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# A LARGE SIEVE INEQUALITY OF ELLIOTT-MONTGOMERY-VAUGHAN TYPE FOR MAASS FORMS ON $GL(n, \mathbb{R})$ WITH APPLICATIONS

YUK-KAM LAU, MING HO NG, EMMANUEL ROYER AND YINGNAN WANG

ABSTRACT. In this paper, we establish a large sieve inequality of Elliott-Montgomery-Vaughan type for Maass forms on  $GL(n, \mathbb{R})$  and explore three applications.

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## 1. INTRODUCTION

Elliott [2], [1], and Montgomery and Vaughan [11] independently developed some sort of large sieve inequalities to study Linnik's problem, which may yield a more general result than the classical Vinogradov's result, cf. [9]. This device, known as the large sieve inequalities of Elliott-Montgomery-Vaughan (EMV) type, was generalized to the setting of primitive holomorphic cusp forms on  $GL(2, \mathbb{R})$  and applied to obtain some statistical results on Hecke eigenvalues of primitive holomorphic cusp forms in [8]. Later, Wang [15] generalized the results to the case of Maass forms on  $GL(2, \mathbb{R})$ .

It is natural to ask for a generalization of large sieve inequalities of EMV type to Maass forms on  $GL(n, \mathbb{R})$  ( $n \geq 3$ ). There are two main difficulties: the first one is that for  $n \geq 3$  the Hecke relations for  $GL(n, \mathbb{R})$  are much more complicated than those of  $GL(2, \mathbb{R})$ , and the trace formula for  $GL(n, \mathbb{R})$  with  $n \geq 3$  is not as simple as the trace formula (say Kuznetsov's and Petersson's trace formulas) on  $GL(2, \mathbb{R})$ . Recently, Xiao and Xu [16], using Kuznetsov's trace formula and Hecke's relations, made a breakthrough and obtained a large sieve inequality of EMV type to Maass forms on  $GL(3, \mathbb{R})$ . Moreover, they also applied their large sieve inequality to get a statistical result of sign changes on the Hecke eigenvalues for  $GL(3, \mathbb{R})$ .

In this paper, we generalize the large sieve inequalities of EMV type to Maass forms on  $GL(n, \mathbb{R})$  for all  $n \geq 3$ , and the result is comparable to the case of automorphic forms on  $GL(2, \mathbb{R})$  (see [8, 15]). Our main tool is the automorphic Plancherel density theorem - a recent great progress due to Matz and Templier [10]. We remark the use of properties of (degenerated) Schur's polynomials instead of Hecke's relations to avoid the complicated calculations as in [16]. More precisely the (degenerated) Schur polynomial is employed to evaluate the main term when applying the truncated trace formula [6, Corollary 3.3] since the main term in [6, Corollary 3.3] is expressed in the form of orbital integral involving the (degenerated)

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Schur polynomial by the work of Matz and Templier [10]. Moreover, we apply our large sieve inequality - Theorem 1.1 on the  $GL(n, \mathbb{R})$  analogue of Linnik's problem, the sign change problems, and the Montgomery-Vaughan conjecture.

Let  $\mathcal{H}^{\natural} = \{\phi_j\}$  be an orthogonal basis consisting of Hecke-Maass cusp forms for  $SL(n, \mathbb{R})$ . Each  $\phi_j$  is associated with a Langlands parameter  $\mu_j \in \mathfrak{a}_{\mathbb{C}}^*/W$  where  $\mathfrak{a}_{\mathbb{C}}^* \cong \{z \in \mathbb{C}^n : \sum_i z_i = 0\}$  and  $W$  is the Weyl group of  $GL(n, \mathbb{R})$ . For  $t \geq 1$ , we let

$$(1.1) \quad \mathcal{H}_t := \{\phi_j \in \mathcal{H}^{\natural} : \|\mu_j\|_2 \leq t, \mu_j \in i\mathfrak{a}^*\}$$

where  $\|\cdot\|_2$  is the standard Euclidean norm, and  $i\mathfrak{a}^* \subset \mathfrak{a}_{\mathbb{C}}^*$  is isomorphic to  $i\mathbb{R}^{n-1}$ . It is known that  $|\mathcal{H}_t| \asymp t^d$  with  $d = n(n+1)/2$ .

Let  $A_{\phi}(m_1, m_2, \dots, m_{n-1})$  be the Fourier coefficient of  $\phi \in \mathcal{H}_t$ . In this paper, we normalize each  $\phi \in \mathcal{H}_t$  such that

$$A_{\phi}(1, 1, \dots, 1) = 1.$$

It is well known that

$$A_{\phi}(m_1, m_2, \dots, m_{n-1}) = \overline{A_{\phi}(m_{n-1}, m_{n-2}, \dots, m_1)}.$$

Moreover, for any  $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$  and any prime  $p$ ,

$$(1.2) \quad \begin{aligned} A_{\phi}(p^{\kappa}) &:= A_{\phi}(p^{\kappa_1}, p^{\kappa_2}, \dots, p^{\kappa_{n-1}}) \\ &= S_{\kappa}(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \dots, \alpha_{\phi,n}(p)) \end{aligned}$$

where  $S_{\kappa}$  is the (degenerate) Schur polynomial (see Section 2 for definition and refer to [3] or [7] for a detailed exposition) and  $\alpha_{\phi}(p) := (\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \dots, \alpha_{\phi,n}(p))$  is (a representative of) the Satake parameter associated to  $\phi$  at  $p$ . Every Satake parameter  $\alpha_{\phi}(p)$  satisfies  $\prod_{i=1}^n \alpha_{\phi,i}(p) = 1$  and

$$\alpha_{\phi,1}(p) + \dots + \alpha_{\phi,n}(p) = A_{\phi}(p, 1, \dots, 1).$$

Put  $\kappa^{\iota} = (\kappa_{n-1}, \dots, \kappa_1)$  if  $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ . Then we have

$$A_{\phi}(p^{\kappa^{\iota}}) = A_{\phi}(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}) = \overline{A_{\phi}(p^{\kappa})},$$

and  $A_{\phi}(p^{\kappa}) \in \mathbb{R}$  if  $\kappa = \kappa^{\iota}$ .

Notation: For  $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ , we denote  $\|\kappa\| := \sum_{j=1}^{n-1} (n-j)\kappa_j$  and  $|\kappa| = \sum_{j=1}^{n-1} \kappa_j$ .

**Theorem 1.1.** *Let  $0 \neq \kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ . Let  $j \geq 1$  be any integer and let  $\{b_p\}_p$  be a sequence of complex numbers indexed by prime numbers such that  $|b_p| \leq B$  for some constant  $B > 0$  and for all primes  $p$ . Then*

$$(1.3) \quad \begin{aligned} &\frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \left| \sum_{P < p \leq Q} b_p \frac{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} \\ &\ll t^{-1/2} \left( \frac{BC_{\kappa} Q^L \|\kappa\|}{\log P} \right)^{2j} + \left( \frac{(BC_{\kappa})^2 j}{P \log P} \right)^j \left\{ 1 + \left( \frac{40j \log P}{P} \right)^{j/3} \right\} \end{aligned}$$

holds uniformly for

$$B > 0, \quad j \geq 1, \quad 2 \leq P < Q \leq 2P,$$

where  $L$  is a positive constant,  $1 \leq C_{\kappa} := 10(1 + |\kappa|)^{n^2-n}$  and the implied constant depends on  $\kappa$  only.

Let  $q \geq 2$  be an integer and let  $\chi$  be a non principal Dirichlet character modulo  $q$ . Then the evaluation of the least integer  $n_\chi$  among all positive integers  $n$  for which  $\chi(n) \neq 0, 1$  is referred as Linnik's problem. One generalization formulated to Maass forms on  $GL(n, \mathbb{R})$  is the evaluation of the smallest integer  $n$  for which

$$A_{\phi_1}(n, 1, \dots, 1) \neq A_{\phi_2}(n, 1, \dots, 1),$$

where  $\phi_1 \neq \phi_2$ . We denote this smallest integer by  $n_{1,2}$ . The first application uses Theorem 1.1 to investigate an analogue of Linnik's problem.

Suppose  $\mathcal{P}$  is a set of prime numbers of positive density in the sense that

$$(1.4) \quad \sum_{\substack{z < p \leq 2z \\ p \in \mathcal{P}}} \frac{1}{p} \geq \frac{\Delta}{\log z} \quad (\forall z \geq z_0),$$

with some fixed constants  $\Delta > 0$  and  $z_0 > 0$ .

**Theorem 1.2.** *Let  $0 \neq \kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$  and assume the set  $\mathcal{P}$  (of primes) satisfies (1.4). Let  $\Lambda = \{\lambda(p)\}_p$  be a fixed complex sequence indexed by prime numbers. For any  $\delta > 0$ , there is a positive constant  $C = C(\delta, \kappa, \mathcal{P})$  such that the number of  $\phi \in \mathcal{H}_t$  satisfying*

$$A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = \lambda(p) \quad \text{for } p \in \mathcal{P}, \quad \text{and} \quad \delta \log t < p \leq 2\delta \log t$$

is bounded by

$$\ll t^d e^{-C \log t / \log_2 t}$$

where  $\log_r$  is the  $r$ -fold iterated logarithm. The implied constant depends at most on  $\delta, \kappa$  and  $\mathcal{P}$ .

**Remark 1.1.** Refer to [8] and [15] for the case of  $GL(2, \mathbb{R})$ .

**Corollary 1.1.** *Let  $\phi_0 \in \mathcal{H}_t$  be fixed and let  $\mathcal{P}$  be as stated in Theorem 1.2. Let  $\ell \in \mathbb{N}$  and let  $\delta > 0$  be any number. Then there is a positive constant  $C = C(\delta, \ell, \mathcal{P})$  such that the number of  $\phi \in \mathcal{H}_t$  satisfying*

$$A_\phi(p^\ell, 1, \dots, 1) = A_{\phi_0}(p^\ell, 1, \dots, 1) \quad \text{for } p \in \mathcal{P}, \quad \text{and} \quad \delta \log t < p \leq 2\delta \log t$$

is bounded by

$$\ll_{\delta, \ell, \mathcal{P}} t^d e^{-C \log t / \log_2 t}.$$

By the corollary, we see that for any fixed  $\phi_1$ , the number of  $\phi_2 \in \mathcal{H}_t$  for which

$$n_{1,2} \ll \log t$$

does not hold is

$$\ll |\mathcal{H}_t| e^{-C \log t / \log_2 t}.$$

The second application concerns the sign changes of Maass forms on  $GL(n, \mathbb{R})$ . In the case of  $GL(2, \mathbb{R})$ , there are fruitful results (for example, see [5], [12], [13]). In the case of  $GL(3, \mathbb{R})$ , Steiger [14] proved that there is a positive proportion of Hecke-Maass forms  $\phi$  with positive real part of  $A_\phi(p, 1)$  for a fixed prime  $p$  and Xiao and Xu [16] gave a statistical result on the signs of  $A_\phi(p^{\kappa_1}, p^{\kappa_2}) + A_\phi(p^{\kappa_2}, p^{\kappa_1})$ . Applying Theorem 1.1, we obtain the following result.

**Theorem 1.3.** *Let  $0 \neq \kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ . Let  $\{\varepsilon_p\}_{p \in \mathcal{P}}$  be a sequence of real numbers with  $\varepsilon_p \in \{\pm 1\}$  where the set of primes  $\mathcal{P}$  satisfies (1.4). For any*

$\delta > 0$ , there is a positive constant  $C = C(\delta, \kappa, \mathcal{P})$  such that the number of  $\phi \in \mathcal{H}_t$  satisfying

$$\varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})) > 0$$

for  $p \in \mathcal{P}$  and  $\delta \log t < p \leq 2\delta \log t$  is bounded by

$$\ll t^d e^{-C \log t / \log_2 t}.$$

The implied constant depends at most on  $\delta, \kappa$  and  $\mathcal{P}$ .

**Remark 1.2.** Refer to [8] and [15] for the case of  $GL(2, \mathbb{R})$  and to [16] for  $GL(3, \mathbb{R})$ .

The size of  $L(1, f)$  for  $L$ -functions over a family of  $f$  has attracted much interest. For  $\phi \in \mathcal{H}_t$ , its associated  $L$ -function is defined as

$$L(s, \phi) := \sum_{m \geq 1} A_\phi(m, 1, \dots, 1) m^{-s},$$

for  $\Re s > (n+1)/2$ , and factors into the Euler product

$$L(s, \phi) = \prod_p \prod_{i=1}^n (1 - \alpha_{\phi, i}(p) p^{-s})^{-1}$$

where  $\alpha_{\phi, i}(p)$ ,  $1 \leq i \leq n$ , are the Satake parameters. It is well known that  $L(s, \phi)$  can be analytically continued to the whole complex plane.

Recently, Lau and Wang [7] proved that for all  $\phi \in \mathcal{H}_t$ , we have

$$\{1 + o(1)\} (2B_n^- \log_2 t)^{-A_n^-} \leq |L(1, \phi)| \leq \{1 + o(1)\} (2B_n^+ \log_2 t)^{A_n^+}.$$

under the Generalized Ramanujan Conjecture and the Generalized Riemann Hypothesis. Here  $B_n^\pm$  are the positive constants in [7, Lemma 5.3] and

$$A_n^+ := n \quad \text{and} \quad A_n^- := \begin{cases} n & \text{if } n \text{ is even,} \\ n \cos(\pi/n) & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, Lau and Wang [7] also proved that there exist  $\phi^\pm \in \mathcal{H}_t$  such that

$$|L(1, \phi^-)| \leq \{1 + o(1)\} (B_n^- \log_2 t)^{-A_n^-}, \quad |L(1, \phi^+)| \geq \{1 + o(1)\} (B_n^+ \log_2 t)^{A_n^+}.$$

The proportion of such exceptional  $\phi^\pm$  in  $\mathcal{H}_t$  is at least  $\exp(-(\log t)/(\log_2 t)^{3+o(1)})$ . In fact, alongside the Montgomery-Vaughan conjecture (cf. Conjecture 1 in [4]), the proportion of  $\phi^\pm$  in  $\mathcal{H}_T$  satisfying  $|L(1, \phi^\pm)|^{\pm 1} \geq (B_n^\pm \log_2 T)^{A_n^\pm}$  is predicted to be  $> \exp(-C \log t / \log_2 t)$  and  $< \exp(-c \log t / \log_2 t)$  respectively for some constants  $C > c > 0$ .

Theorem 1.1 gives an upper bound towards the Montgomery-Vaughan conjecture. Define

$$F_t^+(s) = \frac{1}{|\mathcal{H}_t|} \sum_{\substack{\phi \in \mathcal{H}_t \\ |L(1, \phi)| > (B_n^+ s)^{A_n^+}}} 1$$

and

$$F_t^-(s) = \frac{1}{|\mathcal{H}_t|} \sum_{\substack{\phi \in \mathcal{H}_t \\ |L(1, \phi)| < (B_n^- s)^{A_n^-}}} 1.$$

**Theorem 1.4.** *For any  $\varepsilon > 0$ , there are two positive constants  $c = c(\varepsilon)$  and  $t_0 = t_0(\varepsilon)$  such that*

$$F_t^\pm(\log_2 t + r) \leq \exp \left( -c(|r| + 1) \frac{\log t}{(\log_2 t)(\log_3 t)(\log_4 t)} \right)$$

for  $t \geq t_0$  and  $\log \varepsilon \leq r \leq (9 - \varepsilon) \log_2 t$ .

**Remark 1.3.** *Refer to [8] and [15] for the case of  $GL(2, \mathbb{R})$ .*

## 2. PRELIMINARIES

The Fourier coefficients  $A_\phi(p^\kappa)$  can be expressed in terms of the (degenerate) Schur polynomials and Satake parameters as in (1.2). The degenerate Schur polynomial is defined as

$$(2.1) \quad S_\kappa(x_1, x_2, \dots, x_n) := \frac{\det \left( x_j^{\sum_{l=1}^{n-i} (\kappa_l + 1)} \right)_{1 \leq i, j \leq n}}{\det \left( x_j^{\sum_{l=1}^{n-i} 1} \right)_{1 \leq i, j \leq n}}$$

for  $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ . Matz and Templier established an automorphic equidistribution of the family  $\{A_\phi(p^\kappa) : \phi \in \mathcal{H}^1\}$  – the vertical Sato-Tate law for Hecke-Maass forms. Now we explain a consequence of the equidistribution result.

Let  $\mathfrak{S}_n$  be the symmetric group and let

$$T_0 = \left\{ (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \in (S^1)^n : e^{i(\theta_1 + \theta_2 + \dots + \theta_n)} = 1 \right\}.$$

We define two measures  $d\mu_{\text{ST}}$  and  $d\mu_p$  on  $T_0/\mathfrak{S}_n$  whose integration formulas (over  $[0, 2\pi]^{n-1}$ ) are given by

$$d\mu_{\text{ST}} = \frac{1}{n!} \frac{1}{(2\pi)^{n-1}} \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_{n-1}$$

and

$$d\mu_p = \frac{1}{n!} \prod_{i=2}^n \frac{1 - p^{-i}}{1 - p^{-1}} \cdot \prod_{1 \leq i < j \leq n} \left| \frac{e^{i\theta_i} - p^{-1}e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}} \right|^{-2} \cdot \frac{1}{(2\pi)^{n-1}} d\theta_1 \cdots d\theta_{n-1}.$$

Define  $S_\kappa(1, \dots, 1)$  by taking  $x_i \rightarrow 1$ . By [7, Lemma 7.1 (2)], we have for any  $X \geq 1$  and  $\kappa \in \mathbb{N}_0^{n-1}$ ,

$$(2.2) \quad \max_{|x_i| \leq X, \forall i} |S_\kappa(x_1, \dots, x_n)| \leq X^{\|\kappa\|} S_\kappa(1, \dots, 1) \leq X^{\|\kappa\|} (1 + |\kappa|)^{n^2 - n}.$$

A consequence of Matz and Templier's work on the vertical Sato-Tate is the following, cf. [6, Corollary 3.3].

**Lemma 2.1.** *Let  $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ ,  $\mathcal{H}_t$  and  $A_\phi(p^\kappa) = A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})$  be defined as above, cf. (1.1), (1.2) and (2.1). Then for any  $\ell, m \in \mathbb{N}$ ,*

$$\begin{aligned} & \frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^{u_p} \overline{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}^{v_p} \\ &= \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p + O \left( t^{-1/2} \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} (c_\kappa p^{L\|\kappa\|})^{u_p + v_p} \right) \end{aligned}$$

where  $L$  is a positive constant,  $1 \leq c_\kappa := (1 + |\kappa|)^{n^2 - n}$ .

The product of two Schur polynomials  $S_\kappa$  and  $S_{\kappa'}$  may be evaluated with the Littlewood-Richardson rule:

$$(2.3) \quad S_\kappa S_{\kappa'} = S_\kappa \cdot S_{\kappa'} = \sum_{\xi} d_{\kappa\kappa'}^{\xi} S_{\xi}$$

where the  $d_{\kappa\kappa'}^{\xi}$ 's are nonnegative integers and the summation runs over  $\xi \in \mathbb{N}_0^{n-1}$  satisfying  $\|\xi\| \leq \|\kappa\| + \|\kappa'\|$  and  $\|\xi\| \equiv \|\kappa\| + \|\kappa'\| \pmod{n}$ . (Recall that  $\|\kappa\| := \sum_i (n-i)\kappa_i$ .) Moreover  $\{S_\kappa\}$  form an orthonormal set under the inner product induced by the measure  $d\mu_{\text{ST}}$ ,

$$(2.4) \quad \langle S_\kappa, S_{\kappa'} \rangle = \int_{[0, 2\pi]^{n-1}} S_\kappa(\underline{\theta}) \overline{S_{\kappa'}(\underline{\theta})} d\mu_{\text{ST}} = \delta_{\kappa=\kappa'}.$$

As well, by [7, Proposition 7.4 (1)] we have

$$\int_{T_0/\mathfrak{S}_n} S_\kappa d\mu_p = \prod_{i=1}^{n-1} (1 - p^{-i}) \cdot \sum_{\eta \in \mathbb{N}_0^{n-1}} d_{\kappa\eta}^\eta \cdot p^{-\|\eta\|}$$

where the sum over  $\eta$  is supported on  $|\eta| \geq \|\kappa\|/n$  and with (2.2) and (2.4),

$$0 \leq d_{\kappa\eta}^\eta = \int_{T_0/\mathfrak{S}_n} S_\kappa |S_\eta|^2 d\mu_{\text{ST}} \leq (1 + |\kappa|)^{(n^2-n)}.$$

Consequently, for  $\|\kappa\| \neq 0$  we have

$$\begin{aligned} & \left| \int_{T_0/\mathfrak{S}_n} S_\kappa d\mu_p \right| \\ & \leq (1 + |\kappa|)^{(n^2-n)} \prod_{i=1}^{n-1} (1 - p^{-i}) \max_{\sum_i \eta_i = \lceil \frac{\|\kappa\|}{n} \rceil} \left( \prod_{1 \leq i \leq n-1} \sum_{\ell \geq \eta_i} p^{-i\ell} \right) \\ & \leq (1 + |\kappa|)^{(n^2-n)} \max_{|\eta| = \lceil \frac{\|\kappa\|}{n} \rceil} p^{-\|\eta\|} \\ (2.5) \quad & \leq (1 + |\kappa|)^{(n^2-n)} p^{-1} \end{aligned}$$

where  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ . (Note  $|\eta| \leq \|\eta\|$ .)

By Cauchy-Schwarz's inequality and (2.2), we have

$$\begin{aligned} \sum_{\|\xi\| \leq n|\kappa|} (d_{\kappa\kappa'}^{\xi})^2 &= \langle S_\kappa S_{\kappa'}, S_\kappa S_{\kappa'} \rangle \\ &\leq S_\kappa(1, \dots, 1) S_{\kappa'}(1, \dots, 1) \langle S_\kappa, S_\kappa \rangle^{1/2} \langle S_{\kappa'}, S_{\kappa'} \rangle^{1/2} \\ (2.6) \quad &\leq ((1 + |\kappa|)(1 + |\kappa'|))^{n^2-n} = c_\kappa c_{\kappa'}. \end{aligned}$$

We need an arithmetic function and a result from [8].

**Lemma 2.2.** *Let  $2 \leq P < Q \leq 2P$ ,  $j \geq 1$  and  $n \geq 1$ . Define*

$$a_j(n) = a_j(n; P, Q) = |\{(p_1, \dots, p_j) : p_1 \cdots p_j = n, P < p_1, \dots, p_j \leq Q\}|.$$

For any  $d > 0$ ,  $\sum_n a_j(n) d^{\Omega(n)} / n \ll (3d / \log P)^j$ ; moreover,

$$\begin{aligned} \sum_n a_j(n^2) \frac{d^{\Omega(n)}}{n^2} &\leq \delta_{2|j} \left( \frac{3dj}{P \log P} \right)^{j/2} \\ \sum_n^{\natural} a_j(n) \frac{d^{\Omega(n)}}{n} &\leq \left( \frac{12d^2 j}{P \log P} \right)^{j/2} \left\{ 1 + \left( \frac{j \log P}{54P} \right)^{j/6} \right\} \\ \sum_m^{\flat} \sum_{(m,n)=1}^{\natural} a_j(mn) \frac{d^{\Omega(mn)}}{m^2 n} &\leq \left( \frac{48d^2 j}{P \log P} \right)^{j/2} \left\{ 1 + \left( \frac{20j \log P}{P} \right)^{j/6} \right\} \end{aligned}$$

where  $\Omega(n)$  counts the number of (not necessarily distinct) prime divisors,  $\delta_{2|j} = 1$  if  $2|j$  or 0 otherwise,  $\sum^{\flat}$  and  $\sum^{\natural}$  run over squarefree and squarefull integers respectively.

### 3. PROOF OF THEOREM 1.1

Let  $a_j(\cdot)$  be defined as in Lemma 2.2. Squaring out, we have

$$\begin{aligned} &\left| \sum_{P < p \leq Q} b_p \frac{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} \\ &= \left( \sum_{P < p \leq Q} b_p \frac{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right)^j \left( \sum_{P < p \leq Q} \overline{b_p} \frac{\overline{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}}{p} \right)^j \\ &= \left( \sum_{P^j < \ell \leq Q^j} a_j(\ell) \frac{b_{\ell}}{\ell} \prod_{p^u \parallel \ell} A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^u \right) \\ &\quad \times \left( \sum_{P^j < m \leq Q^j} a_j(m) \frac{\overline{b_m}}{m} \prod_{q^v \parallel m} \overline{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}^v \right) \\ &= \sum_{P^j < \ell, m \leq Q^j} a_j(\ell) a_j(m) \frac{b_{\ell} \overline{b_m}}{\ell m} \\ &\quad \times \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^{u_p} \overline{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}^{v_p} \end{aligned}$$

Averaging over  $\phi \in \mathcal{H}_t$ , it follows from Lemma 2.1 that

$$\begin{aligned} (3.1) \quad &\frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} \dots \\ &= \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} \int_{T_0/\mathfrak{S}_n} S_{\kappa}^{u_p} \overline{S_{\kappa}^{v_p}} d\mu_p + O\left(t^{-1/2} (c_{\kappa} Q^{L\|\kappa\|})^{2j}\right). \end{aligned}$$

Thus the left side of (1.3) can be expressed as follows:

$$(3.2) \quad \frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \left| \sum_{P < p \leq Q} b_p \frac{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} = M + E.$$



The error term  $E$  is

$$\begin{aligned}
 &\ll t^{-1/2} (c_\kappa Q^{L\|\kappa\|})^{2j} \sum_{P^j < \ell, m \leq Q^j} a_j(\ell) a_j(m) \frac{b_\ell \overline{b_m}}{\ell m} \\
 (3.3) \quad &\ll t^{-1/2} (c_\kappa Q^{L\|\kappa\|})^{2j} \left( \frac{3B}{\log P} \right)^{2j}
 \end{aligned}$$

by Lemma 2.2.

Next we evaluate the main term

$$(3.4) \quad M = \sum_{P^j < \ell, m \leq Q^j} a_j(\ell) a_j(m) \frac{b_\ell \overline{b_m}}{\ell m} \prod_{p^{u_p} \|\ell, p^{v_p} \|m} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p.$$

Write  $\ell = \ell_1 \ell'$  and  $m = m_1 m'$  such that  $\ell_1 m_1$  is squarefree,  $\ell' m'$  is squarefull and  $(\ell_1 m_1, \ell' m') = 1$ .<sup>1</sup> (Note  $\ell_1 m_1 = 1$  when  $\ell m$  is squarefull.) Set  $h = \ell_1 m_1$  and  $r = \ell' m'$ . We split the product over prime divisors of  $\ell m$  in (3.4) into a product of two pieces over prime divisors of  $\ell_1 m_1$  and  $\ell' m'$  respectively:

$$\prod_{p^{u_p} \|\ell, p^{v_p} \|m} \dots = \prod_{p^{u_p} \|\ell_1, p^{v_p} \|m_1} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p \prod_{p^{u_p} \|\ell', p^{v_p} \|m'} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p.$$

Inside the second product, we invoke the trivial bound (2.2) and for the first product, (as  $\ell_1 m_1$  is squarefree) we have  $u_p + v_p = 1$  and thus apply (2.5). This leads to

$$\begin{aligned}
 &\left| \prod_{p^{u_p} \|\ell, p^{v_p} \|m} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p \right| \\
 &\leq (1 + |\kappa|)^{\Omega(\ell' m')(n^2 - n)} \prod_{p^{u_p} \|\ell_1, p^{v_p} \|m_1} (1 + |\kappa|)^{n^2 - n} p^{-1} \\
 &\leq (1 + |\kappa|)^{2j(n^2 - n)} h^{-1},
 \end{aligned}$$

and

$$\begin{aligned}
 |M| &\leq (1 + |\kappa|)^{2j(n^2 - n)} \sum_{P^j < \ell_1 \ell', m_1 m' \leq Q^j} a_j(\ell_1 \ell') a_j(m_1 m') \frac{|b_{\ell_1 \ell'} \overline{b_{m_1 m'}}|}{(\ell_1 m_1)^2 \ell' m'} \\
 &\leq (1 + |\kappa|)^{2j(n^2 - n)} B^{2j} \sum_h^b \sum_r^h \frac{1}{h^2 r} \sum_{\substack{P^j < \ell_1 \ell', m_1 m' \leq Q^j \\ \ell_1 m_1 = h, \ell' m' = r}} a_j(\ell_1 \ell') a_j(m_1 m') \\
 &\leq (1 + |\kappa|)^{2j(n^2 - n)} B^{2j} \sum_h^b \sum_r^h \frac{a_{2j}(hr)}{h^2 r} \\
 &\ll (1 + |\kappa|)^{2j(n^2 - n)} B^{2j} \left( \frac{96j}{P \log P} \right)^j \left\{ 1 + \left( \frac{40j \log P}{P} \right)^{j/3} \right\}
 \end{aligned}$$

where the implied constant is independent of  $j$ .

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<sup>1</sup>The decomposition is unique. Assume  $\ell = \ell_1 \ell' = \ell_2 \ell''$  and  $m = m_1 m' = m_2 m''$  are two such decomposition. Every positive integer decomposes uniquely into a product of a squarefree integer and a squarefull integer. From  $(\ell_1 m_1)(\ell' m') = (\ell_2 m_2)(\ell'' m'')$ , we get (\*):  $\ell_1 m_1 = \ell_2 m_2$  and  $\ell' m' = \ell'' m''$ . As  $\ell_1 m_1$  is squarefree, we have  $(\ell_1, m_1) = 1$ ; with  $(\ell_1 m_1, \ell' m') = 1$ , we infer  $(\ell_1, m) = 1$ . So  $(\ell_1, m_2) = 1$ , and  $(\ell_2, m_1) = 1$  by symmetry. By (\*),  $\ell_1 = \ell_2$  and  $m_1 = m_2$ .

## 4. PROOF OF THEOREM 1.2

Let  $\delta \log t \leq P \leq (\log t)^{10}$  and write  $\mathcal{P}_P := \mathcal{P} \cap (P, 2P]$ . Define

$$E(t; P) = \{\phi \in \mathcal{H}_t : A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = \lambda(p) \text{ for } p \in \mathcal{P} \cap (P, 2P]\}.$$

As the Ramanujan Conjecture is open, we consider the exceptional set over each prime

$$\mathcal{E}(t, p) = \left\{ \phi \in \mathcal{H}_t : \log \max_{1 \leq i \leq n} |\alpha_{\phi, i}(p)| > 1 \right\}$$

whose size is under control. Indeed, analogously to Sarnak's bound for the  $GL(2)$  Maass forms, we have  $|\mathcal{E}(t, p)| \ll t^{d-c_0/\log p}$  where  $c_0 > 0$  is a constant, cf. [7, Theorem 7.3]. Hence

$$\left| \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t, p) \right| \ll t^{d-c'/\log P}$$

for some constant  $c'$ . Set

$$E^*(t; P) = E(t; P) \setminus \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t, p).$$

It remains to prove that

$$E^*(t; P) \ll_{\delta, \kappa, \mathcal{P}} t^d e^{-C \log t / \log_2 t}$$

for all  $t > T_0$ , where  $T_0 = T_0(\delta, \kappa, \mathcal{P})$  is a sufficiently large number. We may assume

$$(4.1) \quad |\lambda(p)| < e^{\|\kappa\|} (1 + |\kappa|)^{n^2-n}$$

for all  $P \leq p \leq 2P$ ; otherwise the set  $E(t; P)$  is empty by (2.2). Suppose  $j \in \mathbb{N}$  is chosen such that

$$(4.2) \quad j \leq \frac{P}{40 \log P}.$$

We apply Theorem 1.1 with

$$(4.3) \quad b_p = \begin{cases} \overline{\lambda(p)} & \text{if } p \in \mathcal{P}_P, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\overline{\lambda(p)} A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2$  for  $\phi \in E^*(t; P)$ , it follows that

$$(4.4) \quad \begin{aligned} & \sum_{\phi \in E^*(t; P)} \left| \sum_{p \in \mathcal{P}_P} \frac{|A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2}{p} \right|^{2j} \\ & \leq \sum_{\phi \in \mathcal{H}_t} \left| \sum_{P < p \leq 2P} b_p \frac{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} \\ & \ll t^d \left( \frac{(B_1 C_\kappa)^2 j}{P \log P} \right)^j + t^{d-1/2} \left( \frac{B_1 C_\kappa Q^{L\|\kappa\|}}{\log P} \right)^{2j} \end{aligned}$$

where  $B_1 = e^{\|\kappa\|} (1 + |\kappa|)^{n^2-n}$  and  $Q = 2P$ , in view of (4.1).

The size of  $|A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2$  is about 1 on average. To see it, we firstly deduce from (1.2) and (2.3) that

$$(4.5) \quad \begin{aligned} |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2 &= A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}) \\ &= 1 + \sum_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} d_{\kappa\kappa^\iota}^\xi A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}}) \end{aligned}$$

where  $\kappa^\iota = (\kappa_{n-1}, \dots, \kappa_1)$ . (Then  $\|\kappa^\iota\| = n|\kappa| - \|\kappa\|$ .)

Secondly, we exploit the oscillation among  $A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}})$  by Theorem 1.1 (again). For  $\xi = (\xi_1, \dots, \xi_{n-1})$  with  $1 \leq \|\xi\| \leq n|\kappa| = |n\kappa|$ , we define

$$E^\xi(t; P) = \left\{ \phi \in \mathcal{H}_t : \left| \sum_{\substack{P < p \leq 2P \\ p \in \mathcal{P}}} \frac{A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}})}{p} \right| \geq \frac{\Delta'}{\log P} \right\}$$

where  $\Delta' := \Delta/(2c_\kappa c_{\kappa^\iota}) < \Delta/2$ . Taking  $b_p = 1$  if  $p \in \mathcal{P}$  or 0 otherwise, we get from Theorem 1.1 with  $C_\xi \leq C_{n\kappa}$  that

$$(4.6) \quad |E^\xi(t; P)| \ll t^d \left( \frac{C_{n\kappa}^2 j \log P}{\Delta'^2 P} \right)^j + t^{d-1/2} \left( \frac{C_{n\kappa} Q^L \|\xi\|}{\Delta'} \right)^{2j}.$$

For  $\phi \in E^*(t; P) \setminus \bigcup_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} E^\xi(t; P)$ , the inner sum (over  $p$ ) in (4.4) is, by (4.5),

$$(4.7) \quad \geq \sum_{\substack{P < p \leq 2P \\ p \in \mathcal{P}}} \frac{1}{p} - \sum_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} d_{\kappa\kappa^\iota}^\xi \left| \sum_{\substack{P < p \leq 2P \\ p \in \mathcal{P}}} \frac{A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}})}{p} \right| \geq \frac{\Delta}{2 \log P}$$

Here we have applied that  $c_\kappa c_{\kappa^\iota} \Delta' \leq \Delta/2$  and

$$(4.8) \quad \sum_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} d_{\kappa\kappa^\iota}^\xi \leq \sum_{\|\xi\| \leq n|\kappa|} (d_{\kappa\kappa^\iota}^\xi)^2 \leq c_\kappa c_{\kappa^\iota}$$

by (2.6).

Applying the lower bound (4.7) to the left-hand side of (4.4), we thus infer

$$\begin{aligned} &\left( \frac{\Delta}{2 \log P} \right)^{2j} \left| E^*(t; P) \setminus \bigcup_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} E^\xi(t; P) \right| \\ &\ll t^d \left( \frac{(B_1 C_\kappa)^2 j}{P \log P} \right)^j + t^{d-1/2} \left( \frac{B_1 C_\kappa Q^L \|\kappa\|}{\log P} \right)^{2j} \end{aligned}$$

and, together with (4.6),

$$(4.9) \quad |E^*(t; P)| \ll t^d \left( \frac{(B_1 C_{n\kappa})^2 j \log P}{\Delta'^2 P} \right)^j + t^{d-1/2} \left( \frac{B_1 C_{n\kappa} Q^L \|\kappa\|}{\Delta'} \right)^{2j}.$$

Recall  $\delta \log t \leq P \leq (\log t)^{10}$ . Take

$$j = \left\lceil \Delta^* \frac{\log t}{\log P} \right\rceil$$

with

$$\Delta^* = \min \left( \frac{\delta}{40}, \frac{\delta \Delta'^2}{(2B_1 C_{n\kappa})^2}, \frac{1}{8L \|\kappa\|} \right).$$

Thus (4.2) is valid and the term inside the first bracket of (4.9) is bounded by  $1/4$ . Let  $T_0$  be large enough so that  $1 < j < \delta(\log t)/(\log_2 t)$  and the second term in the right-side of (4.9) is less than  $t^{d-1/6}$  whenever  $t > T_0$ . Then we conclude that

$$|E^*(t; P)| \ll t^d e^{-C \log t / \log_2 t}$$

for some constant  $C > 0$  depending on  $\delta, \kappa$  and  $\mathcal{P}$ . The proof of Theorem 1.2 is complete.

## 5. PROOF OF THEOREM 1.3

The method of proof is the same as Theorem 1.2, starting with the set

$$F(t; P) = \left\{ \phi \in \mathcal{H}_t : \varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})) > 0 \text{ for } p \in \mathcal{P}_P \right\}.$$

The task is to evaluate

$$F^*(t; P) = F(t; P) \setminus \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t, p).$$

Using the positivity of  $\varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}))$  for  $\phi \in F^*(t; P)$ , we have

$$\begin{aligned} & |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})|^2 \\ & \leq 2e^{\|\kappa\|} (1 + |\kappa|)^{n^2-n} \varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})) \end{aligned}$$

by (2.2), and the analogue of (4.5) follows from (2.3) and (2.4):

$$\begin{aligned} & |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})|^2 \\ & = 2A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}) \\ & \quad + A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^2 + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})^2 \\ & = 2(1 + \delta_{\kappa, \kappa^\iota}) + \sum_{\substack{\xi \neq 0 \\ \|\xi\| \leq 2n\|\kappa\|}} (d_{\kappa\kappa^\iota}^\xi + 2d_{\kappa\kappa^\iota}^\xi + d_{\kappa^\iota\kappa^\iota}^\xi) A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}}) \end{aligned}$$

where  $\delta_{\kappa, \kappa^\iota}$  if  $\kappa = \kappa^\iota$  or 0 otherwise, and  $\kappa^\iota = (\kappa_{n-1}, \dots, \kappa_1)$ .

## 6. PROOF OF THEOREM 1.4

Let  $\varepsilon \in (0, 10^{-10}]$  be fixed. We need a short Euler product approximation for a bulk of  $L(1, \phi)$ 's.

**Proposition 6.1.** *There are a constant  $c' > 0$  and a subset  $E^1(z)$  of  $\mathcal{H}_t$  such that*

$$L(1, \phi) = \left\{ 1 + O\left(\frac{1}{\log_2 t}\right) \right\} \prod_{p \leq z} \prod_{i=1}^n \left( 1 - \frac{\alpha_{\phi, i}(p)}{p} \right)^{-1}$$

*uniformly for  $\varepsilon \log t \leq z \leq (\log t)^{10}$  and all Maass forms  $\phi \in \mathcal{H}_t \setminus E^1(z)$ , where the implied constant in the  $O$ -term is absolute and*

$$|E^1(z)| = O_\varepsilon \left( t^d \exp \left( -c' \frac{\log t}{(\log_2 t)(\log_3 t)(\log_4 t)} \right) \right).$$

*Proof.* We follow the same approach as in the proof of [8, Proposition 8.1]. A crucial difference is without the Ramanujan bound now, and thus we exclude the forms outside the set

$$\mathcal{K}_t = \mathcal{K}_t(\eta) := \left\{ \phi \in \mathcal{H}_t : \log \max_{1 \leq i \leq n} |\alpha_{\phi,i}(p)| \leq 1/(\log_3 t)(\log_4 t), \forall p \leq (\log t)^{1/\eta} \right\}$$

where  $\eta > 0$  is any number. The size of the exceptional set, i.e.  $\mathcal{H}_t^- = \mathcal{H}_t \setminus \mathcal{K}_t$ , is small:

$$(6.1) \quad \mathcal{H}_t^- \ll t^d \exp \left( -c \frac{\eta \log t}{(\log_2 t)(\log_3 t)(\log_4 t)} \right)$$

for some constant  $c > 0$ , by [7, Theorem 7.3] (see also [7, (6.1)]). We work on  $\mathcal{K}_t$  with the argument in [8] to complete the proof.  $\square$

Now we prove Theorem 1.4. For  $\phi \in \mathcal{H}_t \setminus E^1(z)$ , we have

$$\begin{aligned} |L(1, \phi)| &\leq \left\{ 1 + O \left( \frac{1}{\log_2 t} \right) \right\} \prod_{p \leq z} \left( 1 - \frac{\alpha'}{p} \right)^{-n} \\ &\leq \left\{ 1 + O \left( \frac{1}{\log_2 t} \right) \right\} (e^\gamma \log z)^{\alpha' n} \\ &\leq \left\{ e^\gamma \left( (e^{\gamma(1-1/\alpha')} \log z)^{\alpha'} + C_0 (\log_2 t)^{\alpha'-1} \right) \right\}^n, \end{aligned}$$

where  $C_0$  is an absolute constant and  $\alpha' = \exp(1/(\log_3 t)(\log_4 t))$ . Taking

$$\begin{aligned} z &= e^{e^{-\gamma(1-1/\alpha')}(\log_2 t + r - C_0(\log_2 t)^{\alpha'-1})^{1/\alpha'}} \\ &= e^{(1+O((\log_4 t)^{-1})(\log_2 t + r - C_0(\log_2 t)^{\alpha'-1}))}, \end{aligned}$$

the proof is complete for  $F_t^+$ . The case of  $F_t^-$  is treated in the same fashion.

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