Long term average cost control problems without ergodicity

Stefan Ankirchner, Stefan Engelhardt

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Abstract

We consider a stochastic control problem with time-inhomogeneous linear dynamics and a long-term average quadratic cost functional. We provide sufficient conditions for the problem to be well-posed. We describe an explicit optimal control in terms of a bounded and non-negative solution of a Riccati equation on $[0, \infty)$, without an initial and terminal condition. We show that, in contrast to the time-homogeneous case, in the inhomogeneous case the optimally controlled state dynamics are not necessarily ergodic.

Keywords: linear-quadratic stochastic control; long-term average cost; Riccati equation; ergodicity; dissipativity.

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Introduction

Suppose that the dynamics of some controlled state satisfy

$$dX_t = (b_t + B_t X_t - \alpha_t)dt + (c_t + C_t X_t)dW_t,$$

where W is a one-dimensional Brownian motion, α is some square-integrable control process and b, B, c, C are real-valued deterministic bounded functions. We consider the problem of minimizing, over all controls α,

$$\limsup_{T\to\infty} \frac{1}{T} E \int_0^T f(s, X_s, \alpha_s)ds, \tag{0.1}$$

where f is a quadratic cost function of the form

$$f(t, x, a) = \beta_{xx}(t)x^2 + \beta_x(t)x + \beta_{xa}(t)ax + \beta_{aa}(t)a^2 + \beta_a(t)a + \beta_0(t)$$

with $\beta_{xx}, \beta_x, \beta_{xa}, \beta_{aa}, \beta_a, \beta_0$ being real-valued, deterministic, right-continuous and bounded functions.

The homogeneous problem version, in which $b, B, c, C, \beta_{xx}, \beta_x, \beta_{xa}, \beta_{aa}, \beta_a, \beta_0$ are all constant functions, is already well-studied in the literature, even for a multidimensional generalization (see, e.g., [3]). The focus of the present article lies on the inhomogenity of the setting. Our aim is to provide sufficient conditions for the inhomogeneous problem to be well-posed and to derive an explicit formula for an optimal control.

[*]University of Jena (Germany), s.ankirchner@uni-jena.de
[†]University of Jena (Germany), engelhardt.stefan@uni-jena.de

As is well-known, the solvability of finite-time inhomogeneous linear-quadratic control problems is strongly linked to the solvabilitity of a related Riccati equation (see e.g. [15] and [16]), which in dimension one has the form

$$U_t' = \frac{(U_t)^2}{2\beta_{aa}(t)} - U_t\left(2B_t + \frac{\beta_{xa}(t)}{\beta_{aa}(t)} + C_t^2\right) - 2\beta_{xx}(t) + \frac{\beta_{xa}^2(t)}{2\beta_{aa}(t)} \tag{0.2}$$

(note that U corresponds to $2P$ in Section 2 of [15]). Given a finite time horizon $T \in (0,\infty)$, a solution of the problem of minimizing $E\int_0^T f(t,X_t,\alpha_t)dt$ can be expressed in terms of the solution of (0.2) with the terminal condition $U_T = 0$.

We show that also the problem of minimizing the long-term cost average functional (0.1) can be reduced to the Riccati equation (0.2). The difficulty in the infinite horizon case, however, is that no terminal condition can be imposed. In order to isolate the solution of (0.2) that determines the minimizer of (0.1), we impose the conditions that the solution is non-negative and bounded from above. Probably the most challenging part of the article is to prove that there exists a unique solution of the Riccati equation (0.2) satisfying these boundedness conditions.

Using the unique bounded non-negative solution of (0.2) on $[0,\infty)$ we define a specific control and show, via a classical verification argument, that it is indeed optimal. In contrast to the homogeneous case, the HJB equation characterizing the control problem does depend on time. This goes in line with the fact that the optimally controlled state dynamics are, again in contrast to the homogeneous case, not necessarily ergodic.

There are many articles that solve long-term average cost control problems with time-homogeneous state dynamics. We refer to [14] for an early survey. In homogeneous models the optimally controlled state dynamics usually are ergodic. Therefore, the literature frequently refers to such problems as ergodic control problems. One message of the current paper is that long-term average cost control problems can be well-posed, even without ergodicity of the optimally controlled state.

A fundamental topic in the field of control theory with long-term average cost functionals is the convergence of the HJB equations of the finite time problem version to an ergodic PDE. More precisely, assume that the HJB equation of a finite time control problem is given by

$$-\partial_t v - \inf_{a\in A}\{\mathcal{L}^a v + f(t,x,a)\} = 0, \tag{0.3}$$

where A is the value set of the controls and $\mathcal{L}^a$ denotes the generator of the controlled state dynamics. There are many contributions providing conditions under which (0.3) transforms into an ergodic PDE of the type

$$\eta - \inf_{a\in A}\left\{\mathcal{L}^a v + \tilde{f}(x,a)\right\} = 0 \tag{0.4}$$

as the time horizon converges to infinity. Notice that a solution of (0.4) consists of a pair $(\eta, v) \in \mathbb{R}\times\mathcal{C}[0,\infty)$. Usually it is assumed that f does not depend on time. Exceptions are [2], [4] assuming a periodicity in time, and [5] assuming that f depends recursively on the value function divided by time-to-maturity.

[1], [10] consider a homogeneous setting and prove convergence, in some sense, of (0.3) to (0.4) under some state periodicity assumptions. [7], [13], [5] use probabilistic representations in terms of backward stochastic differential equations to establish convergence under dissipativity assumptions guaranteeing that the optimally controlled state is ergodic. [8] consider a system of ergodic BSDEs with dissipative forward part and apply them to a long-term utility maximization problem with regime switching.

We stress that in the present article we do not impose any kind of time periodicity assumption. The only assumption on the state coefficients and the cost coefficients is that they are

bounded and right-continuous. As the time horizon converges to infinity, the time dependence in the HJB (0.3) does, in general not disappear, and hence we do not have convergence to (0.4). A time-dependent, but periodic, PDE limit is also described in [4].

Finally, we remark that we do not impose any regularity with respect to time, and hence we can not transform the setting into a 2-dimensional homogeneous setting with time as a new state variable.

1 Main results

In this section we rigorosly describe the model and summarize our main results.

Let W be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t\in[0,\infty)}$ the filtration generated by W, completed by the P-null sets in $\mathcal{F}$.

Assumption 1.1. *Let $\mu, \sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $f : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be of the form*

$$\begin{aligned} \mu(t,x) &= b_t + B_t x, \qquad\qquad \sigma(t,x) = c_t + C_t x, \\ f(t,x,a) &= \beta_{xx}(t)x^2 + \beta_x(t)x + \beta_{xa}(t)ax + \beta_{aa}(t)a^2 + \beta_a(t)a + \beta_0(t) \end{aligned}$$

for $b, B, c, C, \beta_{xx}, \beta_x, \beta_{xa}, \beta_{aa}, \beta_a, \beta_0 : [0,\infty) \to \mathbb{R}$ being deterministic, right-continuous, bounded functions, such that

- $\det(\mathcal{H}(f))(t,\cdot,\cdot) = 4\beta_{aa}(t)\beta_{xx}(t) - \beta_{ax}^2(t) \geqslant \varepsilon_1 > 0$ *for $t \in [0,\infty)$ and some constant $\varepsilon_1 > 0$,*
- $\beta_{aa}(t) \geqslant \varepsilon_2 > 0$ *for $t \in [0,\infty)$ and some constant $\varepsilon_2 > 0$.*

By a control process α we mean a $(\mathcal{F}_t)$-progressively measurable process α such that for all $T \in [0,\infty)$ we have $\int_0^T \alpha_s^2 ds < \infty$. Given a control α, we assume that state process satisfies the SDE

$$dX_t = (\mu(t,X_t) - \alpha_t)dt + \sigma(t,X_t)dW_t. \tag{1.1}$$

Notice that our assumptions imply that for every $x \in \mathbb{R}$ the SDE (1.1) has a unique solution $X^{x,\alpha}$ satisfying $X_0^{x,\alpha} = x$. Moreover, one can show that for all $p \in [1,\infty)$ and $T \in [0,\infty)$ we have $\sup_{t\in[0,T]} E|X_t^{x,\alpha}|^p < \infty$ (see Section 2.5 in [9]).

We say that a control is admissible if for all $x \in \mathbb{R}$ we have $\sup_{t\geqslant 0} E[(X_t^{x,\alpha})^2] < \infty$, and we denote by $\mathcal{A}$ the set of all admissible controls. Notice that if $\alpha \in \mathcal{A}$, then the limsup long term average cost functional

$$\bar{J}(x,\alpha) = \limsup_{T\to\infty} E\frac{1}{T}\int_0^T f(s, X_s^{x,\alpha}, \alpha_s)ds,$$

has only finite values.

We now consider the problem of minimizing $\bar{J}(x,\alpha)$ among all admissible controls. To this end we introduce the value function

$$\bar{V}(x) := \inf_{\alpha\in\mathcal{A}} \bar{J}(x,\alpha), \tag{1.2}$$

for all $x \in \mathbb{R}$. We show below that $\bar{V}$ does not depend on x; but since this is a priorily not known, in the definition of $\bar{V}$ we add the argument x.

We say that $\alpha \in \mathcal{A}$ is an optimal control for (1.2) if for all $x \in \mathbb{R}$ we have $\bar{J}(x,\alpha) = \bar{V}(x)$. Moreover, we say that $\alpha \in \mathcal{A}$ is a closed-loop control if there exists a function $a : [0,\infty) \times \mathbb{R}$ such that for all $x \in \mathbb{R}$ the SDE

$$dX_t = (\mu(t,X_t) - a(t,X_t))dt + \sigma(t,X_t)dW_t. \tag{1.3}$$

has a unique solution $X^{x,a}$ and $\alpha_t = a(t, X_t^{x,a})$, $t \in [0, \infty)$.

We now summarize our main results. First, we describe an optimal control and the value function in terms of a solution of the Riccati equation (0.2). We show that there exists a unique initial condition such that equation (0.2) has on $[0, \infty)$ a solution that is bounded from above and bounded from below by 0.

Proposition 1.2. *There exists exactly one non-negative and bounded solution of* (0.2) *on* $[0, \infty)$.

The result of Proposition 1.2 is proved in Section 2 as a part of Theorem 2.1. In the following we denote by U^∞ the unique non-negative bounded solution of (0.2) described in Proposition 1.2.

In Section 3 we show that there exist constants $\delta_1, \delta_2 > 0$ such that

$$\int_s^t \left(B_r + \frac{\beta_{xa}(r) - U_r^\infty}{2\beta_{aa}(r)} \right) \mathrm{d}r \leqslant -\delta_1(t-s) + \delta_2$$

for all $0 \leqslant s \leqslant t < \infty$. We can thus define a further bounded process

$$\varphi_t^\infty := \int_t^\infty \left[U_s^\infty \left(b_s + c_s C_s + \frac{\beta_a(s)}{2\beta_{aa}(s)} \right) - \frac{\beta_a(s)\beta_{xa}(s)}{2\beta_{aa}(s)} + \beta_x(s) \right] \cdot \exp\left(\int_t^s B_r + \frac{\beta_{xa}(r) - U_r^\infty}{2\beta_{aa}(r)} \, \mathrm{d}r \right) \mathrm{d}s.$$

We next describe a solution of the long term cost minimization problem in terms of U^∞ and φ^∞.

Theorem 1.3. *The closed-loop control with feedback function*

$$a^\infty(t,x) = \frac{\varphi_t^\infty - \beta_a(t) + (U_t^\infty - \beta_{xa}(t))x}{2\beta_{aa}(t)} \tag{1.4}$$

is an optimal control. Moreover,

$$\bar{V}(x) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \left(\varphi_s^\infty b_s + U_s^\infty \frac{c_s^2}{2} + \beta_0(s) - \frac{(\varphi_s^\infty - \beta_a(s))^2}{4\beta_{aa}(s)} \right) \mathrm{d}s. \tag{1.5}$$

Note that (1.5) implies that $\bar{V}$ does not depend on x. In the following we therefore omit the argument x and interpret $\bar{V}$ as a constant.

We prove Theorem 1.3 in Section 3 as a part of Theorem 3.6. We next proceed by comparing the problem of minimizing $\bar{J}(x, \alpha)$ with the problem of minimizing the liminf long term average cost functional

$$\underline{J}(x,\alpha) = \limsup_{T\to\infty} E\frac{1}{T} \int_0^T f(s, X_s^{x,\alpha}, \alpha_s) ds. \tag{1.6}$$

We define also the liminf value

$$\underline{V} := \inf_{\alpha\in\mathcal{A}} \underline{J}(x,\alpha). \tag{1.7}$$

One can show that the feedback function (1.4) is also optimal for (1.7) and that $\underline{V}$ does not depend on x. Moreover, we have

$$\underline{V} = \liminf_{t\to\infty} \frac{1}{t} \int_0^t \left(\varphi_s^\infty b_s + U_s^\infty \frac{c_s^2}{2} + \beta_0(s) - \frac{(\varphi_s^\infty - \beta_a(s))^2}{4\beta_{aa}(s)} \right) \mathrm{d}s. \tag{1.8}$$

In general, $\underline{V}$ is not equal to $\bar{V}$. If $\underline{V} < \bar{V}$, then $X^{\infty,x}$, the state process controlled with the optimal control $\alpha_t^{\infty,x} = a^\infty(t, X_t^{\infty,x})$, is not ergodic, i.e. it does not hold true that the cost time average converges almost surely. More precisely, we have the following.

Proposition 1.4. *If $\underline{V} < \bar{V}$, then for all $x \in \mathbb{R}$ the time average $\frac{1}{T}\int_0^T f(s, X_s^{\infty,x}, \alpha_s^{\infty,x})ds$ does not converge a.s., as $T \to \infty$.*

Proof. We first show that the family $\frac{1}{T}\int_0^T f(s, X_s^{\infty,x}, \alpha_s^{\infty,x})ds$, $T \in [0,\infty)$, is uniformly integrable. To this end let $p \in (1,\infty)$. By Jensen's inequality we have, for some constant K independent of T,

$$\begin{aligned} E\left[\left|\frac{1}{T}\int_0^T f(s, X_s^{\infty,x}, \alpha_s^{\infty,x})ds\right|^p\right] &\leqslant E\left[\frac{1}{T}\int_0^T |f(s, X_s^{\infty,x}, \alpha_s^{\infty,x})|^p ds\right] \\ &\leqslant \left[\frac{1}{T}\int_0^T K(1 + E|X_s^{\infty,x}|^{2p})ds\right] \\ &\leqslant K(1 + \sup_{s\in[0,\infty)} E|X_s^{\infty,x}|^{2p}). \end{aligned}$$

By Lemma 3.3 below there exists a $p > 1$ such that $\sup_{s\in[0,\infty)} E|X_s^{\infty,x}|^{2p} < \infty$. Hence, by the de la Vallee-Poussin theorem, the family $\frac{1}{T}\int_0^T f(s, X_s^{\infty,x}, \alpha_s^{\infty,x})ds$ is uniformly integrable.

Now suppose that $\frac{1}{T}\int_0^T f(s, X_s^{\infty,x}, \alpha_s^{\infty,x})ds$ converges a.s. Then, due to uniform integrability, we also have convergence in L^1. This contradicts however that $\underline{V} < \bar{V}$. ∎

Proposition 1.4 entails, in particular, that if $\underline{V} < \bar{V}$, then the distribution of $X_t^{\infty,x}$ does not converge to a stationary distribution, as $t \to \infty$.

In the homogeneous case where the functions μ, σ and f do not depend on t, the optimally controlled state $X^{\infty,x}$ is ergodic. The homogeneous case is already well studied in the literature (see e.g. [3]). For the convenience of the reader we briefly explain how our results simplify in the homogeneous case and how they can be extended.

The homogeneous case

Suppose that all modelling functions $b, B, c, C, \beta_{xx}, \beta_x, \beta_{xa}, \beta_{aa}, \beta_a, \beta_0$ are constant. In this case also U^∞ and φ^∞ are constant; in particular we have

$$U^\infty = p + \sqrt{p^2 + q}, \tag{1.9}$$

where

$$p = 2B\beta_{aa} + \beta_{xa} + C^2\beta_{aa} \quad \text{and} \quad q = 4\beta_{xx}\beta_{aa} - \beta_{xa}^2.$$

Let $\kappa = B - \frac{U^\infty - \beta_{xa}}{2\beta_{aa}}$, and notice that the optimally controlled state X^∞ satisfies the homogeneous SDE

$$dX_t = \left(b - \frac{\varphi^\infty - \beta_a}{2\beta_{aa}} + \kappa X_t\right)dt + (c + CX_t)dW_t. \tag{1.10}$$

Assumption 1.1 implies that $q > 0$. Thus, with (1.9) we get $U^\infty > p$, and hence

$$\kappa < -\frac{C^2}{2}. \tag{1.11}$$

Property (1.11), sometimes referred to as *dissipativity*, guarantees that (1.10) possesses a unique stationary distribution π (see, e.g., Theorem 8.3 in [12]; use for example the Lyapunov function $W(x) = x^2/2$). Moreover, if $X^{\infty,x}$ denotes the solution of (1.10) with initial condition $x \in \mathbb{R}$, then the distribution of $X_t^{\infty,x}$ converges to the stationary distribution, as $t \to \infty$ (see Remark 8.6 in [12]). This further entails that $\frac{1}{T}\int_0^T f(X_s^{\infty,x}, a^\infty(s, X_s^{\infty,x}))ds$ converges a.s. to $\int f(x, a^\infty(x))\pi(dx)$, as $T \to \infty$.

Dissipativity in the inhomogeneous case

Observe that the optimally controlled state X^∞ satisfies the SDE

$$\mathrm{d}X_t = \left(b_t - \frac{\varphi_t^\infty - \beta_a(t)}{2\beta_{aa}(t)} + \kappa_t X_t\right)\mathrm{d}t + (c_t + C_t X_t)\,\mathrm{d}W_t,$$

where $\kappa_t = B_t - \frac{U_t^\infty - \beta_{xa}(t)}{2\beta_{aa}(t)}$. By Theorem 2.1 below we obtain that there are constants $\delta_1, \delta_2 > 0$ such that for all $0 \leqslant t_1 \leqslant t_2 < \infty$

$$\int_{t_1}^{t_2} \left(\kappa_t + \frac{C_t^2}{2}\right)\mathrm{d}t \leqslant \delta_1(t_2 - t_1) + \delta_2.$$

This implies, that for large enough time intervals $[t_1, t_2]$ we have

$$\int_{t_1}^{t_2} \kappa_t\,\mathrm{d}t < \int_{t_1}^{t_2} -\frac{C_t^2}{2}\,\mathrm{d}t,$$

which seems to be a time-average version of the dissipativity condition (1.11).

However, consider $B_t = 2 \cdot \mathbb{1}_{\{t \in [0,1)\}}$ and all other parameters to be constant with $C = 1$, $\beta_{aa} = \frac{1}{3}$, $\beta_{xx} = \frac{1}{4}$ and $b = \beta_{xa} = \beta_x = \beta_a = \beta_0 = 0$. For $t \geqslant 1$ we have that 1 is a solution of (0.2), and hence $U_t^\infty = 1$ for all $t \geqslant 1$. Furthermore, $\kappa_1 = -\frac{3}{2}$. Since U^∞ is continuous, there is an $\varepsilon > 0$ such that for all $t \in [1 - \varepsilon, 1)$ we have

$$-\frac{1}{2} = -\frac{C_t^2}{2} < 0 < \kappa_t,$$

which means that for at least a short time the condition (1.11) is not satisfied.

A non-ergodic example

Example 1.5. *Consider the control problem with $C = 1$, $\beta_{aa} = \frac{1}{3}$, $\beta_{xx} = \frac{1}{4}$ and $b = B = \beta_{xa} = \beta_x = \beta_a = \beta_0 = 0$. Below we define recursively a sequence of increasing times $0 = t_0 < t_1 < t_2 < \cdots$. Given this sequence we set*

$$c_t = \begin{cases} 1, & \text{if } t \in [t_{2k}, t_{2k+1}) \text{ for a } k \in \mathbb{N}_0, \\ 2, & \text{if } t \in [t_{2k+1}, t_{2k+2}) \text{ for a } k \in \mathbb{N}_0. \end{cases}$$

First, observe that the function constant equal to 1 is a solution of (0.2), and hence $U^\infty = 1$. Suppose that t_{2k} is defined. Observe that

$$\lim_{T\to\infty} \int_{t_{2k}}^{T} e^{-\frac{3}{2}(s - t_{2k})} ds = \frac{2}{3}.$$

Thus, the larger we choose t_{2k+1}, the closer φ_s^∞, $s \in [t_{2k}, (t_{2k} + t_{2k+1})/2]$, gets to $\frac{2}{3}$. Now we choose t_{2k+1} such that

$$\frac{1}{(t_{2k} + t_{2k+1})/2} \int_0^{(t_{2k}+t_{2k+1})/2} (\frac{1}{2} - \frac{3}{4}\varphi_s^\infty) ds \leqslant \frac{1}{k}.$$

We next describe how to choose t_{2k+2}. Observe that

$$\lim_{T\to\infty} \int_{t_{2k+1}}^{T} 2e^{-\frac{3}{2}(s - t_{2k+1})} ds = \frac{4}{3}.$$

Therefore, the larger we choose t_{2k+2}, the closer φ^∞_s, $s \in [t_{2k+1}, (t_{2k+1}+t_{2k+2})/2]$, gets to $\frac{4}{3}$. Now choose t_{2k+2} such that

$$\frac{1}{(t_{2k+1}+t_{2k+2})/2}\int_0^{(t_{2k+1}+t_{2k+2})/2}(2-\frac{3}{4}\varphi^\infty_s)ds \geqslant 1-\frac{1}{k}.$$

We have thus recursively defined the sequence $(t_k)_{k\in\mathbb{N}_0}$.

From (1.5) *and* (1.8) *we now obtain $\bar{V} \geqslant 1$ and $\underline{V} \leqslant 0$.*

Comparison with the finite time control problem

The optimal control in (1.4) has a similar form as the corresponing optimal control with a finite time horizon $T \in (0,\infty)$. Indeed, let $(U^T_t)_{t\in[0,T]}$ be the solution of the Riccati equation (0.2) on $[0,T]$ with terminal condition $U_T = 0$, and let for all $t \in [0,T]$

$$\varphi^T_t = \int_t^T \left[U^T_s(b_s + c_s C_s) + \beta_a(s)\frac{U^T_s - \beta_{xa}(s)}{2\beta_{aa}(s)} + \beta_x(s)\right]\exp\left(\int_t^s B_r + \frac{\beta_{xa}(r)-U^T_r}{2\beta_{aa}(r)}\,\mathrm{d}r\right)\mathrm{d}s.$$

If we replace U^∞ with U^T and φ^∞ with φ^T in (1.4), then we obtain an optimal closed loop control for the problem of minimizing $E\int_0^T f(t, X^{x,\alpha}_t, \alpha_t)dt$ (see, e.g., Theorem 2.4.3 in [15]). Moreover, one can show that U^T and φ^T converge to U^∞ and φ^∞, respectively and hence the optimal feedback function of the finite horizon problem converges to a^∞ as $T \to \infty$ (see Chapter 4 in [6]).

2 Existence and uniqueness of U^∞

In this section we show the existence and uniqueness of U^∞, which is defined as the non-negative bounded solution of (0.2). In fact, we show a little more than that, as can be seen in the following theorem, which contains the main result of this section.

Theorem 2.1. *Let Assumption 1.1 be fulfilled.*

Then there exists exaclty one $u \in \mathbb{R}$ such that (0.2) *with the initial condition $U_0 = u$ has a solution that is on $[0,\infty)$ bounded from below by 0 and bounded form above by $\hat{U} := \hat{p} + \sqrt{\hat{p}^2 + \hat{q}}$, where $\hat{p} := \sup_{s\in[0,\infty)}\big(2B_s\beta_{aa}(s) + \beta_{xa}(s) + C_s^2\beta_{aa}(s)\big)$ and $\hat{q} := \sup_{s\in[0,\infty)}\big(4\beta_{xx}(s)\beta_{aa}(s) - \beta^2_{xa}(s)\big)$.*

Furthermore, there are constants $\delta_1, \delta_2 > 0$ such that for any initial value U_0 yielding $U_r \in [0,\hat{U}]$ for all $0 \leqslant r \leqslant T < \infty$ we have

$$\int_s^t\left(B_r + \frac{\beta_{xa}(r)-U_r}{2\beta_{aa}(r)}\right)\mathrm{d}r \leqslant \int_s^t\left(B_r + \frac{\beta_{xa}(r)-U_r}{2\beta_{aa}(r)} + \frac{C_r^2}{2}\right)\mathrm{d}r \leqslant -\delta_1(t-s) + \delta_2$$

for all $0 \leqslant s \leqslant t \leqslant T$.

We approach this problem by considering a simplified quadratic integral equation, at first for constant and then for picewise constant parameter function. Finally we generalize to right-continuous functions and prove Theorem 2.1 via a time-reversal.

Assumption 2.2. *Let $p, q, a : [0,\infty) \to \mathbb{R}$ be deterministic right-continuous functions such that for all $s \in [0,\infty)$*

$$-\infty < \check{p} \leqslant p_s \leqslant \hat{p} < \infty, \qquad 0 < \check{q} \leqslant q_s \leqslant \hat{q} < \infty, \qquad 0 < \check{a} \leqslant a_s \leqslant \hat{a} < \infty$$

for constants $\check{p}, \hat{p}, \check{q}, \hat{q}, \check{a}, \hat{a} \in \mathbb{R}$.

For the following we define the constants $\check{Y} := \check{p} + \sqrt{\check{p}^2 + \check{q}}$ and $\hat{Y} := \hat{p} + \sqrt{\hat{p}^2 + \hat{q}}$.

Lemma 2.3. *Let Assumption 2.2 be fulfilled and $t \geqslant 0$. Then the integral equation*

$$Y_s = Y_t + \int_t^s -a_r \left(Y_r^2 - 2p_r Y_r - q_r\right) \mathrm{d}r \tag{2.1}$$

for $s \in [t, \infty)$ with starting value $Y_t \in [0, \hat{Y}]$ has a unique solution. Also, the solution $(Y_s)_{s \geqslant t}$ is bounded by

$$\min\left\{Y_t, \check{Y}\right\} \leqslant Y_s \leqslant \hat{Y}$$

for all $s \in [t, \infty)$.

Proof. We define the auxiliary process $\tilde{Y}$ as the solution of the Lipschitz ODE

$$\partial_t \tilde{Y}_s = -a_s \left(\left(\mathcal{T}_0^{\hat{Y}}(\tilde{Y}_s)\right)^2 - 2p_s \tilde{Y}_s - q_s\right), \qquad \tilde{Y}_t = Y_t,$$

where $\mathcal{T}$ is the truncation operator defined by $\mathcal{T}_\alpha^\beta(x) := \max(\alpha, \min(x, \beta))$ for $\alpha \leqslant \beta$. Observe that for $\tilde{Y}_s \in [0, \check{Y})$ we have $-a_s\big(\big(\mathcal{T}_0^{\hat{Y}}(\tilde{Y}_s)\big)^2 - 2p_s\tilde{Y}_s - q_s\big) > 0$ and for $\tilde{Y}_s \in [\hat{Y}, \infty)$ that $-a_s\big(\big(\mathcal{T}_0^{\hat{Y}}(\tilde{Y}_s)\big)^2 - 2p_s\tilde{Y}_s - q_s\big) \leqslant 0$. Hence, for $\tilde{Y}_t < \check{Y}$ we have that $\tilde{Y}_s \geqslant \tilde{Y}_t$ for all $s \in [t, \infty)$, since Y is continuous. By the same argument we also obtain for $\tilde{Y}_t \geqslant \check{Y}$ that $\tilde{Y}_s$ cannot reach any value below $\check{Y}$ and likewise because $\tilde{Y}_t \leqslant \hat{Y}$ that $\tilde{Y}_s \leqslant \hat{Y}$. Thus, the truncation of the quadratic term has no consequence and can be omitted without changing the solution. Hence, the bounds are also valid for Y and the solution of (2.1) is also unique. ∎

In the following we denote by Y the solution of Equation (2.1).

Remark 2.4. *In the proofs of this section we make use of the following hyperbolic identities without explicitly mentioning it:*

- $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ *for* $x \in (-1, 1)$,
- $\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ *for* $|x| > 1$,
- $\cosh(\tanh^{-1}(x)) = (1 - x^2)^{-1/2}$ *for* $x \in (-1, 1)$,
- $\sinh(\coth^{-1}(x)) = (1 - x^2)^{-1/2}$ *for* $x > 1$.

Lemma 2.5. *Let Assumption 2.2 be fulfilled and $t \geqslant 0$. Furthermore, assume that $Y_t \in [0, \hat{Y}]$ and for some $s \in (t, \infty)$ that the functions p, q, a are constant on the interval $[t, s)$, i.e. there are $\bar{p}, \bar{q}, \bar{a} \in \mathbb{R}$ such that $p_r = \bar{p}$, $q_r = \bar{q}$ and $a_r = \bar{a}$ for all $r \in [t, s)$. Then*

$$Y_r = \begin{cases} \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}\tanh\left(\bar{a}\sqrt{\bar{p}^2 + \bar{q}}(r-t) + \tanh^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2+\bar{q}}}\right)\right), & Y_t \in \left[0, \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}\right) \\ \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}, & Y_t = \bar{p} + \sqrt{\bar{p}^2 + \bar{q}} \\ \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}\coth\left(\bar{a}\sqrt{\bar{p}^2 + \bar{q}}(r-t) + \coth^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2+\bar{q}}}\right)\right), & Y_t \in \left(\bar{p} + \sqrt{\bar{p}^2 + \bar{q}}, \infty\right) \end{cases} \tag{2.2}$$

for all $r \in [t, s]$. In particular, Y is monotone on the interval $[t, s]$.

Proof. Observe that the dynamics of Y state that it solves for $r \in [t, s)$ the separable ODE

$$Y_r' = -a_t\left((Y_r - p_t)^2 - p_t^2 - q_t\right).$$

The three cases follow by straightforward calculations. Also, Lemma 2.3 provides uniqueness. The remaining monotonicity follows from the monotonicity of $\tanh$ and $\coth$. ∎

Lemma 2.6. *Let Assumption 2.2 be fulfilled and $[t_1, t_2] \subset [0, \infty)$ with $t_1 < t_2$. Furthermore, assume that $Y_{t_1} \in [0, \hat{Y}]$ and that the functions p, q, a are constant on the interval $[t_1, t_2)$, i.e. there are $\bar{p}, \bar{q}, \bar{a} \in \mathbb{R}$ such that $p_r = \bar{p}$, $q_r = \bar{q}$ and $a_r = \bar{a}$ for all $r \in [t_1, t_2)$. Then, for $t_1 \leqslant t \leqslant s \leqslant t_2$,*

$$\int_t^s -a_r \left(Y_r - p_r\right) \mathrm{d}r = \begin{cases} -\bar{a}\sqrt{\bar{p}^2 + \bar{q}}(s-t), & Y_t = \bar{p} + \sqrt{\bar{p}^2 + \bar{q}} \\ \frac{1}{2} \ln\left(\frac{Y_t^2 - 2\bar{p}Y_t - \bar{q}}{Y_s^2 - 2\bar{p}Y_s - \bar{q}}\right), & Y_t \neq \bar{p} + \sqrt{\bar{p}^2 + \bar{q}} \end{cases} \tag{2.3}$$

and for $Y_t \neq \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}$ we moreover have

$$s - t = \frac{1}{2\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\bar{p}^2 + \bar{q} - (Y_t - \bar{p})^2}{\bar{p}^2 + \bar{q} - (Y_s - \bar{p})^2}\right) + \frac{1}{\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\sqrt{\bar{p}^2 + \bar{q}} + (Y_s - \bar{p})}{\sqrt{\bar{p}^2 + \bar{q}} + (Y_t - \bar{p})}\right).$$

Proof. Rearranging the formula in (2.2) we obtain for $Y_t < \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}$

$$\begin{aligned} s - t &= \frac{1}{\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \left(\tanh^{-1}\left(\frac{Y_s - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right) - \tanh^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right)\right) \\ &= \frac{1}{2\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\sqrt{\bar{p}^2 + \bar{q}} + (Y_s - \bar{p})}{\sqrt{\bar{p}^2 + \bar{q}} - (Y_s - \bar{p})} \frac{\sqrt{\bar{p}^2 + \bar{q}} - (Y_t - \bar{p})}{\sqrt{\bar{p}^2 + \bar{q}} + (Y_t - \bar{p})}\right) \\ &= \frac{1}{2\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\bar{p}^2 + \bar{q} - (Y_t - \bar{p})^2}{\bar{p}^2 + \bar{q} - (Y_s - \bar{p})^2} \frac{\left(\sqrt{\bar{p}^2 + \bar{q}} + (Y_s - \bar{p})\right)^2}{\left(\sqrt{\bar{p}^2 + \bar{q}} + (Y_t - \bar{p})\right)^2}\right) \\ &= \frac{1}{2\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\bar{p}^2 + \bar{q} - (Y_t - \bar{p})^2}{\bar{p}^2 + \bar{q} - (Y_s - \bar{p})^2}\right) + \frac{1}{\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\sqrt{\bar{p}^2 + \bar{q}} + (Y_s - \bar{p})}{\sqrt{\bar{p}^2 + \bar{q}} + (Y_t - \bar{p})}\right), \end{aligned} \tag{2.4}$$

and for $Y_t > \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}$

$$\begin{aligned} s - t &= \frac{1}{\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \left(\coth^{-1}\left(\frac{Y_s - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right) - \coth^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right)\right) \\ &= \frac{1}{2\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\left(-\frac{\sqrt{\bar{p}^2 + \bar{q}} + (Y_s - \bar{p})}{\sqrt{\bar{p}^2 + \bar{q}} - (Y_s - \bar{p})}\right)\left(-\frac{\sqrt{\bar{p}^2 + \bar{q}} - (Y_t - \bar{p})}{\sqrt{\bar{p}^2 + \bar{q}} + (Y_t - \bar{p})}\right)\right) \\ &= \frac{1}{2\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\bar{p}^2 + \bar{q} - (Y_t - \bar{p})^2}{\bar{p}^2 + \bar{q} - (Y_s - \bar{p})^2}\right) + \frac{1}{\bar{a}\sqrt{\bar{p}^2 + \bar{q}}} \ln\left(\frac{\sqrt{\bar{p}^2 + \bar{q}} + (Y_s - \bar{p})}{\sqrt{\bar{p}^2 + \bar{q}} + (Y_t - \bar{p})}\right). \end{aligned}$$

Now we have a look at the integral in (2.3). For $Y_t < \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}$ we get

$$\begin{aligned} \int_t^s -\bar{a}\left(Y_r - \bar{p}\right) \mathrm{d}r &= \int_t^s -\bar{a}\sqrt{\bar{p}^2 + \bar{q}} \tanh\left(\bar{a}\sqrt{\bar{p}^2 + \bar{q}}\,(r-t) + \tanh^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right)\right) \mathrm{d}r \\ &= -\ln\left(\frac{\cosh\left(\tanh^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right)\right)}{\cosh\left(\bar{a}\sqrt{\bar{p}^2 + \bar{q}}\,(s-t) + \tanh^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right)\right)}\right) \\ &= -\ln\left(\frac{\cosh\left(\tanh^{-1}\left(\frac{Y_t - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right)\right)}{\cosh\left(\tanh^{-1}\left(\frac{Y_s - \bar{p}}{\sqrt{\bar{p}^2 + \bar{q}}}\right)\right)}\right) \\ &= \frac{1}{2}\ln\left(\frac{\bar{p}^2 + \bar{q} - (Y_t - \bar{p})^2}{\bar{p}^2 + \bar{q} - (Y_s - \bar{p})^2}\right), \end{aligned}$$

where we use Equation (2.4) in the second to last step. In the case of $Y_t > \bar{p} + \sqrt{\bar{p}^2 + \bar{q}}$ we obtain similarly

$$\begin{aligned}\int_t^s -\bar{a}\,(Y_r - \bar{p})\,\mathrm{d}r &= \int_t^s -\bar{a}\sqrt{\bar{p}^2+\bar{q}}\coth\left(\bar{a}\sqrt{\bar{p}^2+\bar{q}}\,(r-t) + \coth^{-1}\left(\frac{Y_t-\bar{p}}{\sqrt{\bar{p}^2+\bar{q}}}\right)\right)\mathrm{d}r \\ &= -\ln\left(\frac{\sinh\left(\coth^{-1}\left(\frac{Y_t-\bar{p}}{\sqrt{\bar{p}^2+\bar{q}}}\right)\right)}{\sinh\left(\bar{a}\sqrt{\bar{p}^2+\bar{q}}\,(s-t) + \coth^{-1}\left(\frac{Y_t-\bar{p}}{\sqrt{\bar{p}^2+\bar{q}}}\right)\right)}\right) \\ &= \frac{1}{2}\ln\left(\frac{\bar{p}^2+\bar{q}-(Y_t-\bar{p})^2}{\bar{p}^2+\bar{q}-(Y_s-\bar{p})^2}\right).\end{aligned}$$

■

Lemma 2.6 gives us the value of the integral in (2.3) when the parameters are constant all the way. Next, we want to find the value of that integral when the process Y goes up and down ending at the value where it started, which we later call an excursion.

Lemma 2.7. *Let Assumption 2.2 be fulfilled. Furthermore, let* $[t_1,t_2],[t_3,t_4] \subset [0,\infty)$, $Y_{t_1} = Y_{t_4} \in [0,\hat{Y}]$, $Y_{t_2} = Y_{t_3} \in [0,\hat{Y}]$ *and* p,q,a *be constant on* $[t_1,t_2)$ *and also on* $[t_3,t_4)$. *Then*

$$\int_{t_1}^{t_2} -a_s(Y_s-p_s)\,\mathrm{d}s + \int_{t_3}^{t_4} -a_s(Y_s-p_s)\,\mathrm{d}s \leqslant -\min\left\{\check{a}\sqrt{\check{q}}, \frac{\hat{a}(\check{Y}^2+\check{q})}{2\hat{Y}}\right\}((t_2-t_1)+(t_4-t_3)).$$

Proof. First we define $p_1 := p_{t_1}$, $q_1 := q_{t_1}$, $a_1 := a_{t_1}$ and $p_2 := p_{t_3}$, $q_2 := q_{t_3}$, $a_2 := a_{t_3}$ since p, q and a are constant on the intervals $[t_1,t_2)$, $[t_3,t_4)$. Now note that, due to the monotonicity of Y stated in Lemma 2.5, we have one of the three cases

(i) $Y_{t_1} = Y_{t_2} = Y_{t_3} = Y_{t_4} = p_1 + \sqrt{p_1^2+q_1} = p_2 + \sqrt{p_2^2+q_2}$,

(ii) $p_1 + \sqrt{p_1^2+q_1} < Y_{t_1} = Y_{t_4} < Y_{t_2} = Y_{t_3} < p_2 + \sqrt{p_2^2+q_2}$ or

(iii) $p_2 + \sqrt{p_2^2+q_2} < Y_{t_2} = Y_{t_3} < Y_{t_1} = Y_{t_4} < p_1 + \sqrt{p_1^2+q_1}$.

In Case (i) it is straighforward that

$$\begin{aligned}\int_{t_1}^{t_2} -a_s(Y_s-p_s)\,\mathrm{d}s + \int_{t_3}^{t_4} -a_s(Y_s-p_s)\,\mathrm{d}s &= -a_1\sqrt{p_1^2+q_1}(t_2-t_1) - a_2\sqrt{p_2^2+q_2}(t_4-t_3) \\ &\leqslant -\check{a}\sqrt{\check{q}}(t_2-t_1+t_4-t_3).\end{aligned}$$

Now observe for Cases (ii) and (iii) that by Lemma 2.6 and since $Y_{t_1} = Y_{t_4}$, $Y_{t_2} = Y_{t_3}$ we get

$$\begin{aligned}&\int_{t_1}^{t_2} -a_s(Y_s-p_s)\,\mathrm{d}s + \int_{t_3}^{t_4} -a_s(Y_s-p_s)\,\mathrm{d}s \\ &\quad= \frac{1}{2}\ln\left(\frac{Y_{t_1}^2-2p_1Y_{t_1}-q_1}{Y_{t_2}^2-2p_1Y_{t_2}-q_1}\right) + \frac{1}{2}\ln\left(\frac{Y_{t_3}^2-2p_2Y_{t_3}-q_2}{Y_{t_4}^2-2p_2Y_{t_4}-q_2}\right) \\ &\quad= \int_{Y_{t_1}}^{Y_{t_2}} -\frac{x-p_1}{x^2-2p_1x-q_1} + \frac{x-p_2}{x^2-2p_2x-q_2}\,\mathrm{d}x \\ &\quad= \int_{Y_{t_1}}^{Y_{t_2}} \frac{1}{2x}\left(-\frac{x^2-2p_1x-q_1}{x^2-2p_1x-q_1} + \frac{x^2-2p_2x-q_2}{x^2-2p_2x-q_2} - \frac{x^2+q_1}{x^2-2p_1x-q_1} + \frac{x^2+q_2}{x^2-2p_2x-q_2}\right)\mathrm{d}x \\ &\quad= \int_{Y_{t_1}}^{Y_{t_2}} \frac{1}{2x}\left(-\frac{x^2+q_1}{x^2-2p_1x-q_1} + \frac{x^2+q_2}{x^2-2p_2x-q_2}\right)\mathrm{d}x.\end{aligned}$$

Furthermore, note that Case (ii) implies that $0 < x^2 - 2p_1x - q_1$ and $x^2 - 2p_2x - q_2 < 0$, while Case (iii) implies $0 > x^2 - 2p_1x - q_1$ and $x^2 - 2p_2x - q_2 > 0$ for x between Y_{t_1} and Y_{t_2}. Hence we obtain

$$\begin{aligned}\int_{t_1}^{t_2} -a_s(Y_s - p_s)\,\mathrm{d}s + \int_{t_3}^{t_4} -a_s(Y_s - p_s)\,\mathrm{d}s &= \int_{Y_{t_1}}^{Y_{t_2}} \frac{1}{2x}\left(-\frac{x^2+q_1}{x^2-2p_1x-q_1} + \frac{x^2+q_2}{x^2-2p_2x-q_2}\right)\mathrm{d}x \\ &\leqslant -|Y_{t_2} - Y_{t_1}|\,\frac{1}{2\hat{Y}}\left(\frac{\check{Y}^2+\check{q}}{\hat{Y}^2-2\check{p}\hat{Y}-\check{q}} + \frac{\check{Y}^2+\check{q}}{\hat{q}+2\hat{p}\check{Y}-\check{Y}^2}\right) \\ &\leqslant -|Y_{t_2} - Y_{t_1}|\,\frac{1}{\hat{Y}}\,\frac{\check{Y}^2+\check{q}}{\hat{Y}^2-2\check{p}\hat{Y}-\check{q}} \qquad (2.5)\end{aligned}$$

in Case (ii) and (iii).

It remains to estimate the term $|Y_{t_2} - Y_{t_1}|$ with an expression of time difference. To this end, remember the second result from Lemma 2.6 which gives

$$\begin{aligned}t_2 - t_1 &= \frac{1}{2a_1\sqrt{p_1^2+q_1}}\left(\ln\left(\frac{p_1^2+q_1-(Y_{t_1}-p_1)^2}{p_1^2+q_1-(Y_{t_2}-p_1)^2}\right) + 2\ln\left(\frac{\sqrt{p_1^2+q_1}+(Y_{t_2}-p_1)}{\sqrt{p_1^2+q_1}+(Y_{t_1}-p_1)}\right)\right) \\ &= \frac{1}{a_1\sqrt{p_1^2+q_1}}\int_{Y_{t_1}}^{Y_{t_2}} -\frac{x-p_1}{p_1^2+q_1-(x-p_1)^2} - \frac{1}{\sqrt{p_1^2+q_1}-p_1+x}\,\mathrm{d}x \\ &= \frac{1}{a_1\sqrt{p_1^2+q_1}}\int_{Y_{t_1}}^{Y_{t_2}} -\frac{x-p_1}{p_1^2+q_1-(x-p_1)^2} - \frac{\sqrt{p_1^2+q_1}-(x-p_1)}{p_1^2+q_1-(x-p_1)^2}\,\mathrm{d}x \\ &= \frac{1}{a_1}\int_{Y_{t_1}}^{Y_{t_2}} \frac{1}{(x-p_1)^2-p_1^2-q_1}\,\mathrm{d}x\end{aligned}$$

and analogously

$$t_4 - t_3 = \frac{1}{a_2}\int_{Y_{t_3}}^{Y_{t_4}} \frac{1}{(x-p_2)^2-p_2^2-q_2}\,\mathrm{d}x = -\frac{1}{a_2}\int_{Y_{t_1}}^{Y_{t_2}} \frac{1}{(x-p_2)^2-p_2^2-q_2}\,\mathrm{d}x.$$

Hence, by simular arguments as above, we get

$$\begin{aligned}t_2 - t_1 + t_4 - t_3 &= \int_{Y_{t_1}}^{Y_{t_2}} \frac{1}{a_1((x-p_1)^2-p_1^2-q_1)} - \frac{1}{a_2((x-p_2)^2-p_2^2-q_2)}\,\mathrm{d}x \\ &\geqslant |Y_{t_2} - Y_{t_1}|\,\frac{2}{\hat{a}(\hat{Y}^2-2\check{p}\hat{Y}-\check{q})}\end{aligned}$$

and therefore

$$|Y_{t_2} - Y_{t_1}| \leqslant (t_4 - t_3 + t_2 - t_1)\frac{\hat{a}(\hat{Y}^2-2\check{p}\hat{Y}-\check{q})}{2}.$$

Pluging this into Estimate (2.5) we finally obtain in Case (ii) and (iii)

$$\begin{aligned}&\int_{t_1}^{t_2} -a_s(Y_s - p_s)\,\mathrm{d}s + \int_{t_3}^{t_4} -a_s(Y_s - p_s)\,\mathrm{d}s \\ &\qquad\leqslant -\frac{1}{\hat{Y}}\,\frac{\check{Y}^2+\check{q}}{\hat{Y}^2-2\check{p}\hat{Y}-\check{q}}(t_4 - t_3 + t_2 - t_1)\frac{\hat{a}(\hat{Y}^2-2\check{p}\hat{Y}-\check{q})}{2} \\ &\qquad= -\hat{a}\frac{\check{Y}^2+\check{q}}{2\hat{Y}}\big(t_4 - t_3 + t_2 - t_1\big).\end{aligned}$$

■

Lemma 2.8. *Let Assumption 2.2 be fulfilled and assume that on the interval $[t_0, t_1)$ with $0 \leqslant t_0 < t_1 < \infty$ the functions p, q, a are constant and $Y_{t_0} \in [0, \hat{Y}]$. Then*

$$\int_{t_0}^{t_1} -a_s(Y_s - p_s)\,\mathrm{d}s \leqslant -\check{a}\frac{\sqrt{\check{q}}}{\sqrt{2}}(t_1 - t_0) + \frac{2\hat{Y}}{\check{q}}\left(Y_{t_1} - Y_{t_0}\right)\mathbb{1}_{\{Y_{t_1} - Y_{t_0} > 0\}}. \tag{2.6}$$

Proof. To shorten notation we write $\bar{p}, \bar{q}, \bar{a}$ for the constants p_s, q_s, a_s with $s \in [t_0, t_1)$. Also, we set $\delta := \sqrt{\frac{\bar{q}}{2}}$. We derive estimates for the integrand of the integral in (2.6) and for the duration of the "bad" time, where those estimates do not hold true.

First, note that, since Y is monotone and getting nearer to $\bar{p} + \sqrt{\bar{p}^2 + \bar{q}}$ (see Lemma 2.5), we get for any $s \in [t_0, t_1]$ with $-(Y_s - \bar{p}) \leqslant -\delta$ that for all $r \in [s, t_1]$ we have

i) for $Y_s - \left(\bar{p} + \sqrt{\bar{p}^2 + \bar{q}}\right) < 0$

$$-(Y_r - \bar{p}) = -\left(Y_r - \bar{p} - \sqrt{\bar{p}^2 + \bar{q}}\right) - \sqrt{\bar{p}^2 + \bar{q}} \leqslant 0 - \sqrt{\bar{p}^2 + \bar{q}} \leqslant -\sqrt{\frac{\bar{q}}{2}},$$

ii) for $Y_s - \left(\bar{p} + \sqrt{\bar{p}^2 + \bar{q}}\right) = 0$

$$-(Y_r - \bar{p}) = -\left(Y_r - \bar{p} - \sqrt{\bar{p}^2 + \bar{q}}\right) - \sqrt{\bar{p}^2 + \bar{q}} = 0 - \sqrt{\bar{p}^2 + \bar{q}} \leqslant -\sqrt{\frac{\bar{q}}{2}},$$

iii) for $Y_s - \left(\bar{p} + \sqrt{\bar{p}^2 + \bar{q}}\right) > 0$

$$-(Y_r - \bar{p}) \leqslant -(Y_s - \bar{p}) \leqslant -\sqrt{\frac{\bar{q}}{2}}$$

and hence in every case $-(Y_r - \bar{p}) \leqslant -\delta$. Thus, we then obtain

$$\int_s^{t_1} -\bar{a}(Y_r - \bar{p})\,\mathrm{d}r \leqslant \int_s^{t_1} -\bar{a}\delta\,\mathrm{d}r \leqslant -\bar{a}\delta(t_1 - s).$$

Now we have a closer look at the case where $-(Y_{t_0} - \bar{p}) > -\delta$. For this, remember the dynamics of Y which are

$$Y_s = Y_t + \int_t^s -\bar{a}\left((Y_r - \bar{p})^2 - \bar{p}^2 - \bar{q}\right)\mathrm{d}r$$

for $s, t \in [t_0, t_1]$. There are two cases we have to consider. Firstly, $-\delta < -(Y_s - \bar{p}) < \delta$, which implies

$$-\bar{a}\left((Y_s - \bar{p})^2 - \bar{p}^2 - \bar{q}\right) > -\bar{a}\left(\left(\sqrt{\frac{\bar{q}}{2}}\right)^2 - \bar{p}^2 - \bar{q}\right) = \bar{a}\left(\bar{p}^2 + \frac{1}{2}\bar{q}\right) \geqslant \frac{\bar{a}\bar{q}}{2}.$$

And secondly the case of $-(Y_s - \bar{p}) \geqslant \delta$. Note that then $|Y_s - \bar{p}| = -Y_s + \bar{p} \leqslant \bar{p}$ since $Y \geqslant 0$ by Lemma 2.3. This gives us

$$-\bar{a}\left((Y_s - \bar{p})^2 - \bar{p}^2 - \bar{q}\right) \geqslant -\bar{a}\left((-\bar{p})^2 - \bar{p}^2 - \bar{q}\right) = \bar{a}\bar{q}.$$

Hence, for all $s \in [t_0, t_1]$ with $-(Y_s - p) \geqslant \delta$ we have $Y_s' \geqslant \frac{\bar{a}\bar{q}}{2} > 0$. Let

$$\tau := \inf\{t \in [t_0, t_1] | -(Y_t - \bar{p}) \leqslant -\delta\} \wedge t_1$$

be the first time in $[t_0, t_1]$, where $-(Y_\cdot - p) \leqslant -\delta$ or t_1 if there is no such time. Then we obtain

$$Y_\tau - Y_{t_0} = \int_{t_0}^{\tau} Y_t'\,\mathrm{d}t \geqslant \int_{t_0}^{\tau} \frac{\bar{a}\bar{q}}{2}\,\mathrm{d}t = \frac{\bar{a}\bar{q}}{2}(\tau - t_0)$$

and thus

$$\tau - t_0 \leqslant \frac{2}{\bar{a}\bar{q}} (Y_\tau - Y_{t_0}) = \frac{2}{\bar{a}\bar{q}} (Y_\tau - Y_{t_0}) \mathbb{1}_{\{Y_{t_1} - Y_{t_0} > 0\}} \leqslant \frac{2}{\bar{a}\bar{q}} (Y_{t_1} - Y_{t_0}) \mathbb{1}_{\{Y_{t_1} - Y_{t_0} > 0\}}, \tag{2.7}$$

where we use that if $Y_{t_1} - Y_{t_0} \leqslant 0$ we know that $Y_{t_0} \geqslant \bar{p} + \sqrt{\bar{p}^2 + \bar{q}} > \bar{p} + \delta$ and therefore $\tau = t_0$.

Hence, we have the following estimates.

- For the times where $-\bar{a}(Y - \bar{p}) \leqslant -\bar{a}\delta$ we directly estimate the integrand of the left hand side of (2.6) by $-\bar{a}\delta$.
- For the times where $-\bar{a}(Y - \bar{p}) > -\bar{a}\delta$ we can estimate the integrand of the left hand side of (2.6) by $-\bar{a}(Y_{t_0} - \bar{p})$ and the length of this time interval by Estimate (2.7).

To sum up, using that $0 \leqslant \hat{p} + \sqrt{\hat{p}^2 + \hat{q}} \geqslant \bar{p} + \sqrt{\frac{\bar{q}}{2}} - Y_{t_0}$, we derive

$$\begin{aligned}
\int_{t_0}^{t_1} -\bar{a}(Y_r - \bar{p}) \, \mathrm{d}r &\leqslant -\bar{a}\delta(t_1 - \tau) - \bar{a}(Y_{t_0} - \bar{p})(\tau - t_0) \\
&= -\bar{a}\delta(t_1 - t_0) + (\bar{a}\delta - \bar{a}(Y_{t_0} - \bar{p}))(\tau - t_0) \\
&\leqslant -\check{a}\frac{\sqrt{\bar{q}}}{\sqrt{2}}(t_1 - t_0) + \frac{2}{\check{q}} \left(\hat{p} + \sqrt{\hat{p}^2 + \hat{q}} \right) (Y_{t_1} - Y_{t_0}) \mathbb{1}_{\{Y_{t_1} - Y_{t_0} > 0\}}.
\end{aligned}$$

■

Proposition 2.9. *Let Assumption 2.2 be fulfilled, $0 \leqslant t_0 \leqslant t_1 < \infty$ and $Y_{t_0} \in [0, \hat{Y}]$. Then there exist constants $\delta_1, \delta_2 > 0$ independent of t_0 and t_1 such that*

$$\int_{t_0}^{t_1} -a_s(Y_s - p_s) \, \mathrm{d}s \leqslant -\delta_1(t_1 - t_0) + \delta_2.$$

Proof. First, we have a look at the case, where p, q, a are piecewise constant. We split the path of Y into many excursions (as described in Lemma 2.7) and left over time intervals which can not be put together to excursions. Those left over time intervals have to be such that either Y is monotone decreasing or Y is monotone increasing on all of them. Since $0 \leqslant Y \leqslant \hat{Y}$ (see Lemma 2.3) we get from Lemma 2.8 that the contributions of the left over monotone intervals in the estimate are bounded by $2\frac{\hat{Y}}{\check{q}}\hat{Y} =: \delta_2$. Now we set

$$\delta_1 := \min \left(\hat{a}\frac{\check{Y}^2 + \check{q}}{2\hat{Y}}, \frac{\check{a}\sqrt{\bar{q}}}{\sqrt{2}} \right),$$

which is the minimum of the factors that get multiplied with the time increments, given in Lemma 2.7 and Lemma 2.8. Hence, the result holds for all piecewise constant functions p, q, a uniformly.

Since Y depends continuously on a, p and q, for every $\varepsilon_1 > 0$ we can choose piecewise constant approximations $\tilde{a}$, $\tilde{p}$, $\tilde{q}$ fulfilling Assumption 2.2 for the same bounds as a, p, q and generating a $\tilde{Y}$ such that $\max\left(\|a - \tilde{a}\|_{\infty,[t_0,t_1]}, \|p - \tilde{p}\|_{\infty,[t_0,t_1]}, \|q - \tilde{q}\|_{\infty,[t_0,t_1]} \|Y - \tilde{Y}\|_{\infty,[t_0,t_1]}\right) < \varepsilon_1$. Now observe that

$$\begin{aligned}
&\left| \int_{t_0}^{t_1} -a_s(Y_s - p_s) \, \mathrm{d}s - \int_{t_0}^{t_1} -\tilde{a}_s(\tilde{Y}_s - \tilde{p}_s) \, \mathrm{d}s \right| \\
&= \left| \int_{t_0}^{t_1} -(a_s - \tilde{a}_s)(\tilde{Y}_s - \tilde{p}_s) - a_s(Y_s - \tilde{Y}_s - (p_s - \tilde{p}_s)) \, \mathrm{d}s \right| \\
&\leqslant \|a - \tilde{a}\|_\infty \left| \int_{t_0}^{t_1} \tilde{Y}_s - \tilde{p}_s \, \mathrm{d}s \right| + (\|Y - \tilde{Y}\|_\infty + \|p - \tilde{p}\|_\infty) \int_{t_0}^{t_1} a_s \, \mathrm{d}s \\
&\leqslant \|a - \tilde{a}\|_\infty T(\hat{Y} + \max\{|\hat{p}|, |\check{p}|\}) + (\|Y - \tilde{Y}\|_\infty + \|p - \tilde{p}\|_\infty) T\hat{a}.
\end{aligned}$$

Hence, we can choose for every $\varepsilon_2 > 0$ our ε_1 as $\varepsilon_1 = \frac{\varepsilon_2}{3T} \frac{1}{\hat{Y}+\max\{|\hat{p}|,|\check{p}|\}+\hat{a}}$ and obtain

$$\left| \int_{t_0}^{t_1} -a_s(Y_s - p_s)\,\mathrm{d}s - \int_{t_0}^{t_1} -\tilde{a}_s(\tilde{Y}_s - \tilde{p}_s)\,\mathrm{d}s \right| \leqslant \varepsilon_1 T(\hat{Y} + \max\{|\hat{p}|, |\check{p}|\}) + 2\varepsilon_1 T\hat{a} < \varepsilon_2.$$

Thus, the result for piecewise constant functions holds also true for all allowed functions a, p and q. ■

Proposition 2.10. *Let Assumption 2.2 be fulfilled. Denote with $Y^{t,x}$ the solution of the ODE*

$$Y_s^{t,x} = x + \int_t^s -a_r \left(\left(Y_r^{t,x}\right)^2 - 2p_r Y_r^{t,x} - q_r \right) \mathrm{d}r$$

for $0 \leqslant t \leqslant s < \infty$. Then there are constants $K_1, K_2 > 0$ such that for all $x_1, x_2 \in [0, \hat{Y}]$ we have that

$$\left|Y_s^{t,x_1} - Y_s^{t,x_2}\right| \leqslant |x_1 - x_2|\, K_1 e^{-K_2(s-t)}$$

for all $0 \leqslant t \leqslant s < \infty$.

Proof. First, note that for $x_0 \in \mathbb{R}$ the dynamics of Y^{t,x_0} are the same as of Y above. Furthermore, by introducing the function $h(r,x) := -a_r \left(x^2 - 2p_r x - q_r\right)$ for $(r,x) \in [0,\infty) \times \mathbb{R}$ and using differentiation in its weak sense, we can write the dynamics as

$$\partial_s Y_s^{t,x_0} = h(s, Y_s^{t,x_0}), \quad Y_t^{t,x_0} = x_0.$$

By standard theory (see e.g. Theorem 1 in Chapter 2.5 of [11]) it is known that Y^{t,x_0} is also differentiable with respect to its initial value x_0 and that $\partial_{x_0} Y_s^{t,x_0}$ solves the differential equation

$$y'(s) = \partial_x h(s, Y_s^{t,x_0}) y(s), \qquad y_t = 1,$$

which has the solution

$$\partial_{x_0} Y_s^{t,x_0} = y(s) = \exp\left(\int_t^s \partial_x h(r, Y_s^{t,x_0})\,\mathrm{d}r \right) = \exp\left(\int_t^s -2a_r \left(Y_r^{t,x_0} - p_r\right) \mathrm{d}r \right).$$

Therefore,

$$\partial_{x_0} Y_s^{t,x_0} \leqslant \exp\left(-2\delta_1(s-t) + 2\delta_2\right)$$

for some constants $\delta_1, \delta_2 > 0$ by Proposition 2.9. Hence,

$$\begin{aligned} \left|Y_s^{t,x_1} - Y_s^{t,x_2}\right| &= \left| \int_{x_2}^{x_1} \partial_x Y_s^{t,x}\,\mathrm{d}x \right| \\ &\leqslant \left| \int_{x_2}^{x_1} \exp\left(-2\delta_1(s-t) + 2\delta_2\right) \mathrm{d}x \right| \\ &= |x_1 - x_2| \exp\left(-2\delta_1(s-t) + 2\delta_2\right). \end{aligned}$$

Thus, defining $K_1 := e^{2\delta_2}$ and $K_2 := 2\delta_1$ we obtain the claimed result. ■

Now we have all necessary tools in order to prove Theorem 2.1.

Proof of Theorem 2.1. For shorter notation we write

$$\tilde{p}_s := 2B_s\beta_{aa}(s) + \beta_{xa}(s) + C_s^2\beta_{aa}(s), \qquad \tilde{q}_s := 4\beta_{xx}(s)\beta_{aa}(s) - \beta_{xa}^2(s), \qquad \tilde{a}_s := \frac{1}{2\beta_{aa}(s)}$$

and define $p_s^T := \tilde{p}_{T-s}$, $q_s^T := \tilde{q}_{T-s}$ and $a_s^T := \tilde{a}_{T-s}$. Next, we define for $0 \leqslant s \leqslant t \leqslant T$ and $x \in [0, \hat{U}]$ the process $Y^{T,s,x}$ as the solution of the ODE

$$y'(t) = -a_t^T \left((y(t))^2 - 2p_t^T y(t) - q_t^T \right), \qquad y(s) = x,$$

which exists and is unique by Lemma 2.3 and Assumption 1.1. Note that due to the construction of Y we have for $0 \leqslant r \leqslant t \leqslant T$ that

$$Y_t^{T,0,0} = Y_t^{T,r,Y_r^{T,0,0}} = Y_{t-r}^{T-r,0,Y_r^{T,0,0}}$$

and hence for $0 \leqslant t \leqslant T \leqslant \tau$ also

$$Y_{\tau-t}^{\tau,0,0} = Y_{\tau-t}^{\tau,\tau-T,Y_{\tau-T}^{\tau,0,0}} = Y_{T-t}^{T,0,Y_{\tau-T}^{\tau,0,0}}. \tag{2.8}$$

A straightforward calculation yields that $(Y_t^{T,0,U_T})_{t\in[0,T]}$ has exactly the same dynamics as $(U_{T-t}^T)_{t\in[0,T]}$ and hence, by uniqueness, they are equal. Thus, we can apply the results from Proposition 2.9 and Proposition 2.10. Also, for any $0 \leqslant t_0 \leqslant t_1 \leqslant T$ we obtain

$$\begin{aligned}\int_{t_0}^{t_1} -\tilde{a}_r \left(U_r^T - \tilde{p}_r\right) \mathrm{d}r = \int_{t_0}^{t_1} -a_{T-r}^T \left(Y_{T-r}^{T,0,0} - p_{T-r}^T\right) \mathrm{d}r &= \int_{T-t_1}^{T-t_0} -a_r^T \left(Y_r^{T,0,0} - p_r^T\right) \mathrm{d}r \\ &\leqslant -\delta_1(t_1 - t_0) + \delta_2\end{aligned}$$

by applying Proposition 2.9. Replacing $\tilde{a}$ and $\tilde{p}$ by their long forms, gives the stated right hand estimate and noting that $\frac{C^2}{2} \geqslant 0$ the left hand one.

Next, let U^1 and U^2 both fulfill Equation (0.2) and be bounded by 0 and $\hat{U}$ on $[0,T]$. Then due to Proposition 2.10, there are constants $K_1, K_2 > 0$, which are independent of T, such that

$$\left|U_0^1 - U_0^2\right| = \left|Y_T^{T,0,U_T^1} - Y_T^{T,0,U_T^2}\right| \leqslant \left|U_T^1 - U_T^2\right| K_1 e^{-K_2(T-0)} \leqslant \hat{U} K_1 e^{-K_2 T}.$$

Thus, the length of the interval of initial values, yielding by 0 and $\hat{U}$ bounded paths on the time $[0,T]$, is contracting with exponential speed in T. Hence, there exists a unique point u such that for $U_0 = u$ we have that U is bounded by 0 and $\hat{U}$ on $[0,\infty)$. ■

3 Verification of the linear-quadratic non-ergodic control

In this section we first prove a verification result, and then apply it in order to prove Theorem 1.3.

We present the verification result not only for the linear-quadratic model introduced in Section 1 but for a more general model. Indeed, assuming a more general setting does not require new arguments.

Let $\mu : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be Lipschitz continuous in the last two arguments, uniformly in the first argument.

Let the control processes and the set $\mathcal{A}$ of admissible controls be defined as in Section 1. Given a control α, we assume that state process satisfies the SDE

$$dX_t = \mu(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t. \tag{3.1}$$

Notice that our assumptions imply that for every $x \in \mathbb{R}$ the SDE (3.1) has a unique solution $X^{x,\alpha}$ satisfying $X_0^{x,\alpha} = x$. Moreover, one can show that for all $p \in [1,\infty)$ and $T \in [0,\infty)$ we have $\sup_{t\in[0,T]} E|X_t^{x,\alpha}|^p < \infty$ (see Section 2.5 in [9]).

Let $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be measurable and assume that there exists $K \in [0,\infty)$ such that for all $t, x, a \in [0,\infty) \times \mathbb{R} \times \mathbb{R}$ have

$$|f(t,x,a)| \leqslant K(1+|x|^2).$$

Notice that if $\alpha \in \mathcal{A}$, then the limsup cost functional

$$\bar{J}(x,\alpha) = \limsup_{T\to\infty} E\frac{1}{T}\int_0^T f(s, X_s^{x,\alpha}, \alpha_s)ds, \tag{3.2}$$

has only finite values.

We now consider the problem of minimizing $\bar{J}(x,\alpha)$ among all admissible controls. We show that one can characterize the solution in terms of the following PDE

$$\partial_t\psi(t,x) + \inf_{a\in\mathbb{R}}\left\{\mu(t,x,a)\partial_x\psi(t,x) + \frac{1}{2}\sigma^2(t,x,a)\partial_{xx}\psi(t,x) + f(t,x,a)\right\} = 0 \tag{3.3}$$

As a terminal condition we impose that there exists $\eta \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have

$$\limsup_{t\to\infty} -\frac{\psi(t,x)}{t} = \eta. \tag{3.4}$$

Proposition 3.1. *Let $\psi \in \mathcal{C}^{1,2}([0,\infty))$ be a function satisfying* (3.3)*. Suppose that there exists $\eta \in \mathbb{R}$ such that* (3.4) *holds true for all $x \in \mathbb{R}$. Moreover, suppose that there exists $K \in [0,\infty)$ such that for all $t \in [0,\infty)$ and $x \in \mathbb{R}$ we have*

$$|\psi(t,x) - \psi(t,0)| \leqslant K(1+|x|^2), \tag{3.5}$$

and that also the space derivative $\partial_x\psi$ grows at most polynomially in x, uniformly in t. Then $\inf_{\alpha\in\mathcal{A}} \bar{J}(x,\alpha) \geqslant \eta$.

Assume, in addition, that there exists a measurable version of

$$a^*(t,x) \in \operatorname{argmin}_{a\in\mathbb{R}}\left\{\mu(t,x,a)\partial_x\psi(t,x) + \frac{1}{2}\sigma^2(t,x,a)\partial_{xx}\psi(t,x) + f(t,x,a)\right\} \tag{3.6}$$

such that for every $x \in \mathbb{R}$ the SDE

$$dX_t = \mu(t,X_t,a^*(t,X_t))dt + \sigma(t,X_t,a^*(t,X_t))dW_t, \qquad X_0 = x, \tag{3.7}$$

possesses a unique solution $X^{*,x}$ and $\sup_{t\in[0,\infty)} E[(X_t^{*,x})^2] < \infty$. Then $\alpha_t^* = a^*(t, X_t^{*,x})$ satisfies

$$\bar{J}(x,\alpha^*) = \inf_{\alpha\in\mathcal{A}} \bar{J}(x,\alpha) = \eta; \tag{3.8}$$

in particular α^* is an optimal control.

Proof. Let $x \in \mathbb{R}$ and $\alpha \in \mathcal{A}$. We shortly write $X = X^{x,\alpha}$ in the following. The Ito formula and (3.3) imply

$$\begin{aligned}
&\psi(T,X_T) - \psi(0,x) \\
&\quad= \int_0^T \left(\partial_t\psi(t,X_t) + \mu(t,X_t,\alpha_t)\partial_x\psi(t,X_t) + \frac{1}{2}\sigma^2(t,X_t,\alpha_t)\partial_{xx}\psi(t,X_t)\right)dt + M_T \\
&\quad\geqslant -\int_0^T f(t,X_t,\alpha_t)dt + M_T,
\end{aligned} \tag{3.9}$$

where $M_T = \int_0^T \partial_x\psi(t,X_t)\sigma(t,X_t)dW_t$. The assumptions on ψ and on α entail that $\int_0^T(\partial_x\psi(t,X_t)\sigma(t,X_t))^2dt < \infty$, and hence $E(M_T) = 0$. Therefore, taking expectations on both sides of (3.9) and multiplying with $-\frac{1}{T}$ yields

$$\frac{1}{T}E\left(\psi(0,x) - \psi(T,X_T)\right) \leqslant E\frac{1}{T}\int_0^T f(t,X_t,\alpha_t)dt. \tag{3.10}$$

Notice that

$$\frac{1}{T}E\left(\psi(0,x) - \psi(T,X_T)\right) = \frac{\psi(0,x) - \psi(T,x)}{T} + \frac{\mathbb{E}(\psi(T,x) - \psi(T,X_T))}{T}. \tag{3.11}$$

By assumption (3.4), for the first fraction on the RHS of (3.11) we have

$$\limsup_{T\to\infty} \frac{\psi(0,x) - \psi(T,x)}{T} = \eta,$$

and, since $\sup_t E(X_t^2) < \infty$, for the second we have

$$\limsup_T \frac{|E(\psi(T,x) - \psi(T,X_T)|)}{T} \leqslant \limsup_T \frac{K(2 + |x| + \sup_t E(X_t^2))}{T} = 0.$$

Thus, from (3.10) we get

$$\eta \leqslant \limsup_{T\to\infty} E\frac{1}{T}\int_0^T f(t,X_t,\alpha_t)dt = \bar{J}(x,\alpha).$$

Since α is chosen arbitrarily, we also have $\inf_{\alpha\in\mathcal{A}} \bar{J}(x,\alpha) \geqslant \eta$.

Now suppose that $a^* : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is measurable and satisfies (3.6). Moreover, suppose that (3.7) has a unique solution $X^* = X^{*,x}$ and that $\sup_{t\in[0,\infty)} E[(X_t^*)^2] < \infty$. Then the control $\alpha_t^* = a^*(t,X_t^*)$, $t \geqslant 0$, belongs to $\mathcal{A}$. Notice that the inequalities (3.9) and (3.10) become equalities if we replace X with X^*. We thus obtain $\eta = \bar{J}(x,\alpha^*)$. This yields, together with the first part of the proof, the statement (3.8). ■

A verification result for the liminf cost functional

$$\underline{J}(x,\alpha) = \liminf_{T\to\infty} E\frac{1}{T}\int_0^T f(s,X_s^{x,\alpha},\alpha_s)ds, \tag{3.12}$$

can be shown similarly. One simply needs to replace the limsup in (3.4) has to be replaced by a liminf.

Remember that U^∞ is the unique non-negative, bounded solution of (0.2) as described in Theorem 2.1.

Lemma 3.2. *Let Assumption 1.1 be fulfilled. Then the process*

$$\varphi_t^\infty := \int_t^\infty \left[U_s^\infty\left(b_s + c_sC_s + \frac{\beta_a(s)}{2\beta_{aa}(s)}\right) - \frac{\beta_a(s)\beta_{xa}(s)}{2\beta_{aa}(s)} + \beta_x(s)\right]\exp\left(\int_t^s B_r + \frac{\beta_{xa}(r) - U_r^\infty}{2\beta_{aa}(r)}\,\mathrm{d}r\right)\mathrm{d}s$$

for $t \in [0,\infty)$ is well defined and bounded uniformly in time.

Proof. By Theorem 2.1 we obtain

$$\begin{aligned}
&\int_t^\infty \left|\left[U_s^\infty\left(b_s + c_sC_s + \frac{\beta_a(s)}{2\beta_{aa}(s)}\right) - \frac{\beta_a(s)\beta_{xa}(s)}{2\beta_{aa}(s)} + \beta_x(s)\right]\exp\left(\int_t^s B_r + \frac{\beta_{xa}(r) - U_r^\infty}{2\beta_{aa}(r)}\,\mathrm{d}r\right)\right|\mathrm{d}s\\
&\leqslant \sup_{s\in[0,\infty)}\left[\hat{U}\left|b_s + c_sC_s + \frac{\beta_a(s)}{2\beta_{aa}(s)}\right| + \left|-\frac{\beta_a(s)\beta_{xa}(s)}{2\beta_{aa}(s)} + \beta_x(s)\right|\right]\int_t^\infty e^{-\delta_1(s-t)+\delta_2}\,\mathrm{d}s\\
&= \sup_{s\in[0,\infty)}\left[\hat{U}\left|b_s + c_sC_s + \frac{\beta_a(s)}{2\beta_{aa}(s)}\right| + \left|-\frac{\beta_a(s)\beta_{xa}(s)}{2\beta_{aa}(s)} + \beta_x(s)\right|\right]\frac{e^{\delta_2}}{\delta_1}\\
&< \infty,
\end{aligned}$$

which means that φ^∞ is well defined and bounded. ■

In the following we use for $t \in [0,\infty)$, $x \in \mathbb{R}$ the definitions

$$\psi(t,x) := \frac{1}{2}U_t^\infty \cdot x^2 + \varphi_t^\infty \cdot x + \int_0^t -\left(\varphi_s^\infty b_s + U_s^\infty \frac{c_s^2}{2} + \beta_0(s) - \frac{\left(\varphi_s^\infty - \beta_a(s)\right)^2}{4\beta_{aa}(s)}\right) \mathrm{d}s$$

$$a^\infty(t,x) := \frac{\varphi_t^\infty - \beta_a(t) + (U_t^\infty - \beta_{xa}(t))x}{2\beta_{aa}(t)}$$

and

$$\eta := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \varphi_s^\infty b_s + U_s^\infty \frac{c_s^2}{2} + \beta_0(s) - \frac{\left(\varphi_s^\infty - \beta_a(s)\right)^2}{4\beta_{aa}(s)}\,\mathrm{d}s.$$

Lemma 3.3. *Let Assumption 1.1 be fulfilled. Then there exists an $\varepsilon > 0$ such that $\sup_{t\in[0,\infty)} \mathbb{E}\left[\left|X_t^{a^\infty}\right|^p\right] < \infty$ for every $p \in (0, 2+\varepsilon)$ and every initial value $x_0 \in \mathbb{R}$.*

For the proof of Lemma 3.3 we need the following.

Lemma 3.4. *Let $p, q : [0,\infty) \to \mathbb{R}$ be measurable and bounded. The integral equation*

$$h(t) = h(0) + \int_0^t \left[p(s)\cdot h(s) + q(s)\right] \mathrm{d}s,$$

for $h(0) \in \mathbb{R}$ and $t \geqslant 0$, has the unique, explicit solution

$$h(t) = e^{\int_0^t p(s)\,\mathrm{d}s}\left(h(0) + \int_0^t q(s)e^{-\int_0^s p(r)\,\mathrm{d}r}\,\mathrm{d}s\right) = h(0)e^{\int_0^t p(s)\,\mathrm{d}s} + \int_0^t q(s)e^{\int_s^t p(r)\,\mathrm{d}r}\,\mathrm{d}s.$$

Proof. That h solves the integral equation is straightforward by weak differentiation. The uniqueness follows since the integral equation is linear in h with bounded coefficients, which makes it a Lipschitz ODE. ∎

Proof of Lemma 3.3. Observe that

$$\begin{aligned}\mathbb{E}\left[X_t^{\alpha^T}\right] &= x_0 + \mathbb{E}\left[\int_0^t \left(b_s + B_s X_s^{\alpha^T} - \frac{\varphi_s^T - \beta_a(s) + (U_s^T - \beta_{xa}(s))X_s^{\alpha^T}}{2\beta_{aa}(s)}\right)\mathrm{d}s\right] \\ &= x_0 + \int_0^t \left(b_s + \frac{-\varphi_s^T + \beta_a(s)}{2\beta_{aa}(s)}\right)\mathrm{d}s + \int_0^t \left(B_s + \frac{\beta_{xa}(s) - U_s^T}{2\beta_{aa}(s)}\right)\mathbb{E}\left[X_s^{\alpha^T}\right]\mathrm{d}s.\end{aligned}$$

By Lemma 3.4 we get

$$\mathbb{E}\left[X_t^{\alpha^T}\right] = x_0 e^{\int_0^t B_s + \frac{\beta_{xa}(s) - U_s^T}{2\beta_{aa}(s)}\,\mathrm{d}s} + \int_0^t \left(b_s + \frac{-\varphi_s^T + \beta_a(s)}{2\beta_{aa}(s)}\right) e^{\int_s^t B_r + \frac{\beta_{xa}(r) - U_r^T}{2\beta_{aa}(r)}\,\mathrm{d}r}\,\mathrm{d}s$$

and hence, using that

$$\left|b_s + \frac{-\varphi_s^T + \beta_a(s)}{2\beta_{aa}(s)}\right| \leqslant \sup_{r\in[0,\infty)} |b_r| + \frac{\hat{\varphi} + \sup_{r\in[0,\infty)}|\beta_a(r)|}{2\check{\beta}_{aa}} < \infty,$$

and Theorem 2.1 we obtain

$$\begin{aligned}\left|\mathbb{E}\left[X_t^{\alpha^\infty}\right]\right| &\leqslant |x_0|e^{-\delta_1(t-0)+\delta_2} + \sup_{r\in[0,\infty)}\left|b_r + \frac{-\varphi_r^\infty + \beta_a(r)}{2\beta_{aa}(r)}\right| \int_0^t e^{-\delta_1(t-r)+\delta_2}\,\mathrm{d}s \\ &= |x_0|e^{\delta_2}e^{-\delta_1 t} + \left(1 - e^{-\delta_1 t}\right)\frac{e^{\delta_2}}{\delta_1}\sup_{r\in[0,\infty)}\left|b_r + \frac{-\varphi_r^\infty + \beta_a(r)}{2\beta_{aa}(r)}\right| \\ &\leqslant \max\left(|x_0|e^{\delta_2}, \frac{e^{\delta_2}}{\delta_1}\left(\sup_{r\in[0,\infty)}|b_r| + \frac{\hat{\varphi} + \sup_{r\in[0,\infty)}|\beta_a(r)|}{2\check{\beta}_{aa}}\right)\right).\end{aligned}$$

Furthermore, using Itô's Formula

$$\begin{aligned}
&\mathbb{E}\left[\left(X_t^{\alpha^\infty}\right)^2\right] \\
&= x_0^2 + \mathbb{E}\left[\int_0^t 2X_s^{\alpha^\infty}\left(b_s + B_s X_s^{\alpha^\infty} - \frac{\varphi_s^\infty - \beta_a(s) + (U_s^\infty - \beta_{xa}(s))X_s^{\alpha^\infty}}{2\beta_{aa}(s)}\right) + \left(c_s + C_s X_s^{\alpha^\infty}\right)^2 \mathrm{d}s\right] \\
&= x_0^2 + \int_0^t c_s^2 + \mathbb{E}\left[X_s^{\alpha^\infty}\right] 2\left(c_s C_s + b_s - \frac{\varphi_s^\infty - \beta_a(s)}{2\beta_{aa}(s)}\right) \\
&\qquad + \mathbb{E}\left[\left(X_s^{\alpha^\infty}\right)^2\right] 2\left(B_s + \frac{\beta_{xa}(s) - U_s^\infty}{2\beta_{aa}(s)} + \frac{C_s^2}{2}\right)\mathrm{d}s \\
&= x_0^2 \exp\left(2\int_0^t \left(B_s + \frac{\beta_{xa}(s) - U_s^\infty}{2\beta_{aa}(s)}\right) + \frac{C_s^2}{2}\,\mathrm{d}s\right) \\
&\quad + \int_0^t \left\{c_s^2 + 2\mathbb{E}\left[X_s^{\alpha^\infty}\right]\left(c_s C_s + b_s - \frac{\varphi_s^\infty - \beta_a(s)}{2\beta_{aa}(s)}\right)\right\} \\
&\qquad \cdot \exp\left(2\int_s^t \left(B_s + \frac{\beta_{xa}(s) - U_s^\infty}{2\beta_{aa}(s)}\right) + \frac{C_s^2}{2}\,\mathrm{d}r\right)\mathrm{d}s
\end{aligned}$$

due to Lemma 3.4. By Theorem 2.1 we can estimate

$$\begin{aligned}
\mathbb{E}\left[\left(X_t^{\alpha^\infty}\right)^2\right] &\leqslant x_0^2 \exp\left(-2\delta_1 t + 2\delta_2\right) \\
&\qquad + \int_0^t \left\{c_s^2 + 2\mathbb{E}\left[X_s^{\alpha^\infty}\right]\left(c_s C_s + b_s - \frac{\varphi_s^\infty - \beta_a(s)}{2\beta_{aa}(s)}\right)\right\} \exp\left(-2\delta_1(t-s) + 2\delta_2\right)\mathrm{d}s \\
&\leqslant x_0^2 \exp\left(-2\delta_1 t + 2\delta_2\right) \\
&\qquad + \left\{\sup_{s\in[0,\infty)} c_s^2 + 2\sup_{s\in[0,\infty)} \mathbb{E}\left[X_s^{\alpha^\infty}\right]\left(\sup_{s\in[0,\infty)} |c_s C_s| + \sup_{s\in[0,\infty)}\left|b_s - \frac{\varphi_s^\infty - \beta_a(s)}{2\beta_{aa}(s)}\right|\right)\right\} \\
&\qquad\qquad \cdot \frac{\exp(2\delta_2)}{2\delta_1}\left(1 - \exp\left(-2\delta_1 t\right)\right) \\
&< \infty.
\end{aligned}$$

With Jensen's inequality this implies for all $q \in (0,2]$ that

$$\mathbb{E}\left[\left|X_s^{\alpha^\infty}\right|^q\right] \leqslant \mathbb{E}\left[\left|X_s^{\alpha^\infty}\right|^{(q\cdot 2/q)}\right]^{q/2} \leqslant \mathbb{E}\left[\left|X_s^{\alpha^\infty}\right|^2\right]$$

and hence also $\sup_{s\in[0,\infty)} \mathbb{E}\left[\left|X_s^{\alpha^\infty}\right|^q\right] < \infty$.

Furthermore, for $2 \leqslant p \in \mathbb{R}$ we analogously obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|X_t^{\alpha^\infty}\right|^p\right] \\
&\leqslant |x_0|^p \exp\left(-p\left(\delta_1 - \frac{p-2}{2}\sup_{s\in[0,\infty)} C_s^2\right)t + p\delta_2\right) \\
&\qquad + \left\{\frac{p^2-p}{2}\sup_{s\in[0,\infty)} \mathbb{E}\left[\left|X_s^{\alpha^T}\right|^{p-2}\right]\sup_{s\in[0,\infty)} c_s^2\right. \\
&\qquad\qquad \left. + p\sup_{s\in[0,\infty)} \mathbb{E}\left[\left|X_s^{\alpha^\infty}\right|^{p-1}\right]\left((p-1)\sup_{s\in[0,\infty)} |c_s C_s| + \sup_{s\in[0,\infty)}\left|b_s - \frac{\varphi_s^\infty - \beta_a(s)}{2\beta_{aa}(s)}\right|\right)\right\} \\
&\qquad \cdot \frac{\exp(p\delta_2)}{p\left(\delta_1 - \frac{p-2}{2}\sup_{s\in[0,\infty)} C_s^2\right)}\left(1 - \exp\left(-p\left(\delta_1 - \frac{p-2}{2}\sup_{s\in[0,\infty)} C_s^2\right)t\right)\right).
\end{aligned}$$

Since $\delta_1 > 0$ and $\sup_{s\in[0,\infty)} C_s^2 < \infty$, we get that for every p with $2 \leqslant p < 2 + \frac{2\delta_1}{\sup_{s\in[0,\infty)} C_s^2}$ that $\sup_{t\in[0,\infty)} \mathbb{E}\left[\left|X_t^{\alpha^\infty}\right|^p\right] < \infty$. Thus, setting $\varepsilon := \frac{2\delta_1}{\sup_{s\in[0,\infty)} C_s^2}$ we proved the result. ■

Lemma 3.5. *Let Assumption 1.1 be fulfilled. Then ψ fulfills Equation* (3.3) *and a^∞ is a minimizer for this equation.*

Proof. A straightforward calculation yields that

$$\begin{aligned}
&\partial_t\psi(t,x) + (\mu(t,x) - a^\infty(t,x))\partial_x\psi(t,x) + \frac{1}{2}\sigma^2(t,x)\partial_{xx}\psi(t,x) \\
&\qquad = x^2\left[-\frac{(U_t^\infty)^2}{4\beta_{aa}(t)} - \beta_{xx}(t) + \frac{\beta_{xa}^2(t)}{4\beta_{aa}(t)}\right] \\
&\qquad\quad + x\left[-\left(U_t^\infty\frac{\beta_a(t)}{2\beta_{aa}(t)} - \frac{\beta_a(t)\beta_{xa}(t)}{2\beta_{aa}(t)} + \beta_x(t)\right) + U_t^\infty\left(-\frac{\varphi_t^\infty - \beta_a(t)}{2\beta_{aa}(t)}\right)\right] \\
&\qquad\quad + \left[-\beta_0(t) - \beta_a(t)\left(\frac{\varphi_t^\infty - \beta_a(t)}{2\beta_{aa}(t)}\right) - \beta_{aa}(t)\left(\frac{\varphi_t^\infty - \beta_a(t)}{2\beta_{aa}(t)}\right)^2\right] \\
&\qquad = -f(t,x,a^\infty(t,x)).
\end{aligned} \tag{3.13}$$

Next, observe that since f is strictly convex in a and the remainder of Equation (3.3) is affine linear in a, we get that if the derivative with respect to a becomes zero, we are in the unique minimum. Using this, we obtain by

$$\begin{aligned}
&\partial_a\left[\partial_t\psi(t,x) + (\mu(t,x) - a)\partial_x\psi(t,x) + \frac{1}{2}\sigma^2(t,x)\partial_{xx}\psi(t,x) + f(t,x,a)\right]\Big|_{a=a^\infty(t,x)} \\
&= \left[-\partial_x\psi(t,x) + \partial_a f(t,x,a)\right]\Big|_{a=a^\infty(t,x)} \\
&= -U_t^\infty x - \varphi_t^\infty + \beta_{xa}(t)x + \beta_a(t) + 2\beta_{aa}(t)a^\infty(t,x) \\
&= 0
\end{aligned}$$

that a^∞ minimizes Equation (3.3). Therefore, plugging the minimizer a^∞ into Equation (3.3) and using the result of Equation (3.13) we get

$$\begin{aligned}
&\partial_t\psi(t,x) + \inf_{a\in\mathbb{R}}\left\{(\mu(t,x) - a)\partial_x\psi(t,x) + \frac{1}{2}\sigma^2(t,x)\partial_{xx}\psi(t,x) + f(t,x,a)\right\} \\
&= \partial_t\psi(t,x) + (\mu(t,x) - a^\infty(t,x))\partial_x\psi(t,x) + \frac{1}{2}\sigma^2(t,x)\partial_{xx}\psi(t,x) + f(t,x,a^\infty(t,x)) \\
&= 0.
\end{aligned}$$

■

Theorem 3.6. *Let Assumption 1.1 be fulfilled. Then a^∞ is an optimal control, ψ fulfills* (3.3) *and*

$$\bar{J}(x,a^\infty) = \inf_{\alpha\in\mathcal{A}}\bar{J}(x,\alpha) = \eta.$$

Proof. We want to apply Proposition 3.1.

Lemma 3.5 already yields that ψ fulfills (3.3) and that a^∞ is the corresponding minimizer. Next, observe that

$$\begin{aligned}
\limsup_{t\to\infty} -\frac{\psi(t,x)}{t} &= \limsup_{t\to\infty} -\frac{\frac{1}{2}(U_t^\infty)^2x^2 + \varphi_t^\infty x + \int_0^t -\left(\varphi_s^\infty b_s + U_s^\infty\frac{c_s^2}{2} + \beta_0(s) - \frac{(\varphi_s^\infty - \beta_a(s))^2}{4\beta_{aa}(s)}\right)\mathrm{d}s}{t} \\
&= \limsup_{t\to\infty}\frac{1}{t}\int_0^t \varphi_s^\infty b_s + U_s^\infty\frac{c_s^2}{2} + \beta_0(s) - \frac{(\varphi_s^\infty - \beta_a(s))^2}{4\beta_{aa}(s)}\,\mathrm{d}s \\
&= \eta
\end{aligned}$$

since U^∞ and φ^∞ are bounded. Furthermore, for the same reason,

$$\begin{aligned}|\psi(t,x)-\psi(t,0)| &= \left|\frac{1}{2}(U_t^\infty)^2x^2+\varphi_t^\infty x\right| \\ &\leqslant K(1+|x|^2)\end{aligned}$$

and $\partial_x\psi(t,x) = U_t^\infty x + \varphi_t^\infty$ is linear in x with an in time uniformly bounded factor.

Finally, Lemma 3.3 gives the bounded second moment of the controlled process, which means that all conditions of Proposition 3.1 are fulfilled, yielding the statement. ■

4 Conclusion

We have shown that under Assumption (1.1) the problem of minimizing the limsup long-term average cost functional (0.1) is well-posed, and we have described an optimal closed loop control in terms of the unique bounded and non-negative function U^∞. Some further questions arise naturally.

First, is it possible to extend the results to a multi-dimensional setting? Following the same approach, a multi-dimensional Riccati equation on $[0,\infty)$ has to be studied. Notice that some of the comparison arguments of Section 2 can not be simply transferred to a multidimensional setting.

If the drift and diffusion coefficients in the linear state dynamics and the coefficient of the quadratic cost function f are themselves stochastic, then it is natural to assume that the solution of the control problem can be described in terms of a stochastic Riccati equation on $[0,\infty)$. Is is possible to prove existence and uniqueness of a solution and to obtain an optimal control with it?

Finally, we believe that one can generalize the results of Proposition 2.9 and Proposition 2.10 to the more general setting, where the derivative of Y is a strictly concave function having a strictly negative and a strictly positive zero. Also the starting value of Y can be generalized to be greater than any negative zero. A proof for this claim, using abstract arguments instead of the tedious calculations as presented in Section 2, is left for future research.

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