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**A LARGE SIEVE INEQUALITY OF
ELLIOTT-MONTGOMERY-VAUGHAN TYPE FOR MAASS
FORMS ON $GL(n, \mathbb{R})$ WITH APPLICATIONS**

YUK-KAM LAU, MING HO NG, EMMANUEL ROYER AND YINGNAN WANG

ABSTRACT. In this paper, we establish a large sieve inequality of Elliott-Montgomery-Vaughan type for Maass forms on $GL(n, \mathbb{R})$ and explore three applications.

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1. INTRODUCTION

Elliott [2], [1], and Montgomery and Vaughan [11] independently developed some sort of large sieve inequalities to study Linnik's problem, which may yield a more general result than the classical Vinogradov's result, cf. [9]. This device, known as the large sieve inequalities of Elliott-Montgomery-Vaughan (EMV) type, was generalized to the setting of primitive holomorphic cusp forms on $GL(2, \mathbb{R})$ and applied to obtain some statistical results on Hecke eigenvalues of primitive holomorphic cusp forms in [8]. Later, Wang [15] generalized the results to the case of Maass forms on $GL(2, \mathbb{R})$.

It is natural to ask for a generalization of large sieve inequalities of EMV type to Maass forms on $GL(n, \mathbb{R})$ ($n \geq 3$). There are two main difficulties: the first one is that for $n \geq 3$ the Hecke relations for $GL(n, \mathbb{R})$ are much more complicated than those of $GL(2, \mathbb{R})$, and the trace formula for $GL(n, \mathbb{R})$ with $n \geq 3$ is not as simple as the trace formula (say Kuznetsov's and Petersson's trace formulas) on $GL(2, \mathbb{R})$. Recently, Xiao and Xu [16], using Kuznetsov's trace formula and Hecke's relations, made a breakthrough and obtained a large sieve inequality of EMV type to Maass forms on $GL(3, \mathbb{R})$. Moreover, they also applied their large sieve inequality to get a statistical result of sign changes on the Hecke eigenvalues for $GL(3, \mathbb{R})$.

In this paper, we generalize the large sieve inequalities of EMV type to Maass forms on $GL(n, \mathbb{R})$ for all $n \geq 3$, and the result is comparable to the case of automorphic forms on $GL(2, \mathbb{R})$ (see [8, 15]). Our main tool is the automorphic Plancherel density theorem - a recent great progress due to Matz and Templier [10]. We remark the use of properties of (degenerated) Schur's polynomials instead of Hecke's relations to avoid the complicated calculations as in [16]. More precisely the (degenerated) Schur polynomial is employed to evaluate the main term when applying the truncated trace formula [6, Corollary 3.3] since the main term in [6, Corollary 3.3] is expressed in the form of orbital integral involving the (degenerated)

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Schur polynomial by the work of Matz and Templier [10]. Moreover, we apply our large sieve inequality - Theorem 1.1 on the $GL(n, \mathbb{R})$ analogue of Linnik's problem, the sign change problems, and the Montgomery-Vaughan conjecture.

Let $\mathcal{H}^{\mathfrak{h}} = \{\phi_j\}$ be an orthogonal basis consisting of Hecke-Maass cusp forms for $SL(n, \mathbb{R})$. Each ϕ_j is associated with a Langlands parameter $\mu_j \in \mathfrak{a}_{\mathbb{C}}^*/W$ where $\mathfrak{a}_{\mathbb{C}}^* \cong \{z \in \mathbb{C}^n : \sum_i z_i = 0\}$ and W is the Weyl group of $GL(n, \mathbb{R})$. For $t \geq 1$, we let

$$(1.1) \quad \mathcal{H}_t := \{\phi_j \in \mathcal{H}^{\mathfrak{h}} : \|\mu_j\|_2 \leq t, \mu_j \in \mathfrak{ia}^*\}$$

where $\|\cdot\|_2$ is the standard Euclidean norm, and $\mathfrak{ia}^* \subset \mathfrak{a}_{\mathbb{C}}^*$ is isomorphic to $\mathfrak{i}\mathbb{R}^{n-1}$. It is known that $|\mathcal{H}_t| \asymp t^d$ with $d = n(n+1)/2$.

Let $A_{\phi}(m_1, m_2, \dots, m_{n-1})$ be the Fourier coefficient of $\phi \in \mathcal{H}_t$. In this paper, we normalize each $\phi \in \mathcal{H}_t$ such that

$$A_{\phi}(1, 1, \dots, 1) = 1.$$

It is well known that

$$A_{\phi}(m_1, m_2, \dots, m_{n-1}) = \overline{A_{\phi}(m_{n-1}, m_{n-2}, \dots, m_1)}.$$

Moreover, for any $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ and any prime p ,

$$(1.2) \quad \begin{aligned} A_{\phi}(p^{\kappa}) &:= A_{\phi}(p^{\kappa_1}, p^{\kappa_2}, \dots, p^{\kappa_{n-1}}) \\ &= S_{\kappa}(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \dots, \alpha_{\phi,n}(p)) \end{aligned}$$

where S_{κ} is the (degenerate) Schur polynomial (see Section 2 for definition and refer to [3] or [7] for a detailed exposition) and $\alpha_{\phi}(p) := (\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \dots, \alpha_{\phi,n}(p))$ is (a representative of) the Satake parameter associated to ϕ at p . Every Satake parameter $\alpha_{\phi}(p)$ satisfies $\prod_{i=1}^n \alpha_{\phi,i}(p) = 1$ and

$$\alpha_{\phi,1}(p) + \dots + \alpha_{\phi,n}(p) = A_{\phi}(p, 1, \dots, 1).$$

Put $\kappa^{\iota} = (\kappa_{n-1}, \dots, \kappa_1)$ if $\kappa = (\kappa_1, \dots, \kappa_{n-1})$. Then we have

$$A_{\phi}(p^{\kappa^{\iota}}) = A_{\phi}(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}) = \overline{A_{\phi}(p^{\kappa})},$$

and $A_{\phi}(p^{\kappa}) \in \mathbb{R}$ if $\kappa = \kappa^{\iota}$.

Notation: For $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$, we denote $\|\kappa\| := \sum_{j=1}^{n-1} (n-j)\kappa_j$ and $|\kappa| = \sum_{j=1}^{n-1} \kappa_j$.

Theorem 1.1. *Let $0 \neq \kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$. Let $j \geq 1$ be any integer and let $\{b_p\}_p$ be a sequence of complex numbers indexed by prime numbers such that $|b_p| \leq B$ for some constant $B > 0$ and for all primes p . Then*

$$(1.3) \quad \begin{aligned} & \frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \left| \sum_{P < p \leq Q} b_p \frac{A_{\phi}(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} \\ & \ll t^{-1/2} \left(\frac{BC_{\kappa} Q^L \|\kappa\|}{\log P} \right)^{2j} + \left(\frac{(BC_{\kappa})^2 j}{P \log P} \right)^j \left\{ 1 + \left(\frac{40j \log P}{P} \right)^{j/3} \right\} \end{aligned}$$

holds uniformly for

$$B > 0, \quad j \geq 1, \quad 2 \leq P < Q \leq 2P,$$

where L is a positive constant, $1 \leq C_{\kappa} := 10(1 + |\kappa|)^{n^2-n}$ and the implied constant depends on κ only.

Let $q \geq 2$ be an integer and let χ be a non principal Dirichlet character modulo q . Then the evaluation of the least integer n_χ among all positive integers n for which $\chi(n) \neq 0, 1$ is referred as Linnik's problem. One generalization formulated to Maass forms on $GL(n, \mathbb{R})$ is the evaluation of the smallest integer n for which

$$A_{\phi_1}(n, 1, \dots, 1) \neq A_{\phi_2}(n, 1, \dots, 1),$$

where $\phi_1 \neq \phi_2$. We denote this smallest integer by $n_{1,2}$. The first application uses Theorem 1.1 to investigate an analogue of Linnik's problem.

Suppose \mathcal{P} is a set of prime numbers of positive density in the sense that

$$(1.4) \quad \sum_{\substack{z < p \leq 2z \\ p \in \mathcal{P}}} \frac{1}{p} \geq \frac{\Delta}{\log z} \quad (\forall z \geq z_0),$$

with some fixed constants $\Delta > 0$ and $z_0 > 0$.

Theorem 1.2. *Let $0 \neq \kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ and assume the set \mathcal{P} (of primes) satisfies (1.4). Let $\Lambda = \{\lambda(p)\}_p$ be a fixed complex sequence indexed by prime numbers. For any $\delta > 0$, there is a positive constant $C = C(\delta, \kappa, \mathcal{P})$ such that the number of $\phi \in \mathcal{H}_t$ satisfying*

$$A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = \lambda(p) \quad \text{for } p \in \mathcal{P}, \quad \text{and } \delta \log t < p \leq 2\delta \log t$$

is bounded by

$$\ll t^d e^{-C \log t / \log_2 t}$$

where \log_r is the r -fold iterated logarithm. The implied constant depends at most on δ, κ and \mathcal{P} .

Remark 1.1. *Refer to [8] and [15] for the case of $GL(2, \mathbb{R})$.*

Corollary 1.1. *Let $\phi_0 \in \mathcal{H}_t$ be fixed and let \mathcal{P} be as stated in Theorem 1.2. Let $\ell \in \mathbb{N}$ and let $\delta > 0$ be any number. Then there is a positive constant $C = C(\delta, \ell, \mathcal{P})$ such that the number of $\phi \in \mathcal{H}_t$ satisfying*

$$A_\phi(p^\ell, 1, \dots, 1) = A_{\phi_0}(p^\ell, 1, \dots, 1) \quad \text{for } p \in \mathcal{P}, \quad \text{and } \delta \log t < p \leq 2\delta \log t$$

is bounded by

$$\ll_{\delta, \ell, \mathcal{P}} t^d e^{-C \log t / \log_2 t}.$$

By the corollary, we see that for any fixed ϕ_1 , the number of $\phi_2 \in \mathcal{H}_t$ for which

$$n_{1,2} \ll \log t$$

does not hold is

$$\ll |\mathcal{H}_t| e^{-C \log t / \log_2 t}.$$

The second application concerns the sign changes of Maass forms on $GL(n, \mathbb{R})$. In the case of $GL(2, \mathbb{R})$, there are fruitful results (for example, see [5], [12], [13]). In the case of $GL(3, \mathbb{R})$, Steiger [14] proved that there is a positive proportion of Hecke-Maass forms ϕ with positive real part of $A_\phi(p, 1)$ for a fixed prime p and Xiao and Xu [16] gave a statistical result on the signs of $A_\phi(p^{\kappa_1}, p^{\kappa_2}) + A_\phi(p^{\kappa_2}, p^{\kappa_1})$. Applying Theorem 1.1, we obtain the following result.

Theorem 1.3. *Let $0 \neq \kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$. Let $\{\varepsilon_p\}_{p \in \mathcal{P}}$ be a sequence of real numbers with $\varepsilon_p \in \{\pm 1\}$ where the set of primes \mathcal{P} satisfies (1.4). For any*

$\delta > 0$, there is a positive constant $C = C(\delta, \kappa, \mathcal{P})$ such that the number of $\phi \in \mathcal{H}_t$ satisfying

$$\varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})) > 0$$

for $p \in \mathcal{P}$ and $\delta \log t < p \leq 2\delta \log t$ is bounded by

$$\ll t^d e^{-C \log t / \log_2 t}.$$

The implied constant depends at most on δ, κ and \mathcal{P} .

Remark 1.2. Refer to [8] and [15] for the case of $GL(2, \mathbb{R})$ and to [16] for $GL(3, \mathbb{R})$.

The size of $L(1, f)$ for L -functions over a family of f has attracted much interest. For $\phi \in \mathcal{H}_t$, its associated L -function is defined as

$$L(s, \phi) := \sum_{m \geq 1} A_\phi(m, 1, \dots, 1) m^{-s},$$

for $\Re s > (n+1)/2$, and factors into the Euler product

$$L(s, \phi) = \prod_p \prod_{i=1}^n (1 - \alpha_{\phi, i}(p) p^{-s})^{-1}$$

where $\alpha_{\phi, i}(p)$, $1 \leq i \leq n$, are the Satake parameters. It is well known that $L(s, \phi)$ can be analytically continued to the whole complex plane.

Recently, Lau and Wang [7] proved that for all $\phi \in \mathcal{H}_t$, we have

$$\{1 + o(1)\} (2B_n^- \log_2 t)^{-A_n^-} \leq |L(1, \phi)| \leq \{1 + o(1)\} (2B_n^+ \log_2 t)^{A_n^+}.$$

under the Generalized Ramanujan Conjecture and the Generalized Riemann Hypothesis. Here B_n^\pm are the positive constants in [7, Lemma 5.3] and

$$A_n^+ := n \quad \text{and} \quad A_n^- := \begin{cases} n & \text{if } n \text{ is even,} \\ n \cos(\pi/n) & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, Lau and Wang [7] also proved that there exist $\phi^\pm \in \mathcal{H}_t$ such that

$$|L(1, \phi^-)| \leq \{1 + o(1)\} (B_n^- \log_2 t)^{-A_n^-}, \quad |L(1, \phi^+)| \geq \{1 + o(1)\} (B_n^+ \log_2 t)^{A_n^+}.$$

The proportion of such exceptional ϕ^\pm in \mathcal{H}_t is at least $\exp(-(\log t)/(\log_2 t)^{3+o(1)})$. In fact, alongside the Montgomery-Vaughan conjecture (cf. Conjecture 1 in [4]), the proportion of ϕ^\pm in \mathcal{H}_T satisfying $|L(1, \phi^\pm)|^{\pm 1} \geq (B_n^\pm \log_2 T)^{A_n^\pm}$ is predicted to be $> \exp(-C \log t / \log_2 t)$ and $< \exp(-c \log t / \log_2 t)$ respectively for some constants $C > c > 0$.

Theorem 1.1 gives an upper bound towards the Montgomery-Vaughan conjecture. Define

$$F_t^+(s) = \frac{1}{|\mathcal{H}_t|} \sum_{\substack{\phi \in \mathcal{H}_t \\ |L(1, \phi)| > (B_n^+ s)^{A_n^+}}} 1$$

and

$$F_t^-(s) = \frac{1}{|\mathcal{H}_t|} \sum_{\substack{\phi \in \mathcal{H}_t \\ |L(1, \phi)| < (B_n^- s)^{A_n^-}}} 1.$$

Theorem 1.4. *For any $\varepsilon > 0$, there are two positive constants $c = c(\varepsilon)$ and $t_0 = t_0(\varepsilon)$ such that*

$$F_t^\pm(\log_2 t + r) \leq \exp\left(-c(|r| + 1) \frac{\log t}{(\log_2 t)(\log_3 t)(\log_4 t)}\right)$$

for $t \geq t_0$ and $\log \varepsilon \leq r \leq (9 - \varepsilon) \log_2 t$.

Remark 1.3. *Refer to [8] and [15] for the case of $GL(2, \mathbb{R})$.*

2. PRELIMINARIES

The Fourier coefficients $A_\phi(p^\kappa)$ can be expressed in terms of the (degenerate) Schur polynomials and Satake parameters as in (1.2). The degenerate Schur polynomial is defined as

$$(2.1) \quad S_\kappa(x_1, x_2, \dots, x_n) := \frac{\det\left(x_j^{\sum_{l=1}^{n-i} (\kappa_l + 1)}\right)_{1 \leq i, j \leq n}}{\det\left(x_j^{\sum_{l=1}^{n-i} 1}\right)_{1 \leq i, j \leq n}}$$

for $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$. Matz and Templier established an automorphic equidistribution of the family $\{A_\phi(p^\kappa) : \phi \in \mathcal{H}^i\}$ – the vertical Sato-Tate law for Hecke-Maass forms. Now we explain a consequence of the equidistribution result.

Let \mathfrak{S}_n be the symmetric group and let

$$T_0 = \left\{ (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \in (S^1)^n : e^{i(\theta_1 + \theta_2 + \dots + \theta_n)} = 1 \right\}.$$

We define two measures $d\mu_{\text{ST}}$ and $d\mu_p$ on T_0/\mathfrak{S}_n whose integration formulas (over $[0, 2\pi]^{n-1}$) are given by

$$d\mu_{\text{ST}} = \frac{1}{n!} \frac{1}{(2\pi)^{n-1}} \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_{n-1}$$

and

$$d\mu_p = \frac{1}{n!} \prod_{i=2}^n \frac{1-p^{-i}}{1-p^{-1}} \cdot \prod_{1 \leq i < j \leq n} \left| \frac{e^{i\theta_i} - p^{-1}e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}} \right|^{-2} \cdot \frac{1}{(2\pi)^{n-1}} d\theta_1 \cdots d\theta_{n-1}.$$

Define $S_\kappa(1, \dots, 1)$ by taking $x_i \rightarrow 1$. By [7, Lemma 7.1 (2)], we have for any $X \geq 1$ and $\kappa \in \mathbb{N}_0^{n-1}$,

$$(2.2) \quad \max_{|x_i| \leq X, \forall i} |S_\kappa(x_1, \dots, x_n)| \leq X^{|\kappa|} S_\kappa(1, \dots, 1) \leq X^{|\kappa|} (1 + |\kappa|)^{n^2 - n}.$$

A consequence of Matz and Templier's work on the vertical Sato-Tate is the following, cf. [6, Corollary 3.3].

Lemma 2.1. *Let $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$, \mathcal{H}_t and $A_\phi(p^\kappa) = A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})$ be defined as above, cf. (1.1), (1.2) and (2.1). Then for any $\ell, m \in \mathbb{N}$,*

$$\begin{aligned} & \frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \prod_{p^{u_p} \|\ell, p^{v_p} \|m} A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^{u_p} \overline{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^{v_p}} \\ &= \prod_{p^{u_p} \|\ell, p^{v_p} \|m} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p + O\left(t^{-1/2} \prod_{p^{u_p} \|\ell, p^{v_p} \|m} (c_\kappa p^{L\|\kappa\|})^{u_p + v_p}\right) \end{aligned}$$

where L is a positive constant, $1 \leq c_\kappa := (1 + |\kappa|)^{n^2 - n}$.

The product of two Schur polynomials S_κ and $S_{\kappa'}$ may be evaluated with the Littlewood-Richardson rule:

$$(2.3) \quad S_\kappa S_{\kappa'} = S_\kappa \cdot S_{\kappa'} = \sum_{\xi} d_{\kappa\kappa'}^{\xi} S_{\xi}$$

where the $d_{\kappa\kappa'}^{\xi}$'s are nonnegative integers and the summation runs over $\xi \in \mathbb{N}_0^{n-1}$ satisfying $\|\xi\| \leq \|\kappa\| + \|\kappa'\|$ and $\|\xi\| \equiv \|\kappa\| + \|\kappa'\| \pmod{n}$. (Recall that $\|\kappa\| := \sum_i (n-i)\kappa_i$.) Moreover $\{S_\kappa\}$ form an orthonormal set under the inner product induced by the measure $d\mu_{\text{ST}}$,

$$(2.4) \quad \langle S_\kappa, S_{\kappa'} \rangle = \int_{[0,2\pi]^{n-1}} S_\kappa(\underline{\theta}) \overline{S_{\kappa'}(\underline{\theta})} d\mu_{\text{ST}} = \delta_{\kappa=\kappa'}.$$

As well, by [7, Proposition 7.4 (1)] we have

$$\int_{T_0/\mathfrak{S}_n} S_\kappa d\mu_p = \prod_{i=1}^{n-1} (1-p^{-i}) \cdot \sum_{\eta \in \mathbb{N}_0^{n-1}} d_{\kappa\eta}^n \cdot p^{-\|\eta\|}$$

where the sum over η is supported on $|\eta| \geq \|\kappa\|/n$ and with (2.2) and (2.4),

$$0 \leq d_{\kappa\eta}^n = \int_{T_0/\mathfrak{S}_n} S_\kappa |S_\eta|^2 d\mu_{\text{ST}} \leq (1+|\kappa|)^{(n^2-n)}.$$

Consequently, for $\|\kappa\| \neq 0$ we have

$$(2.5) \quad \begin{aligned} & \left| \int_{T_0/\mathfrak{S}_n} S_\kappa d\mu_p \right| \\ & \leq (1+|\kappa|)^{(n^2-n)} \prod_{i=1}^{n-1} (1-p^{-i}) \max_{\sum_i \eta_i = \lceil \frac{\|\kappa\|}{n} \rceil} \left(\prod_{1 \leq i \leq n-1} \sum_{\ell \geq \eta_i} p^{-i\ell} \right) \\ & \leq (1+|\kappa|)^{(n^2-n)} \max_{|\eta| = \lceil \frac{\|\kappa\|}{n} \rceil} p^{-\|\eta\|} \\ & \leq (1+|\kappa|)^{(n^2-n)} p^{-1} \end{aligned}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . (Note $|\eta| \leq \|\eta\|$.)

By Cauchy-Schwarz's inequality and (2.2), we have

$$(2.6) \quad \begin{aligned} \sum_{\|\xi\| \leq n|\kappa|} (d_{\kappa\kappa'}^{\xi})^2 &= \langle S_\kappa S_{\kappa'}, S_\kappa S_{\kappa'} \rangle \\ &\leq S_\kappa(1, \dots, 1) S_{\kappa'}(1, \dots, 1) \langle S_\kappa, S_\kappa \rangle^{1/2} \langle S_{\kappa'}, S_{\kappa'} \rangle^{1/2} \\ &\leq ((1+|\kappa|)(1+|\kappa'|))^{n^2-n} = c_\kappa c_{\kappa'}. \end{aligned}$$

We need an arithmetic function and a result from [8].

Lemma 2.2. *Let $2 \leq P < Q \leq 2P$, $j \geq 1$ and $n \geq 1$. Define*

$$a_j(n) = a_j(n; P, Q) = |\{(p_1, \dots, p_j) : p_1 \cdots p_j = n, P < p_1, \dots, p_j \leq Q\}|.$$

For any $d > 0$, $\sum_n a_j(n) d^{\Omega(n)} / n \ll (3d / \log P)^j$; moreover,

$$\begin{aligned} \sum_n a_j(n^2) \frac{d^{\Omega(n)}}{n^2} &\leq \delta_{2|j} \left(\frac{3dj}{P \log P} \right)^{j/2} \\ \sum_n^{\natural} a_j(n) \frac{d^{\Omega(n)}}{n} &\leq \left(\frac{12d^2 j}{P \log P} \right)^{j/2} \left\{ 1 + \left(\frac{j \log P}{54P} \right)^{j/6} \right\} \\ \sum_m^{\flat} \sum_{(m,n)=1}^{\natural} a_j(mn) \frac{d^{\Omega(mn)}}{m^2 n} &\leq \left(\frac{48d^2 j}{P \log P} \right)^{j/2} \left\{ 1 + \left(\frac{20j \log P}{P} \right)^{j/6} \right\} \end{aligned}$$

where $\Omega(n)$ counts the number of (not necessarily distinct) prime divisors, $\delta_{2|j} = 1$ if $2|j$ or 0 otherwise, \sum^{\flat} and \sum^{\natural} run over squarefree and squarefull integers respectively.

3. PROOF OF THEOREM 1.1

Let $a_j(\cdot)$ be defined as in Lemma 2.2. Squaring out, we have

$$\begin{aligned} &\left| \sum_{P < p \leq Q} b_p \frac{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} \\ &= \left(\sum_{P < p \leq Q} b_p \frac{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right)^j \left(\sum_{P < p \leq Q} \overline{b_p} \frac{\overline{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}}{p} \right)^j \\ &= \left(\sum_{P^j < \ell \leq Q^j} a_j(\ell) \frac{b_\ell}{\ell} \prod_{p^u \parallel \ell} A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^u \right) \\ &\quad \times \left(\sum_{P^j < m \leq Q^j} a_j(m) \frac{\overline{b_m}}{m} \prod_{q^v \parallel m} \overline{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^v} \right) \\ &= \sum_{P^j < \ell, m \leq Q^j} a_j(\ell) a_j(m) \frac{b_\ell \overline{b_m}}{\ell m} \\ &\quad \times \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^{u_p} \overline{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^{v_p}} \end{aligned}$$

Averaging over $\phi \in \mathcal{H}_t$, it follows from Lemma 2.1 that

$$\begin{aligned} (3.1) \quad &\frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} \dots \\ &= \prod_{p^{u_p} \parallel \ell, p^{v_p} \parallel m} \int_{T_0 / \mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p + O\left(t^{-1/2} (c_\kappa Q^{L\|\kappa\|})^{2j}\right). \end{aligned}$$

Thus the left side of (1.3) can be expressed as follows:

$$(3.2) \quad \frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \left| \sum_{P < p \leq Q} b_p \frac{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} = M + E.$$

The error term E is

$$(3.3) \quad \begin{aligned} &\ll t^{-1/2} (c_\kappa Q^{L\|\kappa\|})^{2j} \sum_{P^j < \ell, m \leq Q^j} a_j(\ell) a_j(m) \frac{b_\ell \overline{b_m}}{\ell m} \\ &\ll t^{-1/2} (c_\kappa Q^{L\|\kappa\|})^{2j} \left(\frac{3B}{\log P} \right)^{2j} \end{aligned}$$

by Lemma 2.2.

Next we evaluate the main term

$$(3.4) \quad M = \sum_{P^j < \ell, m \leq Q^j} a_j(\ell) a_j(m) \frac{b_\ell \overline{b_m}}{\ell m} \prod_{p^{u_p} \|\ell, p^{v_p} \| m} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p.$$

Write $\ell = \ell_1 \ell'$ and $m = m_1 m'$ such that $\ell_1 m_1$ is squarefree, $\ell' m'$ is squarefull and $(\ell_1 m_1, \ell' m') = 1$.¹ (Note $\ell_1 m_1 = 1$ when ℓm is squarefull.) Set $h = \ell_1 m_1$ and $r = \ell' m'$. We split the product over prime divisors of ℓm in (3.4) into a product of two pieces over prime divisors of $\ell_1 m_1$ and $\ell' m'$ respectively:

$$\prod_{p^{u_p} \|\ell, p^{v_p} \| m} \dots = \prod_{p^{u_p} \|\ell_1, p^{v_p} \| m_1} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p \prod_{p^{u_p} \|\ell', p^{v_p} \| m'} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p.$$

Inside the second product, we invoke the trivial bound (2.2) and for the first product, (as $\ell_1 m_1$ is squarefree) we have $u_p + v_p = 1$ and thus apply (2.5). This leads to

$$\begin{aligned} &\left| \prod_{p^{u_p} \|\ell, p^{v_p} \| m} \int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p} \overline{S_\kappa^{v_p}} d\mu_p \right| \\ &\leq (1 + |\kappa|)^{\Omega(\ell' m')(n^2 - n)} \prod_{p^{u_p} \|\ell_1, p^{v_p} \| m_1} (1 + |\kappa|)^{n^2 - n} p^{-1} \\ &\leq (1 + |\kappa|)^{2j(n^2 - n)} h^{-1}, \end{aligned}$$

and

$$\begin{aligned} |M| &\leq (1 + |\kappa|)^{2j(n^2 - n)} \sum_{P^j < \ell_1 \ell', m_1 m' \leq Q^j} a_j(\ell_1 \ell') a_j(m_1 m') \frac{|b_{\ell_1 \ell'} \overline{b_{m_1 m'}}|}{(\ell_1 m_1)^2 \ell' m'} \\ &\leq (1 + |\kappa|)^{2j(n^2 - n)} B^{2j} \sum_h^b \sum_r^h \frac{1}{h^2 r} \sum_{\substack{P^j < \ell_1 \ell', m_1 m' \leq Q^j \\ \ell_1 m_1 = h, \ell' m' = r}} a_j(\ell_1 \ell') a_j(m_1 m') \\ &\leq (1 + |\kappa|)^{2j(n^2 - n)} B^{2j} \sum_h^b \sum_r^h \frac{a_{2j}(hr)}{h^2 r} \\ &\ll (1 + |\kappa|)^{2j(n^2 - n)} B^{2j} \left(\frac{96j}{P \log P} \right)^j \left\{ 1 + \left(\frac{40j \log P}{P} \right)^{j/3} \right\} \end{aligned}$$

where the implied constant is independent of j .

¹The decomposition is unique. Assume $\ell = \ell_1 \ell' = \ell_2 \ell''$ and $m = m_1 m' = m_2 m''$ are two such decomposition. Every positive integer decomposes uniquely into a product of a squarefree integer and a squarefull integer. From $(\ell_1 m_1)(\ell' m') = (\ell_2 m_2)(\ell'' m'')$, we get (*): $\ell_1 m_1 = \ell_2 m_2$ and $\ell' m' = \ell'' m''$. As $\ell_1 m_1$ is squarefree, we have $(\ell_1, m_1) = 1$; with $(\ell_1 m_1, \ell' m') = 1$, we infer $(\ell_1, m) = 1$. So $(\ell_1, m_2) = 1$, and $(\ell_2, m_1) = 1$ by symmetry. By (*), $\ell_1 = \ell_2$ and $m_1 = m_2$.

4. PROOF OF THEOREM 1.2

Let $\delta \log t \leq P \leq (\log t)^{10}$ and write $\mathcal{P}_P := \mathcal{P} \cap (P, 2P]$. Define

$$E(t; P) = \{\phi \in \mathcal{H}_t : A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = \lambda(p) \text{ for } p \in \mathcal{P} \cap (P, 2P]\}.$$

As the Ramanujan Conjecture is open, we consider the exceptional set over each prime

$$\mathcal{E}(t, p) = \left\{ \phi \in \mathcal{H}_t : \log \max_{1 \leq i \leq n} |\alpha_{\phi, i}(p)| > 1 \right\}$$

whose size is under control. Indeed, analogously to Sarnak's bound for the $GL(2)$ Maass forms, we have $|\mathcal{E}(t, p)| \ll t^{d-c_0/\log p}$ where $c_0 > 0$ is a constant, cf. [7, Theorem 7.3]. Hence

$$\left| \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t, p) \right| \ll t^{d-c'/\log P}$$

for some constant c' . Set

$$E^*(t; P) = E(t; P) \setminus \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t, p).$$

It remains to prove that

$$E^*(t; P) \ll_{\delta, \kappa, \mathcal{P}} t^d e^{-C \log t / \log_2 t}$$

for all $t > T_0$, where $T_0 = T_0(\delta, \kappa, \mathcal{P})$ is a sufficiently large number. We may assume

$$(4.1) \quad |\lambda(p)| < e^{\|\kappa\|} (1 + |\kappa|)^{n^2 - n}$$

for all $P \leq p \leq 2P$; otherwise the set $E(t; P)$ is empty by (2.2). Suppose $j \in \mathbb{N}$ is chosen such that

$$(4.2) \quad j \leq \frac{P}{40 \log P}.$$

We apply Theorem 1.1 with

$$(4.3) \quad b_p = \begin{cases} \overline{\lambda(p)} & \text{if } p \in \mathcal{P}_P, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\overline{\lambda(p)} A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2$ for $\phi \in E^*(t; P)$, it follows that

$$(4.4) \quad \begin{aligned} & \sum_{\phi \in E^*(t; P)} \left| \sum_{p \in \mathcal{P}_P} \frac{|A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2}{p} \right|^{2j} \\ & \leq \sum_{\phi \in \mathcal{H}_t} \left| \sum_{P < p \leq 2P} b_p \frac{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \right|^{2j} \\ & \ll t^d \left(\frac{(B_1 C_\kappa)^2 j}{P \log P} \right)^j + t^{d-1/2} \left(\frac{B_1 C_\kappa Q^{L\|\kappa\|}}{\log P} \right)^{2j} \end{aligned}$$

where $B_1 = e^{\|\kappa\|} (1 + |\kappa|)^{n^2 - n}$ and $Q = 2P$, in view of (4.1).

The size of $|A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2$ is about 1 on average. To see it, we firstly deduce from (1.2) and (2.3) that

$$(4.5) \quad \begin{aligned} |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2 &= A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}) \\ &= 1 + \sum_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} d_{\kappa\kappa'}^\xi A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}}) \end{aligned}$$

where $\kappa' = (\kappa_{n-1}, \dots, \kappa_1)$. (Then $\|\kappa'\| = n|\kappa| - \|\kappa\|$.)

Secondly, we exploit the oscillation among $A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}})$ by Theorem 1.1 (again). For $\xi = (\xi_1, \dots, \xi_{n-1})$ with $1 \leq \|\xi\| \leq n|\kappa| = |n\kappa|$, we define

$$E^\xi(t; P) = \left\{ \phi \in \mathcal{H}_t : \left| \sum_{\substack{P < p \leq 2P \\ p \in \mathcal{P}}} \frac{A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}})}{p} \right| \geq \frac{\Delta'}{\log P} \right\}$$

where $\Delta' := \Delta/(2c_\kappa c_{\kappa'}) < \Delta/2$. Taking $b_p = 1$ if $p \in \mathcal{P}$ or 0 otherwise, we get from Theorem 1.1 with $C_\xi \leq C_{n\kappa}$ that

$$(4.6) \quad |E^\xi(t; P)| \ll t^d \left(\frac{C_{n\kappa}^2 j \log P}{\Delta'^2 P} \right)^j + t^{d-1/2} \left(\frac{C_{n\kappa} Q^{L\|\xi\|}}{\Delta'} \right)^{2j}.$$

For $\phi \in E^*(t; P) \setminus \bigcup_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} E^\xi(t; P)$, the inner sum (over p) in (4.4) is, by (4.5),

$$(4.7) \quad \geq \sum_{\substack{P < p \leq 2P \\ p \in \mathcal{P}}} \frac{1}{p} - \sum_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} d_{\kappa\kappa'}^\xi \left| \sum_{\substack{P < p \leq 2P \\ p \in \mathcal{P}}} \frac{A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}})}{p} \right| \geq \frac{\Delta}{2 \log P}$$

Here we have applied that $c_\kappa c_{\kappa'} \Delta' \leq \Delta/2$ and

$$(4.8) \quad \sum_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} d_{\kappa\kappa'}^\xi \leq \sum_{\|\xi\| \leq n|\kappa|} (d_{\kappa\kappa'}^\xi)^2 \leq c_\kappa c_{\kappa'}$$

by (2.6).

Applying the lower bound (4.7) to the left-hand side of (4.4), we thus infer

$$\begin{aligned} &\left(\frac{\Delta}{2 \log P} \right)^{2j} \left| E^*(t; P) \setminus \bigcup_{\substack{\xi \neq \mathbf{0} \\ \|\xi\| \leq n|\kappa|}} E^\xi(t; P) \right| \\ &\ll t^d \left(\frac{(B_1 C_\kappa)^2 j}{P \log P} \right)^j + t^{d-1/2} \left(\frac{B_1 C_\kappa Q^{L\|\kappa\|}}{\log P} \right)^{2j} \end{aligned}$$

and, together with (4.6),

$$(4.9) \quad |E^*(t; P)| \ll t^d \left(\frac{(B_1 C_{n\kappa})^2 j \log P}{\Delta'^2 P} \right)^j + t^{d-1/2} \left(\frac{B_1 C_{n\kappa} Q^{L\|\kappa\|}}{\Delta'} \right)^{2j}.$$

Recall $\delta \log t \leq P \leq (\log t)^{10}$. Take

$$j = \left\lceil \Delta^* \frac{\log t}{\log P} \right\rceil$$

with

$$\Delta^* = \min \left(\frac{\delta}{40}, \frac{\delta \Delta'^2}{(2B_1 C_{n\kappa})^2}, \frac{1}{8L\|\kappa\|} \right).$$

Thus (4.2) is valid and the term inside the first bracket of (4.9) is bounded by $1/4$. Let T_0 be large enough so that $1 < j < \delta(\log t)/(\log_2 t)$ and the second term in the right-side of (4.9) is less than $t^{d-1/6}$ whenever $t > T_0$. Then we conclude that

$$|E^*(t; P)| \ll t^d e^{-C \log t / \log_2 t}$$

for some constant $C > 0$ depending on δ, κ and \mathcal{P} . The proof of Theorem 1.2 is complete.

5. PROOF OF THEOREM 1.3

The method of proof is the same as Theorem 1.2, starting with the set

$$F(t; P) = \{ \phi \in \mathcal{H}_t : \varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})) > 0 \text{ for } p \in \mathcal{P}_P \}.$$

The task is to evaluate

$$F^*(t; P) = F(t; P) \setminus \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t, p).$$

Using the positivity of $\varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}))$ for $\phi \in F^*(t; P)$, we have

$$\begin{aligned} & |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})|^2 \\ & \leq 2e^{\|\kappa\|} (1 + |\kappa|)^{n^2 - n} \varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})) \end{aligned}$$

by (2.2), and the analogue of (4.5) follows from (2.3) and (2.4):

$$\begin{aligned} & |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})|^2 \\ & = 2A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}) \\ & \quad + A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^2 + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})^2 \\ & = 2(1 + \delta_{\kappa, \kappa^\iota}) + \sum_{\substack{\xi \neq \mathbf{0} \\ |\xi| \leq 2n|\kappa|}} (d_{\kappa \kappa}^\xi + 2d_{\kappa \kappa^\iota}^\xi + d_{\kappa^\iota \kappa^\iota}^\xi) A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}}) \end{aligned}$$

where $\delta_{\kappa, \kappa^\iota}$ if $\kappa = \kappa^\iota$ or 0 otherwise, and $\kappa^\iota = (\kappa_{n-1}, \dots, \kappa_1)$.

6. PROOF OF THEOREM 1.4

Let $\varepsilon \in (0, 10^{-10}]$ be fixed. We need a short Euler product approximation for a bulk of $L(1, \phi)$'s.

Proposition 6.1. *There are a constant $c' > 0$ and a subset $E^1(z)$ of \mathcal{H}_t such that*

$$L(1, \phi) = \left\{ 1 + O\left(\frac{1}{\log_2 t}\right) \right\} \prod_{p \leq z} \prod_{i=1}^n \left(1 - \frac{\alpha_{\phi, i}(p)}{p} \right)^{-1}$$

uniformly for $\varepsilon \log t \leq z \leq (\log t)^{10}$ and all Maass forms $\phi \in \mathcal{H}_t \setminus E^1(z)$, where the implied constant in the O -term is absolute and

$$|E^1(z)| = O_\varepsilon \left(t^d \exp \left(-c' \frac{\log t}{(\log_2 t)(\log_3 t)(\log_4 t)} \right) \right).$$

Proof. We follow the same approach as in the proof of [8, Proposition 8.1]. A crucial difference is without the Ramanujan bound now, and thus we exclude the forms outside the set

$$\mathcal{K}_t = \mathcal{K}_t(\eta) := \left\{ \phi \in \mathcal{H}_t : \log \max_{1 \leq i \leq n} |\alpha_{\phi, i}(p)| \leq 1/(\log_3 t)(\log_4 t), \forall p \leq (\log t)^{1/\eta} \right\}$$

where $\eta > 0$ is any number. The size of the exceptional set, i.e. $\mathcal{H}_t^- = \mathcal{H}_t \setminus \mathcal{K}_t$, is small:

$$(6.1) \quad \mathcal{H}_t^- \ll t^d \exp \left(-c \frac{\eta \log t}{(\log_2 t)(\log_3 t)(\log_4 t)} \right)$$

for some constant $c > 0$, by [7, Theorem 7.3] (see also [7, (6.1)]). We work on \mathcal{K}_t with the argument in [8] to complete the proof. \square

Now we prove Theorem 1.4. For $\phi \in \mathcal{H}_t \setminus E^1(z)$, we have

$$\begin{aligned} |L(1, \phi)| &\leq \left\{ 1 + O \left(\frac{1}{\log_2 t} \right) \right\} \prod_{p \leq z} \left(1 - \frac{\alpha'}{p} \right)^{-n} \\ &\leq \left\{ 1 + O \left(\frac{1}{\log_2 t} \right) \right\} (e^\gamma \log z)^{\alpha' n} \\ &\leq \left\{ e^\gamma \left((e^{\gamma(1-1/\alpha')} \log z)^{\alpha'} + C_0 (\log_2 t)^{\alpha'-1} \right) \right\}^n, \end{aligned}$$

where C_0 is an absolute constant and $\alpha' = \exp(1/(\log_3 t)(\log_4 t))$. Taking

$$\begin{aligned} z &= e^{-\gamma(1-1/\alpha')(\log_2 t + r - C_0 (\log_2 t)^{\alpha'-1})^{1/\alpha'}} \\ &= e^{(1+O((\log_4 t)^{-1}))(\log_2 t + r - C_0 (\log_2 t)^{\alpha'-1})}, \end{aligned}$$

the proof is complete for F_t^+ . The case of F_t^- is treated in the same fashion.

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