these spaces are zero when $r = 1$ or 3 and n is even, or when $r = 2$ and n is odd.

Proof. The dimensions of the spaces $gr_n^r\mathfrak{grt}$, $gr_n^r\mathfrak{ds}$ and $\mathfrak{ls}_n^r$ in depths are known to be equal to each other in depths $r \leq 3$ ([R], [G]). Since the injective map $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ (10) induces a map $gr\,\mathfrak{ds} \hookrightarrow gr\,\mathfrak{krv}$, Proposition 2 shows that $gr_n^r\mathfrak{krv}$ is sandwiched between $gr_n^r\mathfrak{ds}$ and $\mathfrak{lkv}_n^r$; by the theorem, all five spaces are then equal when $r \leq 3$. (The dimensions of $gr_n^2\mathfrak{krv}$ were also computed in [ALR], without using the comparison with double shuffle.) □

We conjecture that $\mathfrak{lkv}_n^r \simeq \mathfrak{ls}_n^r$ for all n, r, and calculations up to about $n = 15$ bear this conjecture out, but we were not able to prove the isomorphism for any other cases, not even the special case $n \not\equiv r \bmod 2$, where it is well-known that $gr_n^r\mathfrak{grt} = gr_n^r\mathfrak{ds} = gr_n^r\mathfrak{krv} = \mathfrak{ls}_n^r = 0$ (cf. [IKZ], [Br] for classical proofs, or [S2] for the exposition of Écalle's mould-theoretic proof).

Let us end this subsection by giving the *mould-language reformulation* of the definition of $\mathfrak{lkv}$, which will allow us to connect it directly to the definition of the elliptic Kashiwara-Vergne Lie algebra defined in the next subsection, which cannot be defined directly in terms of elements of $\mathfrak{lie}_2$. The mould definition of $\mathfrak{lkv}$ clearly echoes the definition in terms of Lie elements given above; the equivalence is shown in detail in §3.

Definition 5′: Mould-reformulated $\mathfrak{lkv}$. The linearized Kashiwara-Vergne Lie algebra $\mathfrak{lkv}$ is the space of elements $b \in \mathfrak{lie}_C$ of degree ≥ 3 such that writing the depth r part of b as

$$b^r = \sum_{\underline{a}} k_{\underline{a}}\, C_{a_1} \cdots C_{a_r} \tag{15}$$

where the sum runs over tuples $\underline{a} = (a_1, \ldots, a_r)$, $a_i \geq 1$, and setting

$$B^r(u_1, \ldots, u_r) = \sum_{\underline{a}} k_{\underline{a}}\, u_1^{a_1-1} \cdots u_r^{a_r-1} \tag{16}$$

for commutative variables $u_1, \ldots, u_r$ and

$$\bar{B}^r(v_1, \ldots, v_r) = B^r(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2) \tag{17}$$

for commutative variables $v_1, \ldots, v_r$, we have the two properties:

(i) B^r is *push-invariant* for $r \geq 1$, i.e.

$$B(u_0, u_1, \ldots, u_{r-1}) = B(u_1, \ldots, u_r) \tag{18}$$

where $u_0 = -u_1 - \cdots - u_r$, and

(ii) $\bar{B}^r$ is *circ-neutral* for $r > 1$, i.e.

$$\bar{B}^r(v_1, \ldots, v_r) + \bar{B}^r(v_2, \ldots, v_r, v_1) + \cdots + \bar{B}^r(v_r, v_1, \ldots, v_{r-1}) = 0. \tag{19}$$

1.5. **The elliptic Kashiwara-Vergne Lie algebra.** The last section of this article is devoted to the study of the *elliptic Kashiwara-Vergne Lie algebra*. The definition of this algebra is based on that of the linearized Lie algebra $\mathfrak{lkv}$, differing only from Definition 5′ by the denominator appearing in (20), which makes it impossible to express it directly in terms of Lie elements like Definition 5.

Definition 6. The *elliptic Kashiwara-Vergne vector space* $\mathfrak{krv}_{ell}$ is spanned by the elements $b \in \mathfrak{lie}_C$ such that writing the depth r part b^r as in (15) and the associated polynomial B^r as in (16), and setting

$$B^r_*(u_1, \ldots, u_r) = \frac{1}{u_1 \cdots u_r (u_1 + \cdots + u_r)} B^r(u_1, \ldots, u_r) \tag{20}$$

and

$$\bar{B}^r_* = B^r_*(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2),$$

we have

(i) B^r_* is push-invariant as in (18) for $r \geq 1$;

(ii) $\bar{B}^r_*$ is circ-neutral as in (19) for $r > 1$.

The first main result on $\mathfrak{krv}_{ell}$ is of course that it is a bigraded Lie algebra, but this comes from an injective map from $\mathfrak{krv}_{ell}$ into $\mathfrak{oder}_2$ rather than into $\mathfrak{sder}_2$ as for $\mathfrak{lkv}$.

Theorem 5. *(i) The space $\mathfrak{krv}_{ell}$ is bigraded for the weight and the depth.*
(ii) For each $b \in \mathfrak{krv}_{ell}$, there exists a unique polynomial $a \in \mathfrak{lie}_C$, called the partner *of b, such that $D_{b,a} \in \mathfrak{oder}_2$.*
(iii) The image of the injective linear map $b \mapsto D_{b,a}$ is a Lie subalgebra of $\mathfrak{oder}_2$; in other words $\mathfrak{krv}_{ell}$ is a Lie algebra under the Lie bracket

$$\langle b, b' \rangle = D_{b,a}(b') - D_{b',a'}(b) \tag{21}$$

coming from the bracket of derivations as in (1) and (2).

This theorem is proven in §4.1 (Theorem 19); it necessitates the introduction of some more complicated definitions and results from mould theory than those used in §3.

The following result is key to the comparison of $\mathfrak{lkv}$ and $\mathfrak{krv}_{ell}$, and to the proof that $\mathfrak{lkv}$ is a Lie algebra.

Proposition 6. *There is an injective linear map*

$$\begin{aligned} \mathfrak{lkv} &\hookrightarrow \mathfrak{krv}_{ell} \\ b(x,y) &\mapsto [x, b(x,[x,y])] \end{aligned} \tag{22}$$

whose image is a Lie subalgebra of $\mathfrak{krv}_{ell}$. Equivalently, the map can be defined on the family B^r of polynomials in commutative variables associated to b as in (16) by

$$B^r(u_1, \dots, u_r) \mapsto u_1 \cdots u_r(u_1 + \cdots + u_r) B^r(u_1, \dots, u_r).$$

This reflects the fact that by Definition 6, $\mathfrak{krv}_{ell}$ is isomorphic to the space spanned by the polynomials in the commutative variables u_i that become push-invariant and circ-neutral (possibly after adding a constant) after division by $u_1 \cdots u_r(u_1 + \cdots + u_r)$, while $\mathfrak{lkv}$ is isomorphic to space of polynomials that are themselves push-invariant and circ-neutral: thus multiplying by the factor $u_1 \cdots u_r(u_1 + \cdots + u_r)$ maps $\mathfrak{lkv}$ precisely to the subspace of $\mathfrak{krv}_{ell}$ consisting of polynomials that remain polynomial after division by $u_1 \cdots u_r(u_1 + \cdots + u_r)$.

In two independent articles, H. Tsunogai [Ts] and B. Enriquez [E] defined a Lie algebra that Enriquez calls the *elliptic Grothendieck-Teichmüller Lie algebra* $\mathfrak{grt}_{ell}$, based on the idea that just as Ihara had defined $\mathfrak{grt}$ as the algebra of derivations on $\mathfrak{lie}_2$ (identified with the braid Lie algebra on four strands) that extend to a particular type of derivation on the braid Lie algebra on five strands, $\mathfrak{grt}_{ell}$ is the Lie algebra of derivations on $\mathfrak{lie}_2$ (now identified with the genus one braid Lie algebra on two strands) that extend to a very particular type of derivation of the genus one braid Lie algebra on three strands. The construction of $\mathfrak{grt}_{ell}$ shows that it is a Lie subalgebra of $\mathfrak{oder}_2$, and that there is a canonical surjection

$$s : \mathfrak{grt}_{ell} \to \mathfrak{grt}. \tag{23}$$

Let $\mathfrak{r}_{ell}$ denote the kernel. Enriquez [E] showed that there also exists a Lie algebra morphism

$$\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell} \tag{24}$$

that is a section of (23), i.e. such that $\gamma \circ s = id$ on $\mathfrak{grt}$. Thus, there is a semi-direct product isomorphism

$$\mathfrak{grt}_{ell} \simeq \mathfrak{r}_{ell} \rtimes \gamma(\mathfrak{grt}). \tag{25}$$

An elliptic version $\mathfrak{ds}_{ell}$ of the double shuffle Lie algebra $\mathfrak{ds}$ was constructed in [S3] using mould theory, and it is shown there that like $\mathfrak{grt}_{ell}$, $\mathfrak{ds}_{ell}$ is a Lie subalgebra of $\mathfrak{oder}_2$, and that there is an injective Lie morphism $\tilde{\gamma} : \mathfrak{ds} \to \mathfrak{ds}_{ell}$ that makes the diagram

$$\begin{array}{ccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} \\ \gamma \downarrow & & \downarrow \tilde{\gamma} \\ \mathfrak{grt}_{ell} & & \mathfrak{ds}_{ell} \\ & \backslash \quad / & \\ & \mathfrak{oder}_2 & \end{array}$$

commute.

Our second main result on $\mathfrak{krv}_{ell}$ is an analog of the existence of γ and $\tilde{\gamma}$.

Theorem 7. *There is an injective Lie algebra morphism*

$$\hat{\gamma} : \mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}.$$

Based on the known injective Lie morphisms $\mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{krv}$ evoked in §1.3 above, we believe that there are corresponding injective Lie morphisms between the elliptic versions of these Lie algebras. However, we were not able to prove that $\mathfrak{grt}_{ell}$ as defined in [E] injects into $\mathfrak{ds}_{ell}$ or $\mathfrak{krv}_{ell}$. To circumvent this difficulty, we define a Lie subalgebra $\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{grt}_{ell}$, conjecturally isomorphic to $\mathfrak{grt}_{ell}$, as follows.

Definition 7. For $n \geq 0$, let $\delta_{2n} \in \mathfrak{oder}_2$ denote the derivation of $\mathfrak{lie}_2$ defined by

$$\delta_{2n}(x) = ad(x)^{2n}(y), \quad \delta_{2n}([x, y]) = 0.$$

Let $\mathfrak{b}$ be the Lie subalgebra of $\mathfrak{oder}_2$ generated by the δ_{2n}.

Enriquez showed in [E] that $\delta_{2n} \in \mathfrak{r}_{ell}$ for $n \geq 0$, so $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{r}_{ell}$. Let $\mathfrak{B}$ denote the normalization of $\mathfrak{b} \subset \mathfrak{r}_{ell}$ under the semi-direct action of $\gamma(\mathfrak{grt})$ on $\mathfrak{r}_{ell}$ of (25). We set

$$\widetilde{\mathfrak{grt}}_{ell} = \mathfrak{B} \rtimes \gamma(\mathfrak{grt}). \tag{26}$$

Our third main result on $\mathfrak{krv}_{ell}$ relates all these maps via a commutative diagram.

Theorem 8. *We have the following commutative diagram of injective Lie morphisms:*

$$\begin{array}{ccccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} & \hookrightarrow & \mathfrak{krv} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathfrak{grt}}_{ell} & \hookrightarrow & \mathfrak{ds}_{ell} & \hookrightarrow & \mathfrak{krv}_{ell} \\ & \backslash & \downarrow & / & \\ & & \mathfrak{oder}_2. & & \end{array}$$

1.6. **Outline of the article.** In §2, we reformulate the defining conditions of $\mathfrak{krv}$, which lead to the first definition of $\mathfrak{lkv}$ and the proof of Proposition 2. The next section, §3, gives a brief introduction to mould theory and a translation of the defining conditions of $\mathfrak{lkv}$ into that language, and uses mould theory to prove Theorem 3. Finally, the proofs of Theorems 5, Theorem 7 and Theorem 8 are given in the three subsections of §4.

2. Reformulation of the definition of $\mathfrak{krv}$ and definition of the linearized Lie algebra $\mathfrak{lkv}$

In this section, we give a convenient reformulation of the defining conditions of $\mathfrak{krv}$, which leads to a simple definition of the linearized version $\mathfrak{lkv}$ that passes easily into the language of moulds which will be essential for our subsequent proofs in §§3,4.

2.1. **The first defining condition of $\mathfrak{krv}$: specialness.** The first of the two defining conditions of $\mathfrak{krv}$ is that $\mathfrak{krv}$ lies in $\mathfrak{sder}_2$, i.e. elements of $\mathfrak{krv}$ are special tangential derivations having the form $E_{a,b}$ with $E_{a,b}(x) = [x, a]$, $E_{a,b}(y) = [y, b]$ and $[x, a] + [y, b] = 0$.

The following equivalent formulations of the property of specialness as properties of the polynomial b were given in [S1].

Proposition 9. [Schneps, [S1]] *Let $n \geq 3$ and let $b \in \mathfrak{lie}_C$; write $b = b_x x + b_y y = xb^x + yb^y$. Then the following are equivalent:*

(i) There exists a unique element $a \in \mathfrak{lie}_C$ such that $[x, a] + [y, b] = 0$;

(ii) b is push-invariant;

(iii) $b_y = b^y$.

Thanks to this proposition, we can now reformulate the first defining condition of $\mathfrak{krv}$ as follows: the pair of polynomials $a, b \in \mathfrak{lie}_C$ satisfies $[x, a] + [y, b] = 0$ if and only if b is push-invariant and a is its partner.

2.2. **The second defining condition of $\mathfrak{krv}$: divergence.** We now consider the second defining condition of $\mathfrak{krv}$, the divergence condition. Because $\mathfrak{krv}$ is weight-graded, we may restrict attention to derivations $E_{a,b}$ of homogeneous weight n, i.e. such that a and b are Lie polynomials of homogeneous degree $n \geq 3$. The second defining condition (9) then simplifies to the existence of a constant c such that

$$\mathrm{tr}(xa_x + yb_y) = c\ \mathrm{tr}\big((x + y)^n - x^n - y^n\big) \text{ in } \mathfrak{tr}_2.$$

Let us reformulate this as a condition only on b, just as we did for the first defining condition. Since $a \in \mathfrak{lie}_2$, its trace is zero and thus $\mathrm{tr}(xa_x) = \mathrm{tr}(a_x x) = -\mathrm{tr}(a_y y) = -\mathrm{tr}(ya_y)$, so

$$\mathrm{tr}(xa_x + yb_y) = \mathrm{tr}(yb_y - ya_y).$$

Since $E_{a,b} \in \mathfrak{sder}$, we have $[x, a] = [b, y]$. Expanding this in terms of the decompositions of a and b, we obtain

$$xa_x x + xa_y y - xa^x x - ya^y x = xb^x y + yb^y y - yb_x x - yb_y y,$$

from which we deduce that $a_y = b^x$ and $a^y = b_x$. Thus

$$\mathrm{tr}(yb_y - ya_y) = \mathrm{tr}(yb_y - yb^x) = \mathrm{tr}\big(y(b_y - b^x)\big).$$

From Proposition 9, we have $b_y = b^y$, so now, using the circularity of the trace, the divergence condition can be reformulated as

$$\mathrm{tr}\big((b^y - b^x)y\big) = c\ \mathrm{tr}\big((x+y)^n - x^n + y^n\big).$$

We use this to express it as a condition directly on $b^y - b^x$ as follows, using the push-operator defined in (12).

Definition 8. A polynomial $b \in Ass_2$ of homogeneous weight $n > 1$ is said to be *push-constant for the value* c if b does not contain the monomial y^n and for each $1 < r < n$, writing b^r for the depth r part of b, we have

$$\sum_{i=0}^{r} push^i(b^r) = c \sum_w w$$

where the sum in the right-hand factor is over all monomials of weight n and depth r. Equivalently, b is push-constant if it does not contain y^n and for all monomials $w \neq x^n$, we have

$$\sum_{v \in Push(w)} (b|v) = c$$

where $(b|v)$ denotes the coefficient of the monomial v in b, and $Push(w)$ is the list (with possible repetitions) $[w, push(w), \ldots, push^r(w)]$. If $c = 0$, then b is said to be *push-neutral*. If b is a scalar multiple of x^n, then b is push-neutral by default.

Example. The simplest example of a push-constant polynomial is the sum of all monomials of a given depth, for example

$$b = x^a y x^b y x^c + x^c y x^a y x^b + x^b y x^c y x^a + x^a y x^c y x^b + x^b y x^a y x^c + x^c y x^b y x^a.$$

More interesting push-constant polynomials can be obtained from elements $\psi \in \mathfrak{grt}$ by taking the projection of ψ onto the words ending in y and writing this as by. In this way we obtain for example:

$$b = 2x^2y^2 - \frac{11}{2}xyxy + \frac{9}{2}xy^2x - \frac{1}{2}yx^2y + 2yxyx - \frac{1}{2}y^2x^2.$$

The following proposition shows that the divergence condition comes down to requiring that $b^y - b^x$ be push-constant.

Proposition 10. *([S1]) Let b be a push-invariant Lie polynomial of homogeneous degree n. Then b satisfies the divergence condition*

$$\mathrm{tr}((b^y - b^x)y) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big)$$

if and only if $b^y - b^x$ is push-constant for the value nc. Furthermore, if this is the case then

$$c = \frac{1}{n}(b \,|\, x^{n-1}y). \tag{27}$$

Proof. Let w be a monomial of degree n and depth $r \geq 1$, and let C_w denote the list of words obtained from w by cyclically permuting the letters, so that C_w contains exactly n words (with possible repetitions). Let C_w^y denote the list obtained from C_w by removing all words ending in x, so that C_w^y contains exactly r words. Write $C_w^y = [u_1 y, \ldots, u_r y]$. Then we have the equality of lists

$$[u_1, ..., u_r] = Push(u_1).$$

Let $c_w = \mathrm{tr}(w)$, i.e. c_w is the equivalence class of w, which is the set of the words in the list C_w, without repetitions: thus C_w is nothing other than $n/|c_w|$ copies of c_w. The divergence condition

$$\mathrm{tr}\big((b^y - b^x)y\big) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big)$$

translates as the following family of conditions for one word in each equivalence class c_w:

$$\sum_{v \in c_w} \big((b^y - b^x)y \,|\, v\big) = c|c_w|, \tag{28}$$

where each side is the coefficient of the class c_w in the trace, i.e. the sum of the coefficients of the words in c_w in the original polynomial.

If $r > 1$, we can choose a word $uy \in C_w$ that starts in y. Then from (121), the divergence condition on b implies that

$$\begin{aligned} c &= \frac{1}{|c_w|} \sum_{v \in c_w} \big((b^y - b^x)y \,|\, v\big) \\ &= \frac{1}{n} \sum_{v \in C_w} \big((b^y - b^x)y \,|\, v\big) \\ &= \frac{1}{n} \sum_{v \in C_w^y} \big((b^y - b^x)y \,|\, v\big) \\ &= \frac{1}{n} \sum_{u' \in Push(u)} ((b^y - b^x) \,|\, u'). \end{aligned}$$

This is exactly the definition of $b^y - b^x$ being push-constant for the value nc.

If $r = 1$, then w is of depth 1, $|c_w| = n$ and $x^{n-1}y$ is the only word in c_w ending in y. Thus (121) comes down to

$$\big((b^y - b^x)y \,|\, x^{n-1}y\big) = nc.$$

But since b is a Lie polynomial, we have $(b|x^n) = (b^x|x^{n-1}) = 0$, so using $b^y = b_y$ (by Proposition 9), we also have

$$\begin{aligned} \big((b^y - b^x)y \,|\, x^{n-1}y\big) &= (b^y - b^x \,|\, x^{n-1}) = (b^y|x^{n-1}) \\ &= (b_y|x^{n-1}) = (b_y y|x^{n-1}y) = (b|x^{n-1}y), \end{aligned}$$

which proves that $nc = (b|x^{n-1}y)$ as desired. Note that this condition means that if b has no depth 1 part, then $b^y - b^x$ is push-neutral. □

We now have a new way of expressing $\mathfrak{krv}$, which is much easier to translate into the mould language.

Definition 9. Let $V_{\mathfrak{krv}}$ be the vector space spanned by polynomials $b \in \mathfrak{lie}_C$ of homogeneous degree $n \geq 3$ such that

(i) b is push-invariant, and

(ii) $b^y - b^x$ is push-constant for the value $(b \,|\, x^{n-1}y)\big\}$,

equipped with the Lie bracket

$$\{b, b'\} = [b, b'] + E_{a,b}(b') - E_{a',b'}(b)$$

where a and a' are the (unique) partners of b and b' respectively.

Indeed, since Propositions 9 and 10 show that

$$\begin{array}{rcl} \mathfrak{krv} & \xrightarrow{\sim} & V_{\mathfrak{krv}} \\ E_{a,b} & \mapsto & b \end{array} \tag{29}$$

is an isomorphism of vector spaces and $\mathfrak{krv}$ is known to be a Lie subalgebra of $\mathfrak{sder}_2$, the bracket on $V_{\mathfrak{krv}}$ is inherited directly from this and makes $V_{\mathfrak{krv}}$ into a Lie algebra.

2.3. **The linearized Kashiwara-Vergne Lie algebra $\mathfrak{lkv}$.** Using the above isomorphism of $\mathfrak{krv}$ with the vector space $V_{\mathfrak{krv}}$ given by $E_{a,b} \mapsto b$, let us now consider the depth-graded versions of the defining conditions of $V_{\mathfrak{krv}}$, i.e. determine what these conditions say about the lowest-depth parts of elements $b \in V_{\mathfrak{krv}}$. The push-invariance is a depth-graded condition, so it restricts to the statement that the lowest depth part of b is still push-invariant; in particular, by Proposition 9 it admits of a unique partner $a \in \mathfrak{lie}_C$ such that $[x,a]+[y,b]=0$, i.e. such that the associated derivation $E_{a,b}$ lies in $\mathfrak{sder}_2$.

In the second condition, if b is of degree n and depth $r=1$ and b^1 denotes the lowest-depth part of b, then $(b^1)^y = x^{n-1}$, so the push-constance condition on b^1 is empty since $(b^1)^y = (b|x^{n-1}y)x^{n-1}$. If $r>1$, however, then $(b|x^{n-1}y)=0$ and so the push-constance condition on $b^y - b^x$ is actually push-neutrality, which implies the push-neutrality of $(b^r)^y$ alone, since $(b^r)^y$ is the only part of the expression $b^y - b^x$ of minimal depth $r-1$. This leads to the following definition for the depth-graded version of the Kashiwara-Vergne Lie algebra.

Definition 10. The bigraded *linearized Kashiwara-Vergne Lie algebra* is defined by

$$\begin{aligned} \mathfrak{lkv} = \big\{ b \in \mathfrak{lie}_2 \,\big|\, & (i)\ b \text{ is push-invariant} \\ & (ii)\ b \text{ is circ-neutral if } b \text{ is of depth } > 1, \big\}, \end{aligned}$$

equipped with the bracket coming from the bracket of derivations in $\mathfrak{sder}_2$, namely

$$\{b,b'\} = [b,b'] + E_{a,b}(b') - E_{a',b'}(b).$$

The proof of Proposition 1 above, that $\mathfrak{lkv}$ is closed under the proposed Lie bracket, is deferred to the end of §4.1. Proposition 2, however, is proven by the very fact that the defining properties of $\mathfrak{lkv}$ are properties held by the lowest-depth parts of elements of $\mathfrak{krv}$, since this means precisely that there is an injective linear map

$$gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkv},$$

which is a Lie morphism as both spaces are equipped with the Lie bracket coming from $\mathfrak{sder}_2$. It is, however, an open question as to whether these two spaces are equal, since it is not clear that an element satisfying the defining conditions of $\mathfrak{lkv}$ necessarily lifts to an element of $\mathfrak{krv}$. No examples of this are known, and it would be interesting to try to prove equality by starting with a polynomial $\mathfrak{lkv}$ of depth $r>1$ and finding a way to construct a depth by depth lifting to an element of $\mathfrak{krv}$.

3. Rational and polynomial moulds

In this section, we introduce the language of moulds and reformulate the defining conditions of $\mathfrak{lkv}$ in this language. We end the section with the proof of Theorem 3 and its corollary in terms of moulds. We hope that this section and the next one,

which explores the elliptic version of $\mathfrak{krv}$, will illustrate the way in which moulds are powerful tools in this context.

3.1. **Moulds and alternality.** For the purposes of this article, we are concerned only with rational function-valued moulds defined over the rationals. Écalle defines moulds with more general arguments and more general values, but in this article we will use the term *mould* merely to denote a collection $A = \big(A^r(u_1, \dots, u_r)\big)_{r\geq 0}$ where each $A^r(u_1, u_2, ..., u_r) \in \mathbb{Q}(u_1, \dots, u_r)$, i.e. each A^r is a rational function in r commutative variables $u_1, \dots, u_r$ with coefficients in $\mathbb{Q}$. The rational function A^r is the *depth* r part of the mould. When the context is clear we sometimes drop the index and write $A(u_1, \dots, u_r)$ instead of $A^r(u_1, \dots, u_r)$ for the depth r part. In particular we have $A_0 = A(\emptyset) \in \mathbb{Q}$.

Moulds are equipped with addition and multiplication by scalars componentwise; thus they form a vector space. We write ARI for the subspace of (rational) moulds A with $A(\emptyset) = 0$ (keeping in mind that this ARI is only a very small subspace of the full space of moulds studied by Écalle). For convenience, we also define the vector space $\overline{ARI}$ of moulds defined exactly like ARI except on a set of commutative variables $v_1, v_2, \dots$, i.e. $B \in \overline{ARI}$ means $B = \big(B^r\big)_{r\geq 0}$ with $B^r \in \mathbb{Q}[v_1, \dots, v_r]$.

We say that a mould A is *concentrated in depth* r if $A^s = 0$ for all $s \neq r$, and we let $ARI^r \subset ARI$ be the subspace of moulds concentrated in depth r. Thus $ARI = \oplus_{r\geq 1} ARI^r$.

We now introduce Écalle's important *swap* operator on moulds.

Definition 11. The *swap* operator maps ARI to $\overline{ARI}$, and is defined by

$$swap\, B(v_1, \dots, v_r) = B(v_r, v_{r-1} - v_r, \dots, v_1 - v_2)$$

for $B \in ARI$. The inverse operator mapping $\overline{ARI}$ to ARI (which we also denote by *swap*, as the context is clear according to whether *swap* is acting on a mould in ARI or one in $\overline{ARI}$) is given by

$$swap\, C(u_1, \dots, u_r) = C(u_1 + \cdots + u_r, u_1 + \cdots + u_{r-1}, \dots, u_1)$$

for $C \in \overline{ARI}$. Thus it makes sense to write $swap \circ swap = id$.

We also need to consider an important symmetry on moulds, based on the *shuffle* operator on tuples of commutative variables, which is defined by

$$Sh\big((u_1, \dots, u_i)(u_{i+1}, \dots, u_r)\big) = \Big\{(u_{\sigma^{-1}(1)}, \dots, u_{\sigma^{-1}(r)}) \,|\, \sigma \in S_r^i\Big\},$$

where S_r^i is the subset of permutations $\sigma \in S_r$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots \sigma(r)$.

Definition 12. A mould $A \in ARI$ is alternal if in each depth $r \geq 2$ we have

$$\sum_{w\in Sh((u_1,\dots,u_i)(u_{i+1},\dots,u_r))} A^r(w) = 0 \quad \text{for} \;\; 1 \leq i \leq \left[\frac{r}{2}\right].$$

By convention, the alternality condition is void in depth 1, i.e. all depth 1 moulds are considered to be alternal.

Example. In depth 4, there are two alternality conditions, given by

$$A(u_1,u_2,u_3,u_4)+A(u_2,u_1,u_3,u_4)+A(u_2,u_3,u_1,u_4)+A(u_2,u_3,u_4,u_1)=0$$
$$A(u_1,u_2,u_3,u_4)+A(u_3,u_1,u_2,u_4)+A(u_3,u_4,u_1,u_2)+A(u_1,u_3,u_2,u_4)$$
$$+A(u_1,u_3,u_4,u_2)+A(u_3,u_1,u_4,u_2)=0$$

We write ARI_{al} for the subspace of ARI consisting of alternal moulds.

3.2. **Lie elements and alternal moulds.** Alternality is important because alternal polynomial moulds correspond to Lie polynomials in the sense given in the following lemma, whose statements are well-known: the first one is a direct consequence of Lazard elimination (cf. Bourbaki), and for complete elementary proofs of all the statements, see [SST] or [S2].

Lemma 11. *(i) The free associative algebra* Ass_2 *on* x,y *can be decomposed as a direct sum*

$$\mathbb{Q}\langle\langle x,y\rangle\rangle=\mathbb{Q}x\oplus\mathbb{Q}\langle\langle C\rangle\rangle,$$

where $Ass_C=\mathbb{Q}\langle\langle C\rangle\rangle=\mathbb{Q}\langle\langle C_1,C_2,\ldots\rangle\rangle$ *is the free non-commutative polynomial algebra on variables* $C_i=\mathrm{ad}_x^{i-1}(y)$ *for* $i\geq 1$.

(ii) Let Ass_C^r *denote the subspace of* Ass_C *spanned by monomials* $C_{a_1}\cdots C_{a_r}$. *For each* $r\geq 1$, *the map*

$$ma:\begin{array}{rl} Ass_C^r & \to ARI^r \\ C_{a_1}\cdots C_{a_r} & \longmapsto u_1^{a_1-1}\cdots u_r^{a_r-1}\end{array}$$

is a vector space isomorphism.

(iii) For each $r\geq 1$, *the map* ma *restricts to a vector space isomorphism*

$$ma:\mathfrak{lie}_C^r\to ARI_{al}^r,$$

where $\mathfrak{lie}_C^r=\mathfrak{lie}_C\cap\mathbb{Q}^r\langle C\rangle$.

Examples. The mould $ma(C_3)=ma([x,[x,y]])$ is the mould concentrated in depth 1 given by u_1^2. Similarly, $ma(C_2C_1-C_1C_2)=ma([[x,y],y])$ is the mould concentrated in depth 2 given by $u_1^1u_2^0-u_1^0u_2^1=u_1-u_2$.

Definition 13. Let β denote the backwards writing operator on words in x,y, meaning that $\beta(m)$ is obtained from a word m by writing it from right to left. The operator β extends to polynomials by linearity.

We write ARI^{pol} for the vector subspace of polynomial-valued moulds in ARI. Let us give the translation of the restriction of the swap operator to polynomial-valued moulds directly in terms of elements of Ass_2 (cf. [R] or [S2]). Let $f\in Ass_C^r$, and write $f=xf^x+yf^y$. Set $g=\beta(yf^y)$, where β is the backwards operator of Definition 13. Thus all the monomials of g end in y. If we write g explicitly as

$$g=\sum_{\underline{a}=(a_1,\ldots,a_r)}k_{\underline{a}}\,x^{a_1}y\cdots yx^{a_r}y, \tag{30}$$

then (as shown in [S2], (3.2.6)), $swap\big(ma(f)\big)$ is the mould concentrated in depth r given by

$$swap\big(ma(f)\big)(v_1,\ldots,v_r)=\sum_{\underline{a}}k_{\underline{a}}\,v_1^{a_1}...v_r^{a_r}. \tag{31}$$

3.3. **Push-invariance and the first defining relation of $\mathfrak{lev}$.** Let us define the push-operator on moulds in ARI by

$$(push\, B)(u_1, \ldots, u_r) = B(u_0, u_1, ..., u_{r-1})$$

where $u_0 = -u_1 - u_2 - \cdots - u_r$. A mould $B \in ARI$ is push-invariant if $push\,(B) = B$ (in all depths).

The following proposition shows that this definition is precisely the translation into mould terms of the property of push-invariance for a Lie polynomial given in Definition 5 above.

Proposition 12. *Let $b \in \mathfrak{lie}_C$. Then b is a push-invariant polynomial if and only if $ma(b)$ is a push-invariant mould.*

Proof. If $b = y$, then $ma(b)$ is concentrated in depth 1 with value $ma(b)(u_1) = 1$, so these are both clearly push-invariant.

Now let $b \in (\mathfrak{lie}_C)_n^{r-1}$ with $n \geq r \geq 2$. We write

$$b = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, x^{a_1} y \cdots y x^{a_r}.$$

Let $f = yb$, so that $b = f^y$. Recalling that $y = C_1$, the associated moulds are related by the formula

$$ma(f)(u_1, ..., u_r) = ma(C_1 b) = u_1^0 ma(b)(u_2, ..., u_r) = ma(b)(u_2, ..., u_r). \tag{32}$$

Since $b \in (\mathfrak{lie}_C)_n$, we have $\beta(b) = (-1)^{n-1} b$. Set

$$g = \beta(y f^y) = \beta(yb) = (-1)^{n-1} by = (-1)^{n-1} \sum_{\underline{a}} k_{\underline{a}}\, x^{a_1} y ... y x^{a_r} y.$$

By (31), we have

$$swap\big(ma(f)\big)(v_1, \ldots, v_r) = (-1)^{n-1} \sum_{\underline{a}} k_{\underline{a}}\, v_1^{a_1} ... v_r^{a_r}.$$

Looking at

$$push(b)y = \sum_{\underline{a}} k_{\underline{a}}\, x^{a_r} y x^{a_1} y \cdots x^{a_{r-1}} y,$$

we see that $push(b)y$ is obtained from by by cyclically permuting the groups $x^{a_i} y$. Since $b = push(b)$ if and only if $k_{(a_1,\ldots,a_r)} = k_{(a_r,a_1,\ldots,a_{r-1})}$ for each $\underline{a}$, this is equivalent to

$$swap\big(ma(f)\big)(v_1, ..., v_r) = swap\big(ma(f)\big)(v_r, v_1, \ldots, v_{r-1}). \tag{33}$$

Using the definition of the $swap$, we rewrite (33) in terms of $ma(f)$ as

$$ma(f)(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2) = ma(f)(v_{r-1}, v_{r-2} - v_{r-1}, \ldots, v_r - v_1) \tag{34}$$

We now make the change of variables $v_r = u_1 + ... + u_r, v_r - v_1 = u_r$, $v_1 - v_2 = u_{r-1}, \ldots, v_{r-2} - v_{r-1} = u_2$, $v_{r-1} = u_1$ in this equation, obtaining

$$ma(f)(u_1 + \cdots + u_r, -u_2 - \cdots - u_r, u_2, \ldots, u_{r-1}) = ma(f)(u_1, u_2, \ldots, u_r). \tag{35}$$

Finally, using relation (32), we write this in terms of $ma(b)$ as

$$ma(b)(-u_2 - \cdots - u_r, u_2, \ldots, u_{r-1}) = ma(b)(u_2, \ldots, u_r). \tag{36}$$

Making the variable change $u_i \mapsto u_{i-1}$ changes this to

$$ma(b)(-u_1 - \cdots - u_{r-1}, u_1, \ldots, u_{r-2}) = ma(b)(u_1, \ldots, u_{r-1}), \tag{37}$$

which is just the condition of mould push-invariance $ma(b)$ in depth $r-1$. $\square$

3.4. **Circ-neutrality and the second defining relation of $\mathfrak{lev}$.** Let us now show how to reformulate the second defining property of elements of $\mathfrak{lev}$ in terms of moulds.

Definition 14. Let *circ* be the mould operator defined on moulds in $\overline{ARI}$ by

$$circ(B)(v_1,\dots,v_r) = B(v_2,\dots,v_r,v_1).$$

A mould $B \in \overline{ARI}$ is said to be *circ-neutral* if for $r>1$ we have

$$\sum_{i=0}^{r} circ^i(B)(v_1,\dots,v_r) = 0.$$

If B is a polynomial-valued mould of homogeneous degree n (i.e. the polynomial $B(v_1,\dots,v_r)$ is of homogeneous degree $n-r$ for $1 \le r \le n$), we say that B is *circ-constant* if

$$\sum_{i=0}^{r} circ^i(B)(v_1,\dots,v_r) = c\Big(\sum_{\substack{a_1+\cdots+a_r=d\\ a_i\ge 0}} v_1^{a_1}\cdots v_r^{a_r}\Big)$$

for all $1<r\le n$, where $B(v_1)=cv_1^{n-1}$. (If $c=0$, then a circ-constant mould is circ-neutral.) Correspondingly, we also say that a polynomial $b \in Ass_C$ of homogeneous degree n is *circ-constant* if, setting $c=(b|x^{n-1}y)$, we have $b = b_0 + \frac{c}{n}y^n$ where b_0^y is push-constant for the value c (cf. Definition 8). A polynomial-valued mould (resp. a polynomial in Ass_C) is said to be circ-constant if it is a sum of circ-constant homogeneous moulds (resp. polynomials).

Example. Let $\psi \in \mathfrak{grt}$ be homogeneous of degree n. Then as we saw in the example following Definition 8, the polynomial ψ^y is push-constant, so $\psi^y y$ is circ-constant. For example if $n=5$, then $\psi^y y$ is given by

$$\psi^y y = x^4y - 2x^3y^2 + \frac{11}{2}x^2yxy - \frac{9}{2}xyx^2y + 3yx^3y + 2x^2y^3 - \frac{11}{2}xyxy^2 + \frac{9}{2}xy^2xy$$
$$-\frac{1}{2}yx^2y^2 + 2yxyxy - \frac{1}{2}y^2x^2y - xy^4 + 4yxy^3 - 6y^2xy^2 + 4y^3xy$$

which is easily seen to be circ-constant.

For an example of a circ-constant mould, we take $B = swap\big(ma(\psi)\big)$, which has the same coefficients as $\psi^y y$: it is given by

$$\begin{aligned}
B(v_1) &= v_1^4\\
B(v_1,v_2) &= -2v_1^3 + \frac{11}{2}v_1^2v_2 - \frac{9}{2}v_1v_2^2 + 3v_2^3\\
B(v_1,v_2,v_3) &= 2v_1^2 - \frac{11}{2}v_1v_2 - \frac{1}{2}v_2^2 + \frac{9}{2}v_1v_3 + 2v_2v_3 - \frac{1}{2}v_3^2\\
B(v_1,v_2,v_3,v_4) &= -v_1 + 4v_2 - 6v_3 + 4v_4.
\end{aligned}$$

The following result proves that the circ-constance of a polynomial b and that of the associated mould $ma(b)$ are always connected as in the example above. By additivity, it suffices to prove the result for b a homogeneous polynomial of degree n, so that the circ-constance of b is relative to just one constant $c_n = c = (b|x^{n-1}y)$.

Proposition 13. *Let $b \in Ass_C$ be of homogeneous weight $n \geq 3$. Then b is a circ-constant polynomial if and only if $swap\big(ma(b)\big)$ is a circ-constant mould, and b is circ-neutral if and only if $swap\big(ma(b)\big)$ is circ-neutral.*

Proof. Let β be the backwards-writing operator on Ass_C (cf. Definition 13). Write $b = xb^x + yb^y$, and let $g = \beta(yb^y) = \beta(b^y)y$. For $r \geq 1$, let g^r denote the depth r part of g. If we write the polynomial g^r as

$$g^r = \beta\big((b^y)^{r-1}\big)y = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, x^{a_1} y \cdots y x^{a_r} y, \tag{38}$$

then we saw in (30) and (31) that

$$swap\big(ma(b)\big)(v_1,\ldots,v_r) = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, v_1^{a_1} \cdots v_r^{a_r}. \tag{39}$$

Observe that a polynomial is push-constant if and only it is also push-constant written backwards, so in particular, b^y is push-constant if and only if $\beta(b^y)$ is. Suppose that b is circ-constant, i.e. that b^y and thus $\beta(b^y)$ are push-constant for the value $c = (b|x^{n-1}y)$. In view of (38), this means that $\sum_{\underline{a}'} k_{\underline{a}'} = c$ when $\underline{a}'$ runs through the cyclic permutations of $\underline{a} = (a_1,\ldots,a_r)$ for every tuple $\underline{a}$, and this in turns means precisely that the mould $swap\big(ma(b)\big)$ is circ-constant. As for the circ-neutrality equivalence, it follows from the circ-constance, since circ-neutrality is nothing but circ-constance for the constant 0. □

The notion of circ-constance will play a role later in §4.2. In this section we only need circ-neutrality. Indeed, we showed that a polynomial b lies in $\mathfrak{lev}$, i.e. b is a Lie polynomial that is push-invariant and circ-neutral, if and only if the associated mould $ma(b)$ is alternal (by Lemma 11 (iii)), push-invariant (by Proposition 12) and its swap is circ-neutral (by Proposition 13). In other words, we have shown that ma gives a vector space isomorphism

$$ma : \mathfrak{lev} \xrightarrow{\sim} ARI^{pol}_{al+push/circneut}, \tag{40}$$

where the right-hand space is the subspace of ARI of polynomial-valued moulds in ARI that are alternal and push-neutral with circ-neutral swap. In fact this map is an isomorphism

$$\mathfrak{lev}^r_n \simeq ARI^r_{n-r} \cap ARI^{pol}_{al+push/circneut}, \tag{41}$$

of each bigraded piece, where in general we write ARI^r_d for the subspace of polynomial-valued moulds of homogeneous degree d concentrated in depth r.

We will show at the end of §4.1 below that $ARI^{pol}_{al+push/circneut}$ is a Lie algebra under the *ari*-bracket, and thus by the compatibility (104) of the *ari*-bracket with the Poisson bracket given below, we will then be able to conclude that $\mathfrak{lev}$ is also a Lie algebra, proving Proposition 1 of this paper.

3.5. **Proof of Theorem 3.** Recall the statement of Theorem 3.

Theorem 3. *The Lie injection (14) extends to a bigraded Lie injection on the associated linearized spaces, giving the following commutative diagram:*

$$\begin{array}{ccc} gr\,\mathfrak{ds} & \hookrightarrow & gr\,\mathfrak{krv}. \\ \downarrow & & \downarrow \\ \mathfrak{ls} & \hookrightarrow & \mathfrak{lev}. \end{array}$$

For all $n \geq 3$ and $r = 1, 2, 3$, the map is an isomorphism of the bigraded parts

$$\mathfrak{ls}_n^r \simeq \mathfrak{lkv}_n^r.$$

In order to prove this theorem, we first reformulate the statement in terms of moulds and give its proof. Let $ARI_{al/al}$ denote the space of moulds that are alternal and have alternal swap, and following Écalle's notation, let $ARI_{\underline{al}/\underline{al}}$ denote the subspace of $ARI_{al/al}$ of moulds that are even in depth 1. Directly from the definition of $\mathfrak{ls}$, we see that the map ma gives an isomorphism

$$ma : \mathfrak{ls} \xrightarrow{\sim} ARI^{pol}_{\underline{al}/\underline{al}}$$

onto the space of polynomial-valued moulds in $ARI_{\underline{al}/\underline{al}}$. Therefore, Theorem 3 can be stated very simply in terms of moulds as

$$ARI^{pol}_{\underline{al}/\underline{al}} \subset ARI^{pol}_{al+push/circneut}.$$

We will actually prove the more general result without the polynomial hypothesis.

Theorem 14. *There is an inclusion of mould subspaces*

$$ARI_{\underline{al}/\underline{al}} \subset ARI_{al+push/circneut},$$

Moreover in depths $r \leq 3$, we have

$$ARI^r \cap ARI_{\underline{al}/\underline{al}} = ARI^r \cap ARI_{al+push/circneut}.$$

Proof. It is well-known that every alternal mould satisfies

$$A(u_1, \ldots, u_r) = (-1)^{r-1} A(u_r, \ldots, u_1)$$

(cf. [S2], Lemma 2.5.3) and that a mould that is al/al and even in depth 1 is also push-invariant (cf. [S2], Lemma 2.5.5). Thus in particular $ARI_{\underline{al}/\underline{al}} \subset ARI_{al+push}$. It remains only to show that a mould in $ARI_{\underline{al}/\underline{al}}$ is necessarily circ-neutral. In fact, since the circ-neutrality condition is void in depth 1, we will show that even a mould in $ARI_{al/al}$ is circ-neutral; the condition of evenness in depth 1 is there to ensure the push-invariance, but not needed for the circ-neutrality.

The first alternality relation is given by

$$A(u_1, \ldots, u_r) + A(u_2, u_1, \ldots, u_r) + \cdots + A(u_2, \ldots, u_r, u_1) = 0.$$

Since A is push-invariant, this is equal to

$$push^r A(u_1, \ldots, u_r) + push^{r-1} A(u_2, u_1, \ldots, u_r) + \cdots + push\, A(u_2, \ldots, u_r, u_1) = 0.$$

But explicitly considering the action of the push operator on each term shows that

$$\begin{aligned} push^i A(u_2, \ldots, u_{r-i}, u_1, u_{r-i+1}, \ldots, u_r) &= A(u_{i+1}, \ldots, u_r, u_0, u_2, \ldots, u_i) \\ &= circ^{r-i} A(u_0, u_2, \ldots, u_r), \end{aligned}$$

where $u_0 = -u_1 - \cdots - u_r$, so this sum is equal to

$$\sum_{i=0}^{r-1} circ^i A(u_0, u_2, \ldots, u_r) = 0,$$

which proves that A is circ-neutral. This gives the inclusion.

Let us now prove the isomorphism in the cases $r = 1, 2, 3$. The case $r = 1$ is trivial since the alternality conditions are void in depth 1. A polynomial-valued mould concentrated in depth 1 is a scalar multiple of u_1^d, which is automatically in

$ARI_{al/al}$, and lies in $ARI_{\underline{al}/\underline{al}}$ if and only if d is even. Such a mould is automatically alternal and the circ-neutral condition is void; it is push-invariant thanks to the evenness of d. This shows that in depth 1, both spaces are generated by moulds u_1^d for even d, and are thus isomorphic.

Now consider the case $r = 2$. Let $A \in ARI^{pol}_{al+push/circneut}$ be concentrated in depth 2. The circ-neutral property of the swap is explicitly given in depth 2 by $swap(A)(v_1, v_2) + swap(A)(v_2, v_1) = 0$. But this is also the alternality condition on $swap(A)$, so $A \in ARI_{al/al}$. The isomorphism in depth 2 is thus trivial.

Finally, we consider the case $r = 3$. Let $A \in ARI^{pol}_{al+push/circneut}$ be concentrated in depth 3, and let $B = swap(A)$. Again, we only need to show that B is alternal, which in depth 3 means that B must satisfy the single equation

$$B(v_1, v_2, v_3) + B(v_2, v_1, v_3) + B(v_2, v_3, v_1) = 0. \tag{42}$$

The circ-neutrality condition on B is given by

$$B(v_1, v_2, v_3) + B(v_3, v_1, v_2) + B(v_2, v_3, v_1) = 0. \tag{43}$$

It is enough to show that B satisfies the equality

$$B(v_1, v_2, v_3) = B(v_3, v_2, v_1), \tag{44}$$

since applying this to the middle term of (43) immediately yields the alternality property (42) in depth 3. So let us show how to prove (44).

We rewrite the push-invariance condition in the v_i, which gives

$$B(v_1, v_2, v_3) = B(v_2 - v_1, v_3 - v_1, -v_1) \tag{45}$$
$$= B(v_3 - v_2, -v_2, v_1 - v_2) \tag{46}$$
$$= B(-v_3, v_1 - v_3, v_2 - v_3). \tag{47}$$

Making the variable change exchanging v_1 and v_3, this gives

$$B(v_3, v_2, v_1) = B(v_2 - v_3, v_1 - v_3, -v_3) \tag{48}$$
$$= B(v_1 - v_2, -v_2, v_3 - v_2) \tag{49}$$
$$= B(-v_1, v_3 - v_1, v_2 - v_1). \tag{50}$$

By (45), the term $B(v_2 - v_1, v_3 - v_1, -v_1)$ is circ-neutral with respect to the cyclic permutation of v_1, v_2, v_3, so we have

$$B(v_2 - v_1, v_3 - v_1, -v_1) = -B(v_3 - v_2, v_1 - v_2, -v_2) - B(v_1 - v_3, v_2 - v_3, -v_3). \tag{51}$$

But the circ-neutrality of B also lets us cyclically permute the three arguments of B, so we also have

$$-B(v_3 - v_2, v_1 - v_2, -v_2) = B(-v_2, v_3 - v_2, v_1 - v_2) + B(v_1 - v_2, -v_2, v_3 - v_2).$$

Using (45) and substituting this into the right-hand side of (51) yields

$$\begin{aligned} B(v_1, v_2, v_3) &= B(-v_2, v_3 - v_2, v_1 - v_2) \\ &\quad + B(v_1 - v_2, -v_2, v_3 - v_2) - B(v_1 - v_3, v_2 - v_3, -v_3). \end{aligned} \tag{52}$$

Now, exchanging v_1 and v_2 in (50) gives

$$B(v_3, v_1, v_2) = B(-v_2, v_3 - v_2, v_1 - v_2),$$

and doing the same with (48) gives

$$B(v_3, v_1, v_2) = B(v_1 - v_3, v_2 - v_3, -v_3).$$

Substituting these two expressions as well as (49) into the right-hand side of (52), we obtain the desired equality (44). This concludes the proof of Theorem 3. □

Remark. We conjecture that the inclusion of Theorem 14 is an isomorphism. But even the proof of the simple equality (44) is surprisingly complicated in depth 3, let alone in higher depth. Computer calculation does lead to the general conjecture:

Conjecture. *If $A \in ARI_{al+push/circneut}$ and $B = swap(A)$, then for all $r > 1$, we have*

$$B(v_1, \ldots, v_r) = (-1)^{r-1} B(v_r, \ldots, v_1). \tag{53}$$

The identity (53) would also yield the following useful partial result, which is the mould analog for $\mathfrak{lev}$ of a result that is well-known for $\mathfrak{ls}$, namely that the bigraded part $\mathfrak{ls}_n^r = 0$ when $n \not\equiv r \bmod 2$.

Lemma 15. *Fix $1 \leq r \leq n$. Let $A \in ARI_{n-r}^r \cap ARI_{al+push/circneut}^{pol}$ and let $B = swap(A)$. Assume that B satisfies (53). Then if $n - r$ is odd, $A = 0$.*

Proof. Let *mantar* denote the operator on moulds in ARI (resp. $\overline{ARI}$) defined by

$$mantar(A)(u_1, \ldots, u_r) = (-1)^{r-1} A(u_r, \ldots, u_1) \tag{54}$$

(resp. the same expression with v_i instead of u_i). It is easy to check the following identity of operators noted by Écalle:

$$neg \circ push = mantar \circ swap \circ mantar \circ swap,$$

where

$$neg(A)(u_1, \ldots, u_r) = A(-u_1, \ldots, -u_r). \tag{55}$$

Let $A \in ARI_{al+push/circneut}$; then A is push-invariant, so applying the left-hand operator to A gives $neg(A)$. Assuming (53) for $B = swap(A)$, i.e. assuming that $B = mantar(B)$, we see that applying the right-hand operator to A fixes A since on the one hand $swap \circ swap = id$ and on the other, $mantar(A) = A$ for all alternal moulds (cf. [S2], Lemma 2.5.3). Thus A must satisfy $neg(A) = A$, i.e. if $A \neq 0$ then the degree $d = n - r$ of A must be even. □

This implies the following result, which is the analogy for $\mathfrak{lev}$ of the similar well-known result on $\mathfrak{ls}$.

Corollary 16. *If the swaps of all elements of $ARI_{al+push/circneut}^{pol}$ are mantar-invariant, then $ARI_d^r \cap ARI_{al+push/circneut}^{pol} = 0$ whenever d is odd, i.e. by (41),*

$$\mathfrak{lev}_n^r = 0 \quad \text{when } n \not\equiv r \bmod 2$$

4. The elliptic Kashiwara-Vergne Lie algebra

In this section we follow the procedure of [S3] for the double shuffle Lie algebra to define a natural candidate for the elliptic Kashiwara-Vergne Lie algebra, closely related to the linearized Kashiwara-Vergne Lie algebra, and give some of its properties.

4.1. **Definition of the elliptic Kashiwara-Vergne Lie algebra.**

4.1.1. *The Kashiwara-Vergne Lie algebra.* Let Δ be the mould operator given by

$$\Delta(A)(u_1,\dots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)A(u_1,\dots,u_r) \tag{56}$$

for $r\geq 1$, and let ARI^{Δ} denote the space of rational-function moulds A such that $\Delta(A)$ is a polynomial mould (i.e. the denominator of the rational function A is "at worst" $u_1\cdots u_r(u_1+\cdots+u_r)$). We write ARI^{Δ}_{*} for the space of moulds in $ARI^{\Delta}\cap ARI_{*}$, where $*$ may represent any (or no) properties on moulds in ARI.

Recall that earlier we used the notation $ARI_{a/b}$ for the space of moulds having property a and whose swaps have property b; for example, $ARI_{al/al}$ denotes the space of alternal moulds with alternal swap. In this section we introduce a slightly more general notation ARI_{a*b} to denote the space of moulds having property a and whose swap has property b up to adding on a constant-valued mould; thus, we write ARI_{al*al} for the space of alternal moulds whose swaps are alternal up to adding on a constant-valued mould. An example of a mould in ARI_{al*al} is the mould $\Delta^{-1}(A)$, where A is the polynomial mould concentrated in depth 3 given by

$$A(u_1,u_2,u_3) = -\frac{1}{4}u_1^3u_2+\frac{1}{4}u_1^3u_3-\frac{1}{4}u_1^2u_2^2+\frac{1}{2}u_1^2u_3^2+\frac{1}{4}u_1u_3^3-\frac{1}{4}u_2^2u_3^2-\frac{1}{4}u_2u_3^3$$
$$-\frac{1}{12}u_1^2u_2u_3+\frac{1}{6}u_1u_2^2u_3-\frac{1}{12}u_1u_2u_3^2.$$

It is easy to check that $\Delta^{-1}(A)$ is alternal, but its swap is not alternal unless one adds on the constant $1/3$.

Definition 15. The mould elliptic Kashiwara-Vergne vector space is the subspace of polynomial-valued moulds

$$\Delta\big(ARI^{\Delta}_{al+push*circneut}\big).$$

The elliptic Kashiwara-Vergne vector space is the subspace $\mathfrak{krv}_{ell}\subset\mathfrak{lie}_C$ such that

$$ma\big(\mathfrak{krv}_{ell}\big) = \Delta\big(ARI^{\Delta}_{al+push*circneut}\big). \tag{57}$$

The operator Δ trivially respects push-invariance of moulds, so the space $\mathfrak{krv}_{ell}$ lies in the space $\mathfrak{lie}_C^{push}$ of push-invariant elements of $\mathfrak{lie}_C$. We will now show that the subspace $\mathfrak{krv}_{ell}$ is actually a Lie subalgebra of $\mathfrak{lie}_C^{push}$, which is itself a Lie algebra thanks to the following lemma, of which a more explicit version (with a formula for the partner) is proved in [S3] (Lemma 2.1.1).

Lemma 17. *Let $b\in\mathfrak{lie}_C$. Then $b\in\mathfrak{lie}_C^{push}$ if and only if there exists a unique element $a\in\mathfrak{lie}_C$ (the partner of b), such that if $D_{b,a}$ is the derivation of $\mathfrak{lie}_2$ defined by $x\mapsto b$, $y\mapsto a$, then $D_{b,a}$ annihilates $[x,y]$.*

By identifying $\mathfrak{lie}_C^{push}$ with the space of derivations that annihilate $[x, y]$, this lemma shows that $\mathfrak{lie}_C^{push}$ is a Lie algebra under the bracket of derivations. We state this as a corollary.

Corollary 18. *The map $b \mapsto D_{b,a}$ gives an isomorphism*

$$\partial : \mathfrak{lie}_C^{push} \to \mathfrak{oder}_2 \tag{58}$$

whose inverse is $D_{b,a} \mapsto D_{b,a}(x) = b$, and this becomes a Lie isomorphism when $\mathfrak{lie}_C^{push}$ is equipped with the Lie bracket

$$\langle b, b' \rangle = [D_{b,a}, D_{b',a'}](x) = D_{b,a}(b') - D_{b',a'}(b). \tag{59}$$

Thus we know that $\mathfrak{lie}_C^{push}$ is a Lie algebra and it contains the elliptic Kashiwara-Vergne space $\mathfrak{krv}_{ell}$ as a subspace. This leads to our first main result on $\mathfrak{krv}_{ell}$.

Theorem 19. *The subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is a Lie subalgebra.*

In order to prove this theorem, we will make essential use of mould theory, and in particular, of the *ari*-bracket defined by Écalle that makes ARI into a Lie algebra ARI_{ari}. The hairiest definitions and proofs have been relegated to Appendix 1, in order to streamline the exposition of the next paragraph, which contains some basic elements of mould theory that will lead to the proof of the theorem in 4.1.3.

4.1.2. *A few facts about moulds.* In this paragraph we give a few brief reminders about some of the basic operators of mould theory and their connections with the familiar situation of $\mathfrak{lie}_2$; a very concise but self-contained exposition with full definitions is given in Appendix 1, and a complete exposition with proofs can be found in Chapters 2 and 3 of [S2]. In this section, we content ourselves with giving a list of mould operators that generalize the some of the most frequently considered operators on $\mathfrak{lie}_2$ such as the usual and the Poisson bracket, Ihara and special derivations, and the bracket $\langle\, , \rangle$ on $\mathfrak{lie}_C^{push}$. It is important to make the following two observations: (i) all these operators given in mould-theoretic terms can be applied to a much wider class of moulds than merely polynomial-valued moulds, which permits a number of proofs of results on polynomial-valued moulds (and thus polynomials in x, y) that are not accessible otherwise; (ii) there are some very important mould operators that are not translations of anything that can be phrased in the polynomial situation; this is where the real richness of mould theory comes into play. We do not use any of these in this section, but some of them will play a key role in the next subsection (see 4.2.4).

Recall from Lemma 11 that we have an injective linear map ma from $\mathbb{Q}\langle C \rangle$ to polynomial-valued moulds which restricts to an isomorphism from $\mathfrak{lie}_C$ to alternal polynomial-valued moulds, i.e.

$$ma : \mathfrak{lie}_C \xrightarrow{\sim} ARI_{al}^{pol}.$$

The precise definitions of all the Lie brackets and derivations below are given in Appendix 1.

- There is a Lie bracket lu on ARI satisfying

$$ma\big([f, g]\big) = lu\big(ma(f), ma(g)\big)$$

for $f, g \in \mathfrak{lie}_C$. We write ARI_{lu} for the Lie algebra ARI with this bracket.

• For each mould $A \in ARI$, there is a derivation $arit(A)$ of ARI_{lu} that corresponds to the Poisson or Ihara derivation on $\mathfrak{lie}_C$ in the sense that

$$arit\big(ma(f)\big) \cdot ma(g) = -ma\big(d_f(g)\big).$$

• There is a Lie bracket ari on ARI given by

$$ari(A,B) = lu(A,B) - arit(A) \cdot B + arit(B) \cdot A$$

that corresponds to the Poisson or Ihara bracket on $\mathfrak{lie}_C$ in the sense that

$$ari\big(ma(f), ma(g)\big) = ma\big(\{f,g\}\big). \tag{60}$$

We write ARI_{ari} for the Lie algebra with this Lie bracket.

• There is a Lie bracket $\overline{ari}$ on $\overline{ARI}$ which satisfies the following relation with the ari-bracket in the special case where A and B are both push-invariant moulds:

$$\overline{ari}\big(swap(A), swap(B)\big) = swap\big(ari(A,B)\big). \tag{61}$$

• There is a third Lie bracket on ARI, the $Dari$-bracket, which is obtained by transfer by the Δ-operator given in (56), i.e. it is given by

$$Dari(A,B) = \Delta\Big(ari\big(\Delta^{-1}(A), \Delta^{-1}(B)\big)\Big). \tag{62}$$

This means that Δ gives an isomorphism of Lie algebras

$$\Delta : ARI_{ari} \xrightarrow{\sim} ARI_{Dari}. \tag{63}$$

• For each mould $A \in ARI$, there is an associated derivation $Darit(A)$ of ARI_{lu} that preserves ARI^{pol} if A is polynomial-valued and satisfies the following property: the $Dari$-bracket of (62) can also be defined by

$$Dari(A,B) = Darit(A) \cdot B - Darit(B) \cdot A. \tag{64}$$

We end this section by comparing the $Dari$-bracket to the bracket $\langle\ ,\ \rangle$ on $\mathfrak{lie}_C^{push}$ given in Corollary 18.

Proposition 20. *The map*

$$ma : \mathfrak{lie}_C^{push} \to ARI_{Dari},$$

is a Lie algebra morphism, i.e. the Lie brackets $\langle\ ,\ \rangle$ *and* $Dari$ *are compatible in the sense that*

$$ma\big(\langle b, b'\rangle\big) = Dari\big(ma(b), ma(b')\big).$$

Proof. The main point is the following result [BS] (see Theorem 3.5): if D_1 and D_2 lie in $\mathfrak{oder}_2$, then the map

$$\begin{aligned} \mathfrak{oder}_2 &\to ARI_{ari} \\ D &\mapsto \Delta^{-1}\big(ma\big(D(x)\big)\big), \end{aligned} \tag{65}$$

is an injective Lie morphism, i.e.

$$\Delta^{-1}\Big(ma\big([D_1,D_2](x)\big)\Big) = ari\Big(\Delta^{-1}\big(ma(D_1(x))\big), \Delta^{-1}\big(ma(D_2(x))\big)\Big).$$

Applying Δ to both sides of this and using (62), this is equivalent to

$$ma\big([D_1,D_2](x)\big) = Dari\Big(ma\big(D_1(x)\big), ma\big(D_2(x)\big)\Big), \tag{66}$$

which in turn means that

$$ma : \mathfrak{oder}_2 \to ARI_{Dari} \tag{67}$$

is a Lie algebra morphism. We saw in Corollary 18 that we have a Lie isomorphism $\mathfrak{lie}_C^{push} \xrightarrow{\sim} \mathfrak{oder}_2$ when $\mathfrak{lie}_C^{push}$ is equipped with the Lie bracket (59), so by composition, we have an injective Lie morphism

$$b \mapsto D_{b,a} \overset{\Psi}{\mapsto} \Delta^{-1}\big(ma(D_{b,a}(x))\big) \overset{\Delta}{\mapsto} ma(b)$$

is an injective Lie morphism $\mathfrak{lie}_C^{push} \to ARI_{Dari}$, which proves the result. □

4.1.3. *Proof that $\mathfrak{krv}_{ell}$ is a Lie algebra.* This subsection is devoted to the proof of Theorem 19, i.e. that the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is closed under the bracket $\langle\ ,\ \rangle$.

From Proposition 20, ma gives an injective Lie algebra morphism

$$\mathfrak{lie}_C^{push} \to ARI_{Dari}.$$

Thus it is equivalent to prove that the image of the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is closed under the $Dari$-bracket. Since we saw above that

$$\Delta^{-1} : ARI_{Dari} \to ARI_{ari},$$

it is equivalent to show that $ARI^{\Delta}_{al+push*circneut}$ is a Lie subalgebra of ARI_{ari}.

Let $b \in \mathfrak{lie}_C$ be push-invariant and let $D_{b,a} = \partial(b)$ where $\partial : \mathfrak{lie}_C^{push} \to \mathfrak{oder}_2$ is as in (58). It is shown[1] in [BS], Prop. B.1 that for all $b' \in \mathfrak{lie}_C$, we have

$$ma\big(D_{b,a}(b')\big) = Darit\big(ma(b)\big)\big(ma(b')\big). \tag{68}$$

Thus when $b \in \mathfrak{lie}_C^{push}$ and $B = ma(b)$, $Darit(B)$ is nothing but the mould form of $D_{b,a}$; in particular $Darit(B)$ preserves the space of polynomial-valued moulds, $Darit(B) \cdot ma([x,y]) = 0$ and $Darit(B) \cdot ma(y) = ma(a)$. This shows in particular that if $b, b' \in \mathfrak{lie}_C$ and $B = ma(b)$, $B' = ma(b')$, then by (64) and (68), we have

$$\begin{aligned} Dari(B, B') &= Darit(B) \cdot B' - Darit(B') \cdot B \\ &= ma\big(D_{b,a}(b') - D_{b',a'}(b)\big) \\ &= ma\big([D_{b,a}, D_{b',a'}](x)\big) \\ &= ma\big(\langle b, b'\rangle\big). \end{aligned} \tag{69}$$

We use $Darit$ and $Dari$ to prove the desired result in three steps as follows.

Step 1. Since $\mathfrak{lie}_C^{push}$ is the space of push-invariant Lie polynomials, we have

$$ma(\mathfrak{lie}_C^{push}) = ARI^{pol}_{al+push}.$$

But we saw in Proposition 20 that $\mathfrak{lie}_C^{push}$ is a Lie algebra under $\langle\ ,\ \rangle$, so $ARI^{pol}_{al+push}$ is a Lie algebra under $Dari$.

Step 2. The space $ARI^{\Delta}_{al+push}$ is a Lie algebra under ari. Indeed, the definition of Δ shows that this operator does not change the properties of push-invariance or alternality, i.e. $\Delta^{-1}(ARI_{al+push}) = ARI_{al+push}$. Restricted to polynomial-valued moulds, we have $\Delta^{-1}(ARI^{pol}_{al+push}) = ARI^{\Delta}_{al+push}$. Since Δ is an isomorphism from

[1]Note that the notation is slightly different there; we recover this statement by setting $F = b'$, $U = b$, $D_U = D_{b,a}$ and taking care to note that the definition of $Darit_U$ in that article is the conjugation of the definition (107) used here by dar, i.e. it is (107) without the dar terms.

ARI_{ari} to ARI_{Dari} by virtue of (63) and $ARI^{pol}_{al+push}$ is a Lie subalgebra of ARI_{Dari} by Step 1, its image $ARI^{\Delta}_{al+push}$ under Δ^{-1} is thus a Lie subalgebra of ARI_{ari}.

Step 3. We can now complete the proof of Theorem 19 by showing that the space $ARI^{\Delta}_{al+push*circneut}$ is a Lie algebra under ari. For this, we need the following lemma, whose proof is deferred to the end of Appendix 1.

Lemma 21. *The space $\overline{ARI}_{circneut}$ of circ-neutral moulds $A \in \overline{ARI}$ forms a Lie algebra under the $\overline{ari}$-bracket.*

Given this, it is an easy matter to conclude. Let A, B lie in $ARI^{\Delta}_{al+push*circneut}$, and let us show that $ari(A, B)$ lies in the same space. By Step 2, we know that $ari(A, B) \in ARI^{\Delta}_{al+push}$, so we only need to show that $swap\big(ari(A, B)\big)$ is *circ-neutral. But we will show that in fact this mould is actually circ-neutral. To see this, let A_0 and B_0 be the constant-valued moulds such that $swap(A) + A_0$ and $swap(B) + B_0$ are circ-neutral. By Lemma 21, we have

$$\overline{ari}\big(swap(A) + A_0, swap(B) + B_0\big) \in \overline{ARI}_{circneut}.$$

Using the identity $swap\big(ari(M, N)\big) = \overline{ari}\big(swap(M), swap(N)\big)$, valid whenever M and N are push-invariant moulds (cf. [S], (2.5.6)), as well as the fact that constant-valued moulds are both push and swap invariant, we have

$$\begin{aligned}
\overline{ari}\big(swap(A) + A_0, swap(B) + B_0\big) &= \overline{ari}\big(swap(A + A_0), swap(B + B_0)\big) \\
&= swap \cdot ari(A + A_0, B + B_0) \\
&= swap \cdot ari(A, B) + swap \cdot ari(A, B_0) + swap \cdot ari(A_0, B) + swap \cdot ari(A_0, B_0) \\
&= swap \cdot ari(A, B)
\end{aligned}$$

since the definition of the ari-bracket shows that $ari(C, M) = 0$ whenever C is a constant-valued mould. Thus $swap \cdot ari(A, B)$ is circ-neutral, which completes the proof of Theorem 19. □

The following easy corollary provides the promised proof of Proposition 1 stating that $\mathfrak{krv}$ is a Lie algebra.

Corollary 22. *The subspace*

$$ARI^{pol}_{al+push/circneut} \subset ARI^{\Delta}_{al+push*circneut}$$

is a Lie algebra under the ari-bracket. Thus, by (60)*, the space*

$$\mathfrak{krv} = ma^{-1}\big(ARI^{pol}_{al+push/circneut}\big)$$

a Lie algebra under the Poisson bracket.

Proof. By the definition of ari, ARI^{pol} is a Lie subalgebra of ARI. Also, Lemma 21 shows that the space $\overline{ARI}_{circneut}$ of circ-neutral moulds is a Lie subalgebra of $\overline{ARI}_{*circneut}$. Thus $ARI^{\Delta}_{al+push/circneut}$ is a Lie algebra inside $ARI^{\Delta}_{al+push*circneut}$. So the intersection

$$ARI^{pol} \cap ARI^{\Delta}_{al+push/circneut} = ARI^{pol}_{al+push/circneut}$$

is one as well. □

4.2. **The map from $\mathfrak{krv} \to \mathfrak{krv}_{ell}$.** In this subsection we prove our next main result on the elliptic Kashiwara-Vergne Lie algebra, which is analogous to known results on the elliptic Grothendieck-Teichmüller Lie algebra of [E] and the elliptic double shuffle Lie algebra of [S3]. The subsection 4.3 below is devoted to connections between these three situations.

Theorem 23. *There is an injective Lie algebra morphism*

$$\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell} \tag{70}$$

The proof constructs the morphism from $\mathfrak{krv}$ to $\mathfrak{krv}_{ell}$ in four main steps as follows.

Step 1. We first consider a twisted version of the Kashiwara-Vergne Lie algebra, or rather of the associated polynomial space $V_{\mathfrak{krv}}$ of Definition 9, via the map

$$\nu : V_{\mathfrak{krv}} \xrightarrow{\sim} W_{\mathfrak{krv}} \tag{71}$$

$$f \mapsto \nu(f), \tag{72}$$

where ν is the automorphism of Ass_2 defined by

$$\nu(x) = z = -x - y, \quad \nu(y) = y. \tag{73}$$

In paragraph 4.2.1, we prove that $W_{\mathfrak{krv}}$ is a Lie algebra under the Poisson or Ihara bracket, and give a description of $W_{\mathfrak{krv}}$ via two properties, the "twisted" versions of the two defining properties of $V_{\mathfrak{krv}}$ given in Definition 9.

Step 2. In paragraph 4.2.2, we study the mould space $ma\big(W_{\mathfrak{krv}}\big)$. Thanks to the compatibility of the *ari*-bracket with the Poisson bracket (104), this space is a Lie subalgebra of ARI_{ari}. Just as we reformulated the defining properties of $\mathfrak{lkv}$ in mould terms in §3, proving that $ma(\mathfrak{lkv}) = ARI^{pol}_{al+push/circneut}$, here we reformulate the defining properties of $W_{\mathfrak{krv}}$ in mould terms: explicitly, we show that

$$ma\big(W_{\mathfrak{krv}}\big) = ARI^{pol}_{al+sen*circconst}, \tag{74}$$

the space of polynomial-valued moulds that are alternal, satisfy a certain *senary* relation (79) introduced by Écalle (see below), and whose swap is circ-constant up to addition of a constant-valued mould. We observe that if $B \in \overline{ARI}$ is a polynomial-valued mould of homogeneous degree n whose swap is circ-constant up to addition of a constant-valued mould, then the constant-valued mould B_0 is uniquely determined as being the mould concentrated in depth n and taking the value c/n there, where $B(v_1) = cv_1^{n-1}$.

Step 3. For this part we need to introduce Écalle's mould *pal* and its inverse *invpal*, which lie in the Lie group $GARI$ associated to the Lie algebra ARI_{ari}, and study the adjoint operator $Ad_{ari}(invpal)$ on ARI_{ari}. Letting Ξ denote the map

$$Ad_{ari}(invpal) \circ pari : ARI_{ari} \to ARI_{ari},$$

we show that it yields an injective Lie morphism

$$\Xi : ARI^{pol}_{al+sen*circconst} \to ARI^{\Delta}_{al+push*circneut} \tag{75}$$

of subalgebras of ARI_{ari}.

Step 4. The final step is to compose (75) with the Lie morphism $\Delta : ARI_{ari} \to ARI_{Dari}$, obtaining an injective Lie morphism

$$ARI^{pol}_{al+sen*circconst} \to \Delta\big(ARI^{\Delta}_{al+push*circneut}\big),$$

where the left-hand space is a subalgebra of ARI_{ari} and the right-hand one of ARI_{Dari}. Since the right-hand space is equal to $ma(\mathfrak{krv}_{ell})$, the desired injective Lie morphism $\mathfrak{krv} \to \mathfrak{krv}_{ell}$ is obtained by composing all the maps described above, as shown in the following diagram:

$$\begin{array}{ccccc}
\mathfrak{krv} & & & & \\
\text{by (29)} \downarrow & & & & \\
V_{\mathfrak{krv}} & & & & \\
\text{by (71)} \downarrow \nu & & & & \\
W_{\mathfrak{krv}} & & & & \mathfrak{krv}_{ell} \\
\text{by (74)} \downarrow ma & & & & ma^{-1} \uparrow \text{by (57)} \\
ARI^{pol}_{al+sen*circconst} & \xrightarrow[\text{by (75)}]{\Xi} & ARI^{\Delta}_{al+push*circneut} & \xrightarrow{\Delta} & \Delta\big(ARI^{\Delta}_{al+push*circneut}\big)
\end{array}$$

4.2.1. *Step 1: The twisted space $W_{\mathfrak{krv}}$.*

Proposition 24. *Let $W_{\mathfrak{krv}} = \nu(V_{\mathfrak{krv}})$. Then $W_{\mathfrak{krv}}$ is a Lie algebra under the Poisson bracket.*

Proof. The key point is the following lemma on derivations.

Lemma 25. *Conjugation by ν induces an isomorphism of Lie algebras*

$$\begin{aligned} \mathfrak{sder}_2 &\xrightarrow{\sim} \mathfrak{ider}_2 \\ E_{a,b} &\mapsto d_{\nu(b)}. \end{aligned} \tag{76}$$

Proof. Recall that $E_{a,b} \in \mathfrak{sder}_2$ maps $x \mapsto [x,a]$ and $y \mapsto [y,b]$, and $d_{\nu(b)} \in \mathfrak{ider}_2$ is the Ihara derivation defined by $x \mapsto 0$, $y \mapsto [y, \nu(b)]$ (cf. §1.1).

Let us first show that $d_{\nu(b)}$ is the conjugate of $E_{a,b}$ by ν, i.e. $d_{\nu(b)} = \nu \circ E_{a,b} \circ \nu$ (since ν is an involution). It is enough to show they agree on x and y, so we compute

$$\nu \circ E_{a,b} \circ \nu(x) = \nu \circ E_{a,b}(z) = 0 = d_{\nu(b)}(x)$$

and

$$\nu \circ E_{a,b} \circ \nu(y) = \nu \circ E_{a,b}(y) = \nu\big([y,b]\big) = [y, \nu(b)] = d_{\nu(b)}(y).$$

This shows that $\nu \circ E_{a,b} \circ \nu$ is indeed equal to $d_{\nu(b)}$. To show that $d_{\nu(b)}$ lies in $\mathfrak{ider}_2$, we check that $d_{\nu(b)}(z)$ is a bracket of z with another element of $\mathfrak{lie}_2$:

$$d_{\nu(b)}(z) = \nu \circ E_{a,b} \circ \nu(z) = \nu \circ E_{a,b}(x) = \nu([x,a]) = [z, \nu(a)].$$

The same argument goes the other way to show that conjugation by ν maps an element of $\mathfrak{ider}_2$ to an element of $\mathfrak{sder}_2$, which yields the isomorphism (76) as vector spaces. To see that it is also an isomorphism of Lie algebras, it suffices to note that conjugation by ν preserves the Lie bracket of derivations in $\mathfrak{der}_2$, i.e.

$$\nu \circ [D_1, D_2] \circ \nu = [\nu \circ D_1 \circ \nu, \nu \circ D_2 \circ \nu],$$

since ν is an involution. Since the Lie brackets on $\mathfrak{sder}_2$ and $\mathfrak{ider}_2$ are just restrictions to those subspaces of the Lie bracket on the space of all derivations, conjugation by ν carries one to the other. $\square$

We use the lemma to complete the proof of Proposition 24. Write

$$\mathfrak{krv}^\nu = \{\nu \circ E \circ \nu \,|\, E \in \mathfrak{krv}\} \subset \mathfrak{ider}_2.$$

By restricting the isomorphism (76) to the subspace $\mathfrak{krv} \subset \mathfrak{sder}_2$, we obtain a commutative diagram of isomorphisms of vector spaces

$$\begin{array}{ccc} \mathfrak{krv} & \to & \mathfrak{krv}^\nu, \\ \downarrow & & \downarrow \\ V_{\mathfrak{krv}} & \xrightarrow{\nu} & W_{\mathfrak{krv}}, \end{array}$$

where the left-hand vertical arrow is the isomorphism (29) mapping $E_{a,b} \mapsto b$, and the right-hand vertical map sends an Ihara derivation d_f to f. Equipping $W_{\mathfrak{krv}}$ with the Lie bracket inherited from $\mathfrak{krv}^\nu$ makes this into a commutative diagram of Lie isomorphisms. But this bracket is nothing other than the Poisson bracket since $\mathfrak{krv}^\nu \subset \mathfrak{ider}_2$. □

We now give a characterization of $W_{\mathfrak{krv}}$ by two defining properties which are the twists by ν of those defining $V_{\mathfrak{krv}}$. Recall that β is the the backwards operator given in Definition 13.

Proposition 26. *The space $W_{\mathfrak{krv}}$ is the space spanned by polynomials $b \in \mathfrak{lie}_C$, of homogeneous degree $n \geq 3$, such that*

(i) $b_y - b_x$ is anti-palindromic*, i.e. $\beta(b_y - b_x) = (-1)^{n-1}(b_y - b_x)$, and*

(ii) $b + \frac{c}{n}y^n$ is circ-constant*, where $c = (b|x^{n-1}y)$.*

Proof. Let $f = \nu(b)$, so that $f \in V_{\mathfrak{krv}}$. Then the property that $b_y - b_x$ is anti-palindromic is precisely equivalent to the push-invariance of f (this is proved as the equivalence of properties (iv) and (v) of Theorem 2.1 of [S1]). This proves (i).

For (ii), we note that since $f \in V_{\mathfrak{krv}}$, $f^y - f^x$ is push-constant for the value $c = (f|x^{n-1}y) = (-1)^{n-1}(b|x^{n-1}y)$. We have

$$b(x,y) = xb^x(x,y) + yb^y(x,y),$$

so

$$f(x,y) = b(z,y) = zb^x(z,y) + yb^y(z,y) = -xb^x(z,y) - yb^x(z,y) + yb^y(z,y).$$

Thus since $f(x,y) = xf^x(x,y) + yf^y(x,y)$, this gives

$$f^x = -b^x(z,y) \quad \text{and} \quad f^y = -b^x(z,y) + b^y(z,y),$$

so

$$f^y - f^x = b^y(z,y) = \nu(b^y).$$

Thus to prove the result, it suffices to prove that the following statement: if $g \in Ass_C$ is a polynomial of homogeneous degree n that is push-constant for $(-1)^{n-1}c$, then $\nu(g)$ is push-constant for c, since taking $g = f^y - f^x$ then shows that $\nu(g) = b^y$ is push-constant for c. The proof of this statement is straightforward using the substitution $z = -x - y$ (but see the proof of Lemma 3.5 in [S1] for details). Since $c = 0$ if $f \in V_{\mathfrak{krv}}$ is of even degree n (Corollary 4), this proves (ii). □

4.2.2. *Step 2: The mould version $ma(W_{\mathfrak{krv}})$.* The space $ma(W_{\mathfrak{krv}})$ is closed under the *ari*-bracket by (60), since $W_{\mathfrak{krv}}$ is closed under the Poisson bracket.

Let $b \in W_{\mathfrak{krv}}$ and let $B = ma(b)$. Then since b is a Lie polynomial, B is an alternal polynomial mould. Let us give the mould reformulations of properties (i) and (ii) of Proposition 26. The second property is easy since we already showed, in Proposition 13, that a polynomial b is circ-constant if and only if $swap(B)$ is circ-constant.

Expressing the first property in terms of moulds is more complicated and calls for an identity discovered by Écalle. We need to use the mould operator *mantar* defined in (54), as well as the mould operator *pari* defined by

$$pari(B)(u_1, \dots, u_r) = (-1)^r B(u_1, \dots, u_r). \tag{77}$$

The operator *pari* extends the operator $y \mapsto -y$ on polynomials to all moulds, and *mantar* extends the operator $f \mapsto (-1)^{n-1}\beta(f)$. Above all, we need Écalle's mould operator *teru*, defined by taking the mould $teru(B)$ to be equal to B in depths 0 and 1, and for depths $r > 1$, setting

$$\begin{aligned} &teru(B)(u_1, \dots, u_r) = \\ &\quad B(u_1, \dots, u_r) + \frac{1}{u_r}\Big(B(u_1, \dots, u_{r-2}, u_{r-1} + u_r) - B(u_1, \dots, u_{r-2}, u_{r-1})\Big). \end{aligned} \tag{78}$$

Lemma 27. *Let $b \in \mathfrak{lie}_C$. Then the following are equivalent:*

(1) $b_y - b_x$ is anti-palindromic;

(2) $B = ma(b)$ satisfies the senary relation

$$teru \circ pari(B) = push \circ mantar \circ teru \circ pari(B). \tag{79}$$

Proof. The statement is a consequence of the following result, proved in A.3 of the Appendix of [S1]. Let $\tilde{b} \in \mathfrak{lie}_C$ and let $\tilde{B} = ma(\tilde{b})$. Write $\tilde{b} = \tilde{b}_x x + \tilde{b}_y y$ as usual. Then for each depth part $(\tilde{b}_x + \tilde{b}_y)^r$ of the polynomial $\tilde{b}_x + \tilde{b}_y$ $(1 \leq r \leq n-1)$, the anti-palindromic property

$$(\tilde{f}_x + \tilde{f}_y)^r = (-1)^{n-1}\beta(\tilde{f}_x + \tilde{f}_y)^r \tag{80}$$

translates directly to the following relation on $\tilde{B}$:

$$teru(\tilde{B})(u_1, \dots, u_r) = push \circ mantar \circ teru(\tilde{B})(u_1, \dots, u_r). \tag{81}$$

Let us deduce the equivalence of (1) and (2) from that of (80) and (81). Let $\tilde{b}$ be defined by $\tilde{b}(x, y) = b(x, -y)$. This implies that $b_x = (-1)^r \tilde{b}_x$, $b_y = (-1)^{r-1}\tilde{b}_y$, and $\tilde{B} = pari(B)$. Thus $b_y - b_x$ is anti-palindromic if and only if $\tilde{b}_y + \tilde{b}_x$ is, i.e. if and only if (80) holds for $\tilde{b}$, which is the case if and only if (81) holds for $\tilde{B}$, which is equivalent to (79) with $\tilde{B} = pari(B)$. This proves the lemma. □

The following proposition summarizes the mould reformulations of the defining properties (i) and (ii) of $W_{\mathfrak{krv}}$.

Proposition 28. *Let $ARI^{pol}_{al+sen*circconst}$ denote the space of alternal polynomial-valued moulds satisfying the senary relation (79) and having swap that is circ-constant up to addition of a constant-valued mould. Then we have the isomorphism of Lie algebras*

$$ma : W_{\mathfrak{krv}} \xrightarrow{\sim} ARI^{pol}_{al+sen*circconst} \subset ARI_{ari}.$$

4.2.3. *Mould background: Exponential maps from ARI to GARI.* The next stage of our proof, the construction of a Lie algebra morphism

$$ARI^{pol}_{al+sen*circconst} \to ARI^{\Delta}_{al+push*circneut}, \tag{82}$$

is the most difficult, and requires some further definitions from mould-theory. In order to keep it simple, we will make use of the following scheme.

Any vector space $\mathfrak{g}$ equipped with a pre-Lie law $p(f,g)$ is is also automatically equipped with

• a Lie bracket $[f,g] = p(f,g) - p(g,f)$;

• an exponential map $exp_p : \mathfrak{g} \to G$, where $G = exp_p(\mathfrak{g})$ is the associated Lie group, and its inverse map log_p;

• the group law $*$ on G which is given by

$$exp_p(f) * exp_p(g) = exp_p\big(ch_{[,]}(f,g)\big).$$

• an adjoint map of G on $\mathfrak{g}$ defined for $H \in G$ by letting $h = log_p(H)$ and setting

$$Ad_{[,]}(H) \cdot f = exp\big(ad(h)\big) \cdot f = \sum_{n \geq 0} \frac{1}{n!} ad(h)^n \cdot f,$$

where $ad(h) \cdot f = [h,f]$.

When $\mathfrak{g} = ARI$, we have seen that it can be equipped with various pre-Lie laws and Lie brackets. The underlying set of the associated Lie group will always be the set $GARI$ of all moulds with constant term 1, just as ARI is the space of all moulds with constant term 0. (The same holds for $\overline{ARI}$ and $\overline{GARI}$.)

Écalle has studied a large family of different pre-Lie laws on ARI and $\overline{ARI}$, together with all their attendant structures as in the list above. The only ones we need here are the pre-Lie laws

$$\begin{aligned} preari(A,B) &= arit(B) \cdot A + mu(A,B) \qquad \text{on } ARI \\ \overline{preari}(A,B) &= \overline{arit}(B) \cdot A + mu(A,B) \qquad \text{on } \overline{ARI}, \end{aligned}$$

where $arit$ (resp. $\overline{arit}$) are the derivations of ARI_{lu} (resp. $\overline{ARI}_{lu}$) defined in Appendix 1. We will not use these pre-Lie laws in and of themselves, but in the next paragraph we will be using their associated adjoint actions Ad_{ari} and $Ad_{\overline{ari}}$.

We end this paragraph by defining, for any mould $Q \in \overline{GARI}$, an automorphism $\overline{ganit}(Q)$ of the Lie algebra $\overline{ARI}_{lu}$[2]. Set $\mathbf{v} = (v_1, \ldots, v_r)$, and let $W_{\mathbf{v}}$ denote the set of decompositions $d_{\mathbf{v}}$ of $\mathbf{v}$ into chunks

$$d_{\mathbf{v}} = \mathbf{a}_1 \mathbf{b}_1 \cdots \mathbf{a}_s \mathbf{b}_s \tag{83}$$

for $s \geq 1$, where with the possible exception of $\mathbf{b}_s$, the $\mathbf{a}_i$ and $\mathbf{b}_i$ are non-empty. Thus for instance, when $r = 2$ there are two decompositions in $W_{\mathbf{v}}$, namely $\mathbf{a}_1 = (v_1, v_2)$ and $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2)$, and when $r = 3$ there are four decompositions, three for $s = 1$: $\mathbf{a}_1 = (v_1, v_2, v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1, v_2)(v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2, v_3)$, and one for $s = 2$: $\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 = (v_1)(v_2)(v_3)$.

[2]The explicit expression given below does not explain why $\overline{ganit}(Q)$ is an automorphism. The mould-theoretic definition of $\overline{ganit}(Q)$ makes this clear. Let $\overline{anit}$ be the derivation of $\overline{ARI}$ given in Appendix 1. Then $\overline{preani}(A,B) = \overline{anit}(B) \cdot A - mu(A,B)$ is a pre-Lie law on $\overline{ARI}$. Let $log_{\overline{ani}}$ be the associated logarithm map, and set $P = log_{\overline{ani}}(Q) \in \overline{ARI}$. Then $\overline{ganit}(Q)$ is the exponential of the derivation $\overline{anit}(P)$.

Écalle's explicit expression for $\overline{ganit}(Q)$ is given by

$$\big(\overline{ganit}(Q)\cdot T\big)(\mathbf{v}) = \sum_{\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s\in W_{\mathbf{v}}} Q(\lfloor\mathbf{b}_1)\cdots Q(\lfloor\mathbf{b}_s)\ T(\mathbf{a}_1\cdots\mathbf{a}_s), \tag{84}$$

where if $\mathbf{b}_i$ is the chunk $(v_k, v_{k+1}, \ldots, v_{k+l})$, then we use the notation

$$\lfloor\mathbf{b}_i = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1}). \tag{85}$$

4.2.4. *Mould background: The special mould pal and Écalle's fundamental identity.* We are now ready to introduce the fundamental identity of Écalle, which is the key to the construction of the desired map (82).

Definition 16. Let constants $c_r \in \mathbb{Q}$, $r \geq 1$, be defined by setting $f(x) = 1 - e^{-x}$ and expanding $f_*(x) = \sum_{r\geq 1} c_r x^{r+1}$, where $f_*(x)$ is the *infinitesimal generator* of $f(x)$, defined by

$$f(x) = \Big(exp\big(f_*(x)\frac{d}{dx}\big)\Big)\cdot x.$$

Let *lopil* be the mould in $\overline{ARI}_{\overline{ari}}$ defined by the simple expression

$$lopil(v_1, \ldots, v_r) = c_r\,\frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2)\cdots(v_{r-1} - v_r)v_r} \tag{86}$$

Set $pil = exp_{\overline{ari}}(lopil)$ where $exp_{\overline{ari}}$ denotes the exponential map associated to $\overline{preari}$, and set $pal = swap(pil)$.

The mould *lopil* is easily seen to be both alternal and circ-neutral. It is also known (although surprisingly difficult to show) that the mould $lopal = log_{ari}(pal)$ is alternal (cf. [Ec2], or [S2], Chap. 4.). Thus the moulds *pil* and *pal* are both exponentials of alternal moulds; this is called being *symmetral*. The inverses of *pal* (in $GARI$) and *pil* (in $\overline{GARI}$) are given by

$$invpal = exp_{ari}(-lopal), \qquad invpil = exp_{\overline{ari}}(-lopil).$$

The key maps we will be using in our proof are the adjoint operators associated to *pal* and *pil*, given by

$$Ad_{ari}(pal) = exp\big(ad_{ari}(lopal)\big), \qquad Ad_{\overline{ari}}(pil) = exp\big(ad_{\overline{ari}}(lopil)\big), \tag{87}$$

where $ad_{ari}(P)\cdot Q = ari(P, Q)$. The inverses of these adjoint actions are given by

$$Ad_{ari}(invpal) = exp\big(ad_{ari}(-lopal)\big), \quad Ad_{\overline{ari}}(invpil) = exp\big(ad_{\overline{ari}}(-lopil)\big). \tag{88}$$

These adjoint actions produce remarkable transformations of certain mould properties into others, and form the heart of much of Écalle's theory of multizeta values. Écalle's *fundamental identity* relates the two adjoint actions of (87). Valid for all push-invariant moulds M, it is given by

$$swap\cdot Ad_{ari}(pal)\cdot M = \overline{ganit}(pic)\cdot Ad_{\overline{ari}}(pil)\cdot swap(M), \tag{89}$$

where $pic \in \overline{GARI}$ is defined by $pic(v_1, \ldots, v_r) = 1/v_1\cdots v_r$ (see [Ec], or [S2], Theorem 4.5.2 for the complete proof).

For our purposes, it is useful to give a slightly modified version of this identity. Let $poc \in \overline{GARI}$ be the mould defined by

$$poc(v_1, \ldots, v_r) = \frac{1}{v_1(v_1 - v_2)\cdots(v_{r-1} - v_r)}. \tag{90}$$

Then $\overline{ganit}(poc)$ and $\overline{ganit}(pic)$ are inverse automorphisms of $\overline{ARI}_{lu}$ (see [B], Lemma 4.37). Thus, we can rewrite the above identity (89) as

$$\overline{ganit}(poc) \cdot swap \cdot Ad_{ari}(pal) \cdot M = Ad_{\overline{ari}}(pil) \cdot swap(M), \tag{91}$$

and letting $N = Ad_{ari}(pal) \cdot M$, i.e. $M = Ad_{ari}(invpal) \cdot N$, we rewrite it in terms of N as

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot swap(N) = swap \cdot Ad_{ari}(invpal) \cdot N, \tag{92}$$

this identity being valid whenever $M = Ad_{ari}(invpal) \cdot N$ is push-invariant.

4.2.5. *Step 3: Construction of the map* Ξ. In this section we finally arrive at the main step of the construction of our map $\mathfrak{krv} \to \mathfrak{krv}_{ell}$, namely the construction of the map Ξ given in the following proposition.

Proposition 29. *The operator* $\Xi = Ad_{ari}(invpal) \circ pari$ *gives an injective Lie morphism of Lie subalgebras of* ARI_{ari}*:*

$$\Xi : ARI^{pol}_{al+sen*circconst} \longrightarrow ARI^{\Delta}_{al+push*circneut}. \tag{93}$$

Proof. We have already shown that both spaces are Lie subalgebras of ARI_{ari}, the first in Proposition 28 and the second in 4.1.3. Furthermore, since *pari* and $Ad_{ari}(invpal)$ are both invertible and respect the *ari*-bracket, the proposed map is indeed an injective map of Lie subalgebras. Thus it remains only to show that the image of $ARI^{pol}_{al+sen*circconst}$ under Ξ really lies in $ARI^{\Delta}_{al+push*circneut}$. We will show separately that if $B \in ARI^{pol}_{al+sen*circconst}$ and $A = \Xi(B)$, then

(i) A is push-invariant,

(ii) A is alternal,

(iii) $swap(A)$ is circ-neutral up to addition of a constant-valued mould,

(iv) $A \in ARI^{\Delta}$.

Proof of (i). Écalle proved that $Ad_{ari}(pal)$ transforms push-invariant moulds to moulds satisfying the senary relation (81) (see [Ec] (3.58); indeed this is how the senary relation arose). Since B satisfies (79), $\tilde{B} = pari(B)$ satisfies (81), so $Ad_{ari}(invpal)(\tilde{B}) = \Xi(B) = A$ is push-invariant.

Proof of (ii). The subspace of alternal moulds ARI_{al} is closed under *ari* (cf. [SS]), so $exp_{ari}(ARI_{al})$ forms a subgroup of $GARI_{gari}$, which we denote by $GARI^{as}_{gari}$ (the superscript *as* stands for *symmetral*). The *pal* is known to be symmetral (cf. [Ec2], or in more detail [S2], Theorem 4.3.4). Thus, since $GARI^{as}_{gari}$ is a group, the *gari*-inverse mould *invpal* is also symmetral. Therefore the adjoint action $Ad_{ari}(invpal)$ on ARI restricts to an adjoint action on the Lie subalgebra ARI_{al} of alternal moulds. If B is alternal, then $pari(B)$ is alternal, and so $A = \Xi(B)$ is alternal. This completes the proof of (ii).

For the assertions (iii) and (iv), we will make use of Écalle's fundamental identity in the version (92) given in 4.2.4, with $N = pari(B)$ (recall that (92) is valid whenever $Ad_{ari}(invpal) \cdot N$ is push-invariant, which is the case for $pari(B)$ thanks to (i) above). The key point is that the operators $\overline{ganit}(poc)$ and $Ad_{\overline{ari}}(pil)$ on the left-hand side of (92) are better adapted to tracking the circ-neutrality and the denominators than the right-hand operator $Ad_{ari}(invpal)$ considered directly.

Proof of (iii). Let $b \in W_{\mathfrak{ltu}}$, and assume that b is of homogeneous degree n. Let $B = ma(b)$. Then by Proposition 26 and Proposition 13, $swap(B)$ is circ-constant, and even circ-neutral if n is even.

We need to show that $swap \cdot Ad_{ari}(invpal) \cdot pari(B)$ is *circ-neutral. To do this, we use (92) with $N = pari(B)$, and in fact show the result on the left-hand side, which is equal to

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari \cdot swap(B)$$

(noting that *pari* commutes with *swap*). We prove that this mould is *circ-neutral in three steps. First we show that the operator $\overline{ganit}(poc) \cdot pari$ changes a circ-constant mould into one that is circ-neutral (Proposition 30). Secondly, we show that the operator $Ad_{\overline{ari}}(invpil)$ preserves the property of circ-neutrality (Proposition 32). Finally, we show that if M is a mould that is not circ-constant but only *circ-constant, and if M_0 is the (unique) constant-valued mould such that $M + M_0$ is circ-constant, then

$$Ad_{\overline{ari}}(invpil) \cdot ganit(poc) \cdot pari(M) + M_0$$

is circ-neutral. Using (92), this will show that $swap \cdot Ad_{ari}(invpal) \cdot M$ is *circ-neutral.

Proposition 30. *Fix $n \geq 3$, and let $M \in \overline{ARI}$ be a circ-constant polynomial-valued mould of homogeneous degree n. Then $\overline{ganit}(poc) \cdot pari(M)$ is circ-neutral.*

Proof. Let $c = \big(M(v_1) \,|\, v_1^{n-1}\big)$, and let $N = pari(M)$, so that $N(v_1) = -cv_1^{n-1}$. Let $\mathbf{v} = (v_1, \ldots, v_r)$, and let $\mathbf{W_v}$ be the set of decompositions $d_{\mathbf{v}}$ of $\mathbf{v}$ into chunks $d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s$ as in (83). For any decomposition $d_{\mathbf{v}}$, we let its $\mathbf{b}$-part be the unordered set $\{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$, its $\mathbf{a}$-part the unordered set $\{\mathbf{a}_1, \ldots, \mathbf{a}_s\}$, and we write $l_{\mathbf{a}}$ for the number of letters in the $\mathbf{a}$-part, i.e. $l_{\mathbf{a}} = |\mathbf{a}_1| + \cdots + |\mathbf{a}_s|$.

Let

$$\mathbf{W} = \coprod_i \mathbf{W}_{\sigma_r^i(\mathbf{v})},$$

where the $\sigma_r^i(\mathbf{v})$ are the cyclic permutations of $\mathbf{v} = (v_1, \ldots, v_r)$, and let $\mathbf{W^b}$ denote the subset of decompositions in $\mathbf{W}$ having identical $\mathbf{b}$-part. The decompositions in $\mathbf{W}$ having identical $\mathbf{b}$-part to a given decomposition $d_{\mathbf{v}} \in \mathbf{W_v}$ are as follows: there is exactly one decomposition in $\mathbf{W}_{\sigma_r^{i-1}(\mathbf{v})}$ for each i such that v_i is one of the letters in the $\mathbf{a}$-part of $\mathbf{v}$, which is obtained from $d_{\mathbf{v}}$ by placing dividers between the same letters. For example, if $r = 5$ and $d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 = (v_1, v_2)(v_3)(v_4)(v_5)$ then the two other decompositions having the same $\mathbf{b}$-part $\{(v_3), (v_5)\}$ are given by $(v_2)(v_3)(v_4)(v_5)(v_1)$ and $(v_4)(v_5)(v_1, v_2)(v_3)$. Thus if $\mathbf{b}$ denotes the $\mathbf{b}$-part of a given decomposition $d_{\mathbf{v}}$ of $\mathbf{v} = (v_1, \ldots, v_r)$, then $\mathbf{W^b}$ contains exactly $l_{\mathbf{a}}$ decompositions, more precisely exactly one decomposition of each cyclic permutation $(v_i, \ldots, v_r, v_1, \ldots, v_{i-1})$ with v_i in the $\mathbf{a}$-part of $d_{\mathbf{v}}$.

Also, for each $n \geq 1$, let $\mathbf{W}_n^{\mathbf{a}}$ denote the set of monomials w of degree $n - l_{\mathbf{a}}$ in the letters lying in the $\mathbf{a}$-part of $d_{\mathbf{v}}$. For instance in the example above $d_{\mathbf{v}} = (v_1, v_2)(v_3)(v_4)(v_5)$, the $\mathbf{a}$-part is $\{(v_1, v_2), (v_4)\}$ and $\mathbf{W}_5^{\mathbf{a}}$ consists of all monomials of degree 2 in the three letters v_1, v_2, v_4, i.e. $\mathbf{W}_5^{\mathbf{a}} = \{v_1^2, v_2^2, v_4^2, v_1v_2, v_1v_4, v_2v_4\}$. Note in particular that $\mathbf{W}_n^{\mathbf{a}} = \{1\}$ when $|\mathbf{a}| = n$ and $\mathbf{W}_n^{\mathbf{a}} = \emptyset$ when $r > n$.

We now consider the mould $N = pari(M)$, of fixed homogeneous degree n, with $N(v_1) = -cv_1^{n-1}$. Since M is circ-constant for c, we have

$$N(v_1, \dots, v_r) + \cdots + N(v_r, v_1, ..., v_{r-1}) = (-1)^r c \sum_w w. \tag{94}$$

By the explicit formula (84), we have

$$\big(\overline{ganit}(poc) \cdot N\big)(v_1, \dots, v_r) = \sum_{\mathbf{W_v}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) N(\mathbf{a}_1 \cdots \mathbf{a}_s), \tag{95}$$

so adding up over the cyclic permutations of $\mathbf{v}$, we have

$$\begin{aligned}
\sum_{i=0}^{r-1} \big(\overline{ganit}(poc) \cdot N\big)\big(\sigma_r^i(\mathbf{v})\big) &= \sum_{\mathbf{W}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) N(\mathbf{a}_1 \cdots \mathbf{a}_s) \\
&= \sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}} \sum_{\mathbf{W^b}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) N(\mathbf{a}_1 \cdots \mathbf{a}_s) \\
&= \sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) \sum_{j=0}^{l_{\mathbf{a}}-1} N\big(\sigma_{l_{\mathbf{a}}}^j(\mathbf{a}_1 \cdots \mathbf{a}_s)\big) \\
&= (-1)^{l_{\mathbf{a}}} c \sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}} (-1)^{l_{\mathbf{a}}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) \sum_{w \in \mathbf{W}_n^{\mathbf{a}}} w
\end{aligned} \tag{96}$$

where the last equality follows from (94).

If $c = 0$, the expression (96) is trivially equal to zero in all depths $r > 1$, so we obtained the desired result that $ganit(poc) \cdot pari(M)$ is circ-neutral. In order to deal with the case where M is circ-constant for a value $c \neq 0$, we use a trick and subtract off a known mould that is also circ-constant for c.

Lemma 31. *For $n > 1$ and any constant c, let T_c^n be the homogeneous polynomial mould of degree n defined by*

$$T_c^n(v_1, \dots, v_r) = \frac{c}{r} P_r^n,$$

where P_n^r is the sum is over all monomials of degree $n-r$ in the variables $v_1, \dots, v_r$ for $1 \leq r \leq n$. Then T_c^n is circ-constant and $\overline{ganit}(poc) \cdot pari(T_c^n)$ is circ-neutral.

The proof of this lemma is annoyingly technical, so we have relegated it to Appendix 2. Consider the mould $N = M - T_c^n$. The mould N is circ-constant since M and T_c^n both are, but $N(v_1) = 0$, so by the result above, we know that $ganit(poc) \cdot pari(N)$ is circ-neutral. But Lemma 31 shows that $ganit(poc) \cdot pari(T_c^n)$ is circ-neutral, so the mould $ganit(poc) \cdot pari(M)$ is also circ-neutral, as desired. □

We now proceed to the second step, showing that the operator $Ad_{\overline{ari}}(invpil)$ preserves circ-neutrality.

Proposition 32. *If $M \in \overline{ARI}$ is circ-neutral then $Ad_{\overline{ari}}(invpil) \cdot M$ is also circ-neutral.*

Proof. By (88), we have

$$Ad_{\overline{ari}}(invpil) = exp\big(ad_{\overline{ari}}(-lopil)\big) = \sum_{n \geq 0} \frac{(-1)^n}{n} ad_{\overline{ari}}(lopil)^n. \tag{97}$$

The definition of *lopil* in (86) shows that *lopil* is trivially circ-neutral. Thus, since M is circ-neutral, $ad_{\overline{ari}}(lopil)\cdot M = \overline{ari}(lopil, M)$ is also circ-neutral by Lemma 21, and successively so are all the terms $ad_{\overline{ari}}(lopil)^n(M)$. Thus $Ad_{\overline{ari}}(invpil)\cdot M$ is circ-neutral. □

Finally, we now assume that $swap(B)$ is a *circ-neutral polynomial-valued mould in $\overline{ARI}$ of homogeneous degree n. Let B_0 be the (unique) constant-valued mould such that $swap(B)+B_0$ is circ-neutral. Then by Propositions 30 and 32, the mould

$$Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B+B_0)$$

is circ-neutral. This mould breaks up as the sum

$$Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(B) + Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(B_0),$$

but the operator $Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)$ preserves constant-valued moulds (cf. [S], Lemma 4.6.2 for the proof). Thus

$$Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B+B_0) = Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B)+B_0,$$

so

$$Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B) = swap\cdot\Xi(B)$$

is *circ-neutral, completing the proof of (iii).

Proof of (iv). We will again use the left-hand side of (92), this time to track the denominators that appear in the right-hand side. By (92), if B is a polynomial-valued mould satisfying the senary relation, and if $A = \Xi(B) = Ad_{ari}(invpal)\cdot pari(B)$, then A lies in ARI^{Δ} if and only if

$$swap\cdot Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)\cdot swap\big(pari(B)\big)\in ARI^{\Delta}. \tag{98}$$

We will prove that this is the case, by studying the denominators that are produced, first by applying $\overline{ganit}(poc)$ to a polynomial-valued mould, and then by applying $Ad_{\overline{ari}}(invpil)$. The first result is that the denominators introduced by applying $\overline{ganit}(poc)$ are at worst of the form $(v_1-v_2)\cdots(v_{r-1}-v_r)$.

Lemma 33. *Let $M\in\overline{ARI}^{pol}$. Then*

$$swap\cdot\overline{ganit}(poc)\cdot M\in ARI^{\Delta}.$$

Proof. The explicit expression for $\overline{ganit}(Q)$ given in (84) shows that the only denominators that can occur in $\overline{ganit}(poc)\cdot M$ come from the factors

$$poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s) \tag{99}$$

for all decompositions $d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_2$ of $\mathbf{v}=(v_1,\dots,v_r)$ into chunks as in (83), and

$$\lfloor\mathbf{b}_i = (v_k-v_{k-1}, v_{k+1}-v_{k-1},\dots,v_{k+l}-v_{k-1})$$

(for $k>1$) as in (85). Since *poc* is defined as in (90), the only factors that can appear in (99) are (v_l-v_{l-1}) where v_l is a letter in one of $\mathbf{b}_i$, and these factors appear in each term with multiplicity one. Since the sum ranges over all possible decompositions, the only letter of $\mathbf{v}$ that never belongs to any $\mathbf{b}_i$ is v_1; all the other factors (v_i-v_{i-1}) appear. Thus $(v_1-v_2)(v_2-v_3)\cdots(v_{r-1}-v_r)$ is a common denominator for all the terms in the sum defining $\overline{ganit}(poc)\cdot M$. The swap of this common denominator is equal to $u_2\cdots u_r$, so this term is a common denominator for $swap\cdot\overline{ganit}(poc)\cdot M$, which proves the lemma. □

Lemma 34. *Let $M, N \in \overline{ARI}_{*circneut}$ be two moulds such that $swap(M)$ and $swap(N)$ lie in ARI^{Δ}. Then $swap\big(\overline{ari}(M,N)\big)$ also lies in ARI^{Δ}.*

Proof. In Proposition A.1 of the Appendix of [BS], it is shown that if M and N are alternal moulds in $\overline{ARI}$ such that $swap(M)$ and $swap(N)$ lie in ARI^{Δ}, then $swap\big(\overline{ari}(M,N)\big)$ also lies in ARI^{Δ}. In fact, it is shown in Proposition A.2 of that appendix that alternal moulds M whose swap lies in ARI^{Δ} satisfy the following property: setting

$$\check{M}(v_1,\dots,v_r) = v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r\, M(v_1,\dots,v_r),$$

we have

$$\check{M}(0,v_2,\dots,v_r) = \check{M}(v_2,\dots,v_r,0). \tag{100}$$

In fact, the proof that $swap\big(\overline{ari}(M,N)\big)$ lies in ARI^{Δ} does not use the full alternality of M and N, but only (100). Therefore, the same proof goes through when M and N are *circ-neutral moulds such that $swap(M)$ and $swap(N)$ lie in ARI^{Δ}, as long as we check that every *circ-neutral mould M such that $swap(M) \in ARI^{\Delta}$ satisfies (100).

To check this, let M be such a mould; by additivity, we may assume that M is concentrated in a single depth $r > 1$. This means that there is a constant C_M such that

$$M(v_1,\dots,v_r) + M(v_2,\dots,v_r,v_1) + \cdots + M(v_r,v_1,\dots,v_{r-1}) = C_M,$$

which we can also write as

$$\frac{\check{M}(v_1,\dots,v_r)}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r} + \frac{\check{M}(v_2,\dots,v_r,v_1)}{v_2(v_2-v_3)\cdots(v_{r-1}-v_r)(v_r-v_1)v_1} +$$
$$\cdots + \frac{\check{M}(v_r,v_1,\dots,v_{r-1})}{v_r(v_r-v_1)\cdots(v_{r-2}-v_{r-1})v_{r-1}} = C_M$$

where the numerators are polynomials. If we multiply the entire equality by v_1 and set $v_1 = 0$, only the first two terms do not vanish, and they yield precisely the desired relation (100). □

Corollary 35. *If $N \in \overline{ARI}$ is a *circ-neutral mould such that $swap(N) \in ARI^{\Delta}$, then also*

$$swap \cdot Ad_{\overline{ari}}(invpil) \cdot N \in ARI^{\Delta}. \tag{101}$$

Proof. The lemma shows that $swap \cdot \overline{ari}(lopil, N) \in ARI^{\Delta}$ since the mould $lopil$ is circ-neutral and $swap \cdot lopil \in ARI^{\Delta}$ by (86). In fact, applying the lemma successively shows that $swap \cdot ad_{\overline{ari}}(lopil)^n(N) \in ARI^{\Delta}$ for all $n \geq 1$. Since $Ad_{\overline{ari}}(invpil) \cdot N$ is obtained by summing these terms by (97), we obtain (101). □

To conclude, we set $M = swap \cdot pari(B)$; then by Lemma 33 we have

$$swap \cdot \overline{ganit}(poc) \cdot swap \cdot pari(B) \in ARI^{\Delta}.$$

By Proposition 30 this mould is *circ-neutral, so we can apply Corollary 35 with $N = \overline{ganit}(poc) \cdot swap \cdot pari(B)$ to conclude that

$$swap \cdot Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot swap \cdot pari(B) \in ARI^{\Delta}.$$

Thus thus by (92) with $N = pari(B)$, we finally find that

$$Ad_{ari}(invpal) \cdot pari(B) = \Xi(B) \in ARI^{\Delta},$$

which completes the proof of (iv).

We have thus finished proving Proposition 29. Backtracking, this means we have completed the details of Step 3 of the proof of Theorem 23. Step 4, the final step in the proof, is very easy and was explained completely just before paragraph 4.2.1. Thus we have now completed the proof of Theorem 23, i.e. we have completed the construction of the injective Lie algebra morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$. $\square$

4.3. **Relations with elliptic Grothendieck-Teichmüller and double shuffle.** The final result in this paper is the proof of Theorem 8. In fact, this result is simply a consequence of putting together the results of the previous sections with known results. Indeed, the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{grt}}_{ell} & \hookrightarrow & \mathfrak{ds}_{ell} \\ \backslash & & / \\ & \mathfrak{oder}_2, & \end{array} \tag{102}$$

where $Ad_{ari}(invpal) : \mathfrak{ds} \to \mathfrak{ds}_{ell}$ is the right-hand vertical map is shown in [S3].

By (10), the injective map $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ is given by $b(x,y) \mapsto \hat{b} = b(z,-y)$, or more precisely to the derivation in $\mathfrak{krv}$ given by $a \mapsto \hat{b}$, $[a,b] \mapsto 0$.

If $b(x,y) \in \mathfrak{ds}$, then $b(x,-y)$ lies in $W_{\mathfrak{krv}}$ and $b(z,-y)$ lies in $V_{\mathfrak{krv}}$, so this map unpacks to

$$\mathfrak{ds} \overset{y \mapsto -y}{\to} W_{\mathfrak{krv}} \overset{x \mapsto z}{\to} V_{\mathfrak{krv}} \to \mathfrak{krv},$$

where the last map comes from (29). We can thus construct a commutative square

$$\begin{array}{ccc} \mathfrak{ds} & \to & \mathfrak{krv} \\ \downarrow & & \downarrow \\ \mathfrak{ds}_{ell} & \subset & \mathfrak{krv}_{ell} \end{array} \tag{103}$$

given in detail by

$$\begin{array}{ccc} \mathfrak{ds} & \overset{y\mapsto -y}{\to} & W_{\mathfrak{krv}} \simeq \mathfrak{krv} \\ ma \downarrow & & \downarrow ma \\ ARI^{pol}_{al*il} & \overset{pari}{\to} & ARI^{pol}_{al+sen*circconst} \\ Ad_{ari}(invpal) \downarrow & & \downarrow Ad_{ari}(invpal) \circ pari \\ ARI^{\Delta}_{al*al} & \subset & ARI^{\Delta}_{al+push*circneut} \\ \Delta \downarrow & & \downarrow \Delta \\ \Delta(ARI^{\Delta}_{al*al}) & \subset & \Delta(ARI^{\Delta}_{al+push*circneut}) \\ ma^{-1} \downarrow & & \downarrow ma^{-1} \\ \mathfrak{ds}_{ell} & \subset & \mathfrak{krv}_{ell}. \end{array}$$

The first line of this diagram comes from the injection $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ and the definition of $W_{\mathfrak{krv}}$. The second line is the direct mould translation of the top one, as the left-hand space is exactly $ma(\mathfrak{ds})$, the right-hand space is $ma(W_{\mathfrak{krv}})$ by (74), and the map $pari$ restricted to polynomials is nothing other than $y \mapsto -y$. The vertical morphism

$$Ad_{ari}(invpal) : ARI^{pol}_{al*il} \to ARI^{\Delta}_{al*al}$$

is proven in [S3], and the vertical morphism

$$Ad_{ari}(invpal) \circ pari : ARI^{pol}_{al+sen*circconst} \to ARI^{\Delta}_{al+push*circneut}$$

comes from Proposition 29. Since $pari$ is an involution, this proves that the horizontal injection in the third line of the diagram is nothing but an inclusion. Finally, the last line of the diagram comes from the definitions $\mathfrak{ds}_{ell} = \Delta(ARI^{\Delta}_{al*al})$ ([S3]) and $\mathfrak{krv}_{ell} = \Delta(ARI^{\Delta}_{al+push*circneut})$ by Definition 15.

This diagram shows that the diagram (102) above can be completed by the diagram (103) to the commutative diagram of Theorem 8.

5. Appendix 1: Some facts on moulds

In this appendix, we introduce some mould definitions used in some of our proofs, and give the proof of Lemma 21.

Let ARI be the vector space of moulds with constant term 0. There are three different Lie brackets that one can put on the space ARI. We begin by introducing the standard mould multiplication that Écalle denotes $mu(A, B)$:

$$mu(A, B)(u_1, \ldots, u_r) = \sum_{i=0}^{r} A(u_1, \ldots, u_i)B(u_{i+1}, \ldots, u_r).$$

The associated Lie bracket lu is defined by $lu(A, B) = mu(A, B) - mu(B, A)$. We write ARI_{lu} for ARI viewed as a Lie algebra for the lu-bracket. The identical formulas yield a multiplication and Lie algebra (also called mu and lu) on $\overline{ARI}$. If f and g are power series in Ass_C and $A = ma(f)$, $B = ma(g)$, then mu is a

mould translation of the usual non-commutative multiplication, and lu the usual Lie bracket:

$$mu(A,B) = ma(fg), \qquad lu(A,B) = ma\big([f,g]\big).$$

In order to define Écalle's ari-bracket, we first introduce three derivations of ARI_{lu} associated to a given mould $A \in ARI$. It is non-trivial to prove that these operators are actually derivations (cf. [S2], Prop. 2.2.1).

Definition 17. [Ec] Let $B \in ARI$. Then the derivation $amit(B)$ of ARI_{lu} is given by

$$\big(amit(B)\cdot A\big)(u_1,\dots,u_r) = \sum_{1\le i<j<r} A(u_1,\dots,u_i,u_{i+1}+\dots+u_{j+1},u_{j+2},\dots,u_r)B(u_{i+1},\dots,u_j),$$

and the derivation $anit(B)$ is given by

$$\big(anit(B)\cdot A\big)(u_1,\dots,u_r) = \sum_{1<i<j\le r} A(u_1,\dots,u_i+\dots+u_j,u_{j+1},\dots,u_r)B(u_{i+1},\dots,u_j).$$

We also have corresponding derivations $\overline{amit}(B)$ and $\overline{anit}(B)$ of $\overline{ARI}_{lu}$ for $B \in \overline{ARI}$, given by the formulas

$$\big(\overline{amit}(B)\cdot A\big)(v_1,\dots,v_r) = \sum_{1\le i<j<r} A(u_1,\dots,u_i,u_{j+1},\dots,u_r)B(u_{i+1}-u_{j+1},\dots,u_j-u_{j+1}),$$

$$\big(\overline{anit}(B)\cdot A\big)(v_1,\dots,v_r) = \sum_{1<i<j\le r} A(u_1,\dots,u_i,u_{j+1},\dots,u_r)B(u_{i+1}-u_i,\dots,u_j-u_i).$$

Finally, Écalle defines the derivation $arit(B)$ on ARI_{lu} by

$$arit(B) = amit(B) - anit(B),$$

and the ari-bracket on ARI by

$$ari(A,B) = arit(B)\cdot A - arit(A)\cdot B + lu(A,B),$$

as well as the derivation $\overline{arit}$ on $\overline{ARI}_{lu}$ and the bracket $\overline{ari}$ on $\overline{ARI}$ by the same formulas with overlines.

Remark. The definitions of $amit$, $anit$, $arit$ and ari are generalizations to all moulds of familiar derivations of Ass_C. Indeed, if $b, b' \in Ass_C$ and $A = ma(b)$, $B = ma(b')$, then

$$amit(B)\cdot A = ma\big(D^l_g(f)\big)$$

where D^l_g is defined by $x \mapsto 0$, $y \mapsto b'y$,

$$anit(B)\cdot A = ma\big(D^r_{b'}(b)\big)$$

where $D^r_{b'}$ is defined by $x \mapsto 0$, $y \mapsto yb'$, and thus

$$arit(B)\cdot A = ma\big(-d_{b'}(b)\big)$$

where d_g is the Ihara derivation $x \mapsto 0$, $y \mapsto [y,b']$ (see (5)), and

$$ari(A,B) = ma\big([b,b'] + d_b(b') - d_{b'}(b)\big) = ma\big(\{b,b'\}\big). \tag{104}$$

corresponds to the Ihara or Poisson Lie bracket (6) on $\mathfrak{lie}_C$. (See [S2], Corollary 3.3.4).

We now pass to the $Dari$-bracket, which is the Lie bracket on ARI obtained by transfer by the Δ-operator given in (56): it is given by

$$Dari(A,B) = \Delta\Big(ari\big(\Delta^{-1}(A), \Delta^{-1}(B)\big)\Big). \tag{105}$$

This means that Δ gives an isomorphism of Lie algebras

$$\Delta : ARI_{ari} \xrightarrow{\sim} ARI_{Dari}. \tag{106}$$

It is shown in [S3], Prop. 3.2.1 that we have a second definition for the $Dari$-bracket, which is more complicated but sometimes very useful in certain proofs. Let dar denote the mould operator defined by $dar(A)(u_1,\dots,u_r) = u_1\cdots u_r\, A(u_1,\dots,u_r)$. We begin by introducing, for each $A \in ARI$, an associated derivation $Darit(A)$ of ARI_{lu} by the following formula:

$$Darit(A) = dar \circ \Big(-arit\big(\Delta^{-1}(A)\big) + ad\big(\Delta^{-1}(A)\big)\Big) \circ dar^{-1}, \tag{107}$$

where $ad(A)\cdot B = lu(A,B)$. Then $Dari$ corresponds to the bracket of derivations, in the sense that

$$Dari(A,B) = Darit(A)\cdot B - Darit(B)\cdot A. \tag{108}$$

We are now armed to attack Lemma 21, whose statement we recall.

Lemma 21. *The space $\overline{ARI}_{circneut}$ of circ-neutral moulds $A \in \overline{ARI}$ forms a Lie algebra under the $\overline{ari}$-bracket.*

Proof. Let $A, B \in \overline{ARI}_{circneut}$. We need to show that

$$\sum_{i=1}^{r} \overline{ari}(A,B)(v_i,\dots,v_r,v_1,\dots,v_{i-1}) = 0,$$

where the formula for the $\overline{ari}$-bracket on $\overline{ARI}$ is given as in 4.1.3 by the expression

$$\begin{aligned}\overline{ari}(A,B) &= lu(A,B) + \overline{arit}(B)\cdot A - \overline{arit}(A)\cdot B\\ &= lu(A,B) + \overline{amit}(B)\cdot A - \overline{anit}(B)\cdot A - \overline{amit}(A)\cdot B + \overline{anit}(A)\cdot B.\end{aligned}$$

We will show that this expression is circ-neutral because in fact, each of the five terms in the sum is individually circ-neutral. Let us start by showing this for the first term, $lu(A,B)$.

Let σ denote the cyclic permutation of $\{1,\dots,r\}$ defined by

$$\sigma(i) = i+1 \ \text{ for } \ 1 \le i \le r-1, \quad \sigma(r) = 1.$$

By additivity, since the circ-neutrality property is depth-by-depth, we may assume that A is concentrated in depth s and B in depth t, with $s \le t$, $s+t = r$. In this simplifed situation, we have

$$lu(A,B)(v_1,\dots,v_r) = A(v_1,\dots,v_s)B(v_{s+1},\dots,v_r) - B(v_1,\dots,v_t)A(v_{t+1},\dots,v_r).$$

We have

$$\sum_{i=0}^{r-1} lu(A,B)(v_{\sigma^i(1)},\dots,v_{\sigma^i(r)})$$
$$= \sum_{i=0}^{r-1} A(v_{\sigma^i(1)},\dots,v_{\sigma^i(s)})B(v_{\sigma^i(s+1)},\dots,v_{\sigma^i(i)}) - B(v_{\sigma^i(1)},\dots,v_{\sigma^i(t)})A(v_{\sigma^i(t+1)},\dots,v_{\sigma^i(i-1)}),$$
$$= \sum_{i=0}^{r-1} A(v_{\sigma^i(1)},\dots,v_{\sigma^i(s)})B(v_{\sigma^i(s+1)},\dots,v_{\sigma^i(i)}) - A(v_{\sigma^{i+t}(1)},\dots,v_{\sigma^{i+t}(s)})B(v_{\sigma^{i+t}(s+1)},\dots,v_{\sigma^{i+t}(r)})$$
$$= 0$$

as the terms cancel out pairwise.

We now prove that the second term

$$\big(\overline{amit}(B)\cdot A\big)(v_1,\dots,v_r) = \sum_{i=1}^{s} A(v_1,\dots,v_{i-1},v_{i+t},\dots,v_r)B(v_i-v_{i+t},\dots,v_{i+t-1}-v_{i+t})$$

is circ-neutral. Fix $j \in \{1,\dots,s\}$ and consider the term

$$A(v_1,\dots,v_{j-1},v_{j+t},\dots,v_r)B(v_j-v_{j+t},\dots,v_{j+t-1}-v_{j+t}).$$

Thus for each of the other terms

$$A(v_1,\dots,v_{i-1},v_{i+t},\dots,v_r)B(v_i-v_{i+t},\dots,v_{i+t-1}-v_{i+t})$$

in the sum, with $i \in \{1,\dots,s\}$, there is exactly one cyclic permutation, namely σ^{j-i}, that maps this term to

$$A(v_{\sigma^{j-i}(1)},\dots,v_{\sigma^{j-i}(i-1)},v_{\sigma^{j-i}(i+t},\dots,v_{\sigma^{j-i}(r)})B(v_j-v_{j+t},\dots,v_{j+t-1}-v_{j+t}).$$

For fixed $j \in \{1,\dots,s\}$, the values of $k = j-i \bmod s$ as i runs through $\{1,\dots,s\}$ are exactly $\{0,\dots,s-1\}$. Therefore, the coefficient of the term $B(v_j-v_{j+t},\dots,v_{j+t-1}-v_{j+t})$ in the sum of the cyclic permutations of $\overline{amit}(B)\cdot A$ is equal to

$$\sum_{k=0}^{s-1} A(v_{\sigma^k(1)},\dots,v_{\sigma^k(i-1)},v_{\sigma^k(i+t)},\dots,v_{\sigma^k(r)}),$$

which is zero due to the circ-neutrality of A. Thus the coefficient of the term $B(v_j-v_{j+t},\dots,v_{j+t-1}-v_{j+t})$ in the sum of the cyclic permutations of $\overline{amit}(B)\cdot A$ is zero, and this holds for $1 \le j \le s$, so the entire sum is 0, i.e. $\overline{amit}(B)\cdot A$ is circ-neutral. The proof of the circ-neutrality of the term $\overline{anit}(B)\cdot A$ is analogous. By exchanging A and B, this also shows that $\overline{amit}(A)\cdot B$ and $\overline{anit}(A)\cdot B$ are circ-neutral, which concludes the proof of the lemma. □

6. Appendix 2: Proof of Lemma 31

Let us recall the statement of the technical lemma 31.

Lemma 31. *For $n > 1$ and any constant $c \neq 0$, let T_c^n be the homogeneous polynomial mould of degree n defined by*

$$T_c^n(v_1,\dots,v_r) = \frac{c}{r}P_r^n,$$

where P_n^r is the sum is over all monomials of degree $n-r$ in the variables $v_1,\dots,v_r$. Then T_c^n is circ-constant and $\overline{ganit}(poc)\cdot pari(T_c^n)$ is circ-neutral.

Proof. The mould T_c^n is trivially circ-constant. Consider the proof of Proposition 30 with $M = T_c^n$, $N = pari(M)$. In order to show that $ganit(poc)\cdot N$ is circ-neutral, we start by recalling from the proof of Proposition 30 that for each $r > 1$, the cyclic sum

$$\overline{ganit}(poc)\cdot N(v_1,\dots,v_r) + \cdots + \overline{ganit}(poc)\cdot N(v_r,v_1,\dots,v_{r-1})$$

is equal to the expression (96). Thus we need to show that (96) is equal to zero for all $r > 1$. To show this, we will break up the sum

$$\sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}} (-1)^{l_{\mathbf{a}}} poc(\lfloor \mathbf{b}_1)\cdots poc(\lfloor \mathbf{b}_s) \sum_{w\in \mathbf{W}_n^{\mathbf{a}}} w \tag{109}$$

into parts that are simpler to express.

We need a little notation. Let us write $V_j = \{1,\dots,j\}$. If $B \subset V_r$, let P_B^d denote the sum of of all monomials of degree d in the variables $v_i \in B$. We write $P_B^0 = 1$ for all B.

We will break up the sum (109) as the sum of partial sums $S_0 + \cdots + S_r$, where S_0 is the term of (109) corresponding to the empty set and S_i is the sum over the $\mathbf{b}$-parts containing v_i but not $v_{i+1},\dots,v_r$, for each $i \in \{1,\dots,r\}$. Notice that the $\mathbf{b}$-parts containing v_i but not $v_{i+1},\dots,v_r$ are in bijection with the 2^{i-1} subsets $B \subset V_{i-1}$, by taking $\mathbf{b}$ to be the set $B' = B \cup \{v_i\}$, divided into chunks consisting of consecutive integers. For example, if $i = 5$ and $B = \{1,3\}$ then $B' = \{1,3,5\}$ and the associated $\mathbf{b}$-part is $(v_1)(v_3)(v_5)$; if $B = \{1,2\}$ then $B' = \{1,2,5\}$ and the $\mathbf{b}$-part is $(v_1,v_2)(v_5)$, and if $B = \{1,4\}$ then $B' = \{1,4,5\}$ and the $\mathbf{b}$-part is $(v_1)(v_4,v_5)$.

Setting $v_0 = v_r$, this means that $S_0 = P_{V_r}^{n-r}$ and for $1 \le i \le r$,

$$S_i = \sum_{B\subseteq\{1,\dots,i-1\}} \frac{(-1)^{r-|B'|} P_{V_r\setminus B'}^{n-r+|B'|}}{\prod_{j\in B'}(v_{j-1}-v_j)}. \tag{110}$$

In order to prove that (109) is zero, we will give simplified expressions for $S_1,\dots,S_{r-1}$ in Claim 1, a simplified expression for S_r in Claim 2, and then show how to sum them up in Claim 3.

Claim 1. For $1 \le i \le r-1$, we have $v_{i+1},\dots,v_r$. Let $v_0 = v_r$. Then we have

$$S_i = \frac{(-1)^{r-i} P_{\{i-1,i+1,\dots,r-1\}}^{n-r+i}}{(v_r-v_1)(v_1-v_2)\cdots(v_{i-1}-v_i)}. \tag{111}$$

Proof. We will use the following trivial but useful identity. Let $B \subsetneq V_r$, let $v_j \notin B$, and let $B' = B \cup \{v_j\}$. Then

$$P_{B'}^d = P_B^d + v_j P_{B'}^{d-1}. \tag{112}$$

Multiplying by the common denominator, we write (110) as

$$(-1)^{r-i}\prod_{j=1}^{i}(v_{j-1}-v_j)S_i = \sum_{B\subseteq V_{i-1}} (-1)^{i-|B'|} \prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j) P_{V_r\setminus B'}^{n-r+|B'|}. \tag{113}$$

We will show below that for each $1 \le k \le i-1$, the right-hand side of (113) is equal to the expression

$$Q_k = \sum_{v_1,\dots,v_k\notin B\subset V_{i-1}} (-1)^{i-|B|+k-1}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_k\}}(v_{j-1}-v_j)\Bigg) P_{V_{r-1}\setminus B'\cup\{v_1,\dots,v_{k-1}\}}^{n-r+|B'|+k}.$$

Taking $k = i-1$ in this expression, the sum Q_{i-1} reduces to the single term corresponding to $B = \emptyset$, which is just $P^{n-r+i}_{v_{i-1},v_{i+1},\dots,v_{r-1}}$. Thus by (113), we obtain

$$(-1)^{r-i}\prod_{j=1}^{i}(v_{i-1}-v_i)S_i = P^{n-r+i}_{v_{i-1},v_{i+1},\dots,v_{r-1}},$$

which proves (111).

Let us prove that the right-hand side of (113) is equal to Q_k for all $1 \le k \le i-1$. We will use induction on k. Let us do the base case $k = 1$ by showing that the right-hand side of (113) is equal to Q_1. We start by breaking the right-hand side of (113) into $v_1 \in B$ and $v_1 \notin B$, and compute

$$\sum_{v_1\in B}(-1)^{i-|B'|}\prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B'|}_{V_r\setminus B'}+\sum_{v_1\notin B}(-1)^{i-|B'|}\prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B'|}_{V_r\setminus B'}$$

then setting $C = B\setminus\{v_1\}$ in the first sum

$$=\sum_{v_1\notin C}(-1)^{i-|C|}\prod_{j\in V_{i-1}\setminus(C\cup\{v_1\})}(v_{j-1}-v_j)P^{n-r+|C|+2}_{V_r\setminus(C\cup\{v_1,v_i\})}+\sum_{v_1\notin B}(-1)^{i-|B|-1}\prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B'|}_{V_r\setminus B'}$$

then renaming $C = B$ and writing $B_1 = B\cup\{v_1\}$ and $B_1' = B\cup\{v_1,v_i\}$,

$$=\sum_{v_1\notin B}(-1)^{i-|B|}\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)P^{n-r+|B|+2}_{V_r\setminus B_1'}-\sum_{v_1\notin B}(-1)^{i-|B|}\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)\cdot(v_r-v_1)P^{n-r+|B'|}_{V_r\setminus B'}$$

$$=\sum_{v_1\notin B}(-1)^{i-|B|}\Bigg(\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)\Bigg)\Bigg(P^{n-r+|B'|+1}_{V_r\setminus B_1'}-(v_r-v_1)P^{n-r+|B'|}_{V_r\setminus B'}\Bigg)$$

$$=\sum_{v_1\notin B}(-1)^{i-|B|}\Bigg(\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)\Bigg)P^{n-r+|B'|+1}_{V_{r-1}\setminus B'},$$

which is exactly Q_1. The last equality is obtained by using (112) twice on the right-hand factor. This proves the base case $k = 1$.

Now fix $k < i-1$ and assume that $Q_1 = \cdots = Q_k$. We will show by the same method that $Q_k = Q_{k+1}$. We break the expression for Q_k into $v_k \in B$ and $v_k \notin B$, and compute

$$\sum_{v_1,\dots,v_k\notin B,v_{k+1}\in B}(-1)^{i-|B|+k-1}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_k\})}(v_{j-1}-v_j)\Bigg)P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k-1}\}))}$$

$$+\sum_{v_1,\dots,v_{k+1}\notin B}(-1)^{i-|B|+k-1}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_k\})}(v_{j-1}-v_j)\Bigg)P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k-1}\})}$$

then setting $C = B\setminus\{v_{k+1}\}$, $C_{k+1} = C\cup\{v_{k+1}\} = B$, $C_k' = C\cup\{v_{k+1},v_i\} = B'$ in the first sum,

$$=\sum_{v_1,\dots,v_{k+1}\notin C}(-1)^{i-|C|+k}\Bigg(\prod_{j\in V_{i-1}\setminus(C\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)P^{n-r+|C|+2}_{V_{r-1}\setminus(C'\cup\{v_1,\dots,v_{k-1},v_{k+1}\})}$$

$$+\sum_{v_1,\dots,v_{k+1}\notin B}(-1)^{i-|B|+k-1}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_k\})}(v_{j-1}-v_j)\Bigg)P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k-1}\})}$$

and renaming $C = B$,

$$= \sum_{v_1,\dots,v_{k+1}\notin B} (-1)^{i-|B|+k}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)P^{n-r+|B|+2}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k-1},v_{k+1}\})}$$

$$+ \sum_{v_1,\dots,v_{k+1}\notin B} (-1)^{i-|B|+k-1}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)(v_k-v_{k+1})P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k-1}\})}$$

$$= \sum_{v_1,\dots,v_{k+1}\notin B} (-1)^{i-|B|+k}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)\times$$

$$\Bigg(P^{n-r+|B|+2}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k-1},v_{k+1}\})} - (v_k-v_{k+1})P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k-1}\})}\Bigg)$$

$$= \sum_{v_1,\dots,v_{k+1}\notin B} (-1)^{i-|B|+k}\Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)P^{n-r+|B|+2}_{V_{r-1}\setminus(B'\cup\{v_1,\dots,v_{k}\})},$$

again by (112) applied twice. Thus $Q_1 = \cdots = Q_{k+1}$, so by induction, $Q_1 = Q_{i-1}$ and thus the right-hand side of (113) equals Q_{i-1} as desired. □

Unfortunately, the expression in Claim 1 for S_i does not work for $i = r$ due to the fact that when $i = r$ in (113), the subset $B = V_{r-1}$ occurs in the sum and the corresponding polynomial $P_{V_r\setminus B'} = 0$. It turns out that the expression for S_r is actually simpler.

Claim 2. The term S_r is given by

$$S_r = \frac{v_{r-1}^{n-1}}{(v_r-v_1)(v_1-v_2)\cdots(v_{r-2}-v_{r-1})}. \tag{114}$$

Proof. We will show that

$$\prod_{j=1}^{r-1}(v_{j-1}-v_j)S_r = v_{r-1}^{n-1} \tag{115}$$

starting from the equality (113) for $i = r$, slightly rewritten as

$$\prod_{j=1}^{r}(v_{j-1}-v_j)S_r = \sum_{B\subseteq V_{r-1}}(-1)^{r-|B|-1}\prod_{j\in V_{r-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B|+1}_{V_{r-1}\setminus B}. \tag{116}$$

Let us write $C = V_{r-1}\setminus B$; this becomes

$$\prod_{j=1}^{r}(v_{j-1}-v_j)S_r = \sum_{B\subseteq V_{r-1}}(-1)^{r-|B|-1}\prod_{j\in C}(v_{j-1}-v_j)P^{n-r+|B|+1}_{C}. \tag{117}$$

We have $P_\emptyset = 0$, and we may as well sum over the subsets C, so it becomes

$$\prod_{j=1}^{r}(v_{j-1}-v_j)S_r = \sum_{\emptyset\neq C\subseteq V_{r-1}}(-1)^{|C|-1}\prod_{j\in C}(v_{j-1}-v_j)P^{n-|C|-1}_{C}. \tag{118}$$

We will prove the following formula, valid for $1 \le i \le r-1$ and $n \ge 1$:

$$R_i^n = \sum_{\emptyset\neq B\subseteq V_i}(-1)^{|B|-1}\prod_{j\in B}(v_{j-1}-v_j)P_B^{n-|B|} = (v_r-v_i)v_i^{n-1}, \tag{119}$$

where we set $P_B^0 = 1$ and $P_B^m = 0$ if $m < 0$.

This equality suffices to prove the desired result (115). Indeed, taking $i = r-1$, we see that R^n_{r-1} is equal to the right-hand side of (118), so

$$\prod_{j=1}^{r}(v_{j-1}-v_j)S_r = R^n_{r-1} = (v_r - v_{r-1})v_{r-1}^{n-1},$$

and canceling out the factor $(v_r - v_{r-1})$ from both sides yields (115).

Let us prove (119) by induction on i. When $i = 1$, we have $B = \{v_i\}$ and for all $n \geq 1$, we have $R^n_1 = (v_1 - v_r)v_1^{n-1}$, proving the base case. Assume (119) holds for $i-1$ for all $n \geq 1$. Fix n. We break R^n_i into the sum over B containing v_i and B not containing V_i, as follows:

$$\begin{aligned}
R^n_i &= \sum_{\emptyset \neq B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j\in B}(v_{j-1}-v_j)P_B^{n-|B|} \\
&\qquad - \sum_{B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j\in B}(v_{j-1}-v_j)(v_{i-1}-v_i)P_{B,v_i}^{n-|B|-1} \\
&= R^n_{i-1} - \sum_{B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j\in B}(v_{j-1}-v_j)(v_{i-1}-v_i)P_{B,v_i}^{n-|B|-1} \\
&= R^n_{i-1} + (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i) \sum_{\emptyset \neq B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j\in B}(v_{j-1}-v_j)P_{B,v_i}^{n-|B|-1}
\end{aligned}$$

The key point is that for B not containing v_i, we can write

$$P_{B,v_i}^{n-|B|-1} = P_B^{n-|B|-1} + v_i P_B^{n-|B|-2} + v_i^2 P_B^{n-|B|-3} + \cdots + v_i^{n-|B|-2}P_B^1 + v_i^{n-|B|-1}.$$

Using this, the equality becomes

$$\begin{aligned}
&= R^n_{i-1} + (v_{i-1}-v_i)v_i^{n-1} \\
&\qquad - (v_{i-1}-v_i) \sum_{\emptyset \neq B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j\in B}(v_{j-1}-v_j)\big(P_B^{n-|B|-1} + v_i P_B^{n-|B|-2} + \cdots + v_i^{n-|B|-1}\big) \\
&= R^n_{i-1} + (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2} v_i^k R^{n-k-1}_{i-1} \\
&= (v_r - v_{i-1})v_{i-1}^{n-1} + (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2} v_i^k (v_r - v_{i-1})v_{i-1}^{n-k-2}
\end{aligned}$$

$$\begin{aligned}
&= (v_r - v_{i-1})v_{i-1}^{n-1} + (v_{i-1}-v_i)v_i^{n-1} - (v_r - v_{i-1})(v_{i-1}-v_i)\sum_{k=0}^{n-2} v_i^k v_{i-1}^{n-k-2} \\
&= (v_r - v_{i-1})v_{i-1}^{n-1} + (v_{i-1}-v_i)v_i^{n-1} - (v_r - v_{i-1})(v_{i-1}^{n-1} - v_i^{n-1}) \\
&= (v_r - v_{i-1})v_i^{n-1} + (v_{i-1}-v_i)v_i^{n-1} \\
&= (v_r - v_i)v_i^{n-1}.
\end{aligned}$$

This proves (119) and thus completes the proof of Claim 2. □

We can now prove that the expression (109) is equal to zero by showing that $S_0 + \cdots + S_r = 0$.

Claim 3. We have $S_0 + \cdots + S_r = 0$.

Proof. The key point is the following computation of partial sums for $i < r$:

$$S_0 + \cdots + S_i = \frac{(-1)^{r-i} P^{n-r+i}_{v_i,\ldots,v_{r-1}}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)}. \tag{120}$$

We prove it by induction on i. The base case $i = 0$ is just given by the formula for S_0 (with $v_0 = v_r$). Assume (120) up to $i - 1$. Then

$$S_0 + \cdots + S_i = (S_0 + \cdots + S_{i-1}) + S_i \tag{121}$$

$$= \frac{(-1)^{r-i+1} P^{n-r+i-1}_{v_{i-1},\ldots,v_{r-1}}}{(v_r - v_1)\cdots(v_{i-2} - v_{i-1})} + \frac{(-1)^{r-i} P^{n-r+i}_{v_{i-1},v_{i+1},\ldots,v_{r-1}}}{(v_r - v_1)\cdots(v_{i-1} - v_i)}, \tag{122}$$

Using (112) and multiplying (121) by the denominator, we find

$$\begin{aligned}(-1)^{r-i}(v_r - v_1)\cdots(v_{i-1} - v_i)(S_0 + \cdots + S_i) &= P^{n-r+i}_{v_{i-1},v_{i+1},\ldots,v_{r-1}} - (v_{i-1} - v_i) P^{n-r+i-1}_{v_{i-1},\ldots,v_{r-1}} \\ &= P^{n-r+1}_{v_{i-1},v_{i+1},\ldots,v_{r-1}} + v_i P^{n-r+i-1}_{v_{i-1},\ldots,v_{r-1}} - v_{i-1} P^{n-r+i-1}_{v_{i-1},\ldots,v_{r-1}} \\ &= P^{n-r+1}_{v_{i-1},\ldots,v_{r-1}} - v_{i-1} P^{n-r+i-1}_{v_{i-1},\ldots,v_{r-1}} \\ &= P^{n-r+1}_{v_i,\ldots,v_{r-1}}.\end{aligned}$$

Now, taking this equality for $i = r - 1$ yields

$$S_0 + \cdots + S_{r-1} = \frac{-P^{n-1}_{v_{r-1}}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{r-2} - v_{r-1})} = \frac{-v^{n-1}_{r-1}}{(v_r - v_1)\cdots(v_{r-2} - v_{r-1})},$$

which is equal to $-S_r$ by Claim 2. This proves Claim 3. □

Since $S_0 + \cdots + S_r$ is equal to (109), we have finally shown that whatever the value of c, $\overline{ganit}(poc) \cdot N$ is circ-neutral, completing the proof of Proposition 30. □

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