



Dispersion-Current adjoint functions for monitoring accidental sources in 3D transport equations

Adel Hamdi, Antoine Tonnoir

► To cite this version:

Adel Hamdi, Antoine Tonnoir. Dispersion-Current adjoint functions for monitoring accidental sources in 3D transport equations. *Inverse Problems in Science and Engineering*, 2020, 10.1080/17415977.2020.1858076 . hal-02992366

HAL Id: hal-02992366

<https://hal.science/hal-02992366>

Submitted on 6 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Dispersion-Current adjoint functions for monitoring accidental sources in $3D$ transport equations

Adel Hamdi and Antoine Tonnoir

Laboratoire de Mathématiques LMI
Institut National des Sciences Appliquées de Rouen
Avenue de l'Université, 76801 Saint-Etienne-du-Rouvray, Cedex, France

Abstract

The paper deals with the identification of multiple unknown time-dependent point sources occurring in $3D$ dispersion-advection-reaction equations. Based on developed appropriate adjoint functions, we establish a constructive identifiability result depending on the flow nature that yields guidelines leading to a quasi-direct Detection-Identification method. In practice, assuming to be available within a monitored domain some interfaces subdividing it into suspected sections, the developed method goes throughout the entire domain to detect the presence of all active sources. If an activity is detected within a suspected section, the method identifies the total amount discharged in this section and determines whether it is done by a single or multiple unknown occurring sources. Moreover, it localizes the sought position of a detected source as the unique root of a Dispersion-Current vector function defined from the developed adjoint functions. Application to different types of flow and some numerical experiments on surface water pollution are presented.

Keywords: Nonlinear inverse source problems; Dispersion-Current adjoint functions; Data assimilation; $3D$ Advection-Dispersion-Reaction equations; Surface water pollution.

1 Introduction

Inverse source problems are usually tasked with identifying unknown/hidden sources that driven the solution of an associated mathematical model to the measured response. Since the achievement of such task leads to illuminate the unknown causes of certain observed effects, we have seen over the last few decades inverse source problems covering a wide range of applications in science and engineering: In medicine, for example, inverse source problems are used to detect and localize a tumour as well as to identify some biological parameters, like the chemotaxis, that lead to simulate numerically the growth of a tumour in order to predict the adapted treatment to heal the patient [5, 6, 7, 14]. For environmental monitoring, inverse source problems are employed to identify pollution sources in surface water [2, 3, 4, 18], in groundwater [11, 22] and in atmosphere [3]. Some important applications include earthquake source localization [23, 24, 25], source identification in electromagnetics [10] and acoustics [21], among many others.

As far as the identifiability of unknown time-dependent point sources is concerned, in the case of *one dimensional* transport equations, the authors in [13] proved that for a single unknown occurring source, its position and its time-dependent intensity are determined in a unique way from time records of the generated state and of its flux taken at two positions framing the source region. This identifiability result has been improved in [15] to hold using only state time records then, extended firstly in [17] to treat the case of transport equations with spatially varying coefficients and secondly in [16] to address the case of two coupled transport equations. In [4], the author proved the identifiability of a moving time-dependent point source. Besides, regarding *two/three dimensional* transport equations, authors in [2, 3, 20] established identifiability results for multiple unknown time-dependent point sources. These results based on the unique continuation theorem say that the unknown elements defining all finite number of occurring time-dependent point sources are uniquely determined from state/flux time records taken on whatever non-empty part of the boundary. In our view, these results yield an ideal theoretical framework which in practice does neither take into account the flow nature i.e., transport/dispersion dominance in selecting the appropriate measuring part of the boundary nor gives visibility on how to proceed for determining the unknown elements defining the occurring sources.

This study is motivated by the following two points: **1.** We present in section 8 of this paper a *Baby Example* showing that for the one dimensional transport equation, it is not possible to uniquely identify multiple unknown point sources using measurements taken only upstream and/or downstream all occurring point sources. To ensure identifiability, all two distinct point sources should be separated by either two state measuring points or by one state and flux measuring point **2.** The identification method proposed in [3] that consists of minimizing two objective functions (Least squares and Kohn-Vogelius) defined using measurements taken on the whole boundary, encountered serious difficulties starting from the case of two occurring sources: The least squares approach doesn't enable to identify more than one active source whereas the Kohn-Vogelius approach identifies two sources only if they are well separated and the diffusion coefficient is very small in order to avoid rapid intermixing. Otherwise, the authors reported in [3] that sources effects become quickly indistinguishable. The **originality** of the results established in the present paper lies in the developement of direct mathematical techniques to determine the unknown elements defining the occurring sources. Indeed, we develop appropriate adjoint functions that lead to establish a constructive identifiability result depending on the flow nature and transforming the localization of an unknown source position into determining the unique root of a Dispersion-Current vector function defined from the developed adjoint functions. Comparing to the minimization of objective functions, the identification approach established in this paper is expected to improve the accuracy of the identified results as well as to reduce the total identification cost since we do not have to solve two PDEs (State and adjoint state) for each optimization iteration and also do neither face the problem of local minimizers nor of adding regularization terms.

The paper is organized as follows: In section 2, we present the problem statement and introduce some assumptions defining the framework of this study. Section 3 is reserved to develop appropriate adjoint functions and to prove some of their properties for later use. In section 4, we establish a constructive identifiability result based on the developed adjoint functions. Section 5 is devoted to follow the guidelines given by the established identifiability result to develop a Detection-Identification method. Applications, *Baby*

Example on the non-identifiability in the one dimensional case and numerical experiments on the surface water BOD pollution model are presented in the remaining sections.

2 Mathematical modelling and problem statement

Let $T > 0$ be a final monitoring time and Ω be a bounded and connected open subset of \mathbb{R}^3 with a sufficiently smooth boundary $\partial\Omega := \Gamma_{in} \cup \Gamma_{FS} \cup \Gamma_{FA} \cup \Gamma_{out}$, where Γ_{in} is the inflow boundary, Γ_{out} is the outflow boundary, Γ_{FS} regroups the Fluid-Solid boundaries and Γ_{FA} is the Fluid-Air boundary. For example, in surface water pollution, the BOD concentration u within a portion Ω of a river is governed by [8, 9, 12, 30]:

$$L[u](x, t) = F(x, t) \quad \text{in } \Omega \times (0, T), \quad (1)$$

where F represents the set of all occurring pollution sources and L is the second-order linear partial differential operator defined as follows:

$$L[u](x, t) := \partial_t u(x, t) - \operatorname{div}(D(x)\nabla u(x, t)) + V(x) \cdot \nabla u(x, t) + Ru(x, t). \quad (2)$$

The dot product designates the inner product in \mathbb{R}^3 , D is the hydrodynamic dispersion tensor, $V = (V_1, V_2, V_3)^\top$ is the flow velocity field and R is a real number that represents the reaction coefficient. The tensor D is a 3×3 real matrix symmetric, uniformly elliptic and uniformly bounded in Ω . Moreover, D and V satisfy

$$\operatorname{div}(V) = 0 \quad \text{in } \Omega, \quad V = \vec{0} \quad \text{on } \Gamma_{FS} \quad \text{and} \quad V \cdot \nu = 0 \quad \text{on } \Gamma_{FA}. \quad (3)$$

ν is the unit outward vector normal to $\partial\Omega$. In (3), the first condition stands for the incompressibility of water whereas the second one is the so-called no-slip condition [26].

Besides, for appending boundary and initial conditions to (1)-(2), we use [3, 18]:

$$\begin{cases} u(\cdot, 0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \sum_{in} = \Gamma_{in} \times (0, T), \\ D\nabla u \cdot \nu = 0 & \text{on } \sum_{FS} \cup \sum_{FA} \cup \sum_{out}, \end{cases} \quad (4)$$

where $\sum_{FS} = \Gamma_{FS} \times (0, T)$, $\sum_{FA} = \Gamma_{FA} \times (0, T)$ and $\sum_{out} = \Gamma_{out} \times (0, T)$. Notice that due to the linearity of the operator L introduced in (2) and according to the superposition principle, the use of a non-zero initial condition and/or inhomogeneous boundary conditions do not affect the results established in this paper.

In the present study, we are interested in the case of multiple time-dependent point sources occurring in the problem (1)-(4) i.e., F is defined as follows:

$$F(x, t) = \sum_{n=1}^N \lambda_n(t) \delta(x - S_n) \quad \text{in } \Omega \times (0, T), \quad (5)$$

where $N \in \mathbb{N}^*$, δ denotes the Dirac mass, $S_{n=1, \dots, N}$ are N distinct interior locations in Ω that represent the positions of the occurring sources and $\lambda_{n=1, \dots, N} \in L^2(0, T)$ designate their associated time-dependent source intensity functions that satisfy:

$$\int_0^T \lambda_n(t) e^{Rt} dt \neq 0 \quad \text{and} \quad \exists T^0 \in (0, T) / \lambda_n = 0 \quad \text{in } (T^0, T), \quad \forall n \in \{1, \dots, N\}. \quad (6)$$

Then, using the transposition method introduced by Lions [27, 28], it follows that the *forward problem* (1)-(5) admits a unique solution u that belongs to:

$$L^2(0, T; L^2(\Omega)) \cap \mathcal{C}^0(0, T; (H^1)'(\Omega)). \quad (7)$$

We assume to be available $I \in \mathbb{N}^*$ sufficiently smooth interfaces $\Gamma_{i=1,\dots,I}$ subdividing the domain Ω into $I + 1$ subdomains $\Omega_{i=1,\dots,I+1}$ defined by $\partial\Omega_i = \Gamma_{i-1} \cup \Gamma_{FS}^i \cup \Gamma_{FA}^i \cup \Gamma_i$, with $\Gamma_0 = \Gamma_{in}$ and $\Gamma_{I+1} = \Gamma_{out}$ whereas Γ_{FS}^i and Γ_{FA}^i are the parts of the boundaries Γ_{FS} and Γ_{FA} situated between Γ_{i-1} and Γ_i . Moreover, the interface Γ_I is set in a way such that the last subdomain Ω_{I+1} remains a "No source region of Ω ". Therefore, we have

$$\overline{\Omega} = \cup_{i=1}^{I+1} \overline{\Omega}_i, \quad \text{where } \Omega_{I+1} \subset \Omega \setminus \{S_1, \dots, S_N\}. \quad (8)$$

In our study, $\Omega_{i=1,\dots,I}$ represent the suspected sections within the monitored domain Ω wherein sources could occur and the interfaces $\Gamma_i = \partial\Omega_i \cap \partial\Omega_{i+1}$ are available for measuring the state u and its flux $D\nabla u \cdot \nu$ crossing each intersection. Since the source positions $S_{n=1,\dots,N}$ are interior locations in $\Omega_{i=1,\dots,I} \subset \Omega$, it follows that the state u solution of the problem (1)-(5) is smooth enough on $\partial\Omega$ as well as on the interfaces $\Gamma_{i=1,\dots,I}$. That allows us to define the following observation operator:

$$M[F] := \{D\nabla u \cdot \nu \text{ on } \sum_{in}; (u, D\nabla u \cdot \nu) \text{ on } \sum_{i=1,\dots,I}; uD\nu \text{ on } \sum_{FA}\}, \quad (9)$$

where $\sum_i = \Gamma_i \times (0, T)$. The *inverse source problem* in which we are interested here is: Given the records dd_{in} of $D\nabla u \cdot \nu$ on \sum_{in} , the records (d_i, dd_i) of $(u, D\nabla u \cdot \nu)$ on $\sum_{i=1,\dots,I}$ and the records d_{FA} of $uD\nu$ on \sum_{FA} , detect whether there is or not active sources occurring in each suspected subdomain $\Omega_i \subset \Omega$, for $i = 1, \dots, I$. If the presence of active sources is detected within Ω_i , determine the total amount discharged in Ω_i and whether it is done by a single or multiple occurring unknown sources. In the case of a single active source occurring within Ω_i , localize its sought position i.e., find $S_n \in \Omega_i$ that yields

$$(d_i, dd_i) = (u, D\nabla u \cdot \nu) \text{ on } \sum_{i-1} \cup \sum_i \quad \text{and} \quad d_{FA} = u \text{ on } \sum_{FA}^i. \quad (10)$$

Remark 2.1 The last term in (9) defining the observation operator $M[F]$ means that if the dispersion tensor D along the boundary Γ_{FA} is such that $D\nu$ either null or of norm small enough that can be neglected then, the underlined inverse source problem can be solved without any need of state records taken on the Fluid-Air part Γ_{FA} of the boundary.

3 Appropriate adjoint functions

Let $i \in \{1, \dots, I\}$. From multiplying the equations (1)-(2) by a sufficiently regular test function v and integrating by parts over $\Omega_i \times (0, T)$ using Green's formula, we obtain

$$\int_0^T \langle F, v \rangle_{\mathcal{D}(\Omega_i), \mathcal{D}'(\Omega_i)} = \langle L^a[v], u \rangle_{L^2(\Omega_i \times (0, T))} + \mathcal{A}_i^v, \quad (11)$$

where $\langle \cdot, \cdot \rangle_{\mathcal{D}(\Omega_i), \mathcal{D}'(\Omega_i)}$ designates the product in the distribution sense and L^a is the adjoint operator associated to the operator L in (2) i.e.:

$$\begin{aligned} \bullet L^a[v] &:= -\partial_t v - \operatorname{div}(D\nabla v) - V \cdot \nabla v + Rv, \\ \bullet \mathcal{A}_i^v &= \int_{\Omega_i} u(\cdot, T)v(\cdot, T) + \int_{\partial\Omega_i \times (0, T)} (u[D\nabla v + vV] - vD\nabla u) \cdot \nu. \end{aligned} \quad (12)$$

Since the state u is subject to only knowledge of $M[F]$ then, to keep in the right hand side of (11) only known terms, we aim to develop adjoint functions v_i that solve the system

$$\begin{cases} L^a[v_i](x, t) = 0 & \text{in } \Omega_i \times (0, T), \\ (D\nabla v_i + v_i V) \cdot \nu = 0 & \text{on } \Gamma_{FS}^i \times (0, T), \\ u(D\nabla v_i + v_i V) \cdot \nu \text{ known from } M[F] & \text{on } \Gamma_{FA}^i \times (0, T). \end{cases} \quad (13)$$

From (3), the boundary conditions in (13) are equivalent to $D\nabla v_i \cdot \nu = 0$ on $\Gamma_{FS}^i \times (0, T)$ and $uD\nabla v_i \cdot \nu$ known on $\Gamma_{FA}^i \times (0, T)$. Hence, by searching for v_i under the form

$$v_i(x, t) = e^{Rt} v_{0i}(x) \quad \text{in } \Omega_i \times (0, T) \implies -\partial_t v_i + Rv_i = 0 \quad \text{in } \Omega_i \times (0, T), \quad (14)$$

it follows according to (13) that the spatial part v_{0i} in (14) should solve

$$\begin{cases} \operatorname{div}(D\nabla v_{0i}) + V \cdot \nabla v_{0i} = 0 & \text{in } \Omega_i, \\ D\nabla v_{0i} \cdot \nu = 0 & \text{on } \Gamma_{FS}^i, \\ uD\nabla v_{0i} \cdot \nu \text{ known from } M[F] & \text{on } \Gamma_{FA}^i. \end{cases} \quad (15)$$

Therefore, the first kind of adjoint functions is what we refer to as *free space adjoint function* defined from $v_{0i} = 1$ by $v_i(x, t) = e^{Rt}$ in $\Omega_i \times (0, T)$. In the following two subsections, we develop two other kinds of spatial adjoint functions v_{0i} solving (15):

3.1 Full adjoint function

Since from (3) we have $\operatorname{div}(V) = 0$ in Ω_i , it follows that $\operatorname{div}(v_{0i}V) = V \cdot \nabla v_{0i}$ in Ω_i . Afterwards, the first equation in (15) is equivalent to: $\operatorname{div}(D\nabla v_{0i} + v_{0i}V) = 0$ in Ω_i . By looking for v_{0i} under the form $v_{0i}(x) = e^{\psi_i(x)}$ for all $x \in \Omega_i$, we get

$$D\nabla v_{0i} + v_{0i}V = 0 \quad \text{in } \Omega_i \iff D\nabla \psi_i + V = 0 \quad \text{in } \Omega_i. \quad (16)$$

Thus, the scalar potential ψ_i is well defined from the second equation in (16) provided the following compatibility condition on D and V holds true:

$$\overrightarrow{\operatorname{rot}}(D^{-1}V) = \vec{0} \quad \text{in } \Omega_i. \quad (17)$$

For the simplicity of our notations, in the remainder of this paper we denote for $k = 1, 2, 3$ by $(D^{-1})_k$ the k^{th} row of the matrix D^{-1} . Then, we have

$$\overrightarrow{\operatorname{rot}}(D^{-1}V) = \begin{pmatrix} \partial_{x_2}((D^{-1})_3 \cdot V) - \partial_{x_3}((D^{-1})_2 \cdot V) \\ \partial_{x_3}((D^{-1})_1 \cdot V) - \partial_{x_1}((D^{-1})_3 \cdot V) \\ \partial_{x_1}((D^{-1})_2 \cdot V) - \partial_{x_2}((D^{-1})_1 \cdot V) \end{pmatrix}, \quad (18)$$

where $(D^{-1})_k \cdot V$ designates the inner product of the k^{th} row of the matrix D^{-1} and the velocity field V . Hence, provided (17) holds true, the scalar potential ψ_i fulfilling the second equation in (16) is defined for all $x = (x_1, x_2, x_3) \in \Omega_i$ by

$$\begin{aligned} \psi_i(x) = & - \int_{x_3}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta - \int_{b_i}^{x_2} ((D^{-1})_2 \cdot V)(a_i, \xi, x_3) d\xi \\ & - \int_{c_i}^{a_i} ((D^{-1})_3 \cdot V)(a_i, b_i, \zeta) d\zeta, \end{aligned} \quad (19)$$

where $(a_i, b_i, c_i) \in \Omega_i$. In addition, for later use we establish the following property of ψ_i :

Lemma 3.1 *Provided (17) holds, the scalar potential ψ_i defined in (19) satisfies: For all $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in Ω_i , $\psi_i(x) = \psi_i(y)$ implies that*

$$\int_{y_1}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta + \int_{y_2}^{x_2} ((D^{-1})_2 \cdot V)(y_1, \xi, x_3) d\xi + \int_{y_3}^{x_3} ((D^{-1})_3 \cdot V)(y_1, y_2, \zeta) d\zeta = 0.$$

Proof. See the Appendix.

Therefore, in view of (3) and from (16), it follows that $v_{0i}(x) = e^{\psi_i(x)}$ solves the two first equations of the system (15) and reduces to null its last boundary condition. Thus, we obtain the so-called *full adjoint function* defined by

$$v_i(x, t) = e^{Rt + \psi_i(x)} \quad \text{in } \Omega_i \times (0, T). \quad (20)$$

3.2 Separated adjoint functions

The third way of developing v_{0i} that solves the system (15) consists of reducing to zero separately each of the two terms defining the first equation in (15). To this end, let $V = (V_1, V_2, V_3)^\top$ be the velocity field in Ω and $V^\perp = (V_1^\perp, V_2^\perp, V_3^\perp)^\top$ be a vector field defined from the same components of V and perpendicular to it i.e.,

$$V_{k=1,2,3}^\perp \in \{0, \pm V_1, \pm V_2, \pm V_3\} \quad \text{and} \quad V \cdot V^\perp = 0 \quad \text{in } \Omega. \quad (21)$$

Afterwards, we define v_{0i} such that its gradient is collinear to V^\perp i.e.,

$$\nabla v_{0i} = w_i V^\perp \quad \text{in } \Omega_i \quad \implies \quad V \cdot \nabla v_{0i} = 0 \quad \text{in } \Omega_i, \quad (22)$$

where w_i is a weighting function. Besides, in order to satisfy also $\text{div}(D \nabla v_{0i}) = 0$ in Ω_i as well as to ensure the compatibility condition $\overrightarrow{\text{rot}}(\nabla v_{0i}) = \vec{0}$ in Ω_i , we require the weighting function w_i involved in (22) to solve the following system:

$$\begin{cases} \text{div}(w_i D V^\perp) = 0 & \text{in } \Omega_i, \\ \overrightarrow{\text{rot}}(w_i V^\perp) = \vec{0} & \text{in } \Omega_i, \end{cases} \quad \Leftrightarrow \quad \begin{cases} w_i \text{div}(D V^\perp) + \nabla w_i \cdot D V^\perp = 0 & \text{in } \Omega_i, \\ w_i \overrightarrow{\text{rot}}(V^\perp) + \nabla w_i \wedge V^\perp = \vec{0} & \text{in } \Omega_i, \end{cases} \quad (23)$$

where \wedge designates the vector product.

Remark 3.2 *In view of (23), if the dispersion tensor D and the velocity field V satisfy $\text{div}(D V^\perp) = 0$ and $\overrightarrow{\text{rot}}(V^\perp) = \vec{0}$ in Ω_i which is the case, for example, when D and V are defined in Ω_i by mean values, then the weighting function can be taken $w_i = 1$ in Ω_i .*

Moreover, from searching for a weighting function w_i that solves (23) under the form: $w_i = e^{g_i}$, it follows that the unknown function g_i should fulfill in Ω_i :

$$\begin{cases} \nabla g_i \cdot D V^\perp = -\text{div}(D V^\perp), \\ \nabla g_i \wedge V^\perp = -\overrightarrow{\text{rot}}(V^\perp), \end{cases} \quad \Leftrightarrow \quad \begin{cases} \nabla g_i \cdot D V^\perp = -\text{div}(D V^\perp) & \text{"scalar equation",} \\ \begin{pmatrix} 0 & V_3^\perp & -V_2^\perp \\ -V_3^\perp & 0 & V_1^\perp \\ V_2^\perp & -V_1^\perp & 0 \end{pmatrix} \nabla g_i = - \begin{pmatrix} \partial_{x_2} V_3^\perp - \partial_{x_3} V_2^\perp \\ \partial_{x_3} V_1^\perp - \partial_{x_1} V_3^\perp \\ \partial_{x_1} V_2^\perp - \partial_{x_2} V_1^\perp \end{pmatrix} \end{cases} \quad (24)$$

Notice that the 3×3 matrix involved in (24) is antisymmetric and thus, its determinant is equal to zero. Then, we consider the following three particular choices of V^\perp :

- Let $V^{1\perp} = (0, -V_3, V_2)^\top$. From replacing in the system (24) V^\perp by $V^{1\perp}$ and g_i by g_i^1 , it follows under the condition:

$$V_3(\partial_{x_1} V_2) = V_2(\partial_{x_1} V_3) \quad \text{in } \Omega_i, \quad (25)$$

that the *scalar equation* can be taken into account in the 3×3 linear system of (24) as follows: If $V_2 = 0$ or $V_3 = 0$ in Ω_i then, we substitute the null equation in the 3×3 linear system of (24) with the *scalar equation*. If $V_2 V_3 \neq 0$ in Ω_i then, since (25) implies that the two last equations in the 3×3 linear system of (24) are equivalent, we replace one of those two equations by the *scalar equation*. For example, if $V_2 \neq 0$ in Ω_i , replacing the last equation of the 3×3 linear system in (24) by the *scalar equation* gives

$$\begin{pmatrix} 0 & V_2 & V_3 \\ -V_2 & 0 & 0 \\ (DV^{1\perp})_1 & (DV^{1\perp})_2 & (DV^{1\perp})_3 \end{pmatrix} \nabla g_i^1 = \begin{pmatrix} \partial_{x_1} V_1 \\ \partial_{x_1} V_2 \\ -\text{div}(DV^{1\perp}) \end{pmatrix}, \quad (26)$$

where $(DV^{1\perp})_k$ is the k^{th} component of the vector $DV^{1\perp}$ and the first component in the right hand side vector is obtained from applying $\text{div}(V) = 0$ in Ω_i .

- Let $V^{2\perp} = (-V_3, 0, V_1)^\top$. We set in (24) $V^\perp = V^{2\perp}$ and $g_i = g_i^2$. Afterwards, provided the following condition holds:

$$V_3(\partial_{x_2} V_1) = V_1(\partial_{x_2} V_3) \quad \text{in } \Omega_i, \quad (27)$$

the *scalar equation* can be included in the 3×3 linear system of (24) as follows: If $V_1 = 0$ or $V_3 = 0$ in Ω_i , we substitute the null equation in the 3×3 linear system of (24) with the *scalar equation*. If $V_1 V_3 \neq 0$ in Ω_i , then since (27) implies that the first and the last equations in the 3×3 linear system of (24) are equivalent, we replace one of those two equations by the *scalar equation*. For example, if $V_1 \neq 0$ in Ω_i , replacing the last equation of the 3×3 linear system in (24) by the *scalar equation* leads to

$$\begin{pmatrix} 0 & V_1 & 0 \\ -V_1 & 0 & -V_3 \\ (DV^{2\perp})_1 & (DV^{2\perp})_2 & (DV^{2\perp})_3 \end{pmatrix} \nabla g_i^2 = - \begin{pmatrix} \partial_{x_2} V_1 \\ \partial_{x_2} V_2 \\ \text{div}(DV^{2\perp}) \end{pmatrix}. \quad (28)$$

The second component of the right hand side in (28) is obtained from $\text{div}(V) = 0$ in Ω_i .

- Let $V^{3\perp} = (-V_2, V_1, 0)^\top$. From substituting in (24) V^\perp with $V^{3\perp}$ and g_i with g_i^3 , it follows under the condition:

$$V_2(\partial_{x_3} V_1) = V_1(\partial_{x_3} V_2) \quad \text{in } \Omega_i, \quad (29)$$

that the *scalar equation* can be taken into account within the 3×3 linear system of (24) as follows: If $V_1 = 0$ or $V_2 = 0$ in Ω_i , we substitute the null equation in the 3×3 linear system of (24) with the *scalar equation*. If $V_1 V_2 \neq 0$ in Ω_i , then since (29) implies that

the two first equations in the 3×3 linear system of (24) are equivalent, we replace one of those two equations by the *scalar equation*. For example, if $V_1 \neq 0$ in Ω_i , replacing the second equation of the 3×3 linear system by the *scalar equation* gives

$$\begin{pmatrix} 0 & 0 & -V_1 \\ (DV^{3\perp})_1 & (DV^{3\perp})_2 & (DV^{3\perp})_3 \\ V_1 & V_2 & 0 \end{pmatrix} \nabla g_i^3 = \begin{pmatrix} \partial_{x_3} V_1 \\ -\operatorname{div}(DV^{3\perp}) \\ \partial_{x_3} V_3 \end{pmatrix}. \quad (30)$$

The third component of the right hand side in (30) is obtained from $\operatorname{div}(V) = 0$ in Ω_i .

Remark 3.3 For example, if the velocity field $V(x_1, x_2, x_3) = (V_1(x_1, x_2), V_2(x_1, x_2), 0)^\top$ in Ω_i , then all of the three conditions (25), (27) and (29) are well fulfilled.

Thus, according to (22), we determine the function v_{0i}^ℓ associated to g_i^ℓ from solving

$$\nabla v_{0i}^\ell = e^{g_i^\ell} V^{\ell\perp} \quad \text{in } \Omega_i, \quad \text{for } \ell = 1, 2, 3. \quad (31)$$

Moreover, (31) leads to: For all $x = (x_1, x_2, x_3) \in \Omega_i$,

$$\begin{aligned} \bullet v_{0i}^1(x) &= - \int_{b_i}^{x_2} (e^{g_i^1} V_3)(a_i, \eta, x_3) d\eta + \int_{c_i}^{x_3} (e^{g_i^1} V_2)(a_i, b_i, \xi) d\xi, \\ \bullet v_{0i}^2(x) &= - \int_{a_i}^{x_1} (e^{g_i^2} V_3)(\eta, b_i, x_3) d\eta + \int_{c_i}^{x_3} (e^{g_i^2} V_1)(a_i, b_i, \xi) d\xi, \\ \bullet v_{0i}^3(x) &= - \int_{a_i}^{x_1} (e^{g_i^3} V_2)(\eta, x_2, c_i) d\eta + \int_{b_i}^{x_2} (e^{g_i^3} V_1)(a_i, \xi, c_i) d\xi, \end{aligned} \quad (32)$$

where $(a_i, b_i, c_i) \in \Omega_i$. In view of (22)-(23), since $V^{\ell\perp}$ are defined from the components of V and as according to (3), $V = \vec{0}$ on Γ_{FS} , it follows that the *separated adjoint functions* $v_{0i}^{\ell=1,2,3}$ in (32) solve the two first equations of (15) and, according to (31), satisfy

$$D \nabla v_{0i}^\ell \cdot \nu = e^{g_i^\ell} V^{\ell\perp} \cdot D\nu \quad \text{on } \Gamma_{FA}^i. \quad (33)$$

(33) holds since D is a symmetric matrix. Hence, all the developed adjoint functions i.e.,

$$v_i(x, t) = e^{Rt} v_{0i}(x) \quad \text{in } \Omega_i \times (0, T), \quad \forall v_{0i}(x) \in \{1, e^{\psi_i(x)}, v_{0i}^1(x), v_{0i}^2(x), v_{0i}^3(x)\}, \quad (34)$$

solve the two first equations of the system (13). In addition, the *free space adjoint function* $v_i(x, t) = e^{Rt}$ and the *full adjoint function* $v_i(x, t) = e^{Rt + \psi_i(x)}$ reduce to null the last boundary condition of (13) whereas the three *separated adjoint functions* $v_i^\ell(x, t) = e^{Rt} v_{0i}^\ell(x)$ yield $u(D \nabla v_i^\ell + V v_i^\ell) \cdot \nu = e^{g_i^\ell + Rt} V^{\ell\perp} \cdot u D\nu$ on $\Gamma_{FA}^i \times (0, T)$.

3.3 Dispersion-Current vector function

Using the developed adjoint functions in (34) and given two distinct integers m and n from $\{1, 2, 3\}$, we introduce what we will refer to as Dispersion-Current vector function:

$$\begin{aligned} \Psi_i^{m,n} : \quad \Omega_i &\longrightarrow \mathbb{R}^3, \\ x \mapsto \Psi_i^{m,n}(x) &= \begin{pmatrix} e^{\psi_i(x)} \\ v_{0i}^m(x) \\ v_{0i}^n(x) \end{pmatrix}, \end{aligned} \quad (35)$$

where ψ_i is the scalar potential obtained in (19) and v_{0i}^m, v_{0i}^n are two functions from (32). Moreover, to prove the injectivity of $\Psi_i^{m,n}$, we establish the following technical result:

Lemma 3.4 *Let $V = (V_1, V_2, V_3)^\top$ be a vector field such that $V_{k=1,2,3} \in W^{1,\infty}(\Omega)$. For all two elements $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ of Ω_i , we have*

$$\begin{aligned} \bullet v_{0i}^1(x) = v_{0i}^1(y) &\implies \int_{y_1}^{x_1} (e^{g_i^1} V_3)(\eta, x_2, x_3) d\eta = \int_{y_2}^{x_2} (e^{g_i^1} V_2)(a_i, y_2, \xi) d\xi, \\ \bullet v_{0i}^2(x) = v_{0i}^2(y) &\implies \int_{y_1}^{x_1} (e^{g_i^2} V_3)(\eta, b_i, x_3) d\eta = \int_{y_3}^{x_3} (e^{g_i^2} V_1)(y_1, b_i, \xi) d\xi, \\ \bullet v_{0i}^3(x) = v_{0i}^3(y) &\implies \int_{y_1}^{x_1} (e^{g_i^3} V_2)(\eta, x_2, c_i) d\eta = \int_{y_2}^{x_2} (e^{g_i^3} V_1)(y_1, \xi, c_i) d\xi, \end{aligned} \quad (36)$$

where $v_{0i}^{\ell=1,2,3}$ are the three adjoint functions obtained in (32).

Proof. See the Appendix.

That leads to prove the following theorem on the injectivity in a suspected subdomain $\Omega_i \subset \Omega$ of the Dispersion-Current function $\Psi_i^{m,n}$ introduced in (35):

Theorem 3.5 *Let $V = (V_1, V_2, V_3)^\top$ and D be such that Lemmas 3.1 and 3.4 apply. If there exists $k \in \{1, 2, 3\}$ for which D and V yield a.e. in Ω_i one of the two assertions:*

$$\begin{aligned} 1. & V_k > 0 \quad \text{and} \quad V_{m,n} \geq 0 \quad \text{with} \quad (D^{-1})_k \cdot V > 0 \quad \text{and} \quad (D^{-1})_{m,n} \cdot V \geq 0, \\ 2. & V_k < 0 \quad \text{and} \quad V_{m,n} \geq 0 \quad \text{with} \quad (D^{-1})_k \cdot V < 0 \quad \text{and} \quad (D^{-1})_{m,n} \cdot V \geq 0, \end{aligned} \quad (37)$$

where m, n are the two distinct elements of $\{1, 2, 3\} \setminus \{k\}$, then $\Psi_i^{m,n}$ is injective in Ω_i .

Proof. Let $k \in \{1, 2, 3\}$ and m, n be the two distinct elements of $\{1, 2, 3\} \setminus \{k\}$. In view of (35), it follows that for all $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ elements of Ω_i , we have

$$\Psi_i^{m,n}(x) = \Psi_i^{m,n}(y) \quad \Leftrightarrow \quad \begin{cases} \psi_i(x) = \psi_i(y), \\ v_{0i}^m(x) = v_{0i}^m(y), \\ v_{0i}^n(x) = v_{0i}^n(y). \end{cases} \quad (38)$$

• **Assertion 1.** If $V_k > 0$ and $V_{m,n} \geq 0$ a.e. in Ω_i , then since Lemma 3.4 applies, the two last equations in (38) imply, in view of (36), that each element of $\{x_1 - y_1, x_2 - y_2, x_3 - y_3\}$ is either null or admits the same sign as the other non-null element(s) of this set. Therefore, because according to Lemma 3.1, $\psi_i(x) = \psi_i(y)$ is equivalent to

$$\begin{aligned} \int_{y_1}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta + \int_{y_2}^{x_2} ((D^{-1})_2 \cdot V)(y_1, \xi, x_3) d\xi \\ + \int_{y_3}^{x_3} ((D^{-1})_3 \cdot V)(y_1, y_2, \zeta) d\zeta = 0, \end{aligned} \quad (39)$$

and it holds $(D^{-1})_j \cdot V \geq 0$ a.e. in Ω_i , for $j = 1, 2, 3$, then each of the three integrals in (39) is either null or admits the same sign as the other non-null integral(s). Hence, those three integrals are all null. Since $(D^{-1})_k \cdot V > 0$, it follows that $x_k = y_k$. Afterwards,

using $x_k = y_k$ in the two last equations of (38) implies in view of (36) and as $V_k > 0$ that $x_m = y_m$ and $x_n = y_n$. That means $x = y$.

- **Assertion 2.** If $V_k < 0$ and $V_{m,n} \geq 0$ a.e. in Ω_i then, the two last equations in (38) imply in view of (36) that: **1.** If $x_k - y_k \neq 0$, then each of the two terms $x_m - y_m$ and $x_n - y_n$ is either null or admits the opposite sign of $x_k - y_k$. **2.** If $x_k = y_k$, then $x_m = y_m$ and $x_n = y_n$. Since $(D^{-1})_k \cdot V < 0$ whereas $(D^{-1})_{m,n} \cdot V \geq 0$ a.e. in Ω_i , it follows that each of the three integrals in (39) is either null or admits the same sign as the other non-null integral(s). Thus, each of those integrals is null which implies that $x_k = y_k$. Because $V_k < 0$, using $x_k = y_k$ in the two last equations of (38) gives $x_m = y_m$ and $x_n = y_n$. ■

4 Identifiability

We start this section by proving that within all suspected subdomain $\Omega_i \subset \Omega$, there is no a single point source that can generate on the outflow boundary Γ_i state and flux time measurements same as those generated by multiple distinct point sources. Then, we establish that the Dispersion-Current vector function $\Psi_i^{m,n}$ introduced in (35) leads to localize in a unique manner the position of an unknown point source occurring in Ω_i .

Theorem 4.1 *Let $i \in \{1, \dots, I\}$ and $\Omega_i \subset \Omega$, where $\partial\Omega_i = \Gamma_{i-1} \cup \Gamma_{FS}^i \cup \Gamma_{FA}^i \cup \Gamma_i$. We denote by u and $u^{(P)}$ the solutions of the problem (1)-(5) with $F(x, t) = \lambda(t)\delta(x - S)$ and $F(x, t) = \sum_{p=1}^P \lambda_p(t)\delta(x - S_p)$ in $\Omega_i \times (0, T)$, where $P \in \mathbb{N}^*$, $S_{p=1, \dots, P}, S$ are interior points in Ω_i and $\lambda_{p=1, \dots, P}, \lambda$ are functions of $L^2(0, T)$ fulfilling (6).*

$$\begin{cases} u = u^{(P)} & \text{on } \Gamma_i \times (0, T), \\ D\nabla u \cdot \nu = D\nabla u^{(P)} \cdot \nu & \text{on } \Gamma_i \times (0, T). \end{cases} \implies P = 1 \text{ and } S_1 = S. \quad (40)$$

Proof. Let $w = u - u^{(P)}$ in $\Omega_i \times (0, T)$ and $\Omega_i^0 = \Omega_i \setminus \{S, S_{p=1, \dots, P}\}$. In view of the boundary conditions in (40) and the two main equations (1)-(2), it follows that w solves

$$\begin{cases} L[w](x, t) = \lambda(t)\delta(x - S) - \sum_{p=1}^P \lambda_p(t)\delta(x - S_p) & \text{in } \Omega_i \times (0, T), \\ w = D\nabla w \cdot \nu = 0 & \text{on } \Gamma_i \times (0, T). \end{cases} \quad (41)$$

Let ω_i be a connected open subset of Ω such that: $\emptyset \neq (\omega_i \cap \Omega_i) \subset \Omega_i^0$, $\omega_i \cap \Gamma_i \neq \emptyset$ and $\emptyset \neq (\omega_i \cap \Omega_{i+1}) \subset \Omega_{i+1}^0$, where Ω_{i+1}^0 is an open subset of Ω_{i+1} that doesn't contain any source position. Since w fulfills (41), then its extension by zero in $\omega_i \cap \Omega_{i+1}$ solves in $\omega_i \times (0, T)$ the homogeneous parabolic equation obtained from replacing by zero the right hand side of the first equation in (41). Thus, by applying twice the unique continuation Theorem [29], we get $w = 0$ in $\omega_i \times (0, T)$ and then, $w = 0$ in $\Omega_i^0 \times (0, T)$. That implies w is either null a.e. in $\Omega_i \times (0, T)$ or it's a linear combination of the dirac mass and some of its derivatives at the points $S, S_{p=1, \dots, P}$. This second option it's not possible due to the regularity of w . Hence, by setting $w = 0$ a.e. in $\Omega_i \times (0, T)$, it follows from the first equation of (41) that $P = 1$ and $S_1 = S$. ■

Besides, we establish the following constructive identifiability result. This result is said to be constructive since it sets guidelines that lead later on to localize in a unique manner the sought position of an unknown point source occurring in $\Omega_i \subset \Omega$.

Theorem 4.2 *Let $i \in \{1, \dots, I\}$, $\Omega_i \subset \Omega$ and u be the solution of the problem (1)-(5) with $F(x, t) = \lambda(t)\delta(x - S)$ in $\Omega_i \times (0, T)$, where S is an interior point in Ω_i . Provided*

1. D and V are such that Theorem 3.5 applies in Ω_i ,
2. $\lambda \in L^2(0, T)$ and fulfills (6),

boundary records of $(u, D\nabla u \cdot \nu)$ on $(\Gamma_{i-1} \cup \Gamma_i) \times (0, T)$ and of $uD\nu$ on $\Gamma_{FA}^i \times (0, T)$ determine uniquely the unknown source elements S and $\bar{\lambda} = \int_0^T \lambda(t)e^{Rt}dt$.

Proof. Let $u^{(k=1,2)}$ be the solution of the problem (1)-(5) with F in $\Omega_i \times (0, T)$ equal to $F^{(k)}(x, t) = \lambda^{(k)}(t)\delta(x - S^{(k)})$, where $\lambda^{(k)} \in L^2(0, T)$ fulfilling (6) and $S^{(k)}$ is an interior point in Ω_i . We denote by $w = u^{(2)} - u^{(1)}$. Then, from assuming

$$w = D\nabla w \cdot \nu = 0 \quad \text{on } (\Gamma_{i-1} \cup \Gamma_i) \times (0, T) \quad \text{and} \quad wD\nu = \vec{0} \quad \text{on } \Gamma_{FA}^i \times (0, T), \quad (42)$$

it follows that the variable w solves the system:

$$\begin{cases} L[w](x, t) = \lambda^{(2)}(t)\delta(x - S^{(2)}) - \lambda^{(1)}(t)\delta(x - S^{(1)}) & \text{in } \Omega_i \times (0, T), \\ w(\cdot, 0) = 0 & \text{in } \Omega_i, \\ w = 0 & \text{on } \Gamma_{i-1} \times (0, T), \\ D\nabla w \cdot \nu = 0 & \text{on } (\Gamma_{FS}^i \cup \Gamma_{FA}^i \cup \Gamma_i) \times (0, T). \end{cases} \quad (43)$$

From the last equation in (42) and according to (33), it follows that all of the adjoint functions v_i obtained in (34) solve the desired system (13), where its last boundary condition is replaced by $w(D\nabla v_i + v_i V) \cdot \nu = 0$ on $\Gamma_{FA}^i \times (0, T)$. Hence, as in (11)-(12), from multiplying the first equation in (43) by an adjoint function $v_i(x, t) = e^{Rt}v_{0i}(x)$ of (34) and integrating by parts over $\Omega_i \times (0, T)$ using Green's formula, we get

$$\bar{\lambda}^{(2)}v_{0i}(S^{(2)}) - \bar{\lambda}^{(1)}v_{0i}(S^{(1)}) = \mathcal{A}_i^{v_{0i}}, \quad \forall v_{0i}(x) \in \{1, e^{\psi_i(x)}, v_{0i}^1(x), v_{0i}^2(x), v_{0i}^3(x)\}, \quad (44)$$

where $\bar{\lambda}^{(k)} = \int_0^T e^{Rt}\lambda^{(k)}(t)dt$, for $k = 1, 2$ and

$$\mathcal{A}_i^{v_{0i}} = e^{RT} \int_{\Omega_i} w(\cdot, T)v_{0i} + \int_{(\Gamma_{i-1} \cup \Gamma_i) \times (0, T)} e^{Rt} (w[D\nabla v_{0i} + v_{0i}V] - v_{0i}D\nabla w) \cdot \nu. \quad (45)$$

Moreover, using (42) in (44)-(45), gives: For all $v_{0i}(x) \in \{1, e^{\psi_i(x)}, v_{0i}^1(x), v_{0i}^2(x), v_{0i}^3(x)\}$,

$$\bar{\lambda}^{(2)}v_{0i}(S^{(2)}) - \bar{\lambda}^{(1)}v_{0i}(S^{(1)}) = e^{RT} \int_{\Omega_i} w(\cdot, T)v_{0i}. \quad (46)$$

Besides, according to (8), w solves in $\Omega_{I+1} \times (0, T)$ a system similar to (43), where the first equation becomes homogeneous. That implies $w = 0$ in $\Omega_{I+1} \times (0, T)$ which, in view of (6) and using the problem satisfied by w in $\Omega \times (T^0, T)$, gives from applying the unique continuation Theorem [29] that $w(\cdot, T) = 0$ in Ω . Hence, (46) leads to

$$\bar{\lambda}^{(2)}v_{0i}(S^{(2)}) = \bar{\lambda}^{(1)}v_{0i}(S^{(1)}), \quad \forall v_{0i}(x) \in \{1, e^{\psi_i(x)}, v_{0i}^1(x), v_{0i}^2(x), v_{0i}^3(x)\}. \quad (47)$$

Using $v_{0i} = 1$ in (47) gives $\bar{\lambda}^{(2)} = \bar{\lambda}^{(1)}$. Afterwards, from (47) and (35), we get

$$\Psi_i^{m,n}(S^{(2)}) = \Psi_i^{m,n}(S^{(1)}) \implies S^{(2)} = S^{(1)}. \quad (48)$$

The implication in (48) is obtained from applying Theorem 3.5. ■

5 Detection-Identification of sources

Let $i \in \{1, \dots, I\}$, Ω_i be a suspected section of Ω and $N_i \in \{0, \dots, N\}$ be the number of point sources occurring in Ω_i i.e., $S_n \in \Omega_i$, for $n = n_i, \dots, n_i + N_i - 1$, where $n_i \in \{1, \dots, N\}$. Thus, in the problem (1)-(5), $F(x, t) = \sum_{n=n_i}^{n_i+N_i-1} \lambda_n(t) \delta(x - S_n)$ in $\Omega_i \times (0, T)$. Therefore, likewise (11)-(12), from multiplying the equations (1)-(2) by an adjoint function $v_i(x, t) = e^{Rt} v_{0i}(x)$ from (34) and integrating by parts in $\Omega_i \times (0, T)$ using Green's formula, we obtain: For all $v_{0i}(x) \in \{1, e^{\psi_i(x)}, v_{0i}^1(x), v_{0i}^2(x), v_{0i}^3(x)\}$,

$$\mathcal{A}_i^{v_{0i}} = \begin{cases} \sum_{n=n_i}^{n_i+N_i-1} \bar{\lambda}_n v_{0i}(S_n) & \text{if } N_i \geq 1, \\ 0 & \text{if } N_i = 0, \end{cases} \quad (49)$$

where $\bar{\lambda}_n = \int_0^T e^{Rt} \lambda_n(t) dt$ and the coefficient

$$\mathcal{A}_i^{v_{0i}} = e^{RT} \int_{\Omega_i} u(\cdot, T) v_{0i} + \int_{\partial\Omega_i \times (0, T)} e^{Rt} (u [D\nabla v_{0i} + v_{0i} V] - v_{0i} D\nabla u) \cdot \nu. \quad (50)$$

Afterwards, using in (49)-(50) the *free space adjoint function* i.e., $v_{0i} = 1$ leads to

$$\mathcal{A}_i^1 = e^{RT} \int_{\Omega_i} u(\cdot, T) + \int_{(\Gamma_{i-1} \cup \Gamma_i) \times (0, T)} e^{Rt} (uV - D\nabla u) \cdot \nu = \begin{cases} \sum_{n=n_i}^{n_i+N_i-1} \bar{\lambda}_n & \text{if } N_i \geq 1, \\ 0 & \text{if } N_i = 0. \end{cases} \quad (51)$$

Hence, employing the records of $(u, D\nabla u \cdot \nu)$ taken on the intersections $(\Gamma_{i-1} \cup \Gamma_i) \times (0, T)$ to compute the coefficient \mathcal{A}_i^1 in (51) enables to detect the presence of all active sources within Ω_i . Indeed, if $\mathcal{A}_i^1 \neq 0$ then, its value corresponds to the total suspended amount discharged by all sources occurring in Ω_i . Moreover, for example, provided the reaction coefficient $R \geq 0$ and $\sum_{n=n_i}^{n_i+N_i-1} \lambda_n(t) \geq 0$ a.e. in $(0, T)$, it follows from (51) that

$$e^{-RT} \mathcal{A}_i^1 \leq \sum_{n=n_i}^{n_i+N_i-1} \int_0^T \lambda_n(t) dt \leq \mathcal{A}_i^1. \quad (52)$$

Remark 5.1 Since in practice R is usually small, (52) could provide an approximation/estimation of the total amount discharged by all sources occurring in Ω_i without having to identify a priori the historic in $(0, T)$ of each time-dependent intensity λ_n .

5.1 Localization of a detected source

Assume the detection coefficient $\mathcal{A}_i^1 \neq 0$ and there exists one source occurring in Ω_i i.e., $N_i = 1$ and thus, $F(x, t) = \lambda_{n_i}(t) \delta(x - S_{n_i})$ in $\Omega_i \times (0, T)$. From (49)-(50), it follows that the two unknown elements S_{n_i} and λ_{n_i} defining the occurring source are subject to:

$$\begin{cases} \bar{\lambda}_{n_i} = \mathcal{A}_i^1, \\ e^{\psi_i(S_{n_i})} = \frac{\mathcal{A}_i^{e^{\psi_i}}}{\mathcal{A}_i^1}, \\ v_{0i}^\ell(S_{n_i}) = \frac{\mathcal{A}_i^{v_{0i}^\ell}}{\mathcal{A}_i^1}, \quad \text{for } \ell = 1, 2, 3. \end{cases} \quad \implies \quad \Psi_i^{m,n}(S_{n_i}) = \frac{1}{\mathcal{A}_i^1} \begin{pmatrix} \mathcal{A}_i^{e^{\psi_i}} \\ \mathcal{A}_i^{v_{0i}^m} \\ \mathcal{A}_i^{v_{0i}^n} \end{pmatrix}, \quad (53)$$

where $\Psi_i^{m,n}$ is the Dispersion-Current function in (35) defined from the two distinct integers m and n selected in $\{1, 2, 3\}$ such that Theorem 3.5 yields injectivity of $\Psi_i^{m,n}$ in Ω_i . Therefore, the unknown source position S_{n_i} is the unique point of Ω_i that solves the last equation in (53). Besides, according to (50) and in view of (34), we get

$$\begin{aligned}
\bullet \mathcal{A}_i^1 &= e^{RT} \int_{\Omega_i} u(\cdot, T) + \int_{(\Gamma_{i-1} \cup \Gamma_i) \times (0, T)} e^{Rt} (uV - D\nabla u) \cdot \nu, \\
\bullet \mathcal{A}_i^{e^{\psi_i}} &= e^{RT} \int_{\Omega_i} u(\cdot, T) e^{\psi_i} - \int_{(\Gamma_{i-1} \cup \Gamma_i) \times (0, T)} e^{Rt + \psi_i} D\nabla u \cdot \nu, \\
\bullet \mathcal{A}_i^{v_{0i}^{\ell=1,2,3}} &= e^{RT} \int_{\Omega_i} u(\cdot, T) v_{0i}^\ell + \int_{(\Gamma_{i-1} \cup \Gamma_i) \times (0, T)} e^{Rt} (u[e^{g_i^\ell} DV^{\ell\perp} + v_{0i}^\ell V] - v_{0i}^\ell D\nabla u) \cdot \nu \\
&\quad + \int_{\Gamma_{FA}^i \times (0, T)} e^{Rt + g_i^\ell} V^{\ell\perp} \cdot u D\nu.
\end{aligned} \tag{54}$$

From computing the coefficients in (54), we localize the unknown source position S_{n_i} as the unique point within the suspected section Ω_i that solves the last equation in (53).

5.2 Determination of the final state $u(\cdot, T)$

For the computation of the coefficients in (54), we propose the following two different ways to determine the unknown data $u(\cdot, T)$:

- **First way. Least squares:** Since from (8) the subdomain Ω_{I+1} is a "No source region", we use the state records taken on $\Gamma_I \times (0, T)$ to solve the forward problem satisfied by the state u in $\Omega_{I+1} \times (0, T)$. Then, because all sources become inactive in (T^0, T) , we identify $u(\cdot, T)$ as the final condition that leads the solution of the Cauchy problem satisfied by u in $\Omega \times (T^0, T)$ to fit the forward problem solution in $\Omega_{I+1} \times (T^*, T)$, where $T^* \in (T^0, T)$.
- **Second way. Data assimilation:** For $i = 1, \dots, I$, use the state records d_{i-1} on $\Gamma_{i-1} \times (0, T)$ and d_i on $\Gamma_i \times (0, T)$ to determine an approximation $\hat{u}(\cdot, T)$ in Ω_i of $u(\cdot, T)$ from solving the following system:

$$\begin{cases} L[\hat{u}_i](x, t) = 0 & \text{in } \Omega_i \times (0, T), \\ \hat{u}_i(\cdot, 0) = 0 & \text{in } \Omega_i, \\ \hat{u}_i = d_{i-1} & \text{on } \Gamma_{i-1} \times (0, T), \\ \hat{u}_i = d_i & \text{on } \Gamma_i \times (0, T), \\ D\nabla \hat{u}_i \cdot \nu = 0 & \text{on } (\Gamma_{FS}^i \cup \Gamma_{FA}^i) \times (0, T). \end{cases} \tag{55}$$

Then, we establish the following Lemma on how does $\hat{u}_i(\cdot, T)$ approximate $u(\cdot, T)$ in Ω_i :

Lemma 5.2 *Provided (6) holds, there exists $0 < \alpha \in \mathbb{R}$ such that the state \hat{u}_i solution of the system (55) fulfills: For all $t \in (T^0, T)$,*

$$\|\hat{u}_i(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega_i)}^2 \leq \|\hat{u}_i(\cdot, T^0) - u(\cdot, T^0)\|_{L^2(\Omega_i)}^2 e^{-\alpha(t-T^0)}, \tag{56}$$

where T^0 is the final active time introduced in (6) and u is the solution of (1)-(5).

Proof. See the Appendix.

For the clarity of our presentation, we summarize in the following algorithm the main steps defining the Detection-Identification method developed in the present paper:

Algorithm. Detection-Identification method

For $i = 1$ to I do

- Compute the *detection* coefficient \mathcal{A}_i^1 from (54).
- If $|\mathcal{A}_i^1| \leq \varepsilon_0$ then, there is no significant sources occurring in Ω_i . Go to *End do*.
- Otherwise, proceed as follows:
 1. Determine from (51)-(52) the total amount discharged in Ω_i .
 2. Select m and n such that Theorem 3.5 yields injectivity of $\Psi_i^{m,n}$ in Ω_i .
 3. Compute the coefficients in (54) and solve the last equation in (53):
 - If *NO* solution in Ω_i , according to Theorem 4.1, *MULTIPLE* sources occur in Ω_i .
 - If *ONE* solution, from Theorem 4.2, it's the position of the *UNIQUE* source in Ω_i .

End do

6 Application to three types of flow

We apply the developed Detection-Identification method to the following three different types of flow crossing a suspected subdomain $\Omega_i \subset \Omega$: **1.** Flow defined by a mean velocity vector **2.** Flow defined by a unidirectional velocity field **3.** Flow defined by a bidirectional velocity field. We employ Bear's hydrodynamic dispersion tensor, see [8, 9, 33]:

$$D = (D_M + \alpha_T \|V\|_2) \mathbf{I} + \frac{\alpha_L - \alpha_T}{\|V\|_2} V \cdot V^\top, \quad (57)$$

where $D_M > 0$ is a real number that represents the molecular diffusion, \mathbf{I} is the 3×3 identity matrix and $0 < \alpha_T < \alpha_L$ are two real numbers that represent the longitudinal and transverse dispersivities of the porous medium. From (57), it follows that

$$\begin{aligned} \bullet \quad DV &= (D_M + \alpha_L \|V\|_2) V \quad \Leftrightarrow \quad D^{-1}V = \frac{1}{D_M + \alpha_L \|V\|_2} V, \\ \bullet \quad DV^{\ell\perp} &= (D_M + \alpha_T \|V\|_2) V^{\ell\perp}, \quad \text{for } \ell = 1, 2, 3. \end{aligned} \quad (58)$$

Therefore, the scalar potential ψ_i in (19) is given by: For all $x = (x_1, x_2, x_3) \in \Omega_i$,

$$\begin{aligned} \psi_i(x) &= - \int_{x_3}^{x_1} \left(\frac{V_1}{D_M + \alpha_L \|V\|_2} \right) (\eta, x_2, x_3) d\eta - \int_{b_i}^{x_2} \left(\frac{V_2}{D_M + \alpha_L \|V\|_2} \right) (a_i, \xi, x_3) d\xi \\ &\quad - \int_{c_i}^{a_i} \left(\frac{V_3}{D_M + \alpha_L \|V\|_2} \right) (a_i, b_i, \zeta) d\zeta. \end{aligned} \quad (59)$$

6.1 Flow of mean velocity vector

In $\Omega_i \subset \Omega$, assume $V = (V_1, V_2, V_3)^\top$ to be a mean velocity vector and D to be the associated dispersion tensor in (57). Then, (17) is satisfied and from (59), we get

$$\psi_i(x) = - \frac{1}{D_M + \alpha_L \|V\|_2} V \cdot (x - A_i), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i, \quad (60)$$

where $A_i = (a_i, b_i, c_i) \in \Omega_i$. Besides, according to Remark 3.2, we define $v_{0i}^{\ell=1,2,3}$ from solving $\nabla v_{0i}^\ell = V^{\ell\perp}$ in Ω_i . Thus, setting $g_i^\ell = 0$ in (31)-(32) gives: For $\ell = 1, 2, 3$

$$v_{0i}^\ell(x) = V^{\ell\perp} \cdot (x - A_i), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (61)$$

Provided the mean velocity vector V fulfills in Ω_i , for example, $V_1 > 0$, $V_2 \geq 0$ and $V_3 \geq 0$, it follows from (58) that $(D^{-1})_1 V > 0$ and $(D^{-1})_{2,3} V \geq 0$ and thus, the first condition of Theorem 3.5 is fulfilled for $k = 1$, $m = 2$ and $n = 3$. Therefore, from (35), the function

$$\begin{aligned} \Psi_i^{2,3} : \quad \Omega_i &\longrightarrow \mathbb{R}^3, \\ x \mapsto \Psi_i^{2,3}(x) &= \begin{pmatrix} e^{\psi_i(x)} \\ v_{0i}^2(x) \\ v_{0i}^3(x) \end{pmatrix}, \end{aligned} \quad (62)$$

is injective in Ω_i . Afterwards, assuming the detection coefficient $\mathcal{A}_i^1 \neq 0$ and there exists one unknown source $\lambda_{n_i}(t)\delta(x - S_{n_i})$ occurring in Ω_i , it comes from (53)-(54) that the sought source position S_{n_i} is the unique point within Ω_i that satisfies

$$\Psi_i^{2,3}(S_{n_i}) = \frac{1}{\mathcal{A}_i^1} \begin{pmatrix} \mathcal{A}_i^{e^{\psi_i}} \\ \mathcal{A}_i^{v_{0i}^2} \\ \mathcal{A}_i^{v_{0i}^3} \end{pmatrix}, \quad (63)$$

which, in view of (60)-(61), can be written under the following matrix form:

$$\begin{pmatrix} V_1 & V_2 & V_3 \\ -V_3 & 0 & V_1 \\ -V_2 & V_1 & 0 \end{pmatrix} (S_{n_i} - A_i) = \begin{pmatrix} -(D_M + \alpha_L \|V\|_2) \ln\left(\frac{\mathcal{A}_i^{e^{\psi_i}}}{\mathcal{A}_i^1}\right) \\ \frac{\mathcal{A}_i^{v_{0i}^2}}{\mathcal{A}_i^1} \\ \frac{\mathcal{A}_i^{v_{0i}^3}}{\mathcal{A}_i^1} \end{pmatrix}. \quad (64)$$

The determinant of the 3×3 matrix involved in the linear system (64) is $-V_1 \|V\|_2^2 < 0$. That confirms the result announced by Theorem 3.5 on the injectivity of the Dispersion-Current function $\Psi_i^{2,3}$ in (62). Therefore, the unknown position S_{n_i} defining the detected source is the unique point within Ω_i that solves the linear system in (64).

6.2 Flow of unidirectional velocity field

Assume within $\Omega_i \subset \Omega$, there exists two functions φ and ζ such that the velocity field $V(x) = (V_1(x), 0, 0)^\top$, where $V_1(x) = \varphi(x_2)\zeta(x_3) > 0$, for all $x = (x_1, x_2, x_3) \in \Omega_i$. Since the molecular diffusion D_M is usually small enough [8, 9], for the simplicity of our presentation we set $D_M = 0$ in (58). That leads to

$$D^{-1}V = \frac{1}{\alpha_L} (1, 0, 0)^\top \quad \text{in } \Omega_i \quad \implies \quad \overrightarrow{rot}(D^{-1}V) = \vec{0} \quad \text{in } \Omega_i. \quad (65)$$

Thus, the condition (17) on the existence of the scalar potential ψ_i defined from (16) is well fulfilled. Moreover, from using $D_M = 0$ in (59), we obtain

$$\psi_i(x) = -\frac{1}{\alpha_L}(x_1 - a_i), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (66)$$

Since $V_1 > 0$ and $V_2 = V_3 = 0$ in Ω_i , it follows that the first condition of Theorem 3.5 is satisfied for $k = 1$, $m = 2$ and $n = 3$. Therefore, the function $\Psi_i^{2,3}$ defined from (35) is injective in Ω_i . To define $\Psi_i^{2,3}$, we determine the two adjoint functions v_{0i}^2 and v_{0i}^3 .

According to (31), v_{0i}^2 is defined by $\nabla v_{0i}^2 = e^{g_i^2} V^{2\perp}$ in Ω_i , where $V^{2\perp} = (0, 0, V_1)^\top$ and g_i^2 solves (28). Moreover, using (58) in the linear system (28) implies that g_i^2 is subject to:

$$\nabla g_i^2 = -\frac{1}{V_1} G^2, \quad \text{where } G^2 = \begin{pmatrix} 0 \\ \partial_{x_2} V_1 \\ 2\partial_{x_3} V_1 \end{pmatrix}. \quad (67)$$

Because $V_1(x) = \varphi(x_2)\zeta(x_3)$ in Ω_i , it follows that $\overrightarrow{rot}\left(\frac{1}{V_1}G^2\right) = \vec{0}$ in Ω_i and thus, we get

$$g_i^2(x) = -\ln(|\varphi(x_2)|) - 2\ln(|\zeta(x_3)|) = \ln\left(\frac{1}{|\varphi(x_2)|\zeta^2(x_3)}\right), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (68)$$

That leads, in view of (32), to determine v_{0i}^2 as follows:

$$v_{0i}^2(x) = \int_{c_i}^{x_3} (e^{g_i^2} V_1)(b_i, \xi) d\xi = \text{sign}(\varphi(b_i)) \int_{c_i}^{x_3} \frac{1}{\zeta(\xi)} d\xi, \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (69)$$

Besides, v_{0i}^3 is defined from (31) by $\nabla v_{0i}^3 = e^{g_i^3} V^{3\perp}$ in Ω_i , where $V^{3\perp} = (0, V_1, 0)^\top$ and g_i^3 solves (30). From employing (58) in the linear system (30), it comes that g_i^3 solves in Ω_i

$$\nabla g_i^3 = -\frac{1}{V_1} G^3, \quad \text{where } G^3 = \begin{pmatrix} 0 \\ 2\partial_{x_2} V_1 \\ \partial_{x_3} V_1 \end{pmatrix}. \quad (70)$$

Since $V_1(x) = \varphi(x_2)\zeta(x_3)$ in Ω_i , it follows that $\overrightarrow{rot}\left(\frac{1}{V_1}G^3\right) = \vec{0}$ in Ω_i . From (70), we get

$$g_i^3(x) = -2\ln(|\varphi(x_2)|) - \ln(|\zeta(x_3)|) = \ln\left(\frac{1}{\varphi^2(x_2)|\zeta(x_3)|}\right), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (71)$$

Afterwards, from (32), we find

$$v_{0i}^3(x) = \int_{b_i}^{x_2} (e^{g_i^3} V_1)(\eta, c_i) d\eta = \text{sign}(\zeta(c_i)) \int_{b_i}^{x_2} \frac{1}{\varphi(\eta)} d\eta, \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (72)$$

Hence, given $(a_i, b_i, c_i) \in \Omega_i$ and in view (66)-(72), the function $\Psi_i^{2,3}$ is defined by

$$\begin{aligned} \Psi_i^{2,3} : \quad \Omega_i &\longrightarrow \mathbb{R}^3, \\ x \mapsto \Psi_i^{2,3}(x) &= \begin{pmatrix} \exp\left(-\frac{1}{\alpha_L}(x_1 - a_i)\right) \\ \text{sign}(\varphi(b_i)) \int_{c_i}^{x_3} \frac{1}{\zeta(\xi)} d\xi \\ \text{sign}(\zeta(c_i)) \int_{b_i}^{c_i} \frac{1}{\varphi(\eta)} d\eta \end{pmatrix}. \end{aligned} \quad (73)$$

From (73) and since the two functions φ and ζ are such that $\varphi(x_2)\zeta(x_3) > 0$ in Ω_i , we verify readily the injectivity of $\Psi_i^{2,3}$ in Ω_i announced by Theorem 3.5. Thus, assuming the detection coefficient $\mathcal{A}_i^1 \neq 0$ and one unknown source $\lambda_{n_i}(t)\delta(x - S_{n_i})$ is occurring in Ω_i , it follows that the sought source position S_{n_i} is the unique point within Ω_i that solves the last equation in (53), where $\Psi_i^{m,n} = \Psi_i^{2,3}$ obtained in (73).

6.3 Flow of bidirectional velocity field

The third type of flow crossing a suspected section $\Omega_i \subset \Omega$ that we consider for the application of the developed Detection-Identification method is a flow defined by a velocity field: $V(x) = V_0(x_3)(1, \beta, 0)^\top$, where the function $V_0(x_3) > 0$ for all $x = (x_1, x_2, x_3) \in \Omega_i$ and $\beta \geq 0$ is a real number. From setting $D_M = 0$ in (58), it follows that

$$\begin{aligned} \bullet \quad D^{-1}V &= \frac{1}{\alpha_L \sqrt{1 + \beta^2}} (1, \beta, 0)^\top \quad \text{in } \Omega_i \quad \implies \quad \overrightarrow{\text{rot}}(D^{-1}V) = \vec{0} \quad \text{in } \Omega_i, \\ \bullet \quad DV^{\ell\perp} &= \alpha_T \sqrt{1 + \beta^2} V_0 V^{\ell\perp} \quad \text{in } \Omega_i, \quad \text{for } \ell = 1, 2, 3. \end{aligned} \quad (74)$$

The second equation in (74) implies that the condition (17) on the existence of the scalar potential ψ_i defined from (16) is well fulfilled. Moreover, using $D_M = 0$ in (59) gives

$$\psi_i(x) = -\frac{1}{\alpha_L \sqrt{1 + \beta^2}} (x_1 - a_i + \beta(x_2 - b_i)), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (75)$$

Besides, in view of (74), since the function $V_0 > 0$ and the real number $\beta \geq 0$, the first condition of Theorem 3.5 is satisfied for $k = 1$, $m = 2$ and $n = 3$. Thus, the function $\Psi_i^{2,3}$ in (35) is injective in Ω_i . To define $\Psi_i^{2,3}$, we determine the two functions v_{0i}^2 and v_{0i}^3 .

From (31), the adjoint function v_{0i}^2 is defined by $\nabla v_{0i}^2 = e^{g_i^2} V^{2\perp}$ in Ω_i , where $V^{2\perp} = (0, 0, V_0)^\top$ and g_i^2 satisfies (28). Moreover, using (74) in the linear system (28) leads to

$$\nabla g_i^2 = \begin{pmatrix} 0 \\ 0 \\ -2\frac{V_0'}{V_0} \end{pmatrix} \implies g_i^2(x) = \ln\left(\frac{1}{V_0^2(x_3)}\right), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (76)$$

Furthermore, from replacing in (32) g_i^2 by its value obtained in (76), we find

$$v_{0i}^2(x) = \int_{c_i}^{x_3} \frac{1}{V_0(\xi)} d\xi, \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (77)$$

The adjoint function v_{0i}^3 is defined from (31) by $\nabla v_{0i}^3 = e^{g_i^3} V^{3\perp}$ in Ω_i , where $V^{3\perp} = (-\beta V_0, V_0, 0)^\top$ and g_i^3 solves (30). Then, using (74) in the linear system (30) leads to

$$\nabla g_i^3 = \begin{pmatrix} 0 \\ 0 \\ \frac{V_0'}{-V_0} \end{pmatrix} \implies g_i^3(x) = \ln \left(\frac{1}{V_0(x_3)} \right), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (78)$$

Afterwards, from substituting in (32) the function g_i^3 by its value in (78), we get

$$v_{0i}^3(x) = -\beta(x_1 - a_i) + (x_2 - b_i), \quad \forall x = (x_1, x_2, x_3) \in \Omega_i. \quad (79)$$

Therefore, given $(a_i, b_i, c_i) \in \Omega_i$, $\Psi_i^{2,3}$ is defined in Ω_i from (75), (77) and (79) as follows:

$$\begin{aligned} \Psi_i^{2,3} : \quad \Omega_i &\longrightarrow \mathbb{R}^3, \\ x \mapsto \Psi_i^{2,3}(x) &= \begin{pmatrix} \exp \left(-\frac{1}{\alpha_L \sqrt{1 + \beta^2}} (x_1 - a_i + \beta(x_2 - b_i)) \right) \\ \int_{c_i}^{x_3} \frac{1}{V_0(\xi)} d\xi \\ -\beta(x_1 - a_i) + (x_2 - b_i) \end{pmatrix}. \end{aligned} \quad (80)$$

Since the function $V_0 > 0$ in Ω_i , we check readily from (80) the injectivity of $\Psi_i^{2,3}$ in Ω_i announced by Theorem 3.5. Hence, if the detection coefficient $\mathcal{A}_i^1 \neq 0$ due to the presence in Ω_i of an unknown source $\lambda_{n_i}(t)\delta(x - S_{n_i})$, the sought source position S_{n_i} is the unique point within Ω_i that solves the last equation in (53), where $\Psi_i^{m,n} = \Psi_i^{2,3}$ given in (80).

7 Numerical experiments

We carry out numerical experiments in the case of a domain Ω defined by

$$\Omega = \{x = (x_1, x_2, x_3) \in (0, 10) \times (0, 1) \times (0, 1)\}, \quad (81)$$

wherein, we assume to be available the following $I = 4$ interfaces:

$$\Gamma_i = \{x = (2i, x_2, x_3) \in (0, 1)^2, \text{ for } i = 1, \dots, I\}. \quad (82)$$

We generate synthetic measurements on the interfaces $\Gamma_{i=1,\dots,I} \times (0, T)$ from solving the problem (1)-(5), where D is Bear's dispersion tensor introduced in (57) for two different types of flow: Flow of mean velocity vector $V = (0.5, 0, 0)^\top$ and flow of unidirectional velocity field $V(x) = (x_2(1 - x_2)x_3, 0, 0)^\top$. We used the coefficients $R = 0$, $\alpha_T = 0.02$, $\alpha_L = 0.3$ and $D_M = 0$. Regarding the source F in (5), we considered $N = 2$ point sources located at $S_1 = (1.2, 0.4, 0.3) \in \Omega_1$, $S_2 = (3.0, 0.6, 0.2) \in \Omega_2$ and loading in $(0, T)$:

$$\lambda_1(t) = \sin \left(\frac{2\pi}{T} t \right) \chi_{(0, \frac{T}{2})}(t) \quad \text{and} \quad \lambda_2(t) = \frac{5}{2} \sin \left(\frac{2\pi}{T} (t - \frac{T}{4}) \right) \chi_{(\frac{T}{4}, \frac{3T}{4})}(t). \quad (83)$$

Since along $\Gamma_{FA} = \{x = (x_1, x_2, 1) \in (0, 10) \times (0, 1)\}$ we have $\nu = (0, 0, 1)^\top$ and thus, $V \cdot \nu = 0$, it follows from (57) that $D\nu = \alpha_T \|V\|_2 \nu$ on Γ_{FA} . Therefore, in the carried out numerical experiments, we neglected the term $D\nu$ which means we do not use state measurements on $\Gamma_{FA} \times (0, T)$ in the computation of the coefficients v_{0i}^ℓ from (54).

We computed from (54) the detection coefficient \mathcal{A}_i^1 associated to all suspected sections $\Omega_{i=1,\dots,I}$. If $\mathcal{A}_i^1 \neq 0$, we localize the unknown position defining the detected source occurring in Ω_i from: **1.** Solving the linear system in (64), for the flow of mean velocity vector. **2.** Solving the last equation in (53), where $\Psi_i^{m,n} = \Psi_i^{2,3}$ given in (73), for the flow defined by the unidirectional velocity field. Besides, since we use $R = 0$, it follows from (53) that if there exists a source $\lambda_{n_i}(t)\delta(x - S_{n_i})$ occurring in Ω_i , then $\mathcal{A}_i^1 = \int_0^T \lambda_{n_i}(t)dt$. However, if there is no source occurring in Ω_i , then $\mathcal{A}_i^1 = 0$. We set $T = 20$ and using (83), we calculate the exact values of \mathcal{A}_i^1 in all suspected sections $\Omega_{i=1,\dots,I}$:

$$\mathcal{A}_1^1 = \frac{T}{\pi} \approx 6.37, \quad \mathcal{A}_2^1 = \frac{5T}{2\pi} \approx 15.92 \quad \text{and} \quad \mathcal{A}_3^1 = \mathcal{A}_4^1 = 0. \quad (84)$$

As far as the unknown data $u(\cdot, T)$ involved in (54) is concerned, we used data assimilation to determine an approximation $\hat{u}_i(\cdot, T)$ of $u(\cdot, T)$ in Ω_i . To this end, we solved the forward problem (55) whose, according to Lemma 5.2, the solution $\hat{u}_i(\cdot, t)$ converges to $u(\cdot, T)$ in Ω_i when t tends to T . We start by presenting the behaviour of the detection coefficient \mathcal{A}_i^1 computed from (54), where we replaced $u(\cdot, T)$ by $\hat{u}_i(\cdot, t)$ for $t \in (0, T)$.

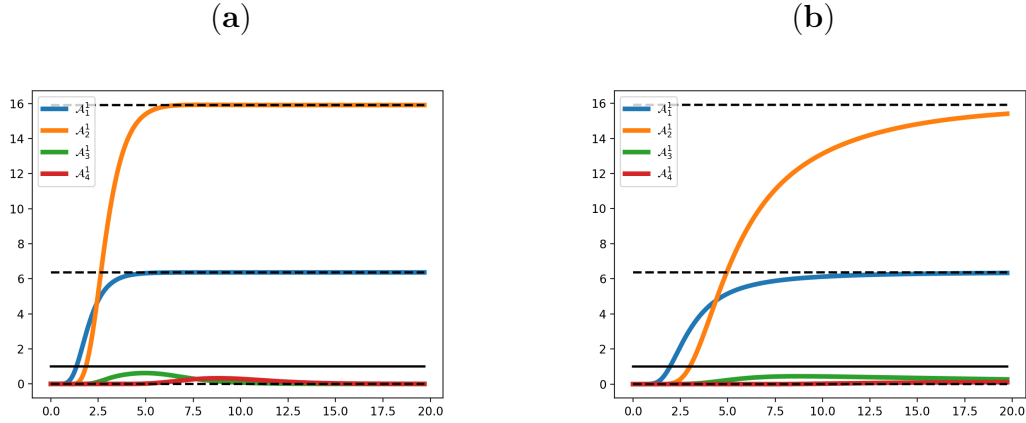


Figure 1: Behaviour of \mathcal{A}_i^1 : **(a)** Mean velocity vector **(b)** Unidirectional velocity field.

The results in Figure 1 show that for the flow of mean velocity vector, the detection coefficient \mathcal{A}_i^1 calculated from (54) converges quickly to its exact values given in (84) for all suspected sections $\Omega_{i=1,\dots,I}$. For the flow of velocity field, the calculated $\mathcal{A}_{i=1,2}^1$ in the two sections $\Omega_{i=1,2}$, wherein there is an occurring source, converge to their exact values when t is sufficiently close to $T = 20$. This observation could be explained, in view of Lemma 5.2, by the convergence rate of $\hat{u}_i(\cdot, t)$ to the unknown final state $u(\cdot, T)$.

For the two considered types of flow, we computed the coefficients in (54) and localized the positions of the detected unknown sources. In the sequel, we present the obtained results, where the dashed lines give the exact coordinates of each sought source position:

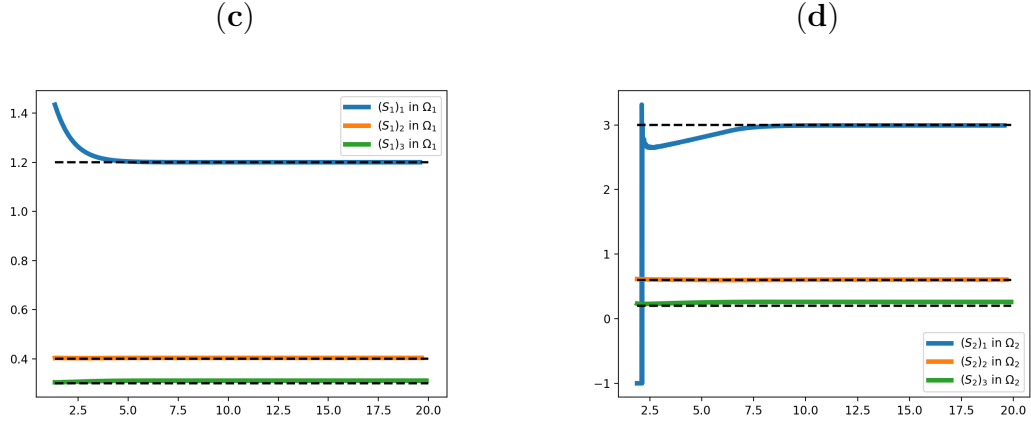


Figure 2: Flow of mean velocity: (c) Localized source in Ω_1 (d) Localized source in Ω_2 .

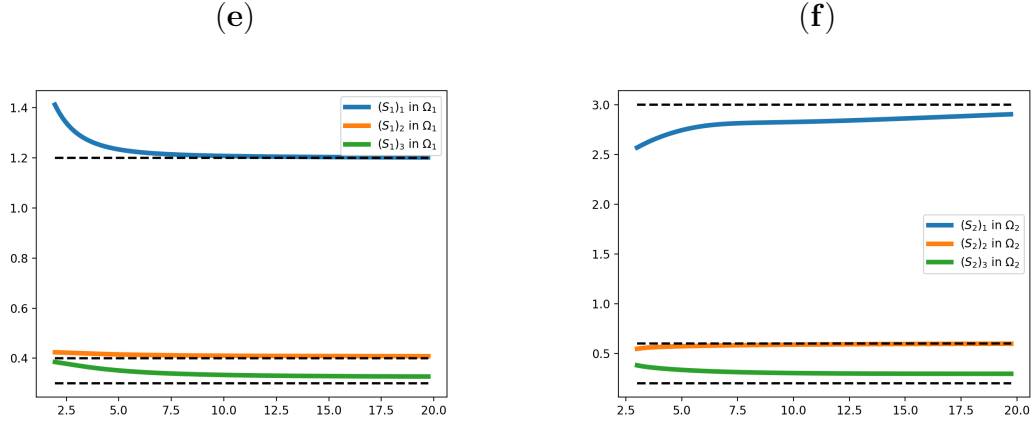


Figure 3: Flow of velocity field: (e) Localized source in Ω_1 (f) Localized source in Ω_2 .

The analysis of the numerical results presented in Figures 2-3 shows that for t sufficiently close to T , the used $\hat{u}_i(\cdot, t)$ yields a good approximation in Ω_i of $u(\cdot, T)$ which leads the developed method to localize accurately the unknown positions of all detected sources.

8 Baby Example

Here, we illustrate in the one dimensional case of the underlined inverse source problem the non-identifiability of multiple unknown point sources from measurements taken only upstream and/or downstream all occurring point sources.

Let $N \in \mathbb{N}^*$, $0 < \ell \in \mathbb{R}$ and u be the solution of the following problem:

$$-Du''(x) + Vu'(x) + Ru(x) = \sum_{n=1}^N \lambda_n \delta(x - S_n) \quad \text{in } (0, \ell) \quad \text{and} \quad u(0) = u'(\ell) = 0. \quad (85)$$

Provided $DV \neq 0$ and $V^2 + 4RD > 0$, the characteristic equation $-Dr^2 + Vr + R = 0$ admits the roots: $r_1 = (V/(2D))(1 + \sqrt{1 + 4RD/V^2})$ and $r_2 = (V/(2D))(1 - \sqrt{1 + 4RD/V^2})$. Let $\psi(x) = -(e^{r_1 x} - e^{r_2 x})/(D(r_1 - r_2))$, $\forall x \in (0, \ell)$. The solution u of (85) is given by

$$u(x) = \sum_{n=1}^N \lambda_n \left(\psi(x - S_n) \mathcal{H}(x - S_n) - \frac{\psi'(\ell - S_n)}{\psi'(\ell)} \psi(x) \right), \quad \forall x \in (0, \ell), \quad (86)$$

where \mathcal{H} is the Heaviside function. We introduce the following observation operator:

$$\begin{aligned} M : \quad & \left(\mathbb{R}^* \times (0, \ell) \right)^N \longrightarrow \mathbb{R}^{2N}, \\ & F = \left(\lambda_1, S_1, \dots, \lambda_N, S_N \right) \mapsto M[F] = \left(u(p_1), u(P_1), \dots, u(p_N), u(P_N) \right), \end{aligned} \quad (87)$$

where $p_{i=1, \dots, N}$ and $P_{i=1, \dots, N}$ are $2N$ distinct state measuring points within $(0, \ell)$.

Theorem 8.1 *Let D , V and R be real numbers such that $DV \neq 0$ and $V^2 + 4RD > 0$. The $2N \times 2N$ Jacobian matrix J_M of the observation operator $M[F]$ in (85)-(87) fulfills*

1. For $N = 1$: $\mathcal{H}(p_1 - S_1) \neq \mathcal{H}(P_1 - S_1) \iff \det(J_M) \neq 0$.
2. For $N = 2$: Let $0 < p_1 < P_1 < p_2 < P_2 < \ell$ and, for example, $S_1 < S_2$. We have

$$\begin{aligned} \text{(i) Cardinality} & \left(\{p_{i=1,2}, P_{i=1,2}\} \cap]S_1, S_2[\right) \leq 1 \implies \det(J_M) = 0, \\ \text{(ii)} & \quad 0 < p_1 < S_1 < P_1 < p_2 < S_2 < P_2 < \ell \implies \det(J_M) \neq 0. \end{aligned} \quad (88)$$

Proof. See the Appendix.

Theorem 8.1 indicates that for $N = 1$ point source $\lambda \delta(x - S)$ occurring in $(0, \ell)$, two state measuring points p_1, P_1 framing the source region i.e., $0 < p_1 < S < P_1 < \ell$ yield local injectivity of the observation operator $M[F]$ in (85)-(87) and thus, local identifiability of the two source elements λ, S . However, for $N = 2$ point sources occurring in $(0, \ell)$, the local identifiability of the four source elements holds only if in addition to the two state measuring points framing the sources region, two other state measuring points are set between the two source positions. Moreover, using similar computations, we prove that the two additional state measuring points that should separate the two occurring source positions can be replaced by one state and flux measuring point whereas the two measuring points p_1 and P_1 framing the sources region can be placed at the two boundary points i.e., $p_1 = 0$ and $P_1 = \ell$ if at these points both the state and its flux are known.

9 Conclusion, discussion and comparison

In this paper, we developed appropriate adjoint functions that led to establish a constructive Detection-Identification method for solving the nonlinear inverse source problem of identifying multiple unknown time-dependent point sources occurring in $3D$ transport equations. In the literature, many authors have addressed similar inverse source problems in different PDEs, for instance [2, 3, 4, 18, 19, 20, 31]. In these works, the developed identification approaches are mainly either iterative based on the minimisation of cost functions such as least squares and Kohn-Vogelius or quasi-direct such as the algebraic method especially used for elliptic equations [1, 32]. Besides, the underlined inverse source problem becomes more challenging in $n = 2, 3$ dimensions where the involved PDE admits an advection term. Indeed, this asymmetric term creates hydrodynamic dispersion as a consequence of molecular diffusion and mechanical dispersion caused by non-uniform velocities [8, 9]. Consequently, in such PDEs the effectiveness of a developed identification approach relies on how much it takes into account the impact of these physical phenomena on the used measurements: For example, in the case of an advection dominant flow, using measurements taken only upstream all occurring unknown sources do not yield identifiability as well as in the case of high dispersion coefficients, the signals emitted by different sources might get rapidly intermixed and thus, without using measurements taken between the occurring sources these latest become indistinguishable. These assertions are illustrated in the one dimensional case by the Baby Example of section 8. Moreover, the authors in [3] used measurements taken on the whole boundary to address the identification of multiple time-dependent point sources in $2D$ advection-diffusion equation from minimising two different cost functions i.e., least squares and Kohn-Vogelius. They reported that the least squares approach doesn't enable to identify more than one active source whereas the Kohn-Vogelius approach identifies two sources only if they are well separated and the diffusion coefficient is very small in order to avoid rapid intermixing.

Therefore, comparing with identification approaches from the literature, it follows that:

1. The constructive Detection-Identification method developed in the present paper takes into account the velocity field effects on the flow nature, is quasi-direct and thus easy to implement and does not have to deal always with the classic issues of iterative approaches i.e., questions related to the choices of an initial iterate and a regularisation term as well as to ensure convergence.
2. The herein developed method is based on a direct detection procedure that determines whether within the monitored domain a single or rather multiple unknown point sources occur and provides a framing/approximation of the total amount discharged by all occurring sources. These results lead to *i) Reducing the total identification cost*: In the case of no significant unknown sources i.e., sources discharging a total amount smaller than a certain tolerance or the case where the immediate aim is rather to know how important is the total discharged amount in order to take prompt appropriate actions, we can choose to not go ahead with the identification of the remaining unknown elements defining the occurring sources. *ii) Avoiding misinterpretations*: In the case of multiple unknown point sources, the procedure detects the presence of more than one source and suggests to add measurements taken between the occurring sources in order to ensure their identifiability. However, as

reported in [3], in this case usually the iterative approaches give identification results that could be inaccurate and thus, leading to misinterpretations.

3. For the identification approach developed in this paper, the construction of the appropriate adjoint functions defining the dispersion-current function $\Psi_i^{m,n}$ relies on some conditions that should be fulfilled by the velocity field V and the dispersion tensor D . In practice, failing those conditions, one could employ rather approximations of V and D that ensure the applicability of the developed approach.

Acknowledgement: This work was supported by M2SINUM project co-financed by the European fund(ERDF,18P03390/18E01750/18P02733) and the Normandie Regional Council.

Appendix.

A. We start by establishing the proof of the results announced in Lemmas 3.1, 3.4 and 5.2.

I. Proof of Lemma 3.1: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two elements of Ω_i . In view of (19), from writing $\psi_i(x) = \psi_i(y)$ under the form $\psi_i(y) - \psi_i(x) = 0$ and treating in this latest equation separately each two integrals following the same axis, we obtain:

- The two integrals with respect to the variable η in the equation $\psi_i(y) - \psi_i(x) = 0$:

$$\begin{aligned} \int_{a_i}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta - \int_{a_i}^{y_1} ((D^{-1})_1 \cdot V)(\eta, y_2, y_3) d\eta &= \int_{y_1}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta \\ + \int_{a_i}^{y_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta - \int_{a_i}^{y_1} ((D^{-1})_1 \cdot V)(\eta, y_2, y_3) d\eta. \end{aligned} \quad (89)$$

The first integral in the right hand side of the equation (89) is obtained from using Chasles's relation by introducing y_1 on the integral over (a_i, x_1) in the left hand side of (89). Moreover, by adding and substracting the following integral:

$$\int_{a_i}^{y_1} ((D^{-1})_1 \cdot V)(\eta, y_2, x_3) d\eta, \quad (90)$$

to the right hand side of the equation in (89), we get

$$\begin{aligned} \int_{a_i}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta - \int_{a_i}^{y_1} ((D^{-1})_1 \cdot V)(\eta, y_2, y_3) d\eta &= \int_{y_1}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta \\ + \int_{a_i}^{y_1} \left[((D^{-1})_1 \cdot V)(\eta, \xi, x_3) \right]_{\xi=y_2}^{\xi=x_2} d\eta + \int_{a_i}^{y_1} \left[((D^{-1})_1 \cdot V)(\eta, y_2, \zeta) \right]_{\zeta=y_3}^{\zeta=x_3} d\eta. \end{aligned} \quad (91)$$

Thus, using Fubini's Theorem in the two last integrals of the right hand side in (91) gives

$$\begin{aligned} \int_{a_i}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta - \int_{a_i}^{y_1} ((D^{-1})_1 \cdot V)(\eta, y_2, y_3) d\eta &= \int_{y_1}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta \\ + \int_{y_2}^{x_2} \int_{a_i}^{y_1} \partial_{\xi}((D^{-1})_1 \cdot V)(\eta, \xi, x_3) d\eta d\xi + \int_{y_3}^{x_3} \int_{a_i}^{y_1} \partial_{\zeta}((D^{-1})_1 \cdot V)(\eta, y_2, \zeta) d\eta d\zeta. \end{aligned} \quad (92)$$

In addition, since $\overrightarrow{\text{rot}}(D^{-1}V) = \vec{0}$ in Ω_i , it follows from (18) that

$$\begin{aligned} * \partial_{\xi}((D^{-1})_1 \cdot V)(\eta, \xi, x_3) &= \partial_{\eta}((D^{-1})_2 \cdot V)(\eta, \xi, x_3), \\ * \partial_{\zeta}((D^{-1})_1 \cdot V)(\eta, y_2, \zeta) &= \partial_{\eta}((D^{-1})_3 \cdot V)(\eta, y_2, \zeta). \end{aligned} \quad (93)$$

Hence, from replacing the partial derivatives in (92) by their values in (93), we obtain

$$\begin{aligned} \int_{a_i}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta - \int_{a_i}^{y_1} ((D^{-1})_1 \cdot V)(\eta, y_2, y_3) d\eta &= \int_{y_1}^{x_1} ((D^{-1})_1 \cdot V)(\eta, x_2, x_3) d\eta \\ + \int_{y_2}^{x_2} \left[((D^{-1})_2 \cdot V)(\eta, \xi, x_3) \right]_{\eta=a_i}^{\eta=y_1} d\xi + \int_{y_3}^{x_3} \left[((D^{-1})_3 \cdot V)(\eta, y_2, \zeta) \right]_{\eta=a_i}^{\eta=y_1} d\zeta. \end{aligned} \quad (94)$$

- The two integrals with respect to the variable ξ in the equation $\psi_i(y) - \psi_i(x) = 0$:

$$\begin{aligned} \int_{b_i}^{x_2} ((D^{-1})_2 \cdot V)(a_i, \xi, x_3) d\xi - \int_{b_i}^{y_2} ((D^{-1})_2 \cdot V)(a_i, \xi, y_3) d\xi \\ = \int_{y_2}^{x_2} ((D^{-1})_2 \cdot V)(a_i, \xi, x_3) d\xi + \int_{b_i}^{y_2} \left[((D^{-1})_2 \cdot V)(a_i, \xi, \zeta) \right]_{\zeta=y_3}^{\zeta=x_3} d\xi, \\ = \int_{y_2}^{x_2} ((D^{-1})_2 \cdot V)(a_i, \xi, x_3) d\xi + \int_{y_3}^{x_3} \int_{b_i}^{y_2} \partial_{\zeta}((D^{-1})_2 \cdot V)(a_i, \xi, \zeta) d\xi d\zeta. \end{aligned} \quad (95)$$

The right hand side of the first equality in (95) is obtained from its left hand side term by introducing, according to Chasles's relation, y_2 on the integral over (b_i, x_2) . Moreover, since $\overrightarrow{rot}(D^{-1}V) = \vec{0}$ in Ω_i and in view of (18), it follows that

$$\partial_\zeta((D^{-1})_2 \cdot V)(a_i, \xi, \zeta) = \partial_\xi((D^{-1})_3 \cdot V)(a_i, \xi, \zeta). \quad (96)$$

Thus, using (96) to replace the partial derivative in the last integral of (95) leads to

$$\begin{aligned} & \int_{b_i}^{x_2} ((D^{-1})_2 \cdot V)(a_i, \xi, x_3) d\xi - \int_{b_i}^{y_2} ((D^{-1})_2 \cdot V)(a_i, \xi, y_3) d\xi \\ &= \int_{y_2}^{x_2} ((D^{-1})_2 \cdot V)(a_i, \xi, x_3) d\xi + \int_{y_3}^{x_3} \left[((D^{-1})_3 \cdot V)(a_i, \xi, \zeta) \right]_{\xi=b_i}^{\xi=y_2} d\zeta. \end{aligned} \quad (97)$$

- The two integrals with respect to the variable ζ in the equation $\psi_i(y) - \psi_i(x) = 0$:

$$\int_{c_i}^{x_3} ((D^{-1})_3 \cdot V)(a_i, b_i, \zeta) d\zeta - \int_{c_i}^{y_3} ((D^{-1})_3 \cdot V)(a_i, b_i, \zeta) d\zeta = \int_{y_3}^{x_3} ((D^{-1})_3 \cdot V)(a_i, b_i, \zeta) d\zeta. \quad (98)$$

Therefore, since $\psi_i(y) - \psi_i(x)$ is equal to the sum of the left hand side terms in the three equations (94), (97) and (98) then, from setting the sum of the right hand side terms of those equations to null, we find the result announced in Lemma 3.1. \blacksquare

II. Proof of Lemma 3.4: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two elements of Ω_i . In view of (32), it follows that:

- The first equation $v_{0i}^1(x) = v_{0i}^1(y)$ is equivalent to

$$\begin{aligned} & \int_{y_3}^{x_3} (e^{g_i^1} V_2)(a_i, b_i, \xi) d\xi = \int_{b_i}^{x_2} (e^{g_i^1} V_3)(a_i, \eta, x_3) d\eta - \int_{b_i}^{y_2} (e^{g_i^1} V_3)(a_i, \eta, y_3) d\eta, \\ &= \int_{b_i}^{x_2} (e^{g_i^1} V_3)(a_i, \eta, x_3) d\eta + \int_{b_i}^{y_2} \left[(e^{g_i^1} V_3)(a_i, \eta, \xi) \right]_{\xi=y_3}^{\xi=x_3} d\eta, \\ &= \int_{y_2}^{x_2} (e^{g_i^1} V_3)(a_i, \eta, x_3) d\eta + \int_{y_3}^{x_3} \int_{b_i}^{y_2} \partial_\xi \left((e^{g_i^1} V_3)(a_i, \eta, \xi) \right) d\eta d\xi, \\ &= \int_{y_2}^{x_2} (e^{g_i^1} V_3)(a_i, \eta, x_3) d\eta - \int_{y_3}^{x_3} \left(\int_{b_i}^{y_2} \partial_\eta \left((e^{g_i^1} V_2)(a_i, \eta, \xi) \right) d\eta \right) d\xi. \end{aligned} \quad (99)$$

The second equality in (99) is obtained using Chasles's relation by adding the point y_2 on the interval of integration (b_i, x_2) . The third equality is found using Fubini's theorem by integrating first with respect to the variable η then, with respect to ξ . The Fourth equality is obtained from the compatibility condition $\overrightarrow{rot}(e^{g_i^1} V^{1\perp}) = \vec{0}$ in Ω_i which implies that $\partial_{x_3}(e^{g_i^1} V_3) = -\partial_{x_2}(e^{g_i^1} V_2)$ in Ω_i . Therefore, by computing the integral over $d\eta$ in the right hand side of the last equation of (99) and then, cancelling out the two same terms occurring in both sides of this equation, we obtain

$$v_{0i}^1(x) = v_{0i}^1(y) \Leftrightarrow \int_{y_2}^{x_2} (e^{g_i^1} V_3)(a_i, \eta, x_3) d\eta = \int_{y_3}^{x_3} (e^{g_i^1} V_2)(a_i, y_2, \xi) d\xi. \quad (100)$$

- The second equation $v_{0i}^2(x) = v_{0i}^2(y)$ is equivalent to

$$\begin{aligned} & \int_{y_3}^{x_3} (e^{g_i^2} V_1)(a_i, b_i, \xi) d\xi = \int_{a_i}^{x_1} (e^{g_i^2} V_3)(\eta, b_i, x_3) d\eta - \int_{a_i}^{y_1} (e^{g_i^2} V_3)(\eta, b_i, y_3) d\eta, \\ &= \int_{a_i}^{x_1} (e^{g_i^2} V_3)(\eta, b_i, x_3) d\eta + \int_{a_i}^{y_1} \left[(e^{g_i^2} V_3)(\eta, b_i, \xi) \right]_{\xi=y_3}^{\xi=x_3} d\eta, \\ &= \int_{y_1}^{x_1} (e^{g_i^2} V_3)(\eta, b_i, x_3) d\eta + \int_{y_3}^{x_3} \int_{a_i}^{y_1} \partial_\xi \left((e^{g_i^2} V_3)(\eta, b_i, \xi) \right) d\eta d\xi, \\ &= \int_{y_1}^{x_1} (e^{g_i^2} V_3)(\eta, b_i, x_3) d\eta - \int_{y_3}^{x_3} \left(\int_{a_i}^{y_1} \partial_\eta \left((e^{g_i^2} V_1)(\eta, b_i, \xi) \right) d\eta \right) d\xi. \end{aligned} \quad (101)$$

The second equality in (101) is obtained using Chasles's relation by adding the point y_1 on the interval of integration (a_i, x_1) . The third equality is found using Fubini's theorem by integrating first with respect to the variable η then, with respect to ξ . The Fourth equality is obtained from the compatibility condition $\overrightarrow{rot}(e^{g_i^2} V^{2\perp}) = \vec{0}$ in Ω_i which implies that $\partial_{x_3}(e^{g_i^2} V_3) = -\partial_{x_1}(e^{g_i^2} V_1)$ in Ω_i . Hence, by computing the integral over $d\eta$ in the right hand side of the last equation of (101) and then, cancelling out the two same terms occurring in both sides of this equation, we get

$$v_{0i}^2(x) = v_{0i}^2(y) \Leftrightarrow \int_{y_1}^{x_1} (e^{g_i^2} V_3)(\eta, b_i, x_3) d\eta = \int_{y_3}^{x_3} (e^{g_i^2} V_1)(y_1, b_i, \xi) d\xi. \quad (102)$$

• The third equation $v_{0i}^3(x) = v_{0i}^3(y)$ is equivalent to

$$\begin{aligned} \int_{y_2}^{x_2} (e^{g_i^3} V_1)(a_i, \xi, c_i) d\xi &= \int_{a_i}^{x_1} (e^{g_i^3} V_2)(\eta, x_2, c_i) d\eta - \int_{a_i}^{y_1} (e^{g_i^3} V_2)(\eta, y_2, c_i) d\eta, \\ &= \int_{y_1}^{x_1} (e^{g_i^3} V_2)(\eta, x_2, c_i) d\eta + \int_{a_i}^{y_1} \left[(e^{g_i^3} V_2)(\eta, \xi, c_i) \right]_{\xi=y_2}^{\xi=x_2} d\eta, \\ &= \int_{y_1}^{x_1} (e^{g_i^3} V_2)(\eta, x_2, c_i) d\eta + \int_{y_2}^{x_2} \int_{a_i}^{y_1} \partial_\xi \left((e^{g_i^3} V_2)(\eta, \xi, c_i) \right) d\eta d\xi, \\ &= \int_{y_1}^{x_1} (e^{g_i^3} V_2)(\eta, x_2, c_i) d\eta - \int_{y_2}^{x_2} \left(\int_{a_i}^{y_1} \partial_\eta \left((e^{g_i^3} V_1)(\eta, \xi, c_i) \right) d\eta \right) d\xi. \end{aligned} \quad (103)$$

The second equality in (103) is obtained using Chasles's relation by adding the point y_1 on the interval of integration (a_i, x_1) . The third equality is found using Fubini's theorem by integrating first with respect to the variable η then, with respect to ξ . The Fourth equality is obtained from the compatibility condition $\overrightarrow{rot}(e^{g_i^3} V^{3\perp}) = \vec{0}$ in Ω_i which implies that $\partial_{x_2}(e^{g_i^3} V_2) = -\partial_{x_1}(e^{g_i^3} V_1)$ in Ω_i . Thus, by computing the integral over $d\eta$ in the right hand side of the last equation of (103) and then, cancelling out the two same terms occurring in both sides of this equation, we find

$$v_{0i}^3(x) = v_{0i}^3(y) \Leftrightarrow \int_{y_1}^{x_1} (e^{g_i^3} V_2)(\eta, x_2, c_i) d\eta = \int_{y_2}^{x_2} (e^{g_i^3} V_1)(y_1, \xi, c_i) d\xi. \quad (104)$$

Hence, (100), (102) and (104) lead to the result announced in (36). ■

III. Proof of Lemma 5.2: Let $\tilde{u}_i = \hat{u}_i - u$ in $\Omega_i \times (0, T)$. In view of (6), \tilde{u}_i solves

$$\begin{cases} L[\tilde{u}_i] = 0 & \text{in } \Omega_i \times (T^0, T), \\ \tilde{u}_i(\cdot, T^0) = \hat{u}_i(\cdot, T^0) - u(\cdot, T^0) & \text{in } \Omega_i, \\ \tilde{u}_i = 0 & \text{on } (\Gamma_{i-1} \cup \Gamma_i) \times (T^0, T), \\ D\nabla \tilde{u}_i \cdot \nu = 0 & \text{on } (\partial\Omega_i \setminus (\Gamma_{i-1} \cup \Gamma_i)) \times (T^0, T). \end{cases} \quad (105)$$

Multiplying the first equation of (105) by \tilde{u}_i and integrating by parts in Ω_i using Green's formula and (3) gives: For all $t \in (T^0, T)$,

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}^2 + R \|\tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}^2 + \int_{\Omega_i} \left(D\nabla \tilde{u}_i(\cdot, t) - \tilde{u}_i(\cdot, t) V \right) \cdot \nabla \tilde{u}_i(\cdot, t) = 0. \quad (106)$$

Since, from (3), $\text{div}(V) = 0$ in Ω , it follows that $\text{div}(V\tilde{u}_i) = V \cdot \nabla \tilde{u}_i$ in Ω . Afterwards, by

applying an integration by parts and using Green's formula, we get

$$\begin{cases} \int_{\Omega_i} \tilde{u}_i(\cdot, t) V \cdot \nabla \tilde{u}_i(\cdot, t) = \int_{\Omega_i} \operatorname{div}(V \tilde{u}_i(\cdot, t)) \tilde{u}_i(\cdot, t) = - \int_{\Omega_i} \tilde{u}_i(\cdot, t) V \cdot \nabla \tilde{u}_i(\cdot, t) + \int_{\partial\Omega_i} \tilde{u}_i^2(\cdot, t) V \cdot \nu, \\ \int_{\partial\Omega_i} \tilde{u}_i^2(\cdot, t) V \cdot \nu = \int_{\Gamma_{i-1} \cup \Gamma_i} \tilde{u}_i^2(\cdot, t) V \cdot \nu + \int_{\partial\Omega_i \setminus (\Gamma_{i-1} \cup \Gamma_i)} \tilde{u}_i^2(\cdot, t) V \cdot \nu = 0, \end{cases} \quad (107)$$

$$\implies \int_{\Omega_i} \tilde{u}_i(\cdot, t) V \cdot \nabla \tilde{u}_i(\cdot, t) = 0, \quad \forall t \in (T^0, T). \quad (108)$$

Because the matrix D is uniformly elliptic in Ω and using Poincaré inequality, we obtain

$$\begin{cases} \exists \gamma > 0, \quad \gamma \|\nabla \tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}^2 \leq \int_{\Omega_i} D \nabla \tilde{u}_i(\cdot, t) \cdot \nabla \tilde{u}_i(\cdot, t), \\ \exists C_p > 0, \quad \|\tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)} \leq C_p \|\nabla \tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}, \end{cases} \quad (109)$$

$$\implies \frac{\gamma}{C_p^2} \|\tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}^2 \leq \int_{\Omega_i} D \nabla \tilde{u}_i(\cdot, t) \cdot \nabla \tilde{u}_i(\cdot, t), \quad \forall t \in (T^0, T). \quad (110)$$

Therefore, in view of (108)-(110), it follows from (106) that

$$\frac{d}{dt} \|\tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}^2 \leq -2(R + \frac{\gamma}{C_p^2}) \|\tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}^2, \quad \forall t \in (T^0, T). \quad (111)$$

Afterwards, by applying Gronwall's Lemma on (111), we get

$$\|\tilde{u}_i(\cdot, t)\|_{L^2(\Omega_i)}^2 \leq \|\tilde{u}_i(\cdot, T^0)\|_{L^2(\Omega_i)}^2 \exp\left(-2(R + \frac{\gamma}{C_p^2})(t - T^0)\right), \quad \forall t \in (T^0, T). \quad (112)$$

That is the result announced in (56) for $\alpha = 2(R + \frac{\gamma}{C_p^2})$. ■

B. Baby Example: To establish the proof of Theorem 8.1, we prove the following two technical lemmas:

Lemma 9.1 *Let ψ be the function involved in (86) and $(n, m) \in \{1, \dots, N\}^2$. The following three functions defined in $(0, \ell)^2$ from ψ by*

$$\begin{aligned} \bullet \varphi_{\lambda S}^{n,m}(x, y) &= \frac{\psi(y)}{\psi'(\ell)} \left(\psi(x - S_n) \psi''(\ell - S_m) - \psi'(x - S_m) \psi'(\ell - S_n) \right) - \psi(x - S_n) \psi'(y - S_m), \\ \bullet \varphi_{SS}^{n,m}(x, y) &= \frac{\psi(y)}{\psi'(\ell)} \left(\psi'(x - S_m) \psi''(\ell - S_n) - \psi'(x - S_n) \psi''(\ell - S_m) \right) + \psi'(x - S_n) \psi'(y - S_m), \\ \bullet \varphi_{\lambda\lambda}^{n,m}(x, y) &= \frac{\psi(y)}{\psi'(\ell)} \left(\psi(x - S_m) \psi'(\ell - S_n) - \psi(x - S_n) \psi'(\ell - S_m) \right) + \psi(x - S_n) \psi(y - S_m), \end{aligned}$$

are symmetric: $\varphi_{\lambda S}^{n,m}(x, y) = \varphi_{\lambda S}^{n,m}(y, x), \forall (x, y) \in (0, \ell)^2$. The same holds for $\varphi_{SS}^{n,m}, \varphi_{\lambda\lambda}^{n,m}$.

Proof. Let $\zeta_{i,j}^{n,m}(x, y) = D^2(r_1 - r_2)^2 \psi^{(i)}(x - S_n) \psi^{(j)}(y - S_m)$, where $(n, m) \in \{1, \dots, N\}^2$, $(i, j) \in \mathbb{N}^2$ and $\psi^{(k=i,j)}$ is the k^{th} derivative of the function ψ in (86). That leads to

$$\zeta_{i,j}^{n,m}(x, y) = \left(r_1^i e^{r_1(x-S_n)} - r_2^i e^{r_2(x-S_n)} \right) \left(r_1^j e^{r_1(y-S_m)} - r_2^j e^{r_2(y-S_m)} \right). \quad (113)$$

Afterwards, from setting $y = \ell$ in (113), it follows that: For all $i \in \mathbb{N}$ and $j \in \mathbb{N}^*$,

$$\zeta_{i,j}^{n,m}(x, \ell) - \zeta_{j-1,i+1}^{m,n}(x, \ell) = \beta_{i,j}^{n,m} \left(r_2 e^{r_1 x + r_2 \ell} - r_1 e^{r_2 x + r_1 \ell} \right), \quad (114)$$

where $\beta_{i,j}^{n,m} = r_1^{j-1} r_2^i e^{-r_1 S_m - r_2 S_n} - r_1^i r_2^{j-1} e^{-r_1 S_n - r_2 S_m}$. Then, using (114), we get

$$\begin{aligned} \frac{\psi(y)}{\psi'(\ell)} \left(\zeta_{i,j}^{n,m}(x, \ell) - \zeta_{j-1,i+1}^{m,n}(x, \ell) \right) &= \hat{\beta}_{i,j}^{n,m} \left(r_2 e^{r_1(x+y)+r_2 \ell} + r_1 e^{r_2(x+y)+r_1 \ell} \right) \\ &- \hat{\beta}_{i,j}^{n,m} \left(r_1 e^{r_1 y + r_2 x + r_1 \ell} + r_2 e^{r_1 x + r_2 y + r_2 \ell} \right), \end{aligned} \quad (115)$$

with $\hat{\beta}_{i,j}^{n,m} = \beta_{i,j}^{n,m} / (r_1 e^{r_1 \ell} - r_2 e^{r_2 \ell})$. Furthermore, from (113), we verify that

$$\begin{aligned} \zeta_{i,j-1}^{n,m}(x, y) &= \left(r_1^i e^{r_1(x-S_n)} - r_2^i e^{r_2(x-S_n)} \right) \left(r_1^{j-1} e^{r_1(y-S_m)} - r_2^{j-1} e^{r_2(y-S_m)} \right) \\ &= r_1^{i+j-1} e^{r_1(x+y-S_n-S_m)} + r_2^{i+j-1} e^{r_2(x+y-S_n-S_m)} \\ &- r_1^i r_2^{j-1} e^{r_1 x + r_2 y - r_1 S_n - r_2 S_m} - r_1^{j-1} r_2^i e^{r_1 y + r_2 x - r_1 S_m - r_2 S_n}. \end{aligned} \quad (116)$$

Therefore, for all $(i, j) \in \mathbb{N} \times \mathbb{N}^*$ and $(x, y) \in (0, \ell)^2$, it follows from (115)-(116) that

$$\begin{aligned} \frac{\psi(y)}{\psi'(\ell)} \left(\zeta_{i,j}^{n,m}(x, \ell) - \zeta_{j-1,i+1}^{m,n}(x, \ell) \right) - \zeta_{i,j-1}^{n,m}(x, y) &= \hat{\beta}_{i,j}^{n,m} \left(r_2 e^{r_1(x+y)+r_2 \ell} + r_1 e^{r_2(x+y)+r_1 \ell} \right) \\ &- \left(r_1^{i+j-1} e^{r_1(x+y-S_n-S_m)} + r_2^{i+j-1} e^{r_2(x+y-S_n-S_m)} \right) \\ &+ \frac{r_1^{i+1} r_2^{j-1} e^{r_1 \ell - r_1 S_n - r_2 S_m} - r_1^{j-1} r_2^{i+1} e^{r_2 \ell - r_1 S_m - r_2 S_n}}{r_1 e^{r_1 \ell} - r_2 e^{r_2 \ell}} \left(e^{r_1 x + r_2 y} + e^{r_1 y + r_2 x} \right). \end{aligned} \quad (117)$$

Besides, using $\zeta_{i,j}^{n,m}$, the functions introduced in Lemma 9.1 can be rewritten as follows:

$$\begin{aligned} * \varphi_{\lambda S}^{n,m}(x, y) &= \frac{1}{D^2(r_1 - r_2)^2} \left(\frac{\psi(y)}{\psi'(\ell)} \left(\zeta_{0,2}^{n,m}(x, \ell) - \zeta_{1,1}^{m,n}(x, \ell) \right) - \zeta_{0,1}^{n,m}(x, y) \right), \\ * \varphi_{SS}^{n,m}(x, y) &= -\frac{1}{D^2(r_1 - r_2)^2} \left(\frac{\psi(y)}{\psi'(\ell)} \left(\zeta_{1,2}^{n,m}(x, \ell) - \zeta_{1,2}^{m,n}(x, \ell) \right) - \zeta_{1,1}^{n,m}(x, y) \right), \\ * \varphi_{\lambda\lambda}^{n,m}(x, y) &= -\frac{1}{D^2(r_1 - r_2)^2} \left(\frac{\psi(y)}{\psi'(\ell)} \left(\zeta_{0,1}^{n,m}(x, \ell) - \zeta_{0,1}^{m,n}(x, \ell) \right) - \zeta_{0,0}^{n,m}(x, y) \right). \end{aligned} \quad (118)$$

Hence, the symmetry of the functions $\varphi_{\lambda S}^{n,m}$, $\varphi_{SS}^{n,m}$ and $\varphi_{\lambda\lambda}^{n,m}$ in (118) is an immediate consequence of the symmetry with respect to x and y of the right hand side in (117). ■

Lemma 9.2 Let $(n, m) \in \{1, \dots, N\}^2$ and $(p_i, P_i) \in (0, \ell)^2$. If p_i and P_i lie in a same side with respect to $\{S_n, S_m\}$ i.e., $\mathcal{H}(p_i - S_n) = \mathcal{H}(p_i - S_m) = \mathcal{H}(P_i - S_n) = \mathcal{H}(P_i - S_m)$, then we have

$$\begin{aligned} &\bullet \partial_{\lambda_p} u(p_i) \partial_{S_q} u(P_i) - \partial_{S_q} u(p_i) \partial_{\lambda_p} u(P_i) = 0, \quad \forall (p, q) \in \{n, m\}^2, \\ &\bullet \partial_{S_n} u(p_i) \partial_{S_m} u(P_i) - \partial_{S_m} u(p_i) \partial_{S_n} u(P_i) = 0, \\ &\bullet \partial_{\lambda_n} u(p_i) \partial_{\lambda_m} u(P_i) - \partial_{\lambda_m} u(p_i) \partial_{\lambda_n} u(P_i) = 0, \end{aligned} \quad (119)$$

where \mathcal{H} is the Heaviside function and u is the solution of (85) given in (86).

Proof. Let $(p_i, P_i) \in (0, \ell)^2$ be such that $\mathcal{H}(p_i - S_n) = \mathcal{H}(p_i - S_m)$ and $\mathcal{H}(P_i - S_n) = \mathcal{H}(P_i - S_m)$. Then, from (86), it follows that: For all $(p, q) \in \{n, m\}^2$,

$$\begin{aligned} * \frac{1}{\lambda_q} \left(\partial_{\lambda_p} u(p_i) \partial_{S_q} u(P_i) - \partial_{S_q} u(p_i) \partial_{\lambda_p} u(P_i) \right) &= \\ &\left(\varphi_{\lambda S}^{p,q}(p_i, P_i) + \psi(p_i - S_p) \psi'(P_i - S_q) [1 - \mathcal{H}(P_i - S_p)] \right) \mathcal{H}(p_i - S_p) \\ &- \left(\varphi_{\lambda S}^{p,q}(P_i, p_i) + \psi(P_i - S_p) \psi'(p_i - S_q) [1 - \mathcal{H}(p_i - S_p)] \right) \mathcal{H}(P_i - S_p), \end{aligned} \quad (120)$$

$$\begin{aligned}
* \frac{1}{\lambda_n \lambda_m} & \left(\partial_{S_n} u(p_i) \partial_{S_m} u(P_i) - \partial_{S_m} u(p_i) \partial_{S_n} u(P_i) \right) = \\
& \left(\varphi_{SS}^{n,m}(p_i, P_i) - \psi'(p_i - S_n) \psi'(P_i - S_m) [1 - \mathcal{H}(P_i - S_n)] \right) \mathcal{H}(p_i - S_n) \quad (121) \\
& - \left(\varphi_{SS}^{n,m}(P_i, p_i) - \psi'(P_i - S_n) \psi'(p_i - S_m) [1 - \mathcal{H}(p_i - S_n)] \right) \mathcal{H}(P_i - S_n),
\end{aligned}$$

$$\begin{aligned}
* \partial_{\lambda_n} u(p_i) \partial_{\lambda_m} u(P_i) - \partial_{\lambda_m} u(p_i) \partial_{\lambda_n} u(P_i) = \\
\left(\varphi_{\lambda\lambda}^{n,m}(p_i, P_i) - \psi(p_i - S_n) \psi(P_i - S_m) [1 - \mathcal{H}(P_i - S_n)] \right) \mathcal{H}(p_i - S_n) \quad (122) \\
- \left(\varphi_{\lambda\lambda}^{n,m}(P_i, p_i) - \psi(P_i - S_n) \psi(p_i - S_m) [1 - \mathcal{H}(p_i - S_n)] \right) \mathcal{H}(P_i - S_n),
\end{aligned}$$

where $\varphi_{\lambda S}^{n,m}$, $\varphi_{SS}^{n,m}$ and $\varphi_{\lambda\lambda}^{n,m}$ are the three symmetric functions introduced in Lemma 9.1. Therefore, if p_i and P_i are both upstream S_n and S_m i.e., $\mathcal{H}(p_i - S_{n,m}) = \mathcal{H}(P_i - S_{n,m}) = 0$ then, all terms in (120)-(122) vanish. Moreover, if p_i and P_i are rather both downstream S_n and S_m i.e., $\mathcal{H}(p_i - S_{n,m}) = \mathcal{H}(P_i - S_{n,m}) = 1$ then, the terms in (120)-(122) vanish too due the symmetry of the three functions $\varphi_{\lambda S}^{n,m}$, $\varphi_{SS}^{n,m}$ and $\varphi_{\lambda\lambda}^{n,m}$. ■

Proof of Theorem 8.1:

1. For $N = 1$, the determinant of the 2×2 Jacobian matrix J_M is

$$\det(J_M) = \partial_{\lambda_1} u(p_1) \partial_{S_1} u(P_1) - \partial_{S_1} u(p_1) \partial_{\lambda_1} u(P_1). \quad (123)$$

From (123) and by applying (120) for $i = p = q = 1$, it follows due to the symmetry of the function $\varphi_{\lambda S}^{1,1}$ introduced in Lemma 9.1 that if $\mathcal{H}(p_1 - S_1) = \mathcal{H}(P_1 - S_1) = 0$ or $\mathcal{H}(p_1 - S_1) = \mathcal{H}(P_1 - S_1) = 1$ then, $\det(J_M) = 0$. However, if $\mathcal{H}(p_1 - S_1) \neq \mathcal{H}(P_1 - S_1)$ i.e., for example $0 < p_1 < S_1 < P_1 < \ell$ then, from applying (120) with $i = p = q = 1$ and since $r_1 > 0$ whereas $r_2 < 0$, we obtain

$$\det(J_M) = -\frac{\lambda_1}{D^2(r_1 - r_2)} \frac{e^{r_1 p_1} - e^{r_2 p_1}}{r_1 e^{r_1 \ell} - r_2 e^{r_2 \ell}} \left(r_2 e^{r_1 P_1 + r_2 \ell} - r_1 e^{r_2 P_1 + r_1 \ell} \right) e^{-S_1(r_1 + r_2)} \neq 0. \quad (124)$$

2. For $N = 2$: From computing the determinant of the 4×4 Jacobian matrix J_M by developing firstly with respect to its last row then, with respect to its third row, we find

$$\begin{aligned}
\det(J_M) &= \left(\partial_{\lambda_2} u(p_2) \partial_{S_2} u(P_2) - \partial_{S_2} u(p_2) \partial_{\lambda_2} u(P_2) \right) \left(\partial_{\lambda_1} u(p_1) \partial_{S_1} u(P_1) - \partial_{S_1} u(p_1) \partial_{\lambda_1} u(P_1) \right) \\
&- \left(\partial_{S_1} u(p_2) \partial_{S_2} u(P_2) - \partial_{S_2} u(p_2) \partial_{S_1} u(P_2) \right) \left(\partial_{\lambda_1} u(p_1) \partial_{\lambda_2} u(P_1) - \partial_{\lambda_2} u(p_1) \partial_{\lambda_1} u(P_1) \right) \\
&- \left(\partial_{\lambda_2} u(p_2) \partial_{S_1} u(P_2) - \partial_{S_1} u(p_2) \partial_{\lambda_2} u(P_2) \right) \left(\partial_{\lambda_1} u(p_1) \partial_{S_2} u(P_1) - \partial_{S_2} u(p_1) \partial_{\lambda_1} u(P_1) \right) \\
&- \left(\partial_{\lambda_2} u(p_1) \partial_{S_1} u(P_1) - \partial_{S_1} u(p_1) \partial_{\lambda_2} u(P_1) \right) \left(\partial_{\lambda_1} u(p_2) \partial_{S_2} u(P_2) - \partial_{S_2} u(p_2) \partial_{\lambda_1} u(P_2) \right) \\
&- \left(\partial_{S_1} u(p_1) \partial_{S_2} u(P_1) - \partial_{S_2} u(p_1) \partial_{S_1} u(P_1) \right) \left(\partial_{\lambda_1} u(p_2) \partial_{\lambda_2} u(P_2) - \partial_{\lambda_2} u(p_2) \partial_{\lambda_1} u(P_2) \right) \\
&+ \left(\partial_{\lambda_2} u(p_1) \partial_{S_2} u(P_1) - \partial_{S_2} u(p_1) \partial_{\lambda_2} u(P_1) \right) \left(\partial_{\lambda_1} u(p_2) \partial_{S_1} u(P_2) - \partial_{S_1} u(p_2) \partial_{\lambda_1} u(P_2) \right). \quad (125)
\end{aligned}$$

- (i) If **Cardinality** $\left(\{p_{i=1,2}, P_{i=1,2}\} \cap]S_1, S_2[\right) \leq 1$ and since $0 < p_1 < P_1 < p_2 < P_2 < \ell$ then, there exists $i \in \{1, 2\}$ such that p_i and P_i lie both in a same side with respect to $\{S_1, S_2\}$. Hence, by applying Lemma 9.2 for $n = 1$ and $m = 2$, it follows from (119) that all terms in (125) vanish which leads to $\det(J_M) = 0$.

- (ii) From setting $0 < p_1 < S_1 < P_1 < p_2 < S_2 < P_2 < \ell$ and using (86), it follows that the last term in (125) vanishes. Besides, by computing the other five terms, we obtain

$$\begin{aligned}
\det(J_M) &= \lambda_1 \lambda_2 \frac{\psi(p_1)}{\psi'(\ell)} \left(\psi'(P_1 - S_1) \psi(p_2 - S_1) - \psi(P_1 - S_1) \psi'(p_2 - S_1) \right) \\
&\quad \left(\psi'(\ell - S_2) \psi'(P_2 - S_2) - \psi''(\ell - S_2) \psi(P_2 - S_2) \right). \\
&= \lambda_1 \lambda_2 \frac{\psi(p_1)}{D^2 \psi'(\ell)} e^{-(r_1+r_2)(S_1+S_2)} \left(r_1 e^{r_1 \ell + r_2 P_2} - r_2 e^{r_2 \ell + r_1 P_2} \right) \left(e^{r_2 P_1 + r_1 P_2} - e^{r_1 P_1 + r_2 P_2} \right).
\end{aligned} \tag{126}$$

Since $r_1 > 0$ and $r_2 < 0$, it follows from the last equation in (126) that $\det(J_M) \neq 0$. ■

References

- [1] Abdelaziz B., El Badia A. and El Hajj A. (2015) Direct algorithms for solving some inverse source problems in 2D elliptic equations, *Inverse Problems* **31** 105002.
- [2] Andrle M., Ben Belgacem F. and El Badia A., (2011) Identification of Moving Pointwise Sources in an Advection-Dispersion-Reaction Equation, *Inverse Problems*, 27-025007.
- [3] Andrle M. and El Badia A. (2012) Identification of multiple moving pollution sources in surface waters or atmospheric media with boundary observations, *Inverse Problems*, 28 075009.
- [4] Ben Belgacem F. (2012) Uniqueness for an ill-posed reaction-dispersion model. Application to organic pollution in stream-waters (One-dimensional model), *Inverse Problems and Imaging* 6-2, pp. 163-81.
- [5] Andrea Hawkins-Daarud, Kristoffer G van der Zee, and J Tinsley Oden (2012) Numerical simulation of a thermodynamically consistent four-species tumor growth model. *International journal for numerical methods in biomedical engineering*, 28(1):324.
- [6] Andrés Quiroga, Damin Fernndez, Germn Torres, and Cristina Turner (2015) Adjoint method for a tumor invasion Pde-constrained optimization problem in 2D using adaptive finite element method. *Applied Mathematics and Computation*, 270:358368.
- [7] Amir Gholami, Andreas Mang, and George Biros (2016) An inverse problem formulation for parameter estimation of a reactiondiffusion model of low grade gliomas. *Journal of mathematical biology*, 72(1-2):409433.
- [8] Bear J. and Bachmat Y. (1991) *Introduction to Modeling of Transport Phenomena in Porous Media*, Kluwer Academic Publishers.
- [9] Bear J. and Cheng A.H.-D (2010) *Modeling Groundwater Flow and Contaminant Transport*, Springer.
- [10] Benoit J., Chauviere C. and Bonnet P. (2012) Source identification in time domain electromagnetics, *Journal of Computational Physics* 231(8):3446-3456.
- [11] Cheng-Hung Huang, Jia-Xun Li and Sin Kim (2008) An inverse problem in estimating the strength of contaminant source for groundwater systems, *Applied Mathematical Modelling* 32:417-431.

- [12] Cox B. A. (2003) A review of currently available in-stream water-quality models and their applicability for simulating dissolved oxygen in lowland rivers, *The Science of the Total Environment* 314-316, pp. 335-77.
- [13] El Badia A., Ha-Duong T., Hamdi A. (2005) Identification of a point source in a linear advection-dispersion-reaction equation: application to a pollution source problem, *Inverse Problems*, Vol. 21, No. 3, pp. 1121-39.
- [14] Fokas A., Kurylev Y. and Marinakis V. (2004) The unique determination of neuronal currents in the brain via magnetoencephalography, *Inverse Problems*, Vol. 20, pp. 1067-82.
- [15] Hamdi A. (2009) The recovery of a time-dependent point source in a linear transport equation: application to surface water pollution, *Inverse Problems*, Vol. 25, No. 7, pp. 75006-23.
- [16] Hamdi A. (2009) Identification of a time-varying point source in a system of two coupled linear diffusion-advection-reaction equations: application to surface water pollution, *Inverse Problems*, Vol. 25, No. 11, pp. 115009-29.
- [17] Hamdi A., Mahfoudhi I. (2013) Inverse source problem in a one-dimensional evolution linear transport equation with spatially varying coefficients: application to surface water pollution, *Inverse Problems in Science & Engineering*, Vol. 21, No. 6, pp. 1007-31.
- [18] Hamdi A. (2017) Detection-identification of multiple unknown time-dependent point sources in a 2D transport equation: Application to accidental pollution, *Inverse Problems in Science and Engineering*, Vol. 25, Issue 10, pp. 1423-1447.
- [19] Hasanov A. and Mukanova B. (2015) Fourier collocation algorithm for identifying the spacewise-dependent source in the advection-diffusion equation from boundary data measurements, *Applied Numerical Mathematics*, **97**: 1-14.
- [20] Isakov V. (1990) *Inverse Source Problems*, American Mathematical Society, Providence, Rhode Island.
- [21] Idemen M. and Alkumru A. (2012) On an inverse source problem connected with photo-acoustic and thermo-acoustic tomographies, *Wave Motion* 49(6):595-604.
- [22] Gurhan Gurarslan and Halil Karahan (2015) Solving inverse problems of groundwater-pollution-source identification using a differential evolution algorithm, *Hydrogeology Journal* 23(6):1109-1119.
- [23] Larmat C., Tromp J., Liu Q. and Montagner J.P. (2002) Time reversal location of glacial earthquakes, *Journal of Geophysical Research*.
- [24] Kawakatsu H. and Montagner J.P. (2008) Time reversal seismic imaging and moment tensor, *Geophys. J. Int.*
- [25] Koketsu K (2000) Inverse Problems. *Inverse Problems in Seismology*, Bulletin of the Japan Society for Industrial and Applied Mathematics, 10(2):110-120.
- [26] Lauga E., Brenner P., Stone A. (2005) Microfluidics: The No-Slip Boundary Condition, Ch. 15 in *Handbook of Experimental Fluid Dynamics*, Springer, New-York.
- [27] Lions J. L. (1992) Pointwise control for distributed systems in control and estimation in distributed parameters systems, Edited by Banks H. T. SIAM.

- [28] Lions J. L. (1981) Some methods in the mathematical analysis of systems and their control, Science Press, Beijing, Gordon and Breach, New York.
- [29] Lin F. H. (1990) A uniqueness theorem for parabolic equations, Communications on Pure and Applied Mathematics, Vol. 43, pp. 125-36, MR 90j:35106
- [30] Linfield C. Brown and Thomas O. Branwell Jr. (1987) The enhanced stream water quality models QUAL2E and QUAL2E-UNCAS: Documentation and user manual, EPA: 600/3-87/007.
- [31] Machado T.J., Angelo J.S. and Novotny A.A. (2017) A new One-Shot Pointwise Source Reconstruction Method. Mathematical Methods in the Applied Sciences, 40(5):1367-1381.
- [32] Nara T. and Ando S. (2003) A projective method for an inverse source problem of the Poisson equation, Inverse Problems, Vol. 19, Issue 2, pp. 355-369.
- [33] Oelkers E.H. (1996) Physical and chemical properties of rocks and fluids for chemical mass transport calculations, Rev. Mineral Geochem. 34, pp. 131-191.