Multiplicities and Plancherel formula for the space of nondegenerate Hermitian matrices

Raphaël Beuzart-Plessis

Version of Record: https://www.sciencedirect.com/science/article/pii/S0022314X21001815
Manuscript_7f9b6959f8866d342fd883ab8cd58a4e

Multiplicities and Plancherel formula for the space of nondegenerate Hermitian matrices

Raphaël Beuzart-Plessis *

June 22, 2021

Abstract

This paper contains two results concerning the spectral decomposition, in a broad sense, of the space of nondegenerate Hermitian matrices over a local field of characteristic zero. The first is an explicit Plancherel decomposition of the associated L^2 space thus confirming a conjecture of Sakellaridis-Venkatesh in this particular case. The second is a formula for the multiplicities of generic representations in the p-adic case that extends previous work of Feigon-Lapid-Offen. Both results are stated in terms of Arthur-Clozel's quadratic local base-change and the proofs are based on local analogs of two relative trace formulas previously studied by Jacquet and Ye and known as (relative) Kuznetsov trace formulas.

Contents

*The project leading to this publication has received funding from Excellence Initiative of Aix-Marseille University-A*MIDEX, a French "Investissements d'Avenir" programme.

1 Introduction

Let E/F be a quadratic extension of local fields and let $n \geqslant 1$ be a positive integer. Set $G = \mathrm{GL}_n(E)$ and let $X = X_n$ be the space of nondegenerate Hermitian matrices i.e.

$$X = \left\{x \in G \mid {}^t x^c = x\right\}$$

where c is the non-trivial Galois involution of E/F. There is a natural right action of G on X and X carries an (unique up to a scalar) invariant measure for this action. We also set $G' = \mathrm{GL}_n(F)$ and $\mathrm{BC} : \mathrm{Irr}(G') \to \mathrm{Irr}(G)$ to be Arthur-Clozel's base-change map [AC] between the smooth duals of G' and G. The image of BC is then the set of irreducible smooth representations π of G that are Galois invariant i.e. satisfying $\pi \simeq \pi^c$.

The main theme of this paper is, roughly speaking, the "spectrum" of the space X. More precisely, we will consider the following two specific questions:

(1) L^2 version: give an explicit decomposition of $L^2(X)$ into a direct integral of unitary irreducible representations (Plancherel decomposition);

(2) Smooth version: compute the multiplicity function $\pi \in \mathrm{Irr}(G) \mapsto m(\pi) = \dim \mathrm{Hom}_G(\pi, C^\infty(X))$ where $C^\infty(X)$ is the space of smooth functions on X and $\mathrm{Hom}_G(.,.)$ stands for the space of G-equivariant (continuous)[1] linear maps.

Note that for $x \in X$, the stabilizer G_x is the unitary group preserving the Hermitian form naturally associated to x and, by Frobenius reciprocity, we have

$$m(\pi) = \sum_{x \in X/G} \dim \mathrm{Hom}_{G_x}(\pi, \mathbb{C})$$

[1]The continuity requirement is only meaningful in the Archimedean case where π should run over the *Casselman-Wallach globalizations* of irreducible Harish-Chandra modules and these naturally come with a (Fréchet) topology. However, in this paper we will only consider the multiplicities $m(\pi)$ when F is non-Archimedean in which case these subtleties will not intervene.

where x runs overs G-orbits in X (or, equivalently, equivalence classes of Hermitian forms on E^n) and $\mathrm{Hom}_{G_x}(\pi, \mathbb{C})$ is the space of (continuous) G_x-invariant linear forms on π (so-called *local unitary periods*). The second problem has first been considered by Jacquet [Jac01] who proved, using a global method, that when $n = 3$ and π is supercuspidal, $m(\pi) \neq 0$ if and only if $\pi \simeq \pi^c$ (i.e. π is in the image of BC) in which case each of the space $\mathrm{Hom}_{G_x}(\pi, \mathbb{C})$ is one-dimensional (so that $m(\pi) = 2$ sinc X has two G-orbits in this case). Following the same global approach and combining it with local methods, Feigon-Lapid-Offen [FLO] have obtained extremely fine results on the multiplicities $m(\pi)$. In this paper, we will only propose a modest improvement on their work for generic representations. On the other hand, our solution to problem (1) seems new as it hasn't been adressed in the litterature yet but, again, to work it out we will make an extensive use of the work [FLO] (which is again a generalization, and refinement, of Jacquet's work for $n = 3$ [Jac01]). The answers we obtain for both problems rely heavily on the base-change map BC.

1.1 Plancherel decomposition

Our main result on problem (1) (Theorem 6.1.1) can be stated as follows.

Theorem 1. *There is a (natural) isomorphism of unitary G-representations*

$$L^2(X) \simeq \int_{\mathrm{Temp}(G')}^{\oplus} \mathrm{BC}(\sigma) d\mu_{G'}(\sigma)$$

where $\mathrm{Temp}(G') \subset \mathrm{Irr}(G')$ *is the tempered dual of* G' *and* $d\mu_{G'}$ *the Plancherel measure for the group* G'.

This theorem confirms, in the particular case at hand, a general conjecture of Sakellaridis-Venkatesh on the L^2-spectrum of spherical varieties [SV, Conjecture 16.2.2]. More precisely, Sakellaridis and Venkatesh associate to X a dual group $\check{G}_X = \mathrm{GL}_n(\mathbb{C}) = \check{G}'$ together with a "distinguished morphism" $\check{G}_X \to \check{G}$ to the Langlands dual group of G (seen as an algebraic group over F). In [SV], only splits groups are considered so that there is no need to consider L-groups. This is not precisely the case here (since the group G is not split over F) but the distinguished morphism naturally extends to the base-change map between L-groups ${}^L G' \to {}^L G$ and an obvious extrapolation[2] of [SV, Conjecture 16.2.2] predicts a decomposition like the one of Theorem 1.

An immediate consequence of Theorem 1 is to the determination of the so-called "relative discrete series" for X i.e. of the unitary representations of G that embed in the space $L^2(X, \chi)$ for some character χ of the center: these are precisely the base-change of discrete series of G' (see Corollary 6.1.1). Note that these representations are always tempered but not necessarily discrete series of the group G. It was already shown by Jerrod Smith [Smith] that these representations are indeed relative discrete series but he didn't prove that they actually exhaust all of them.

The proof of Theorem 1 actually gives more information. Namely, we define G-invariant semi-definite scalar products $\langle ., . \rangle_{X,\sigma}$ on $C_c^\infty(X)$, that are indexed by the irreducible tempered representations σ of G' and factorize through a quotient isomorphic to $\mathrm{BC}(\sigma)^\vee$ (for technical reasons, we prefer to take the smooth contragredient of the base-change), such that

$$\langle \varphi_1, \varphi_2 \rangle_X = \int_{\mathrm{Temp}(G')} \langle \varphi_1, \varphi_2 \rangle_{X,\sigma} d\mu_{G'}(\sigma) \tag{1.1.1}$$

[2] That the "L-group" of X should really be ${}^L G'$ equipped with the base-change map ${}^L G' \to {}^L G$ is also consistent with a conjecture of Jacquet on distinction of irreducible representations by unitary groups. A refined version of this conjecture, due to Feigon-Lapid-Offen, will be discussed below.

for every $\varphi_1, \varphi_2 \in C_c^\infty(X)$ where $\langle ., . \rangle_X$ stands for the L^2-scalar product on X. That such a formula implies a decomposition like the one of Theorem 1 follows from Bernstein [Ber3] interpretation of abstract Plancherel decompositions. The scalar products $\langle ., . \rangle_{X,\sigma}$ are built on certain canonical G-equivariant embeddings $\mathcal{W}(\mathrm{BC}(\sigma)) \to C^\infty(X)$, where $\mathcal{W}(\mathrm{BC}(\sigma))$ denotes the Whittaker model of $\mathrm{BC}(\sigma)$ (for a certain choice of Whittaker datum), that have been introduced by Feigon-Lapid-Offen [FLO] in their work on the factorization of global unitary periods. By Frobenius reciprocity, these embeddings are equivalent to the data of G_x-invariant functionals $\alpha_x^\sigma : \mathcal{W}(\mathrm{BC}(\sigma)) \to \mathbb{C}$ for $x \in X$ satisfying $\alpha_{xg}^\sigma = \alpha_x^\sigma \circ \mathrm{BC}(\sigma)(g)$ for $g \in G$. We call the α_x^σ, $x \in X$, the *FLO functionals* associated to σ. The definition of those functionals by Feigon-Lapid-Offen is actually implicit: these are characterized by a series of identities between *relative Bessel distributions* through a certain transfer of functions $\varphi \in C_c^\infty(X) \mapsto f' \in C_c^\infty(G')$ that was established by Jacquet [Jac03]. One of the main result of [FLO] is that these functionals give a factorization of global unitary periods of (cuspidal) automorphic forms on GL_n (thus generalizing a result of Jacquet [Jac01] in the case $n = 3$). In Section 6.3, we will reinterpret their result in a form that make the relation to the local scalar products $\langle ., . \rangle_{X,\sigma}$ more transparent. This simple cosmetic exercise has the pleasant feature of being remarkably aligned with certain general speculations of Sakellaridis-Venkatesh on relations between global automorphic periods and local Plancherel formulas [SV, §17].

1.2 Multiplicities

As already said, the multiplicity $m(\pi)$ has already been extensively studied by Jacquet [Jac01] and Feigon-Lapid-Offen [FLO]. Their most complete result are for generic representations: when π is generic, [FLO, Theorem 0.2] gives a lower bound for $m(\pi)$ which is attained for "almost all" generic π. We henceforth assume that F is a p-adic field. In order to state the result of [FLO] and our (small) improvement on it, we find it convenient to equip the sets $\mathrm{Irr}(G')$ and $\mathrm{Irr}(G)$ with structures of algebraic varieties over $\mathbb{C}$. This construction is surely well-known, it is simply based on Langlands classification, but in lack of a proper reference we explain it in Section 5.1 (see however [Pras] for a similar construction on the Galois side). For these extra structures, the map BC is a finite morphism of algebraic varieties and we denote by $\deg \mathrm{BC} : \mathrm{Irr}(G) \to \mathbb{N}$ the associated degree function (it sends a representation $\pi \in \mathrm{Irr}(G)$ to the sum of the degrees of BC at the elements in the fiber $\mathrm{BC}^{-1}(\pi)$). Since we are in the p-adic case, G has two orbits in X (corresponding to the two isomorphism classes of Hermitian spaces of dimension n). The following result is a restatement of [FLO, Theorem 0.2].

Theorem 2 (Feigon-Lapid-Offen)**.** *Suppose that $\pi \in \mathrm{Irr}(G)$ is generic. Then, we have $m(\pi) \geqslant \deg \mathrm{BC}(\pi)$. More precisely, for each $x \in X$ we have*

$$\dim \mathrm{Hom}_{G_x}(\pi, \mathbb{C}) \geqslant \begin{cases} \lceil \frac{\deg \mathrm{BC}(\pi)}{2} \rceil & \text{if } G_x \text{ is quasi-split,} \\ \lfloor \frac{\deg \mathrm{BC}(\pi)}{2} \rfloor & \text{otherwise.} \end{cases} \tag{1.2.1}$$

Moreover, if BC *is unramified at (every point in the fiber of) π then equality holds in* (1.2.1)*.*

Our main result is that the above lower bound is actually always attained. More precisely, we show.

Theorem 3. *Let $\pi \in \mathrm{Irr}(G)$ be generic. Then, we have $m(\pi) = \deg \mathrm{BC}(\pi)$. In particular, equality always holds in* (1.2.1)*.*

This result has been conjectured Feigon-Lapid-Offen [FLO, Conjecture 13.17] and it also confirms (in this particular case) a general conjecture of Prasad for Galois pairs [Pras]. Let us insist here that the above formula for the multiplicity $m(\pi)$ is only proved here in the p-adic case although the results from [FLO] are also valid for $F = \mathbb{R}$ and Theorem 3 is expected to hold in this case too. The main reason for this restriction is the following. As we shall explain in the next paragraph, the proof of Theorem 3 is based on two main ingredients: local versions of relative trace formulas of Jacquet-Ye and a certain scalar Paley-Wiener theorem for Bessel distributions on GL_n (or on a general quasi-split group, cf. Theorem 2.4.1). Although the former is established regardless of the base field, the aforementioned Paley-Wiener theorem is restricted to p-adic fields and I do not know how to prove an analog of it when $F = \mathbb{R}$. This actually seems an interesting problem on its own and certainly not as straightforward as in the p-adic case. Moreover, as written here, the proof of Theorem 3 also uses other particular features of p-adic groups (such as a natural algebraic structure on their admissible duals) but we believe that once a suitable analog of Theorem 2.4.1 is available in the Archimedean case, the arguments can be adapted to cover this case too.

1.3 Tools: local trace formulas and Whittaker Paley-Wiener theorem

The main new tools we introduce to prove Theorems 1 and 3 are certain local analogs of relative trace formulas first introduced in a global setting by Jacquet and Ye [Ye88], [Ye89], [JY90], [JY92], [JY96], [Jac01]. A comparison of these global relative trace formulas, that was established in general by Jacquet [Jac03], [Jac04], [Jac05], led to a complete characterization of cuspidal automorphic representations that are *distinguished* by a given unitary group by Jacquet [Jac10] (for the quasi-split unitary group) and then in general by Feigon-Lapid-Offen [FLO]. This can be seen as a solution to the global analog of problem (2) above. Roughly speaking, we will deduce Theorems 1 and 3 through a similar local comparison. We note that local versions of the Jacquet-Ye trace formulas have been developed by Feigon [Fe] in the case $n = 2$ so that our treatment can be seen as a generalization of her work to arbitrary rank.

To be more precise, we develop local analogs of both the Kuznetsov trace formula (for an arbitrary quasi-split group) and of the relative Kuznetsov trace formula for X: these are identities relating so-called (relative) *Bessel distributions* (the spectral side) to (relative) orbital integrals (the geometric side). We refer the reader to the core of the text for details and precise statements (see in particular Theorems 2.3.2 and 4.2.2). We content ourself here to mention that these relative trace formulas are easy to establish. Namely, contrary to other formulas of the same sort, we can completely avoid analytic difficulties by using a regularization process of certain divergent oscillatory integrals due to Sakellaridis-Venkatesh [SV, Corollary 6.3.3] and generalized by Lapid-Mao in [LM, Proposition 2.11] (this last result roughly says that integration over a maximal unipotent subgroup against a generic character of the latter behaves, in some respect, as a compact integration).

Another result that we will need to establish Theorem 3 is a certain scalar Whittaker Paley-Wiener theorem describing, in the case of a quasi-split reductive p-adic group G, the image by some "Bessel transform" of the space of test functions $C_c^\infty(G)$ (see Section 2.4 and Theorem 2.4.1 for a precise statement). The result is far simpler to state than for the usual trace Paley-Wiener theorem [BDK] and it is moreover an easy consequence of the theory of Jacquet's functionals. However, we have not seen this theorem stated elsewhere in the litterature (maybe because of simplicity).

1.4 Acknowledgement

The results of this paper have been announced by the author in his “Cours Peccot" in April-May 2017. I would like to thank the Collège de France for giving me the opportunity to give this course and I apologize for the delay in writing the details of this work.

I am very grateful to Jean-Loup Waldspurger for a thorough reading of a first version of this manuscript allowing for many improvements and in particular for catching a fatal error in the proof of Proposition 5.1.1.

The project leading to this publication has received funding from Excellence Initiative of Aix-Marseille University-A*MIDEX, a French “Investissements d'Avenir" programme.

1.5 General notation

- In the whole paper, F denotes a local field of characteristic zero (Archimedean or non-Archimedean). In some specific sections (in particular, in the whole of Chapter 5), F will be assumed to be p-adic but such restriction will always be explicitely stated.

- For a smooth manifold X, we denote by $C_c^\infty(X)$ the usual space of test functions on X. For a totally disconnected locally compact space X, we denote by $C_c^\infty(X)$ the space of locally constant compactly supported complex functions on X.

- If f and g are two positive functions on a set X, we write

$$f(x) \ll g(x), \quad x \in X,$$

  to mean that there exists a constant $C > 0$ such that $f(x) \leqslant Cg(x)$ for every $x \in X$. If we want to emphasize that the implicit constant depends on auxilliary parameters $y_1, \dots, y_k$ we write $f(x) \ll_{y_1,\dots,y_k} g(x)$ instead.

- The symbol $\widehat{\otimes}$ stands for the projective completed tensor product of locally convex topological vector spaces (cf. [Tr, Chap. 43]; this will only be used for Fréchet spaces).

- When a group G acts on the right (resp. on the left) of a set X, we denote by R (resp. L) the corresponding action by translation on the space of functions on X.

- If G is a group and S a subset of it, we write $\mathrm{Norm}_G(S)$ for the normalizer of S in G.

- For every integer $n \geqslant 0$, we denote by $\mathfrak{S}_n$ the symmetric group in n letters.

- If G is a Lie group, we write $\mathfrak{g}$ for its Lie algebra and $\mathcal{U}(\mathfrak{g})$ for the corresponding enveloping algebra.

- Let G be a real or p-adic reductive group. By a *smooth representation* of G we mean a representation over a complex vector space with open stabilizers in the p-adic case, a smooth admissible Fréchet representation of moderate growth in the sense of Casselman-Wallach in the real case [Cas2], [WallII, Chap. 11]. If π is a smooth irreducible representation of G, we denote by $\pi^\vee$ its smooth contragredient (that is the Casselman-Wallach globalization of the admissible dual of the underlying Harish-Chandra module in the real case).

- We denote the set of isomorphism classes of smooth irreducible representations of G by $\mathrm{Irr}(G)$ and we write $\mathrm{Temp}(G) \subset \mathrm{Irr}(G)$ for the subset of tempered representations.

- If G is a p-adic reductive group, H is a closed subgroup and π, σ are smooth representations of G and H respectively, we write $\mathrm{Hom}_H(\pi,\sigma)$ for the space of H-equivariant linear maps $\pi \to \sigma$.
- Still in the p-adic case, if P is a parabolic subgroup of G and σ a smooth representation of one of its Levi component, we denote by $I_P^G(\sigma)$ the normalized smooth parabolic induction of σ.

2 Local Kuznetsov trace formula and a scalar Whittaker Paley-Wiener theorem

Let F be a local field of characteristic zero (Archimedean or p-adic) and $\underline{G}$ be a quasi-split connected reductive group defined over F. The main goal of this chapter is to develop a local Kuznetsov trace formula for $\underline{G}(F)$ in the spirit of the work of Feigon [Fe] for the group $\mathrm{PGL}_2(F)$.

More precisely, let $\underline{B} = \underline{TN}$ be a Borel subgroup of $\underline{G}$ (defined over F) and $\underline{B}^- = \underline{TN}^-$ be the opposite Borel subgroup (with respect to $\underline{T}$). We set $G = \underline{G}(F)$, $B = \underline{B}(F)$, $T = \underline{T}(F)$, $N = \underline{N}(F)$ and $N^- = \underline{N}^-(F)$. We denote by δ_B the modular character of B and we fix an element $w \in G$ such that $N^- = w^{-1}Nw$. Let $\xi : N \to \mathbb{S}^1$ be a non-degenerate character (i.e. whose stabilizer in T is reduced to the center of G). We define a non-degenerate unitary character $\xi^- : N^- \to \mathbb{S}^1$ by $\xi^-(u^-) = \xi(wu^-w^{-1})$ for every $u^- \in N^-$.

For $f_1, f_2 \in C_c^\infty(G)$, we consider the kernel K_{f_1,f_2} of the biregular action of $f_1 \otimes \overline{f_2}$ on $L^2(G)$. Then, the distribution of interest is obtained, formally, by integrating this kernel over $N^- \times N$ against the character $(u^-, u) \mapsto \xi^-(u^-)^{-1}\xi(u)$. This expression is usually divergent and needs to be suitably regularized (see Section 2.2). Once this is done, the resulting distribution admits two natural and distinct expansions: one geometric, in terms of relative orbital integrals, and one spectral, in terms of Bessel distributions also called relative characters. The equality between the two expansions is the aforementioned local Kuznetsov trace formula (cf. Theorem 2.3.2).

The statements and proofs of these two expansions are given in Sections 2.2 and 2.3 respectively. For technical reasons, it will be more convenient to work with the Harish-Chandra Schwartz space $\mathcal{C}(G)$ rather than $C_c^\infty(G)$. We recall the definition as well as basic properties of $\mathcal{C}(G)$ and related function spaces in Section 2.1. Finally, in Section 2.4 we give a scalar Paley-Wiener theorem for Bessel distributions in the p-adic case whose proof is an easy consequence of the theory of Jacquet's functionals (although we will rather work with the more convenient tool of the regularized ξ-integral introduced by Lapid-Mao [LM]).

We equip N and N^- with Haar measures such that the isomorphism $N \simeq N^-$, $u \mapsto w^{-1}uw$, is measure-preserving. We also endow G and T with Haar measures such that the following integration formula

$$\int_G f(g)dg = \int_{N^- \times T \times N} f(u^-tu)\delta_B(t)du^-dtdu \tag{2.0.1}$$

is satisfied for every $f \in L^1(G)$.

2.1 Reminder on Harish-Chandra Schwartz space

Let Ξ^G be the Harish-Chandra basic spherical function of G (see [Wald1, §II.1], [Var, §II.8.5]). It depends on the choice of a maximal compact subgroup K of G that we assume fixed from now

on. The function Ξ^G is K-biinvariant and we have [Wald1, Lemme II.1.3], [Var, Proposition 16(iii) p.329]

$$\int_K \Xi^G(g_1 k g_2) dk = \Xi^G(g_1)\Xi^G(g_2) \tag{2.1.1}$$

for every $g_1, g_2 \in G$ and where the Haar measure on K is normalized to have total mass 1.

Let σ_G be a log-norm on G (see [Beu1, §1.2]). We assume that σ_G is bi-K-invariant and satisfies $\sigma_G(g^{-1}) = \sigma_G(g)$. There exists $d_0 > 0$ such that ([Wald1, Lemme II.1.5, Proposition II.4.5], [Var, Proposition 31 p.340, Theorem 23 p.360])

$$\int_G \Xi^G(g)^2 \sigma_G(g)^{-d_0} dg < \infty \tag{2.1.2}$$

and

$$\int_{N^-} \Xi^G(u^-) \sigma_G(u^-)^{-d_0} du^- < \infty. \tag{2.1.3}$$

Let $\mathcal{C}(G)$ be the *Harish-Chandra Schwartz space* of G. It is the space of functions $f : G \to \mathbb{C}$ which are C^∞ in the Archimedean case, biinvariant by a compact-open subgroup in the p-adic case, and satisfy inequalities

$$|f(g)| \ll_d \Xi^G(g)\sigma_G(g)^{-d}, \quad g \in G$$

for every $d > 0$ in the p-adic case;

$$|(R(X)L(Y)f)(g)| \ll_{d,X,Y} \Xi^G(g)\sigma_G(g)^{-d}, \quad g \in G$$

for every $d > 0$ and every $X, Y \in \mathcal{U}(\mathfrak{g})$ in the Archimedean case.

There is a natural topology on $\mathcal{C}(G)$ making it into a Fréchet space in the Archimedean case and a strict LF space in the p-adic case [Beu1, §1.5]. The Harish-Chandra Schwartz space $\mathcal{C}(G \times G)$ of $G \times G$ is defined similarly. We will need the following, probably well-known, result.

Lemma 2.1.1. *Assume that F is Archimedean. Then, there is a topological isomorphism $\mathcal{C}(G)\widehat{\otimes}\mathcal{C}(G) \simeq \mathcal{C}(G \times G)$ sending a pure tensor $f_1 \otimes f_2$ to the function $(g_1, g_2) \mapsto f_1(g_1)f_2(g_2)$.*

Proof. The bilinear map

$$\mathcal{C}(G) \times \mathcal{C}(G) \to \mathcal{C}(G \times G)$$
$$(f_1, f_2) \mapsto ((g_1, g_2) \mapsto f_1(g_1)f_2(g_2))$$

is continuous and therefore induces a continuous linear map

$$\mathcal{C}(G)\widehat{\otimes}\mathcal{C}(G) \to \mathcal{C}(G \times G). \tag{2.1.4}$$

By [Ber3, end of Section 3.5], $\mathcal{C}(G)$ is nuclear. Hence, by Grothendieck's weak-strong principle [Gro, théorème 13, Chap. II §3, n.3], the map (2.1.4) is injective with image the space of all functions $f : G \times G \to \mathbb{C}$ satisfying the following condition:

For every $g \in G, T \in \mathcal{C}(G)'$ the functions $g' \mapsto f(g, g')$ and $g \mapsto \langle f(g, .), T\rangle$ belong to $\mathcal{C}(G)$.

But it is easy to see that every $f \in \mathcal{C}(G \times G)$ satisfies this condition. Therefore, the linear map (2.1.4) is bijective and thus, by the open mapping theorem [Tr, Theorem 17.1], a topological isomorphism. □

Remark 2.1.1. *Assume that F is non-Archimedean. Let J be a compact-open subgroup of G and denote by $\mathcal{C}(J\backslash G/J)$, $\mathcal{C}(J\times J\backslash G\times G/J\times J)$ the subspaces of J and $J\times J$ biinvariant functions in $\mathcal{C}(G)$ and $\mathcal{C}(G\times G)$ respectively. We can show similarly the existence of a natural topological isomorphism $\mathcal{C}(J\backslash G/J)\widehat{\otimes}\mathcal{C}(J\backslash G/J)\simeq\mathcal{C}(J\times J\backslash G\times G/J\times J)$ but such isomorphism does no longer exist without fixing "the level". Indeed, there is a natural algebraic isomorphism $\mathcal{C}(G)\widehat{\otimes}\mathcal{C}(G)\simeq\mathcal{C}(G\times G)$ which is however not topological. We refer the reader to [Gro, Exemple 4, Chap. II §3 n.3 p.84] for a detailed discussion of a similar issue for the projective tensor product $C_c^\infty(M)\widehat{\otimes}C_c^\infty(N)$ where M and N are infinitely differentiable real manifolds.*

We let $\mathcal{C}^w(G)$ be the space of *tempered* functions on G that is functions $f:G\to\mathbb{C}$ which are C^∞ in the Archimedean case, biinvariant by a compact-open subgroup in the p-adic case, and for which there exists $d>0$ such that

$$|f(g)|\ll\Xi^G(g)\sigma_G(g)^d,\ \ g\in G,$$

in the p-adic case;

$$|(R(X)L(Y)f)(g)|\ll_{X,Y}\Xi^G(g)\sigma_G(g)^d,\ \ g\in G,$$

for every $X,Y\in\mathcal{U}(\mathfrak{g})$ in the Archimedean case. The space $\mathcal{C}^w(G)$ is naturally equipped with a structure of LF space for which the subspace $C_c^\infty(G)$ is dense.

By [SV, Corollary 6.3.3], [Beu1, Proposition 7.1.1][3] the linear form

$$f\in C_c^\infty(G)\mapsto\int_N f(u)\xi(u)du$$

extends continuously to $\mathcal{C}^w(G)$. As in [Beu1, §7.1], we denote by

$$f\in\mathcal{C}^w(G)\mapsto\int_N^* f(u)\xi(u)du$$

this unique continuous extension that we will call the (N,ξ)-*regularized integral*. Let $\varphi\in C_c^\infty(T)$ and $f\in\mathcal{C}^w(G)$. Define $\mathrm{Ad}(\varphi)f\in\mathcal{C}^w(G)$ by

$$(\mathrm{Ad}(\varphi)f)(g)=\int_T\varphi(t)f(t^{-1}gt)dt,\quad g\in G.$$

We also set

$$\widehat{\varphi}(u)=\int_T\varphi(t)\delta_B(t)\xi(tut^{-1})dt,\quad u\in N.$$

Note that $\widehat{\varphi}$ is invariant by the derived subgroup N' of N and that it is "rapidly decreasing" (and even compactly supported in the non-Archimedean case) on N/N' by usual properties of the Fourier transform. By the same argument as [Beu1, Lemma 7.1.2(ii)] we have

$$\int_N^*(\mathrm{Ad}(\varphi)f)(u)\xi(u)du=\int_N f(u)\widehat{\varphi}(u)du \tag{2.1.5}$$

where the second integral is absolutely convergent. More precisely, for every $d>0$ we have

$$\int_N\Xi^G(u)\sigma_G(u)^d|\widehat{\varphi}(u)|du<\infty. \tag{2.1.6}$$

Actually (2.1.5) can be taken as a definition of the (N,ξ)-regularized integral since, by Dixmier-Malliavin [DM], any function of $\mathcal{C}^w(G)$ is a finite sum of functions of the form $\mathrm{Ad}(\varphi)f$.

[3]Strictly speaking in *loc. cit.* only the case of unitary groups is treated but the arguments extend verbatim to the general case.

2.2 Geometric expansion

Let $f_1, f_2 \in \mathcal{C}(G)$. We set

$$K_{f_1,f_2}(x,y) := \int_G f_1(x^{-1}gy)\overline{f_2(g)}dg, \quad x,y \in G.$$

Note that this expression is absolutely convergent by (2.1.2). More precisely, let $d_0 > 0$ be such that (2.1.2) is satisfied. Then, from (2.1.1), (2.1.2) and the inequality $\sigma_G(g_1g_2) \ll \sigma_G(g_1)\sigma_G(g_2)$ for every $g_1, g_2 \in G$, it is easy to infer that

$$|K_{f_1,f_2}(x,y)| \ll_d \Xi^G(x)\Xi^G(y)\sigma_G(x)^{-d}\sigma_G(y)^d, \quad x,y \in G, \tag{2.2.1}$$

for every $d > 0$. Therefore by (2.1.3) the expression

$$K_{f_1,f_2}^{N^-,\xi^-}(x) := \int_{N^-} K_{f_1,f_2}(u^-,x)\xi^-(u^-)^{-1}du^-$$

is absolutely convergent for any $x \in G$. We claim that this function is tempered i.e.

$$K_{f_1,f_2}^{N^-,\xi^-} \in \mathcal{C}^w(G). \tag{2.2.2}$$

Indeed, in the p-adic case it is clear as $K_{f_1,f_2}^{N^-,\xi^-}$ is biinvariant by a compact-open subgroup and by (2.2.1) it satisfies

$$|K_{f_1,f_2}^{N^-,\xi^-}(x)| \ll \Xi^G(x)\sigma_G(x)^{d_0}, \quad x \in G,$$

where d_0 is chosen such that the integral (2.1.3) converges. In the Archimedean case, by differentiating under the integral sign (which is justified here by the absolute convergence of the resulting expression), we see that $K_{f_1,f_2}^{N^-,\xi^-}$ is C^∞ and that

$$R(X)L(Y)K_{f_1,f_2}^{N^-,\xi^-} = K_{R(X)f_1,R(\overline{Y})f_2}^{N^-,\xi^-}$$

for every $X, Y \in \mathcal{U}(\mathfrak{g})$. Thus, by (2.2.1), we have

$$|R(X)L(Y)K_{f_1,f_2}^{N^-,\xi^-}(x)| \ll_{X,Y} \Xi^G(x)\sigma_G(x)^{d_0}, \quad x \in G$$

for every $X, Y \in \mathcal{U}(\mathfrak{g})$ where d_0 is again chosen such that the integral (2.1.3) converges. This proves the claim (2.2.2).

By (2.2.2), we can now define the following expression

$$I(f_1,f_2) := \int_N^* K_{f_1,f_2}^{N^-,\xi^-}(u)\xi(u)du = \int_N^* \int_{N^-} K_{f_1,f_2}(u^-,u)\xi^-(u^-)^{-1}du^-\xi(u)du.$$

Remark 2.2.1. *By being slightly more careful, we can show that $K_{f_1,f_2}^{N^-,\xi^-}$ is a Harish-Chandra Schwartz function (i.e. $K_{f_1,f_2}^{N^-,\xi^-} \in \mathcal{C}(G)$) so that the integral over N above is actually absolutely convergent. However, the final expression is only convergent as an iterated double integral and we will not use this fact in the sequel.*

For $t \in T$ and $f \in \mathcal{C}(G)$ we set

$$O(t,f) = \int_{N\times N^-} f(u^- tu)\xi(u)\xi^-(u^-)du^- du.$$

Lemma 2.2.1. *The expression defining $O(t,f)$ is absolutely convergent locally uniformly in t and f.*

Proof. After the change of variable $u \mapsto tut^{-1}$, we see that it suffices to show the existence of $d > 0$ such that

$$\int_{N\times N^-} \Xi^G(u^- u)\sigma_G(u^- u)^{-d}du^- du < \infty.$$

By the Iwasawa decomposition, there exist functions $t_B : G \to T$, $u_B : G \to N$ and $k_B : G \to K$ such that $g = k_B(g)t_B(g)u_B(g)$ for every $g \in G$. As Ξ^G and σ_G are K-invariant, we have

$$\int_{N\times N^-} \Xi^G(u^- u)\sigma_G(u^- u)^{-d}du^- du = \int_{N\times N^-} \Xi^G(t_B(u^-)u)\sigma_G(t_B(u^-)u)^{-d}du^- du.$$

By [Wald1, Proposition II.4.5] and [Var, Theorem 23 p.360] for any $d' > 0$ we can choose d such that the above expression is essentially bounded by

$$\int_{N^-} \delta_B(t_B(u^-))^{-1/2}\sigma_G(t_B(u^-))^{-d'}du^-.$$

Finally by [Wald1, Lemme II.3.4, Lemme II.4.2] and [WallI, Theorem 4.5.4] for d' sufficiently large the last integral above converges. This proves the lemma. □

Set

$$I_{\mathrm{geom}}(f_1,f_2) = \int_T O(t,f_1)\overline{O(t,f_2)}\delta_B(t)dt.$$

The main result of this section is the following.

Theorem 2.2.1. *The expression defining $I_{\mathrm{geom}}(f_1,f_2)$ is absolutely convergent and moreover we have*

$$I(f_1,f_2) = I_{\mathrm{geom}}(f_1,f_2).$$

Proof. We extend the association $(f_1,f_2) \mapsto K_{f_1,f_2}$ to every Harish-Chandra Schwartz function $F \in \mathcal{C}(G\times G)$ by setting

$$K_F(x,y) = \int_G F(x^{-1}gy,g)dg, \quad \text{for } x,y \in G.$$

We have $K_{f_1,f_2} = K_{f_1\otimes\overline{f_2}}$ where $f_1 \otimes \overline{f_2} \in \mathcal{C}(G\times G)$ is the function given by $(f_1\otimes\overline{f_2})(g_1,g_2) = f_1(g_1)\overline{f_2(g_2)}$. The same argument as before shows that

$$|K_F(x,y)| \ll_{d,F} \Xi^G(x)\Xi^G(y)\sigma_G(x)^{-d}\sigma_G(y)^d, \quad x,y \in G \tag{2.2.3}$$

for every $d > 0$ and $F \in \mathcal{C}(G\times G)$. Therefore, we can define

$$K_F^{N^-,\xi^-}(x) := \int_{N^-} K_F(u^-,x)\xi^-(u^-)^{-1}du^-$$

for any $x \in G$ and $F \in \mathcal{C}(G \times G)$ and by the same argument as for (2.2.2) we have $K_F^{N^-,\xi^-} \in \mathcal{C}^w(G)$. Denote by R^Δ the right diagonal action of T on $\mathcal{C}(G \times G)$.

In the p-adic case, we choose a compact-open subgroup K_T of T by which both f_1 and f_2 are right-invariant and we set $\varphi = \mathrm{vol}(K_T)^{-1}\mathbf{1}_{K_T} \in C_c^\infty(T)$, $F = f_1 \otimes \overline{f_2} \in \mathcal{C}(G \times G)$. Then, we have $f_1 \otimes \overline{f_2} = R^\Delta(\varphi)F$. In the Archimedean case, by Dixmier-Malliavin [DM], $f_1 \otimes \overline{f_2}$ is a finite sum of functions of the form $R^\Delta(\varphi)F$ where $\varphi \in C_c^\infty(T)$ and $F \in \mathcal{C}(G \times G)$. For notational simplicity we will assume that $f_1 \otimes \overline{f_2} = R^\Delta(\varphi)F$ for some functions $(\varphi, F) \in C_c^\infty(T) \times \mathcal{C}(G \times G)$, the modifications needed to treat the general case being obvious.

In both cases, we have $K_{f_1,f_2}^{N^-,\xi^-} = K_{R^\Delta(\varphi)F}^{N^-,\xi^-}$ and a simple change of variable shows that

$$K_{R^\Delta(\varphi)F}^{N^-,\xi^-} = \mathrm{Ad}(\varphi)K_F^{N^-,\xi^-}$$

(where the operator $\mathrm{Ad}(\varphi)$ was introduced in Section 2.1). Hence, by (2.1.5) we have

$$I(f_1, f_2) = \int_N^* (\mathrm{Ad}(\varphi)K_F^{N^-,\xi^-})(u)\xi(u)du = \int_N K_F^{N^-,\xi^-}(u)\widehat{\varphi}(u)du$$

where the function $\widehat{\varphi}$ is defined as in Section 2.1. Unfolding all the definitions, we arrive at the following equality:

$$I(f_1, f_2) = \int_N \int_{N^-} \int_G F(u^- gu, g)dg\xi^-(u^-)du^-\widehat{\varphi}(u)du.$$

As follows readily from (2.2.3), (2.1.3) and (2.1.6) this last expression is absolutely convergent. By (2.0.1), we have

$$\begin{aligned} I(f_1, f_2) &= \int_{N\times N^-} \int_{N^-\times T\times N} F(u^- v^- tvu, v^- tv)\delta_B(t)dvdtdv^-\xi^-(u^-)du^-\widehat{\varphi}(u)du \\ &= \int_T \int_{N^2\times(N^-)^2} F(u^- tu, v^- tv)\xi^-(u^-)\xi^-(v^-)^{-1}\widehat{\varphi}(v^{-1}u)du^-dv^-dudv\delta_B(t)dt. \end{aligned}$$

Set

$$O(t, F) = \int_{N^2\times(N^-)^2} F(u^- tu, v^- tv)\xi^-(u^-)\xi^-(v^-)^{-1}\xi(u)\xi(v)^{-1}du^-dv^-dudv$$

for every $t \in T$ and $F \in \mathcal{C}(G \times G)$. By the same argument as for Lemma 2.2.1, this expression is absolutely convergent locally uniformly in t and F. Note that

$$O(t, f_1 \otimes \overline{f_2}) = O(t, f_1)\overline{O(t, f_2)}, \quad t \in T.$$

We have (where all the manipulations are justified since $O(t, F)$ converges locally uniformly in t and F)

$$\begin{aligned} &\int_{N^2\times(N^-)^2} F(u^- tu, v^- tv)\xi^-(u^-)\xi^-(v^-)^{-1}\widehat{\varphi}(v^{-1}u)du^-dv^-dudv \\ &= \int_{N^2\times(N^-)^2} F(u^- tu, v^- tv)\xi^-(u^-)\xi^-(v^-)^{-1}\int_T \varphi(a)\delta_B(a)\xi(av^{-1}ua^{-1})dadu^-dv^-dudv \\ &= \int_T \varphi(a)\delta_B(a)^{-1}\int_{N^2\times(N^-)^2} F(u^- ta^{-1}ua, v^- ta^{-1}va)\xi^-(u^-)\xi^-(v^-)^{-1}\xi(u)\xi(v)^{-1}du^-dv^-dudvda \\ &= \int_T \varphi(a)\delta_B(a)^{-1}O(a^{-1}t, R^\Delta(a)F)da. \end{aligned}$$

Thus, the above computations show that the expression

$$\int_T \int_T \varphi(a)\delta_B(a)^{-1} O(a^{-1}t, R^\Delta(a)F)da\delta_B(t)dt \tag{2.2.4}$$

is convergent as an iterated integral for any $F \in \mathcal{C}(G \times G)$ and $\varphi \in C_c^\infty(T)$ and moreover that

$$I(f_1, f_2) = \int_T \int_T \varphi(a)\delta_B(a)^{-1} O(a^{-1}t, R^\Delta(a)F)da\delta_B(t)dt \tag{2.2.5}$$

whenever $f_1 \otimes \overline{f_2} = R^\Delta(\varphi)F$. We are now going to show that this last expression is absolutely convergent. In the p-adic case it is clear when $f_1 = f_2$ as the integrand is nonnegative and the general case follows by Cauchy-Schwarz. In the Archimedean case, the argument is essentially the same but slightly less direct. We actually show the following:

(2.2.6) The expression (2.2.4) converges absolutely for any $F \in \mathcal{C}(G \times G)$ and $\varphi \in C_c^\infty(T)$.

Let $\varphi \in C_c^\infty(T)$. As $|\varphi|$ is bounded by φ' for some $\varphi' \in C_c^\infty(T)$, we may assume that $\varphi \geqslant 0$. Let $(T_n)_n$ be an increasing sequence of compact subsets of T such that $T = \bigcup_n T_n$. It suffices to show that for every $\phi \in L^\infty(T \times T)$ the sequence of continuous linear forms

$$L_{n,\phi} : F \in \mathcal{C}(G \times G) \mapsto \int_{T_n \times T_n} \phi(a,t)\varphi(a) O(a^{-1}t, R^\Delta(a)F)\delta_B(a^{-1}t)dadt$$

converges pointwise. By Lemma 2.1.1 and [Beu1, (A.5.3)], it suffices to show that for any $f_1, f_2 \in \mathcal{C}(G)$ the sequence $(L_{n,\phi}(f_1 \otimes \overline{f_2}))_n$ converges for all $\phi \in L^\infty(T \times T)$ or what amounts to the same that the integral

$$\int_{T\times T} \varphi(a) O(a^{-1}t, R(a)f_1 \otimes R(a)\overline{f_2})\delta_B(a^{-1}t)dadt = \int_{T\times T} \varphi(a) O(a^{-1}t, R(a)f_1)\overline{O(a^{-1}t, R(a)f_2)}\delta_B(a^{-1}t)dadt$$

is absolutely convergent. By Cauchy-Schwarz again, we just need to check that

$$\int_{T\times T} \varphi(a)|O(a^{-1}t, R(a)f)|^2\delta_B(a^{-1}t)dadt < \infty$$

for every $f \in \mathcal{C}(G)$. Letting $F = f \otimes \overline{f}$, we have $O(a^{-1}t, R^\Delta(a)F) = |O(a^{-1}t, R(a)f)|^2$. Thus, for this particular choice of F and φ the integrand in (2.2.4) is nonnegative hence this expression, which is the same as above, is absolutely convergent. This proves the claim.

By (2.2.5) and (2.2.6), we now have

$$\begin{aligned} I(f_1, f_2) &= \int_T \varphi(a) \int_T O(a^{-1}t, R^\Delta(a)F)\delta_B(a^{-1}t)dtda = \int_T \varphi(a) \int_T O(t, R^\Delta(a)F)\delta_B(t)dtda \\ &= \int_T \int_T \varphi(a) O(t, R^\Delta(a)F)da\delta_B(t)dt = \int_T O(t, R^\Delta(\varphi)F)\delta_B(t)dt \\ &= \int_T O(t, f_1 \otimes \overline{f_2})\delta_B(t)dt = I_{\mathrm{geom}}(f_1, f_2) \end{aligned}$$

where all the above expressions are absolutely convergent. This proves the theorem. □

2.3 Spectral expansion

Let $\mathrm{Temp}(G)$ denote the set of isomorphism classes of irreducible tempered representations of G. This set carries a natural topology (see [Beu3, Section 2.6]). Let $\pi \in \mathrm{Temp}(G)$. The representation π is unitary and we fix an invariant scalar product $(.,.)$ on its space. Then, to every $f \in \mathcal{C}(G)$ we can associate an operator $\pi(f)$ such that for u, v smooth vectors in the space of π we have

$$(\pi(f)u, v) = \int_G f(g)(\pi(g)u, v)vdg$$

where the integral converges absolutely. This operator is of trace-class (it is even of finite rank in the p-adic case) and the function

$$f_\pi : g \in G \mapsto \mathrm{Trace}(\pi(g^{-1})\pi(f))$$

belongs to $\mathcal{C}^w(G)$ [Beu1, (2.2.5)]. According to Harish-Chandra [H-C], [Wald1] (see also [Ber3]) there exists a unique measure $d\mu_G(\pi)$ on $\mathrm{Temp}(G)$ such that

$$f(g) = \int_{\mathrm{Temp}(G)} f_\pi(g) d\mu_G(\pi)$$

for every $f \in \mathcal{C}(G)$ and $g \in G$ where the right-hand side is an absolutely convergent integral.

For any $\pi \in \mathrm{Temp}(G)$ we define a *Bessel distribution* by

$$f \in \mathcal{C}(G) \mapsto I_\pi(f) := \int_N^* f_\pi(w^{-1}u)\xi(u)du = \int_N^* \mathrm{Trace}(\pi(w)\pi(f)\pi(u^{-1}))\xi(u)du.$$

Let $f_1, f_2 \in \mathcal{C}(G)$. We set

$$I_{\mathrm{spec}}(f_1, f_2) := \int_{\mathrm{Temp}(G)} I_\pi(f_1)\overline{I_\pi(f_2)} d\mu_G(\pi).$$

The main result of this section is the following.

Theorem 2.3.1. *The expression defining $I_{\mathrm{spec}}(f_1, f_2)$ is absolutely convergent and moreover we have*

$$I(f_1, f_2) = I_{\mathrm{spec}}(f_1, f_2).$$

Proof. First we consider the convergence of $I_{\mathrm{spec}}(f_1, f_2)$. By [Beu3, Proposition 2.131] the functions $\pi \in \mathrm{Temp}(G) \mapsto I_\pi(f_1)$ and $\pi \mapsto I_\pi(f_2)$ are continuous and compactly supported in the p-adic case whereas there are continuous and essentially bounded by $N(\pi)^{-k}$ for any $k > 0$ in the Archimedean case where $N(.)$ is the "norm" on $\mathrm{Temp}(G)$ introduced in [Beu3, §2.6]. Combining this with [Beu3, (2.7.4)] we see that the integral defining $I_{\mathrm{spec}}(f_1, f_2)$ is absolutely convergent. Actually, using the full strength of [Beu3, Proposition 2.131] we even have that $(f_1, f_2) \in \mathcal{C}(G)^2 \mapsto I_{\mathrm{spec}}(f_1, f_2)$ is a continuous sesquilinear form. By making the arguments for (2.2.1) and (2.2.2) effective, we have similarly that $(f_1, f_2) \in \mathcal{C}(G)^2 \mapsto I(f_1, f_2)$ is a (separately) continuous sesquilinear form. Therefore we just need to show the equality of the theorem for a dense subset of $\mathcal{C}(G)$. In particular, we may assume that the operator-valued Fourier transform $\pi \in \mathrm{Temp}(G) \mapsto \pi(f_1)$ is compactly supported [Beu1, Theorem 2.6.1]. In this case the identity of the theorem is just a reformulation of [Beu1, Lemma 7.2.2(v)][4]. □

[4]Once again only the case of unitary groups was considered in *loc. cit.* but the proof works equally well in the more general situation considered here.

Combining Theorem 2.2.1 with Theorem 2.3.1 we arrive at the following.

Theorem 2.3.2 (Local Kuznetsov trace formula)**.** *For any $f_1, f_2 \in \mathcal{C}(G)$ we have*

$$I_{\mathrm{geom}}(f_1, f_2) = I_{\mathrm{spec}}(f_1, f_2).$$

Remark 2.3.1. *Although not transparent from the notation, both sides depend on the choice of w: this dependence is quite transparent for $I_{\mathrm{spec}}(f_1, f_2)$ from the definition whereas for $I_{\mathrm{geom}}(f_1, f_2)$ the dependence is hidden in the definition of ξ^- (given at the beginning of this chapter).*

2.4 A scalar Whittaker Paley-Wiener theorem

In this subsection we assume that F is a p-adic field. Let $\widehat{\mathcal{Z}}(G)$ be the Bernstein center of G [Ber2]. Then $\widehat{\mathcal{Z}}(G)$ is a direct product of integral domains indexed by the Bernstein components of G. We let $\mathcal{Z}(G)$ be the corresponding direct sum. Let $\mathrm{Cusp}(G)$ be the set of pairs (M, σ) where M is a semi-standard Levi subgroup of G and σ is the isomorphism class of a supercuspidal representation of M. There is a natural action of the Weyl group $W = \mathrm{Norm}_G(T)/T$ on $\mathrm{Cusp}(G)$ and the maximal spectrum of $\mathcal{Z}(G)$ is in natural bijection with the quotient $\mathrm{Cusp}(G)/W$.

A smooth representation π of G is said to be (N, ξ)-*generic* if $\mathrm{Hom}_N(\pi, \xi) \neq 0$. For M a semi-standard Levi subgroup, we define similarly the notion of (N^M, ξ^M)-generic smooth representation of M where $N^M = N \cap M$ and ξ^M denotes the restriction of ξ to N^M. We let $\mathrm{Cusp}_{\mathrm{gen}}(G)$ be the subset of $(M, \sigma) \in \mathrm{Cusp}(G)$ such that σ is (N^M, ξ^M)-generic. It is known that a pair $(M, \sigma) \in \mathrm{Cusp}(G)$ belongs to $\mathrm{Cusp}_{\mathrm{gen}}(G)$ if and only if for one, or equivalently every, parabolic subgroup P with Levi component M the normalized smooth induction $I_P^G(\sigma)$ is (ξ, N)-generic in which case it contains a unique (N, ξ)-generic irreducible subquotient. Moreover, $\mathrm{Cusp}_{\mathrm{gen}}(G)$ is stable by the action of W and $\mathrm{Cusp}_{\mathrm{gen}}(G)/W$ is a disjoint union of connected components of $\mathrm{Cusp}(G)/W$. We denote by $\mathcal{Z}_{\mathrm{gen}}(G)$ the algebra of regular functions on $\mathrm{Cusp}_{\mathrm{gen}}(G)/W$ (thus, it is a direct factor of $\mathcal{Z}(G)$).

Let $C^\infty(G)$ be the space of functions $G \to \mathbb{C}$ which are bi-invariant by some compact-open subgroup of G. It has a natural topology of LF space (for every compact-open subgroup J we endow $C(J\backslash G/J)$ with the topology of pointwise convergence) for which the subspace $C_c^\infty(G)$ is dense. We will use the following very nice extension of [SV, Corollary 6.3.3] which is due to Lapid and Mao [LM, Proposition 2.11]: the linear form

$$f \in C_c^\infty(G) \mapsto \int_N f(u)\xi(u)du$$

extends continuously to $C^\infty(G)$. As in Section 2.1, we denote by

$$f \in C^\infty(G) \mapsto \int_N^* f(u)\xi(u)du$$

this unique continuous extension. Note that its restriction to $\mathcal{C}^w(G)$ coincides with the (N, ξ)-regularized integral of Section 2.1 as the embedding $\mathcal{C}^w(G) \subset C^\infty(G)$ is continuous.

Let $\mathrm{Sm}^{\mathrm{fl}}(G)$ be the category of smooth complex representations of G which are of finite length. Let $\pi \in \mathrm{Sm}^{\mathrm{fl}}(G)$. To $f \in C_c^\infty(G)$ we associate the operator $\pi(f)$ such that for every vectors v, $v^\vee$ in the spaces of π and $\pi^\vee$ (the smooth contragredient of π) we have

$$\langle \pi(f)v, v^\vee \rangle = \int_G f(g)\langle \pi(g)v, v^\vee \rangle dg.$$

This operator is of finite rank and the function $g \in G \mapsto \mathrm{Trace}(\pi(g)\pi(f))$ belongs to $C^\infty(G)$. We define the *Bessel distribution* I_π by

$$I_\pi(f) := \int_N^* \mathrm{Trace}(\pi(w)\pi(f)\pi(u^{-1}))\xi(u)du, \quad f \in C_c^\infty(G).$$

Obviously, when $\pi \in \mathrm{Temp}(G)$ this definition coincides with the restriction to $C_c^\infty(G)$ of the distribution defined in Section 2.3. Note that I_π only depends on the semi-simplification of π (as it only depends on the distributional character of π). Thus, for $(M,\sigma) \in \mathrm{Cusp}_{\mathrm{gen}}(G)$ we can set $I_{M,\sigma} = I_{I_P^G(\sigma)}$ where P is any parabolic subgroup with Levi component M.

Let $\mathcal{I}_{\mathrm{gen}}(G)$ be the space of functions on $\mathrm{Cusp}_{\mathrm{gen}}(G)$ of the form $(M,\sigma) \mapsto I_{M,\sigma}(f)$ where $f \in C_c^\infty(G)$. The main result of this section is the following.

Theorem 2.4.1. *We have*

$$\mathcal{I}_{\mathrm{gen}}(G) = \mathcal{Z}_{\mathrm{gen}}(G).$$

Proof. The inclusion $\mathcal{I}_{\mathrm{gen}}(G) \subset \mathcal{Z}_{\mathrm{gen}}(G)$ follows from [LM, Proposition 2.8] and usual properties of the Jacquet functionals. Moreover, the action of the Bernstein center on $C_c^\infty(G)$ shows that $\mathcal{I}_{\mathrm{gen}}(G)$ is an ideal of $\mathcal{Z}_{\mathrm{gen}}(G)$. On the other hand, for any $(M,\sigma) \in \mathrm{Cusp}_{\mathrm{gen}}(G)$ the functional $I_{M,\sigma}$ is nonzero by [LM, Proposition 2.10]. Hence, $\mathcal{I}_{\mathrm{gen}}(G)$ is an ideal of $\mathcal{Z}_{\mathrm{gen}}(G)$ which is not contained in any maximal ideal so that finally $\mathcal{I}_{\mathrm{gen}}(G) = \mathcal{Z}_{\mathrm{gen}}(G)$. □

3 The symmetric space X and FLO invariant functionals

3.1 Groups and normalization of measures

In this chapter we let E/F be a quadratic extension of local fields of characteristic zero. We denote by $\mathrm{Tr}_{E/F} : E \to F$ the trace map and by η be the quadratic character of $F^\times$ associated to this extension. We also fix a non-trivial unitary additive character $\psi' : F \to \mathbb{S}^1$ and we let $\psi = \psi' \circ \mathrm{Tr}_{E/F}$.

Let $n \geqslant 1$. We set $G = \mathrm{GL}_n(E)$ and $G' = \mathrm{GL}_n(F)$. Let T_n, N_n and B_n be the algebraic subgroups of diagonal, unipotent upper triangular and upper triangular matrices of GL_n respectively. We set $T = T_n(E)$, $T' = T_n(F)$, $N = N_n(E)$, $N' = N_n(F)$, $B = B_n(E)$, $B' = B_n(F)$ and we denote by δ_B, $\delta_{B'}$ the modular characters of B and B' respectively.

We denote by c the non-trivial Galois automorphism of E over F and by $g \mapsto g^c$ the natural extension of c to G. For $g \in G$, we also write ${}^t g$ for the transpose of g.

Using ψ' and ψ we define in the usual way non-degenerate characters ψ'_n and ψ_n of N' and N respectively: for every $u = (u_{i,j})_{1 \leqslant i,j \leqslant n} \in N'$ we have

$$\psi'_n(u) = \psi'(\sum_{i=1}^{n-1} u_{i,i+1})$$

and similarly for ψ_n. Set

$$w = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Then we have $\psi'_n(wu^-w^{-1}) = \psi'_n({}^t u^-)$ for every $u^- \in {}^t N'$.

We denote by $\mathrm{Irr}^{\mathrm{gen}}(G) \subseteq \mathrm{Irr}(G)$ (resp. $\mathrm{Irr}^{\mathrm{gen}}(G') \subseteq \mathrm{Irr}(G')$) the subset of generic irreducible representations and for $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$ (resp. $\sigma \in \mathrm{Irr}^{\mathrm{gen}}(G')$) by $\mathcal{W}(\pi, \psi_n)$ (resp. $\mathcal{W}(\sigma, \psi'_n)$) the corresponding Whittaker model.

We equip N', T' and G' with Haar measures such that the following integration formula

$$\int_{G'} f(g)dg = \int_{N' \times T' \times N'} f({}^t u_1 t u_2)\delta_{B'}(t)du_1 dt du_2$$

is valid for every $f \in L^1(G')$.

Let P_n be the mirabolic subgroup of GL_n (i.e. the subgroup of matrices with last row $(0, \dots, 0, 1)$) and set $P = P_n(E)$, $P' = P_n(F)$. We equip P (resp. P') with a right Haar measure normalized such that setting

$$W_f(g_1, g_2) = \int_N f(g_1^{-1} u g_2)\psi_n(u)^{-1}du, \quad g_1, g_2 \in G$$

$$(\text{resp. } W_{f'}(g_1, g_2) = \int_{N'} f'(g_1^{-1} u g_2)\psi'_n(u)^{-1}du, \quad g_1, g_2 \in G'),$$

we have the Fourier inversion formulas

$$f(1) = \int_{N\backslash P} W_f(p, p)dp \ (\text{resp. } f'(1) = \int_{N'\backslash P'} W_{f'}(p, p)dp) \tag{3.1.1}$$

for every $f \in C_c^\infty(G)$ (resp. $f' \in C_c^\infty(G')$) see [LM, Lemma 4.4]. Actually, the definition of W_f and $W_{f'}$ extend to any $f \in \mathcal{C}^w(G)$ and $f' \in \mathcal{C}^w(G')$ by replacing the integrals over N and N' by the *regularized* one introduced in Section 2.1. Then, the right-hand side of (3.1.1) is still absolutely convergent (this follows from [Beu3, Lemma 2.14.1 and Lemma 2.15.1] in the degenerate case $E = F \times F$) and defines a continuous linear form on $\mathcal{C}^w(G)$ or $\mathcal{C}^w(G')$. Therefore, by density of $C_c^\infty(G)$ or $C_c^\infty(G')$ in $\mathcal{C}^w(G)$ or $\mathcal{C}^w(G')$, the inversion formula (3.1.1) continues to hold for every $f \in \mathcal{C}^w(G)$ and $f' \in \mathcal{C}^w(G')$.

For every $\sigma \in \mathrm{Temp}(G')$, the expression

$$\langle W, W' \rangle_{\mathrm{Whitt}} = \int_{N'\backslash P'} W(p)\overline{W'(p)}dp, \quad W, W' \in \mathcal{W}(\sigma, \psi'_n), \tag{3.1.2}$$

is absolutely convergent and defines a nonzero G'-invariant inner product on $\mathcal{W}(\sigma, \psi'_n)$ by [Ber1], [Bar]. This pairing allows to identify $\overline{\mathcal{W}(\sigma, \psi'_n)} = \mathcal{W}(\sigma^\vee, {\psi'_n}^{-1})$ with the smooth contragredient of $\mathcal{W}(\sigma, \psi'_n)$. With our normalization of Haar measures, we have

$$\int_{N'}^* \langle R(u)W, W' \rangle_{\mathrm{Whitt}} \psi'_n(u)^{-1}du = W(1)\overline{W'(1)} \tag{3.1.3}$$

for every $\sigma \in \mathrm{Temp}(G')$ and $W, W' \in \mathcal{W}(\sigma, \psi'_n)$ where the above regularized integral is taken in the sense of Section 2.1. Indeed, the function $f(g) = \langle R(g)W, W' \rangle_{\mathrm{Whitt}}$, being a smooth matrix coefficient of a tempered representation, belongs to $\mathcal{C}^w(G')$ and by unicity of the Whittaker model, there exists a constant c (independent of W and W') such that $W_f(g_1, g_2) = cW(g_1)\overline{W'(g_2)}$. Applying the inversion formula (3.1.1), we get

$$\langle W, W' \rangle_{\mathrm{Whitt}} = f(1) = c \int_{N'\backslash P'} W(p)\overline{W'(p)}dp = c\langle W, W' \rangle_{\mathrm{Whitt}}.$$

As this is true for every $W, W' \in \mathcal{W}(\sigma, \psi'_n)$, this shows that $c = 1$ and the claim (3.1.3) is proved. Of course, a similar formula is valid for G.

3.2 The symmetric space X

Let $h_V : E^n \times E^n \to E$ be a nondegenerate Hermitian form (our convention is that Hermitian forms are always linear in the first variable and antilinear in the second one). We denote by $V = (E^n, h_V)$ the associated Hermitian space and by $U(V) \subseteq G$ the corresponding unitary group defined by

$$U(V) = \{g \in G \mid h_V(gv, gv') = h_V(v, v') \ \forall v, v' \in E^n\}.$$

We also set $X_V = U(V)\backslash G$. Let $\mathcal{V}$ be a set of representatives of the isomorphism classes of Hermitian spaces of dimension n over E with underlying space E^n (this set is finite and has two elements if F is p-adic, $n+1$ if $F = \mathbb{R}$).

$$X = \bigsqcup_{V \in \mathcal{V}} X_V. \tag{3.2.1}$$

Let

$$\mathrm{Herm}_n^* = \{h \in G \mid {}^t h^c = h\}$$

be the variety of invertible Hermitian matrices of size n. For each $V \in \mathcal{V}$ we identify h_V with the unique element of Herm_n^* such that

$$h_V(v, v') = {}^t v'^c h_V v, \quad v, v' \in E^n.$$

Then, there is an isomorphism $X \simeq \mathrm{Herm}_n^*$ given by

$$x \in X_V \mapsto {}^t x^c h_V x, \quad V \in \mathcal{V}\,.$$

This isomorphism sends the action by right translations of G on X to the right action of G on Herm_n^* given by $h \cdot g = {}^t g^c h g$. Besides, Herm_n^* admits a commuting left $F^\times$-action simply given by scalar multiplication. We denote by $(\lambda, x) \in F^\times \times X \mapsto \lambda x$ the corresponding action on X. Note that, when n is odd or $F = \mathbb{R}$, this extra action permutes certain components of the decomposition (3.2.1).

Note that $T' \subseteq \mathrm{Herm}_n^*$. We let T_X be the subvariety of X corresponding to T' by the above isomorphism and we endow this set with the image of the Haar measure that we have fixed on T'. We also denote by δ_X the composition of the isomorphism $T_X \simeq T'$ with the modular character $\delta_{B'}$. Note that T_X is invariant by translation by T and consists of finitely many T-orbits. We equip N with a Haar measure and X with a G-invariant measure such that the following integration formula

$$\int_X \varphi(x)dx = \int_N \int_{T_X} \varphi(tu)\delta_X(t)dtdu \tag{3.2.2}$$

is valid for every $\varphi \in L^1(X)$.

Whenever convergent, we denote by

$$\langle \varphi, \varphi' \rangle_X = \int_X \varphi(x)\overline{\varphi'(x)}dx$$

the L^2-inner product of two functions $\varphi, \varphi' \in C^\infty(X)$.

By [GO, Corollary 1.2], for every $V \in \mathcal{V}$ the pair $(G, U(V))$ is *tempered* in the sense of [Beu2, §2.7] that is:

(3.2.3) There exists $d > 0$ such that the integral $\displaystyle\int_{U(V)} \Xi^G(h)\sigma_G(h)^{-d}dh$ is convergent.

As in the proof of [Beu2, Proposition 1.7.1], this implies the following:

(3.2.4) For every $\varphi, \varphi' \in C_c^\infty(X)$ the function

$$g \in G \mapsto \langle R(g)\varphi, \varphi' \rangle_X$$

belongs to $\mathcal{C}^w(G)$.

3.3 Jacquet-Ye's transfer

For $\varphi \in C_c^\infty(X)$, $f' \in C_c^\infty(G')$, $t \in T_X$ and $a \in T'$ we define the orbital integrals

$$O(t, \varphi) = \int_N \varphi(tu)\psi_n(u)du \quad \text{and} \quad O(a, f') = \int_{N' \times N'} f'({}^t u_1 a u_2)\psi_n'(u_1 u_2)du_1 du_2.$$

Note that these integrals are absolutely convergent as the integrand are compactly supported. For every $a \in T'$, we set

$$\gamma(a) := \prod_{k=1}^{n-1} \eta(a_k)^k$$

where $a_1, \ldots, a_n$ denote the diagonal entries of a. We say that the functions $\varphi \in C_c^\infty(X)$ and $f' \in C_c^\infty(G')$ *match* and we will write $\varphi \leftrightarrow f'$ if

$$\gamma(a)O(a, f') = O(t, \varphi)$$

whenever $t \in T_X$ maps to $a \in T'$ via the isomorphism $T_X \simeq T'$.

The following theorem is due to Jacquet [Jac03] (in the p-adic case) and Aizenbud-Gourevitch [AG] (in the Archimedean case).

Theorem 3.3.1 (Jacquet, Aizenbud-Gourevitch)**.** *Every $\varphi \in C_c^\infty(X)$ matches a function $f' \in C_c^\infty(G')$. Conversely, every $f' \in C_c^\infty(G')$ matches a function $\varphi \in C_c^\infty(X)$.*

3.4 Feigon-Lapid-Offen's functionals

Let $\pi \in \mathrm{Temp}(G)$. We denote by $\mathcal{E}_G(X, \mathcal{W}(\pi, \psi_n)^*)$ the set of all maps

$$\alpha : X \times \mathcal{W}(\pi, \psi_n) \to \mathbb{C}$$

which are G-invariant for the diagonal G-action i.e. satisfying $\alpha(xg, R(g^{-1})W) = \alpha(x, W)$ for every $x \in X$, $W \in \mathcal{W}(\pi, \psi_n)$ and $g \in G$, and such that $W \in \mathcal{W}(\pi, \psi_n) \mapsto \alpha(x, W)$ is a continuous linear functional for every $x \in X$ (the continuity condition is only for the Archimedean case). Let $x_1, \ldots, x_k$ be a family of representatives for the G-orbits in X, then we have an isomorphism

$$\mathcal{E}_G(X, \mathcal{W}(\pi, \psi_n)^*) \simeq \bigoplus_{i=1}^{k} \mathrm{Hom}_{G_{x_i}}(\mathcal{W}(\pi, \psi_n), \mathbb{C}),$$

$$\alpha \mapsto (\alpha(x_i, .))_{1 \leqslant i \leqslant k}.$$

To any $\alpha \in \mathcal{E}_G(X, \mathcal{W}(\pi, \psi_n)^*)$ we associate a *relative Bessel distribution* $J_\pi^\alpha : C_c^\infty(X) \to \mathbb{C}$ by

$$J_\pi^\alpha(\varphi) := \langle \varphi \cdot \alpha, \lambda_1^\vee \rangle, \quad \varphi \in C_c^\infty(X),$$

where $\varphi \cdot \alpha$ is the smooth functional

$$W \in \mathcal{W}(\pi, \psi_n) \mapsto \int_X \varphi(x)\alpha(x, W)dx$$

that we identify with an element of $\overline{\mathcal{W}(\pi, \psi_n)} = \mathcal{W}(\pi^\vee, \psi_n^{-1})$ via the invariant inner product $\langle ., . \rangle_{\mathrm{Whitt}}$ defined by (3.1.2) and $\lambda_1^\vee$ denotes the functional $W^\vee \mapsto W^\vee(1)$ on $\mathcal{W}(\pi^\vee, \psi_n^{-1})$. Similarly for any $\sigma \in \mathrm{Temp}(G')$, we define a *Bessel distribution* I_σ on $C_c^\infty(G')$ by

$$I_\sigma(f') := \langle f' \cdot \lambda_w, \lambda_1^\vee \rangle, \quad f' \in C_c^\infty(G'),$$

where $f' \cdot \lambda_w$ is the smooth functional

$$W \in \mathcal{W}(\sigma, \psi'_n) \mapsto \int_{G'} f'(g)W(wg)dg$$

that we again identify with an element of $\mathcal{W}(\sigma^\vee, {\psi'_n}^{-1})$ via the pairing $\langle ., . \rangle_{\mathrm{Whitt}}$ and $\lambda_1^\vee$ denotes the functional $W^\vee \mapsto W^\vee(1)$ on $\mathcal{W}(\sigma^\vee, {\psi'_n}^{-1})$. We have

(3.4.1) The above Bessel distribution I_σ coincides with the one defined in Section 2.3.

Indeed, since both functionals are continuous on $C_c^\infty(G')$ we just need to show the equality between them for functions $f' \in C_c^\infty(G')$ which are right-K'-finite. Let $f' \in C_c^\infty(G')$ which transforms for the right action according to a finite dimensional representation ρ of K'. Let $\mathcal{B}[\rho^\vee]$ be a basis of the $\rho^\vee$-isotypic component $\mathcal{W}(\sigma, \psi'_n)[\rho^\vee]$ that is orthonormal with respect to the inner product $\langle ., . \rangle_{\mathrm{Whitt}}$. Then, denoting temporarily by I'_σ the Bessel functional defined in Section 2.3, by (3.1.3) we have

$$\begin{aligned} I'_\sigma(f') &= \int_{N'}^* \sum_{W \in \mathcal{B}[\rho^\vee]} \langle R(uw)R(f')W, W \rangle_{\mathrm{Whitt}} \psi'_n(u)^{-1} du \\ &= \sum_{W \in \mathcal{B}[\rho^\vee]} (f' \cdot \lambda_w)(W)\lambda_1^\vee(\overline{W}) = I_\sigma(f'). \end{aligned}$$

The following is [FLO, Theorem 12.4].

Theorem 3.4.1 (Feigon-Lapid-Offen)**.** *Let $\sigma \in \mathrm{Temp}(G')$. Then, there exists a unique element*

$$\alpha^\sigma \in \mathcal{E}_G(X, \mathcal{W}(\mathrm{BC}(\sigma), \psi_n)^*)$$

such that we have the identity

$$J^{\alpha^\sigma}_{\mathrm{BC}(\sigma)}(\varphi) = I_\sigma(f')$$

for every pair of matching test functions $(\varphi, f') \in C_c^\infty(X) \times C_c^\infty(G')$.

Let $\sigma \in \mathrm{Temp}(G')$ and $\alpha^\sigma \in \mathcal{E}_G(X, \mathcal{W}(\mathrm{BC}(\sigma), \psi_n)^*)$ be as in the theorem above. We set $\alpha^\sigma_x = \alpha^\sigma(x, .) \in \mathrm{Hom}_{G_x}(\mathcal{W}(\pi, \psi_n), \mathbb{C})$ for every $x \in X$ and we call them *the FLO functionals* associated to σ. By abuse of language, we shall also call α^σ the FLO functional associated to σ. For notational simplicity, we set

$$J_\sigma := J^{\alpha^\sigma}_{\mathrm{BC}(\sigma)}$$

and call it *the FLO relative character* associated to σ.

Let $\lambda \in F^\times$ and $\varphi \in C_c^\infty(X)$. Then, for any matching test function $f' \in C_c^\infty(G')$ it is easy to see that the left translates $L(\lambda)\varphi = \varphi(\lambda^{-1}.)$ and $L(\lambda)f' = f'(\lambda^{-1}.)$ also match. From this and the characterization of the FLO functional, we readily infer that

$$(L(\lambda)\varphi) \cdot \alpha^\sigma = \omega_\sigma(\lambda)\varphi \cdot \alpha^\sigma, \quad \text{for every } \sigma \in \mathrm{Temp}(G'). \tag{3.4.2}$$

3.5 Harish-Chandra Schwartz and tempered functions on X

In this section and the next, we assume that F is a p-adic field.

For every $x \in X$ we set

$$\Xi^X(x) = \mathrm{vol}_X(xK)^{-1/2}.$$

Let σ_X be a log-norm on X (see [Beu1, §1.2]). We define the *Harish-Chandra Schwartz space* $\mathcal{C}(X)$ as the space of functions $\varphi : X \to \mathbb{C}$ which are right invariant by a compact-open subgroup of G and such that for every $d > 0$ we have

$$|\varphi(x)| \ll \Xi^X(x)\sigma_X(x)^{-d}, \quad x \in X. \tag{3.5.1}$$

For every compact-open subgroup $J \subset G$, the subspace $\mathcal{C}(X)^J \subset \mathcal{C}(X)$ of right J-invariant functions is naturally a Fréchet space and therefore $\mathcal{C}(X) = \bigcup_J \mathcal{C}(X)^J$ is a strict LF space (that is a countable inductive limit of Fréchet spaces with closed embeddings as connecting morphisms). We have:

(3.5.2) The subspace $C_c^\infty(X)$ is dense in $\mathcal{C}(X)$.

Indeed, let $J \subset G$ be a compact-open subgroup and $\varphi \in \mathcal{C}(X)^J$. Let $(X_k)_{k\geqslant 1}$ be an increasing and exhausting sequence of J-invariant compact subsets of X. Then, the sequence $\varphi_k = \mathbf{1}_{X_k}\varphi$ belongs to $C_c^\infty(X)^J$ and converges to φ in the Fréchet space $\mathcal{C}(X)^J$ as can easily be seen from the fact that $\sigma_X(x) \to \infty$ as $x \to \infty$.

We also define $\mathcal{C}^w(X)$ as the space of *tempered* functions on X i.e. functions $\varphi : X \to \mathbb{C}$ which are right invariant by a compact-open subgroup of G and satisfying the inequality

$$|\varphi(x)| \ll \Xi^X(x)\sigma_X(x)^{d}, \quad x \in X, \tag{3.5.3}$$

for some $d > 0$. For every compact-open subgroup $J \subset G$ and $d > 0$, the subspace $\mathcal{C}^w_d(X)^J \subset \mathcal{C}^w(X)$ of right J-invariant functions which satisfy the temperedness estimate (3.5.3) for the given exponent d is naturally a Fréchet space. Therefore $\mathcal{C}^w(X) = \bigcup_{J,d>0} \mathcal{C}^w_d(X)^J$ is a LF space (that is a countable inductive limit of Fréchet spaces).

By [Beu2, Proposition 3.1.1(iii)], for every $\varphi \in \mathcal{C}(X)$ and $\varphi' \in \mathcal{C}^w(X)$ the inner product $\langle \varphi, \varphi' \rangle_X$ converges absolutely.

Proposition 3.5.1. *(i) For every $(\varphi, \varphi') \in \mathcal{C}(X) \times \mathcal{C}^w(X)$ the function*

$$g \in G \mapsto \langle R(g)\varphi, \varphi' \rangle_X$$

belongs to $\mathcal{C}^w(G)$ and the resulting sesquilinear map $\mathcal{C}(X) \times \mathcal{C}^w(X) \to \mathcal{C}^w(G)$ is separately continuous.

(ii) The action by right convolution

$$C_c^\infty(G) \times \mathcal{C}(X) \to \mathcal{C}(X)$$

$$(f, \varphi) \mapsto R(f)\varphi$$

extends to a separately continuous bilinear map $\mathcal{C}(G) \times \mathcal{C}(X) \to \mathcal{C}(X)$.

(iii) Let $\pi \in \mathrm{Temp}(G)$ and $\iota : \pi \to C^\infty(X)$ be a G-equivariant linear map. Then, the image of ι lands in $\mathcal{C}^w(X)$.

Proof. (i) According to [Ber3, Key Lemma, §3.4] we have equalities of topological vector spaces

$$\mathcal{C}(X) = \bigcap_{d>0} L^2(X, \sigma_X(x)^d dx)^\infty \text{ and } \mathcal{C}^w(X) = \bigcup_{d>0} L^2(X, \sigma_X(x)^{-d} dx)^\infty \tag{3.5.4}$$

where for $d \in \mathbb{R}$, $L^2(X, \sigma_X(x)^d dx)$ stands for the space of smooth (that is right-invariant by a compact-open subgroup) square-integrable functions on X with respect to the measure $\sigma_X(x)^d dx$. Let $\|.\|_{X,d}$ be the Hilbert norm on $L^2(X, \sigma_X(x)^d dx)$ and set $\|.\|_X = \|.\|_{X,0}$. We may assume, without loss in generality, that the log-norm σ_X is right K-invariant.

Recall that for every $V \in \mathcal{V}$, the pair $(G, U(V))$ is *tempered* in the sense of [Beu2, §2.7] (see (3.2.3)). Hence, by [Beu2, Proposition 2.7.1], the unitary G-representation $L^2(X)$ is tempered meaning that its Plancherel support is included in the set of irreducible tempered representations. From [CHH, Theorem 2], it follows that for every compact-open subgroup $J \subset K$, there exists a constant $C_J > 0$ such that

$$\langle R(g)\varphi_1, \varphi_2 \rangle_X \leqslant C_J \Xi^G(g) \|\varphi_1\|_X \|\varphi_2\|_X \tag{3.5.5}$$

for every $\varphi_1, \varphi_2 \in L^2(X)^J$ and $g \in G$.

Let now $d > 0$, $J \subset K$ be a compact-open subgroup and $(\varphi_1, \varphi_2) \in L^2(X, \sigma_X(x)^d dx)^J \times L^2(X, \sigma_X(x)^{-d} dx)^J$. Then, we have $\sigma_X^{d/2}|\varphi_1| \in L^2(X)^J$ and $\sigma_X^{-d/2}|\varphi_2| \in L^2(X)^J$. Moreover, there exists a constant $C_0 > 0$ such that $\sigma_X(x) \leqslant C_0 \sigma_X(xg)\sigma_G(g)$ for every $(x, g) \in X \times G$. Therefore, using (3.5.5), we obtain

$$\begin{aligned}
|\langle R(g)\varphi_1, \varphi_2 \rangle_X| &\leqslant \int_X |\varphi_1|(xg)|\varphi_2|(x)dx = \int_X \sigma_X(x)^{d/2}|\varphi_1|(xg)\sigma_X(x)^{-d/2}|\varphi_2|(x)dx \\
&\leqslant C_0 \sigma_G(g)^{d/2} \int_X \sigma_X(xg)^{d/2}|\varphi_1|(xg)\sigma_X(x)^{-d/2}|\varphi_2|(x)dx \\
&= C_0 \sigma_G(g)^{d/2} \langle R(g)\sigma_X^{d/2}|\varphi_1|, \sigma_X^{-d/2}|\varphi_2| \rangle_X \\
&\leqslant C_0 C_J \Xi^G(g) \sigma_G(g)^{d/2} \|\varphi_1\|_{X,d} \|\varphi_2\|_{X,-d}
\end{aligned}$$

for every $g \in G$. Combined with (3.5.4), this implies part (i) of the proposition.

(ii) Let $\varphi \in \mathcal{C}(X)$. We need to show that the linear map $f \in C_c^\infty(G) \mapsto R(f)\varphi \in \mathcal{C}(X)$ extends continuously to $\mathcal{C}(G)$. The equalities (3.5.4) imply that, through the integration pairing $\langle ., . \rangle_X$, $\mathcal{C}(X)$ gets identified with the space of smooth continuous anti-linear forms on $\mathcal{C}^w(X)$. Let $f \in \mathcal{C}(G)$. By (i), the anti-linear form

$$\varphi' \in \mathcal{C}^w(X) \mapsto \int_G f(g) \langle R(g)\varphi, \varphi' \rangle_X dg$$

is well-defined and continuous. It is also smooth as f is biinvariant by a compact-open subgroup. Therefore, there exists a unique element $R(f)\varphi \in \mathcal{C}(X)$ such that

$$\int_G f(g) \langle R(g)\varphi, \varphi' \rangle_X dg = \langle R(f)\varphi, \varphi' \rangle_X$$

for every $\varphi' \in \mathcal{C}^w(X)$. Moreover, this definition is easily seen to coincides with the action by right convolution when $f \in C_c^\infty(G)$. Finally, the linear map $f \in \mathcal{C}(G) \mapsto R(f)\varphi \in \mathcal{C}(X)$ is continuous by the closed graph theorem [Tr, Corollary 4, §17] since, by definition, for every $\varphi' \in \mathcal{C}^w(X)$ the linear form $f \in \mathcal{C}(G) \mapsto \langle R(f)\varphi, \varphi' \rangle_X$ is continuous.

(iii) The argument is similar to the proof of [Beu2, Lemma 4.2.1] so we only sketch it. The idea, which goes back to Lagier [Lag] and Kato-Takano [KT], is to relate functions in the image of ι to smooth matrix coefficients of π and then deduce the result from the known asymptotics for smooth matrix coefficients of tempered representations. More precisely, for each $V \in \mathcal{V}$, denoting by $x_V \in X_V = U(V)\backslash G$ the canonical base-point, using the weak Cartan decomposition of Benoist-Oh [BO] and Delorme-Sécherre [DS] (see also [SV, Lemma 5.3.1] for a different proof) we can construct as in [SV, Corollary 5.3.2] a subset $G_V^+ \subset G$ such that

$$X_V = x_V G_V^+ \tag{3.5.6}$$

and (the so-called "wave-front lemma")

(3.5.7) For every compact-open subgroup $J_1 \subset G$, there exists another compact-open subgroup $J_2 \subset G$ such that

$$x_V J_2 g \subset x_V g J_1$$

for every $g \in G_V^+$.

Moreover, by [Beu2, Proposition 3.3.1 (ii)] (which holds as the pair $(G, U(V))$ is tempered in the sense of [Beu2, §2.7], see (3.2.3)) there also exists $d > 0$ such that

$$\Xi^G(g) \ll \Xi^X(x_V g)\sigma_X(x_V g)^d,\ g \in G_V^+. \tag{3.5.8}$$

Let $e \in \pi$ and $J_1 \subset G$ be a compact-open subgroup leaving e invariant. Let $J_2 \subset G$ be as in (3.5.7) (for every $V \in \mathcal{V}$). Then, by equivariance of ι, for every $k_2 \in J_2$ there exists $k_1 \in J_1$ such that

$$\iota(e)(x_V g) = \iota(e)(x_V g k_1) = \iota(e)(x_V k_2 g) = \iota(\pi(k_2 g)e)(x_V)$$

for every $V \in \mathcal{V}$ and $g \in G_V^+$. Therefore,

$$\iota(e)(x_V g) = \int_{K_2} \iota(\pi(k_2 g)e)(x_V)dk_2 = \langle \pi(g)e, e_V^\vee \rangle \text{ for } V \in \mathcal{V}, g \in G_V^+$$

where $e_V^\vee$ is a certain vector in the smooth contragredient of π. By the asymptotic of smooth coefficients of tempered representations [CHH], we have $|\langle \pi(g)e, e_V^\vee \rangle| \ll \Xi^G(g)$ for $g \in G$, hence by (3.5.6) and (3.5.8) we get

$$|\iota(e)(x)| \ll \Xi^X(x)\sigma_X(x)^d$$

for every $x \in X = \bigsqcup_{V \in \mathcal{V}} X_V$. As the function $\iota(e)$ is also smooth, this shows that $\iota(e) \in \mathcal{C}^w(X)$ and the proposition is proved.

□

3.6 Abstract tempered relative characters

In this section, we continue to assume that F is a p-adic field. Let $\pi \in \mathrm{Temp}(G)$. We denote by $C_c^\infty(X)_\pi$ the $\pi^\vee$-coinvariant space of $C_c^\infty(X)$ i.e. the maximal quotient which is G-isomorphic to a direct sum of copies of $\pi^\vee$. We define the space of *abstract relative characters supported on* π as the space

$$\mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n)$$

of (N, ψ_n)-equivariant functionals on $C_c^\infty(X)_\pi$. Note that $J_\sigma \in \mathrm{Hom}_N(C_c^\infty(X)_{\mathrm{BC}(\sigma)}, \psi_n)$ for every $\sigma \in \mathrm{Temp}(G')$.

Lemma 3.6.1. *Let $J \in \mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n)$. Then, J extends by continuity to $\mathcal{C}(X)$ and moreover there exists a function $F \in \mathcal{C}^w(X)$ such that*

$$J(\varphi) = \int_N^* \langle R(u)\varphi, F\rangle_X \psi_n(u)^{-1} du \tag{3.6.1}$$

for every $\varphi \in \mathcal{C}(X)$.

Remark 3.6.1. *Note that by Proposition* (3.5.1)*(i) the above "regularized" integral makes sense for every $\varphi \in \mathcal{C}(X)$ and $F \in \mathcal{C}^w(X)$.*

Proof. By Frobenius reciprocity and unicity of the Whittaker model, J induces a G-equivariant linear map

$$W_J : C_c^\infty(X) \to \mathcal{W}(\pi^\vee, \psi_n)$$

satisfying that $J(\varphi) = W_J(\varphi)(1)$ for every $\varphi \in C_c^\infty(X)$. Let $W_J^* : \mathcal{W}(\pi^\vee, \psi_n) \to C^\infty(X)$ be the smooth adjoint of W_J with respect to the invariant inner products $\langle ., . \rangle_X$ and $\langle ., . \rangle_{\mathrm{Whitt}}$. By (3.1.3), we have

$$J(\varphi)\overline{w(1)} = W_J(\varphi)(1)\overline{w(1)} = \int_N^* \langle R(u)W_J(\varphi), w\rangle_{\mathrm{Whitt}} \psi_n(u)^{-1} du$$

for every $\varphi \in C_c^\infty(X)$ and $w \in \mathcal{W}(\pi^\vee, \psi_n)$. Choose $w \in \mathcal{W}(\pi^\vee, \psi_n)$ such that $w(1) = 1$ and set $F = W_J^*(w)$. By Proposition (3.5.1)(iii), we have $F \in \mathcal{C}^w(X)$. On the other hand, by adjunction we have $\langle R(u)W_J(\varphi), w\rangle_{\mathrm{Whitt}} = \langle R(u)\varphi, F\rangle_X$ for every $\varphi \in C_c^\infty(X)$ and $u \in N$. Therefore the function F satisfies (3.6.1) for every $\varphi \in C_c^\infty(X)$. That J extends continuously to $\mathcal{C}(X)$ and (3.6.1) is still satisfied for $\varphi \in \mathcal{C}(X)$ now follows from Proposition (3.5.1)(i). □

4 Jacquet-Ye's local trace formula

In this chapter, we develop a local trace formula for the symmetric variety X. More precisely, we consider a *relative local Kuznetsov trace formula* for X which is obtained by applying the (N, ψ_n)-regularized integral of Section 2.1 to a matrix coefficient for $L^2(X)$. The resulting 'distribution' (a sesquilinear form on $C_c^\infty(X)$) admits both a geometric expansion, in terms of relative orbital integrals, and a spectral expansion, in terms of the FLO relative characters of Section 3.4. The equality between the two expansions is the aforementioned local trace formula (Theorem 4.2.2). It will be applied in Chapters 5 and 6 to finish the computation of multiplicities of generic representations with respect to X and to the Plancherel decomposition of X respectively. In Section 4.1, we define the relevant distribution on $C_c^\infty(X)$ and we establish a geometric expansion for it. In Section 4.2, we state and prove the spectral expansion and the resulting trace formula identity (Theorem 4.2.2).

We note here that a similar formula has been developed by Feigon [Fe, Sect. 4] in the context of the symmetric variety $X = \mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(E)$. One main difference between the two formulas is that the spectral side of Feigon's identity is given in terms of explicit invariant linear forms on tempered representations whereas the spectral side of Theorem 4.2.2 is given in terms of the FLO functionals J_σ (see the definition at the beginning of §4.2) which are in turn only defined implicitely through the Jacquet-Ye transfer (see §3.4).

4.1 Geometric expansion

Let $\varphi_1, \varphi_2 \in C_c^\infty(X)$. By (3.2.4), we can define the following expression

$$J(\varphi_1, \varphi_2) = \int_N^* \langle R(u)\varphi_1, \varphi_2 \rangle_X \psi_n(u)^{-1} du \tag{4.1.1}$$

where the right-hand side is an (N, ψ_n^{-1})-regularized integral as defined in Section 2.1.

For $t \in T_X$ and $\varphi \in C_c^\infty(X)$ we set

$$O(t, \varphi) = \int_N \varphi(tu)\psi_n(u)^{-1} du.$$

Lemma 4.1.1. *The expression defining $O(t, \varphi)$ is absolutely convergent locally uniformly in t and φ.*

Proof. This follows from the fact that the morphism $T_X \times N \to T_X \times X$, $(t, u) \mapsto (t, tu)$ is a closed embedding (hence proper). □

Set

$$J_{\mathrm{geom}}(\varphi_1, \varphi_2) = \int_{T_X} O(t, \varphi_1)\overline{O(t, \varphi_2)}\delta_X(t) dt.$$

The main result of this section is the following.

Theorem 4.1.1. *The expression defining $J_{\mathrm{geom}}(\varphi_1, \varphi_2)$ is absolutely convergent and we have*

$$J(\varphi_1, \varphi_2) = J_{\mathrm{geom}}(\varphi_1, \varphi_2).$$

Proof. The proof is very similar to that of Theorem 2.2.1 so we will be brief and not give all the details. First we extend the definition of $J(\varphi_1, \varphi_2)$ to $\Phi \in C_c^\infty(X \times X)$ by

$$J(\Phi) := \int_N^* \int_X \Phi(xu, x) dx \psi_n(u)^{-1} du.$$

Note that this expression makes sense since we can show similarly to (3.2.4) that the function

$$K_\Phi : g \in G \mapsto \int_X \Phi(xg, x) dx$$

belongs to $\mathcal{C}^w(G)$. We have $J(\varphi_1, \varphi_2) = J(\varphi_1 \otimes \overline{\varphi_2})$ where $\varphi_1 \otimes \overline{\varphi_2} \in C_c^\infty(X \times X)$ is the function given by $(\varphi_1 \otimes \overline{\varphi_2})(x_1, x_2) = \varphi_1(x_1)\overline{\varphi_2(x_2)}$.

Let R^Δ be the right diagonal action of T on $C_c^\infty(X \times X)$. In the p-adic case, we choose a compact-open subgroup K_T of T by which both φ_1 and φ_2 are right-invariant and we set $\phi = \mathrm{vol}(K_T)^{-1}\mathbf{1}_{K_T}$, $\Phi = \varphi_1 \otimes \overline{\varphi_2}$ so that $\varphi_1 \otimes \overline{\varphi_2} = R^\Delta(\phi)\Phi$. In the Archimedean case, by Dixmier-Malliavin [DM], we may assume that $\varphi_1 \otimes \overline{\varphi_2} = R^\Delta(\phi)\Phi$ for some $\phi \in C_c^\infty(T)$ and $\Phi \in C_c^\infty(X \times X)$. Then, by (2.1.5), in both cases we have

$$\begin{aligned} J(\varphi_1, \varphi_2) &= \int_N^* K_{R^\Delta(\phi)\Phi}(u)\psi_n(u)^{-1} du = \int_N^* (\mathrm{Ad}(\phi)K_\Phi)(u)\psi_n(u)^{-1} du \\ &= \int_N K_\Phi(u)\widehat{\phi}(u) du = \int_N \int_X \Phi(xu, x) dx \widehat{\phi}(u) du \end{aligned} \tag{4.1.2}$$

where

$$\widehat{\phi}(u) = \int_T \phi(a)\psi_n(aua^{-1})^{-1}\delta_B(a)da, \quad u \in N.$$

It follows readily from (3.2.4) and (2.1.6) that the last expression in (4.1.2) is absolutely convergent. By (3.2.2), we have

$$\begin{aligned}\int_N \int_X \Phi(xu,x)dx\widehat{\phi}(u)du &= \int_N \int_{T_X} \int_N \Phi(tvu,tv)dv\delta_X(t)dt\widehat{\phi}(u)du \\ &= \int_{T_X} \int_{N^2} \Phi(tvu,tv)\widehat{\phi}(u)dudv\delta_X(t)dt.\end{aligned}$$

Set

$$O(t,\Phi) = \int_{N^2} \Phi(tu,tv)\psi_n(u)^{-1}\psi_n(v)dudv$$

for every $\Phi \in C_c^\infty(X\times X)$ and $t \in T_X$. The same arguments as for Lemma 4.1.1 show that this expression is absolutely convergent locally uniformly in t and Φ. Note that $O(t,\varphi_1\otimes\overline{\varphi_2}) = O(t,\varphi_1)\overline{O(t,\varphi_2)}$ for every $t \in T_X$. Simple manipulations (which are justified by the absolute convergence of $O(t,\Phi)$ uniformly in t and Φ) show that

$$\int_{N^2} \Phi(tvu,tv)\widehat{\phi}(u)dudv = \int_T \phi(a)\delta_B(a)^{-1}O(ta^{-1},R^\Delta(a)\Phi)da$$

for every $t \in T_X$. Thus, the above computations imply that the expression

$$\int_{T_X}\int_T \phi(a)\delta_B(a)^{-1}O(ta^{-1},R^\Delta(a)\Phi)da\delta_X(t)dt \tag{4.1.3}$$

is convergent as an iterated integral for every $\phi \in C_c^\infty(T)$ and $\Phi \in C_c^\infty(X\times X)$ and moreover that

$$J(\varphi_1,\varphi_2) = \int_{T_X}\int_T \phi(a)\delta_B(a)^{-1}O(ta^{-1},R^\Delta(a)\Phi)da\delta_X(t)dt \tag{4.1.4}$$

whenever $\varphi_1\otimes\overline{\varphi_2} = R^\Delta(\phi)\Phi$. The argument at the end of the proof of Theorem 2.2.1, in particular for the claim (2.2.6), adapts almost verbatim to this situation to show that (4.1.3) is actually absolutely convergent. (Here, we recall that, in the Archimedean case for any compact subset $L \subset X$, denoting by $C_L^\infty(X)$ the subspace of smooth functions supported in L, we have $C_L^\infty(X)\widehat{\otimes}C_L^\infty(X) \simeq C_{L\times L}^\infty(X\times X)$ [Gro, Exemple 1, Chap. II §3 n.3]). Using (4.1.4), simple manipulations now allow to get the identity

$$J(\varphi_1,\varphi_2) = J_{\mathrm{geom}}(\varphi_1,\varphi_2)$$

and the fact that the expression defining $J_{\mathrm{geom}}(\varphi_1,\varphi_2)$ is absolutely convergent. □

4.2 Spectral expansion

Recall from Section 3.4 that to every $\sigma \in \mathrm{Temp}(G')$ is associated a relative character J_σ which is a functional on $C_c^\infty(X)$. For every $\varphi_1,\varphi_2 \in C_c^\infty(X)$ we set

$$J_{\mathrm{spec}}(\varphi_1,\varphi_2) = \int_{\mathrm{Temp}(G')} J_\sigma(\varphi_1)\overline{J_\sigma(\varphi_2)}d\mu_{G'}(\sigma) \tag{4.2.1}$$

where $\mu_{G'}$ denotes the Plancherel measure of G' (see Section 2.3).

Theorem 4.2.1. *For every $\varphi_1, \varphi_2 \in C_c^\infty(X)$, the expression defining $J_{\mathrm{spec}}(\varphi_1, \varphi_2)$ is absolutely convergent and we have*

$$J(\varphi_1, \varphi_2) = J_{\mathrm{spec}}(\varphi_1, \varphi_2).$$

Proof. Let $f_1, f_2 \in C_c^\infty(G')$ be test functions matching φ_1, φ_2 respectively in the sense of Section 3.3. By Theorem 4.1.1, the definition of the transfer, and the fact that the isomorphism $T_X \simeq T'$ is measure preserving, we have

$$J(\varphi_1, \varphi_2) = \int_{T_X} O(t, \varphi_1)\overline{O(t, \varphi_2)}\delta_X(t)dt = \int_{T'} O(a, f_1)\overline{O(a, f_2)}\delta_{B'}(a)da$$

where the "transfer factors" disappear as $\gamma(a)^2 = 1$. By Theorem 2.3.2, this last expression is equal to

$$\int_{\mathrm{Temp}(G')} I_\sigma(f_1)\overline{I_\sigma(f_2)}d\mu_{G'}(\sigma). \tag{4.2.2}$$

By definition of the FLO relative characters J_σ, this is further equal to

$$\int_{\mathrm{Temp}(G')} J_\sigma(\varphi_1)\overline{J_\sigma(\varphi_2)}d\mu_{G'}(\sigma) = J_{\mathrm{spec}}(\varphi_1, \varphi_2).$$

Moreover, as (4.2.2) is absolutely convergent (by Theorem 2.2.1), the above expression is also convergent and this proves the theorem. □

From Theorem 4.2.1 and Theorem 4.1.1, we deduce:

Theorem 4.2.2 (Local Kuznetsov trace formula for X)**.** *For every $\varphi_1, \varphi_2 \in C_c^\infty(X)$, we have*

$$J_{\mathrm{geom}}(\varphi_1, \varphi_2) = J_{\mathrm{spec}}(\varphi_1, \varphi_2).$$

Assume now that F is a p-adic field. By Proposition 3.5.1(i), the definition (4.1.1) of $J(\varphi_1, \varphi_2)$ extends to any $\varphi_1, \varphi_2 \in \mathcal{C}(X)$ and moreover, J is a separately continuous Hermitian form on $\mathcal{C}(X)$. On the other hand, by Lemma 3.6.1 the FLO relative characters J_σ, $\sigma \in \mathrm{Temp}(G')$, extend by continuity to $\mathcal{C}(X)$. Hence, the definition (4.2.1) of $J_{\mathrm{spec}}(\varphi_1, \varphi_2)$ still makes sense, formally, for every $\varphi_1, \varphi_2 \in \mathcal{C}(X)$. In this context, Theorem 4.2.1 admits the following extension.

Theorem 4.2.3. *For every $\varphi_1, \varphi_2 \in \mathcal{C}(X)$, the expression defining $J_{\mathrm{spec}}(\varphi_1, \varphi_2)$ is absolutely convergent and we have*

$$J(\varphi_1, \varphi_2) = J_{\mathrm{spec}}(\varphi_1, \varphi_2).$$

Proof. Let $J \subset G$ be a compact-open subgroup and $\varphi \in \mathcal{C}(X)^J$. Let $(\varphi_k)_{k\geqslant 1}$ a sequence in $C_c^\infty(X)^J$ converging to φ in $\mathcal{C}(X)^J$ (such sequence exists by (3.5.2)). Since separately continuous bilinear forms on Fréchet spaces are automatically continuous [Tr, Corollary 34.2], by Theorem 4.2.1 and the continuity of J we deduce that the sequence

$$J(\varphi_k, \varphi_k) = \int_{\mathrm{Temp}(G')} |J_\sigma(\varphi_k)|^2 d\mu_{G'}(\sigma)$$

converges to $J(\varphi, \varphi)$. Hence, by Fatou's lemma and the continuity of J_σ on $\mathcal{C}(X)$, the integral

$$\int_{\mathrm{Temp}(G')} |J_\sigma(\varphi)|^2 d\mu_{G'}(\sigma)$$

converges and is bounded by $J(\varphi, \varphi)$. By Cauchy-Schwarz, it follows that $J_{\mathrm{spec}}(\varphi_1, \varphi_2)$ is absolutely convergent and defines a continuous sesquilinear form on $\mathcal{C}(X)^J$. The theorem follows by the continuity of J and the density of $C_c^\infty(X)$ in $\mathcal{C}(X)$. □

5 Multiplicities

In this chapter, we keep the notation introduced in the Chapters 3 and 4 and we moreover assume that:

$$F \text{ is a } p-\text{adic field.}$$

The goal of this chapter is to complement results of Feigon-Lapid-Offen on the computations of the *multiplicity*

$$m(\pi) = \dim \mathrm{Hom}_G(\pi, C^\infty(X))$$

for $\pi \in \mathrm{Irr}(G)$ *generic*. This multiplicity is always finite by a general result of Delorme [Del, Theorem 4.5] and naturally decomposes as a sum over $V \in \mathcal{V}$ of individual multiplicities

$$m_V(\pi) = \dim \mathrm{Hom}_G(\pi, C^\infty(U(V)\backslash G)) = \dim \mathrm{Hom}_{U(V)}(\pi, \mathbb{C})$$

where the last equality follows from Frobenius reciprocity.

In [FLO, Theorem 0.2], Feigon, Lapid and Offen gives a lower bound for $m_V(\pi)$ in terms of the (cardinality of the) general fibers of Arthur and Clozel's base-change map $\mathrm{BC} : \mathrm{Irr}(G') \to \mathrm{Irr}(G)$ [AC]. They moreover show that this lower bound is actually equal to the multiplicity when BC is "unramified at π" (in a sense that will be made precise in the next section). The new result obtained here is that equality *always* holds as conjectured by Feigon-Lapid-Offen [FLO, Conjecture 13.17]. The main ingredients entering into the proof are the local trace formula for X developed in the last chapter as well as the scalar Whittaker-Paley-Wiener theorem of Section 2.4 for the group G'.

In order to state the main result in the appropriate context, in Section 5.1 we explain how to endow $\mathrm{Irr}(G)$ and $\mathrm{Irr}(G')$ with natural structures of algebraic varieties and we study related properties of the base-change map BC and the map λ associating to an irreducible representation its cuspidal support. Using these extra structures, we state in Section 5.2 the main result whose proof occupies Sections 5.3 to 5.5. More precisely, in Section 5.3, we make a reduction to tempered representations following [FLO, §6]. In Section 5.4, we relate the multiplicity $m(\pi)$ to the FLO functionals of Section 3.4 via the local trace formula developed in the previous chapter. Once this relation is established, the theorem readily follows from the scalar Whittaker Paley-Wiener theorem and the necessary arguments are given in Section 5.5.

Here is a list of notation and conventions that we shall use in this chapter (besides the one introduced in previous sections):

- A *semi-standard Levi* of G (resp. G') means a Levi subgroup containing T (resp. T'). Similarly, a *standard parabolic subgroup* of G (resp. G') is a parabolic subgroup containing B (resp. B') and a standard Levi subgroup is the unique semi-standard Levi component of a standard parabolic subgroup.

- For M a Levi subgroup of G or G', we denote by $X(M)$, $X_{\mathrm{unit}}(M)$, $X_{\mathrm{unr}}(M)$ and $X_{\mathrm{alg}}(M)$ the groups of smooth, unitary, unramified and algebraic (defined over F) characters of M respectively. Recall that $X_{\mathrm{unr}}(M)$ is a complex torus whose index in $X(M)$ is countable. Therefore, $X(M)$ has a natural structure of algebraic variety over $\mathbb{C}$ (with countably many components). We set $\mathcal{A}_M^* = X_{\mathrm{alg}}(M) \otimes \mathbb{R}$. There is an injective homomorphism $\mathcal{A}_M^* \to X(M)$ sending $\lambda \otimes x$ to the character $m \in M \mapsto |\lambda(m)|_F^x$. The image of this homomorphism is the subgroup of positive valued characters of M. Therefore, if $\chi \in X(M)$, its absolute value $|\chi|$ corresponds to an element of $\mathcal{A}_M^*$ that we denote by $\Re(\chi)$. More generally, if σ is an

irreducible smooth representation of M with central character ω_σ, $|\omega_\sigma|$ extends uniquely to a positive valued character of M and we set $\Re(\sigma) = \Re(|\omega_\sigma|)$.

- If $L \subset M$ is another Levi subgroup, there is a natural inclusion $\mathcal{A}_M^* \subset \mathcal{A}_L^*$ with a natural section $\mathcal{A}_L^* \twoheadrightarrow \mathcal{A}_M^*$ whose kernel we denote by $(\mathcal{A}_L^M)^*$. The inclusion $T' \subset T$ induces an identification $\mathcal{A}_{T'}^* = \mathcal{A}_T^*$ and we just write $\mathcal{A}^*$ for this real vector space.

- Still for M a Levi subgroup of G (resp. of G'), we set $W(G,M) = \mathrm{Norm}_G(M)/M$ (resp. $W(G',M) = \mathrm{Norm}_{G'}(M)/M$) for the corresponding Weyl group and $W^M = W(M,T)$ (resp. $W^M = W(M,T')$) for the Weyl group of T (resp. T') in M. Then, $W(G,M)$ acts naturally on $\mathcal{A}_M^*$. We have again a natural identification $W^{G'} = W^G$ and we simply write W for this Weyl group. We fix on $\mathcal{A}^*$ a W-invariant Euclidean norm $\|.\|$. Note that for every pair $L \subset M$ of semi-standard Levi subgroups, the subspaces $\mathcal{A}_L^*$ and $(\mathcal{A}_L^M)^*$ are orthogonal for the resulting Euclidean structure.

- We denote by $\mathrm{Irr}(G)$ (resp. $\mathrm{Irr}(G')$) the set of isomorphism classes of smooth irreducible representations of G (resp. G') and by $\mathrm{Irr}^{\mathrm{gen}}(G)$, $\mathrm{Temp}(G)$, $\Pi_2(G)$, $\Pi_{2,\mathrm{ess}}(G)$, $\Pi_{\mathrm{cusp}}(G)$ (resp. $\mathrm{Irr}^{\mathrm{gen}}(G')$, $\mathrm{Temp}(G')$, $\Pi_2(G')$, $\Pi_{2,\mathrm{ess}}(G')$, $\Pi_{\mathrm{cusp}}(G')$) the subsets of generic, tempered, square-integrabl, essentially square-integrable and supercuspidal irreducible representations respectively.

- If $P = MU$ is a parabolic subgroup of G and τ a smooth representation of M, we denote by $I_P^G(\tau)$ the smooth unitarily normalized parabolic induction of τ. If moreover P is standard and M decomposes in diagonal blocks as

$$M = \mathrm{GL}_{n_1}(E) \times \ldots \times \mathrm{GL}_{n_k}(E)$$

and τ is of the form $\tau = \tau_1 \boxtimes \ldots \boxtimes \tau_k$ and we write

$$\tau_1 \times \ldots \times \tau_k$$

for $I_P^G(\tau)$. Similar notation apply to representations of G'.

5.1 Algebraic structure on $\mathrm{Irr}(G)$, the Bernstein center and base-change

Let $\mathrm{Sqr}(G)$ be the set of pairs (M,σ) where M is a semi-standard Levi of G and $\sigma \in \Pi_{2,\mathrm{ess}}(M)$ is an irreducible essentially square-integrable representation of M. We equip $\mathrm{Sqr}(G)$ with its unique structure of algebraic variety over $\mathbb{C}$ (with infinitely many components) such that for every $(M,\sigma) \in \mathrm{Sqr}(G)$, the map

$$X_{\mathrm{unr}}(M) \to \mathrm{Sqr}(G),\ \chi \mapsto (M, \sigma \otimes \chi)$$

is a finite covering over a connected component of $\mathrm{Sqr}(G)$. The Weyl group W is acting on $\mathrm{Sqr}(G)$ by regular automorphisms and we denote by $\mathrm{Sqr}(G)/W$ the GIT quotient. By the special form of the Levi subgroups of G and their associated Weyl groups, the connected components of $\mathrm{Sqr}(G)/W$ are all isomorphic to products of varieties of the form $(\mathbb{C}^\times)^t/\mathfrak{S}_t$ where $\mathfrak{S}_t$ acts on $(\mathbb{C}^\times)^t$ by permutation of the entries. This implies in particular that $\mathrm{Sqr}(G)/W$ is smooth.

To $(M,\sigma) \in \mathrm{Sqr}(G)$ we associate the unique irreducible quotient of $I_P^G(\sigma)$ where P is any parabolic subgroup with Levi component M such that $\Re(\sigma)$ is (non-strictly) dominant with respect

to P. By the Langlands classification this induces a bijection $\mathrm{Sqr}(G)/W \simeq \mathrm{Irr}(G)$ and we use this bijection to transfer the structure of algebraic variety on $\mathrm{Sqr}(G)/W$ to $\mathrm{Irr}(G)$.

We will use this bijection to identify $\mathrm{Sqr}(G)/W$ and $\mathrm{Irr}(G)$, thus for $(M,\sigma) \in \mathrm{Sqr}(G)$ its image $[M,\sigma] \in \mathrm{Sqr}(G)/W$ is identified with the corresponding Langlands quotient in $\mathrm{Irr}(G)$. Also, for $(M,\sigma) \in \mathrm{Sqr}(G)$ we will write $\mathrm{Irr}_{M,\sigma}(G)$ for the image in $\mathrm{Irr}(G)$ of the subset

$$\{(M, \sigma \otimes \chi) \mid \chi \in X(M)\}$$

of $\mathrm{Sqr}(G)$. Setting

$$W'_\sigma = \{(\chi, w) \in X(M) \rtimes W(G,M) \mid w\sigma \simeq \sigma \otimes \chi\}$$

(a finite group) the map $\chi \in X(M) \mapsto [M, \sigma \otimes \chi]$ induces a regular isomorphism $X(M)/W'_\sigma \simeq \mathrm{Irr}_{M,\sigma}(G)$. We emphasize here that, as $X(M)$ stands for the group of all *smooth* characters of M (not necessarily unramified), $\mathrm{Irr}_{M,\sigma}(G)$ is only a countable union of connected components of $\mathrm{Irr}(G)$.

For $(M,\sigma) \in \mathrm{Sqr}(G)$, we also set

$$\mathrm{Temp}_{M,\sigma}(G) = \mathrm{Irr}_{M,\sigma}(G) \cap \mathrm{Temp}(G) \text{ and } \mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G) = \mathrm{Irr}_{M,\sigma}(G) \cap \mathrm{Irr}^{\mathrm{gen}}(G).$$

Assuming that σ is square-integrable (which we may up to a twist), $\mathrm{Temp}_{M,\sigma}(G)$ is the image of $X_{\mathrm{unit}}(M)$ by the surjective regular map $X(M) \to \mathrm{Irr}_{M,\sigma}(G)$, $\chi \mapsto [M, \sigma \otimes \chi]$. Since $X_{\mathrm{unit}}(M)$ is Zariski dense in $X(M)$ this shows:

(5.1.1) $\qquad \mathrm{Temp}(G) \text{ is Zariski-dense in } \mathrm{Irr}(G).$

Let $\pi = [M,\sigma] \in \mathrm{Irr}^{\mathrm{gen}}(G)$. Then, for every parabolic subgroup P with Levi component M we have $\pi \simeq I_P^G(\sigma)$ [Ze, Theorem 9.7]. Conversely, if $[M,\sigma] \in \mathrm{Sqr}(G)/W$ is such that for one parabolic subgroup P with Levi component M, $I_P^G(\sigma)$ is irreducible then its image in $\mathrm{Irr}(G)$ is generic. Therefore, by [Ren, Proposition VI.8.4] we have

(5.1.2) $\qquad \mathrm{Irr}^{\mathrm{gen}}(G) \text{ is Zariski open in } \mathrm{Irr}(G).$

Let $\mathcal{Z}(G)$ be the "finite" Bernstein center (as defined in Section 2.4) and let $\mathcal{B}(G)$ be its maximal spectrum which is an algebraic variety over $\mathbb{C}$. Then, we have an identification $\mathcal{B}(G) \simeq \mathrm{Cusp}(G)/W$ of algebraic varieties where $\mathrm{Cusp}(G)$ is the set of pairs (L,τ) with L a semi-standard Levi subgroup and $\tau \in \Pi_{\mathrm{cusp}}(L)$ (the isomorphism class of) an irreducible supercuspidal representation of L that we endow with a structure of algebraic variety the same way we did for $\mathrm{Sqr}(G)$. For $(L,\tau) \in \mathrm{Cusp}(G)$, we denote by $\mathcal{B}_{L,\tau}(G)$ the subset

$$\{[L, \tau \otimes \chi] \mid \chi \in X(L)\}$$

of $\mathcal{B}(G)$. As before, $\mathcal{B}_{L,\tau}(G)$ is a union of connected component and the map $X(L) \to \mathcal{B}_{L,\tau}(G)$, $\chi \mapsto [L, \tau \otimes \chi]$ induces an isomorphism $X(L)/W'_\tau \simeq \mathcal{B}_{L,\tau}(G)$.

The natural inclusion $\mathrm{Cusp}(G) \subset \mathrm{Sqr}(G)$ descends to an open-closed immersion $\mathcal{B}(G) \hookrightarrow \mathrm{Irr}(G)$ and in particular $\mathcal{B}(G)$ is also smooth. This embedding admits a left-inverse

$$\lambda : \mathrm{Irr}(G) \to \mathcal{B}(G)$$

which associates to $\pi \in \mathrm{Irr}(G)$ its supercuspidal support (i.e. the unique element $[L,\tau] \in \mathcal{B}(G)$ such that π is a subquotient of $I_Q^G(\tau)$ for one, or equivalently every, parabolic with Levi component L).

Lemma 5.1.1. *λ is a regular finite morphism.*

Proof. Let $(M,\sigma) \in \mathrm{Sqr}(G)$. It suffices to show that the restriction of λ to $\mathrm{Irr}_{M,\sigma}(G)$ is regular and finite. Choose $(L,\tau) \in \mathrm{Cusp}(G)$ in the cuspidal support of σ. We have a commutative diagram

$$\begin{array}{ccc} X(M) & \xrightarrow{\quad\mathrm{Res}\quad} & X(L) \\ \downarrow & & \downarrow \\ X(M)/W'_\sigma \simeq \mathrm{Irr}_{M,\sigma}(G) & \xrightarrow{\lambda} & \mathcal{B}_{L,\tau}(G) \simeq X(L)/W'_\tau \end{array}$$

where the two vertical maps are $\chi \mapsto [M, \sigma\otimes\chi]$ and $\chi \mapsto [L,\tau\otimes\chi]$ respectively. Moreover, the restriction map $\mathrm{Res} : X(M) \to X(L)$ is a closed immersion and in particular finite. By the universal property of GIT quotients, the bottom map is therefore regular and finite. □

Let $(M,\sigma) \in \mathrm{Sqr}(G)$. We denote by $\mathrm{Irr}_{M,\sigma}(G)^\lambda$ and $\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)^\lambda$ the respective images of $\mathrm{Irr}_{M,\sigma}(G)$ and $\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)$ by λ. By the previous lemma, $\mathrm{Irr}_{M,\sigma}(G)^\lambda$ is closed in $\mathcal{B}(G)$.

Proposition 5.1.1. *$\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)^\lambda$ is open in $\mathrm{Irr}_{M,\sigma}(G)^\lambda$ and $\lambda : \mathrm{Irr}_{M,\sigma}(G) \to \mathrm{Irr}_{M,\sigma}(G)^\lambda$ restricts to an isomorphism over $\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)^\lambda$.*

Proof. Without loss in generality, we may assume that $\sigma \in \Pi_2(M)$. First we prove

(5.1.3) For $\pi \in \mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)$ and $\pi' \in \mathrm{Irr}_{M,\sigma}(G)$ if $\lambda(\pi) = \lambda(\pi')$ then $\pi = \pi'$.

Indeed, let $\pi \in \mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)$ and $\pi' \in \mathrm{Irr}_{M,\sigma}(G)$ and assume that $\lambda(\pi) = \lambda(\pi')$. There exist $\chi,\chi' \in X(M)$ and a parabolic subgroup P with Levi component M such that $\pi = I_P^G(\sigma\otimes\chi)$ and π' is the Langlands quotient of $I_P^G(\sigma\otimes\chi')$. Since σ is generic, $I_P^G(\sigma\otimes\chi')$ admits an irreducible generic subquotient [Rod, Théorème 4] with the same cuspidal support as π'. As there is a unique irreducible generic representation with a given cuspidal support, this shows that $\pi = I_P^G(\sigma\otimes\chi)$ is a subquotient of $I_P^G(\sigma\otimes\chi')$. Moreover, it follows from the geometric lemma of Bernstein-Zelevinsky and Casselman (see [BZ, Geometric Lemma] and [Cas1, §6.3]) that for every parabolic subgroup $Q \subset G$ the length of the supercuspidal parts of the Jacquet modules $J_Q I_P^G(\sigma\otimes\chi)$ and $J_Q I_P^G(\sigma\otimes\chi')$ are the same. By exactness of the Jacquet functor J_Q, this shows that if $\pi' \neq \pi$ then the supercuspidal part of the Jacquet module $J_Q\pi'$ is zero for every parabolic subgroup Q but this is impossible by [Ren, lemme VI.7.2 (iii)]. Therefore $\pi = \pi'$.

We now prove the proposition. As finite morphisms are closed, by (5.1.2), Lemma 5.1.1 and (5.1.3), we see that $\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)^\lambda$ is open in $\mathrm{Irr}_{M,\sigma}(G)^\lambda$ and moreover the restriction of λ to $\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)^\lambda$ is a finite bijective map $\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G) \to \mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)^\lambda$. Therefore, by [Stacks, Tag 04XV], it only remains to check that λ is unramified on $\mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)$.

Let $\chi_0 \in X(M)$ be such that $[M,\sigma\otimes\chi_0] \in \mathrm{Irr}^{\mathrm{gen}}_{M,\sigma}(G)$ and $(L,\tau) \in \mathrm{Cusp}(G)$ be in the cuspidal support of $\sigma_0 = \sigma\otimes\chi_0$. Let $W^0_{\sigma_0} \subset W(G,M)$ and $W^0_\tau \subset W(G,L)$ be the stabilizers of σ_0 and τ respectively. We have:

(5.1.4) The restriction map $\mathrm{Res} : X_{\mathrm{unr}}(M) \to X_{\mathrm{unr}}(L)$ descends to a regular morphism

$$X_{\mathrm{unr}}(M)/W^0_{\sigma_0} \to X_{\mathrm{unr}}(L)/W^0_\tau.$$

Indeed, let $w \in W^0_{\sigma_0}$ and take any lift $\widetilde{w} \in \mathrm{Norm}_G(M)$. The pair $(\widetilde{w}L\widetilde{w}^{-1}, \widetilde{w}\tau)$ is also in the cuspidal support of σ_0 and so, up to multiplying $\widetilde{w}$ by an element of M we have $\widetilde{w} \in \mathrm{Norm}_G(L)$ and $\widetilde{w}\tau \simeq \tau$. Then, denoting by w' the image of $\widetilde{w}$ in W^0_τ, we have $\mathrm{Res}(w\chi) = w'\,\mathrm{Res}(\chi)$ for every $\chi \in X_{\mathrm{unr}}(M)$ and (5.1.4) follows.

The maps $\chi \in X_{\mathrm{unr}}(M) \mapsto [M, \sigma_0 \otimes \chi] \in \mathrm{Irr}(G)$ and $\chi \in X_{\mathrm{unr}}(L) \mapsto [L, \tau \otimes \chi] \in \mathcal{B}(G)$ descend to regular morphisms $X_{\mathrm{unr}}(M)/W^0_{\sigma_0} \to \mathrm{Irr}(G)$ and $X_{\mathrm{unr}}(L)/W^0_\tau \to \mathcal{B}(G)$ which are local isomorphisms near 1 and such that the following diagram commutes

$$\begin{array}{ccc} X_{\mathrm{unr}}(M)/W^0_{\sigma_0} & \longrightarrow & X_{\mathrm{unr}}(L)/W^0_\tau \\ \downarrow & & \downarrow \\ \mathrm{Irr}(G) & \xrightarrow{\lambda} & \mathcal{B}(G). \end{array}$$

Consequently, it only remains to prove that $X_{\mathrm{unr}}(M)/W^0_{\sigma_0} \to X_{\mathrm{unr}}(L)/W^0_\tau$ is unramified at 1. Actually, we are going to show that this map is a closed immersion.

We may decompose M as

$$M = \mathrm{GL}_{n_1}(E) \times \ldots \times \mathrm{GL}_{n_k}(E),$$

where $n_1, \ldots, n_k$ are positive integers such that $n_1 + \ldots + n_k = n$, and we may accordingly decompose σ_0 as a tensor product

$$\sigma_0 = \nu_1 \boxtimes \ldots \boxtimes \nu_k$$

where, for each $1 \leqslant i \leqslant k$, ν_i is an essentially square-integrable representation of $\mathrm{GL}_{n_i}(E)$. Let Σ be the set of all isomorphism classes among $\nu_1, \ldots, \nu_k$ and for each $\nu \in \Sigma$ set

$$m(\nu) = \left|\{1 \leqslant i \leqslant k \mid \nu \simeq \nu_i\}\right|.$$

Regrouping the ν_i's according to their isomorphism classes, we get an isomorphism $X_{\mathrm{unr}}(M) \simeq \prod_{\nu\in\Sigma}(\mathbb{C}^\times)^{m(\nu)}$ which descends to an isomorphism

$$X_{\mathrm{unr}}(M)/W^0_{\sigma_0} \simeq \prod_{\nu\in\Sigma}(\mathbb{C}^\times)^{m(\nu)}/\mathfrak{S}_{m(\nu)}. \tag{5.1.5}$$

According to the classification by Bernstein and Zelevinsky of the essentially square-integrable representations of general linear groups [Ze, Theorem 9.3], for each $\nu \in \Sigma$ there is a *segment* Δ_ν, that is a set of the form $\Delta_\nu = \{\rho_\nu|\det|_E^{a_\nu}, \rho_\nu|\det|_E^{a_\nu+1}, \ldots, \rho_\nu|\det|_E^{b_\nu}\}$ where ρ_ν is (the isomorphism class of) a supercuspidal representation of some $\mathrm{GL}_{d_\nu}(E)$ and a_ν, b_ν are real numbers with $b_\nu - a_\nu \in \mathbb{N}$, such that ν is isomorphic to the unique irreducible quotient of

$$\rho_\nu|\det|_E^{a_\nu} \times \rho_\nu|\det|_E^{a_\nu+1} \times \ldots \times \rho_\nu|\det|_E^{b_\nu}.$$

Set $T = \bigcup_{\nu\in\Sigma} \Delta_\nu$ and for each $\rho \in T$ let

$$\ell(\rho) = \sum_{\nu\in\Sigma;\rho\in\Delta_\nu} m(\nu).$$

Then, up to the ordering, τ is isomorphic to $\boxtimes_{\rho\in T}\, \rho^{\boxtimes\ell(\rho)}$. Therefore, there is an isomorphism $X_{\mathrm{unr}}(L) \simeq \prod_{\rho\in T}(\mathbb{C}^\times)^{\ell(\rho)}$ that descends to an isomorphism

$$X_{\mathrm{unr}}(L)/W^0_\tau \simeq \prod_{\rho\in T}(\mathbb{C}^\times)^{\ell(\rho)}/\mathfrak{S}_{\ell(\rho)}$$

such that combined with the isomorphism (5.1.5), the map $X_{\mathrm{unr}}(M)/W^0_{\sigma_0} \to X_{\mathrm{unr}}(L)/W^0_\tau$ becomes

$$\prod_{\nu\in\Sigma}(\mathbb{C}^\times)^{m(\nu)}/\mathfrak{S}_{m(\nu)} \to \prod_{\rho\in T}(\mathbb{C}^\times)^{\ell(\rho)}/\mathfrak{S}_{\ell(\rho)}, \tag{5.1.6}$$

$$(z_\nu)_{\nu\in\Sigma} \mapsto (\bigtimes_{\nu\in\Sigma;\rho\in\Delta_\nu} z_\nu)_{\rho\in T}$$

where $\times_{\nu\in\Sigma;\rho\in\Delta_\nu} z_\nu$ denotes the "concatenation" of the z_ν with $\rho \in \Delta_\nu$ (whose image in $(\mathbb{C}^\times)^{m(\nu)}/\mathfrak{S}_{m(\nu)}$ does not depend on the ordering).

Therefore, it only remains to show that (5.1.6) is a closed immersion. By Zelevinsky's classification of generic representations of $\mathrm{GL}_n(E)$ [Ze, Theorem 9.7], for every $\nu, \nu' \in \Sigma$, if $\Delta_\nu \cup \Delta_{\nu'}$ is again a segment then $\Delta_\nu \subseteq \Delta_{\nu'}$ or $\Delta_{\nu'} \subseteq \Delta_\nu$. In particular, it follows that for $\nu \in \Sigma$ the union

$$\bigcup_{\nu'\in\Sigma;\Delta_{\nu'}\subsetneq\Delta_\nu} \Delta_{\nu'}$$

is strictly smaller than Δ_ν. Let $\rho_\nu \in \Delta_\nu$ be in the complement of this subset. Then, for every $\nu, \nu' \in \Sigma$, $\rho_\nu \in \Delta_{\nu'}$ implies $\Delta_\nu \subseteq \Delta_{\nu'}$. Moreover, for each $\nu \in \Sigma$ the map

$$\prod_{\nu'\in\Sigma;\Delta_\nu\subseteq\Delta_{\nu'}}(\mathbb{C}^\times)^{m(\nu')}/\mathfrak{S}_{m(\nu')} \to \prod_{\nu'\in\Sigma;\Delta_\nu\subsetneq\Delta_{\nu'}}(\mathbb{C}^\times)^{m(\nu')}/\mathfrak{S}_{m(\nu')} \times (\mathbb{C}^\times)^{\ell(\rho_\nu)}/\mathfrak{S}_{\ell(\rho_\nu)},$$

$$(z_{\nu'})_{\Delta_\nu\subseteq\Delta_{\nu'}} \mapsto \left((z_{\nu'})_{\Delta_\nu\subsetneq\Delta_{\nu'}}, \bigtimes_{\Delta_\nu\subseteq\Delta_{\nu'}} z_{\nu'}\right)$$

is a closed immersion e.g. because it admits a left inverse. Therefore, that the map (5.1.6) is a closed immersion follows from the next lemma.

Lemma 5.1.2. *Let I, J be finite sets and $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$ be families of algebraic varieties over $\mathbb{C}$. Let $f : \prod_{i\in I} X_i \to \prod_{j\in J} Y_j$ be a regular morphism. Let also $i \mapsto j_i \in J$ be an injective map and $\preceq$ be an order on I such that the following condition is satisfied:*

(5.1.7) For each $i_0 \in I$, the composition of f with the projection $\prod_{j\in J} Y_j \to Y_{j_{i_0}}$ factorizes through the projection $\prod_{i\in I} X_i \to \prod_{i_0\preceq i} X_i$ and the product

$$\prod_{i_0\preceq i} X_i \to \prod_{i_0\prec i} X_i \times Y_{j_{i_0}}$$

of the induced morphism $\prod_{i_0\preceq i} X_i \to Y_{j_{i_0}}$ with the projection $\prod_{i_0\preceq i} X_i \to \prod_{i_0\prec i} X_i$ is a closed immersion.

Then, f is a closed immersion.

Proof. It is easy to see that the condition (5.1.7) is still satisfied for any order finer than $\preceq$. In particular, we may assume that $\preceq$ is a total order. Then, we can write $I = \{i_1, \dots, i_d\}$ such that $i_k \preceq i_l$ if and only if $k \leqslant l$. By descending induction on $1 \leqslant k \leqslant d$, (5.1.7) implies that the morphism $\prod_{l\geqslant k} X_{i_l} \to \prod_{l\geqslant k} Y_{j_{i_l}}$ is a closed immersion. In particular, for $k = 1$ we get that the map $\prod_{i\in I} X_i \to \prod_{i\in I} Y_{j_i}$ is a closed immersion from which it follows that so does f. □

□

Of course, all the above constructions and results apply similarly to G'. Let $\mathrm{BC} : \mathrm{Irr}(G') \to \mathrm{Irr}(G)$ be the quadratic base-change map constructed by Arthur and Clozel [AC]. By [AC, Lemma 6.10], BC restricts to a map $\mathcal{B}(G') \to \mathcal{B}(G)$. Moreover, by [AC, Lemma 6.12], the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Irr}(G') & \xrightarrow{\mathrm{BC}} & \mathrm{Irr}(G) \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} \\ \mathcal{B}(G') & \xrightarrow{\mathrm{BC}} & \mathcal{B}(G). \end{array} \tag{5.1.8}$$

Lemma 5.1.3. *(i) For each connected component $\Omega \subset \mathrm{Irr}(G)$, there exists $(M, \sigma) \in \mathrm{Sqr}(G')$ such that $\mathrm{BC}^{-1}(\Omega) \subseteq \mathrm{Irr}_{M,\sigma}(G')$. Moreover, for every connected components $\Omega, \Omega' \subset \mathrm{Irr}(G')$ we either have $\mathrm{BC}(\Omega) = \mathrm{BC}(\Omega')$ or that $\mathrm{BC}(\Omega)$ and $\mathrm{BC}(\Omega')$ lie in distinct connected components of $\mathrm{Irr}(G)$.*

(ii) BC *is a finite regular map which is flat over its image.*

Proof. (i) This follows rather easily from the description of the fibers of the base-change map [AC, Proposition 6.7] and its compatibility with parabolic induction.

(ii) Let $(M, \sigma) \in \mathrm{Sqr}(G')$. By the compatibility between base-change and parabolic induction, there exist $(L, \tau) \in \mathrm{Sqr}(G)$ and a closed embedding $X(M) \to X(L)$ such that the following diagram commutes

$$\begin{array}{ccc} X(M) & \longrightarrow & X(L) \\ \downarrow & & \downarrow \\ \mathrm{Irr}_{M,\sigma}(G') & \xrightarrow{\mathrm{BC}} & \mathrm{Irr}(G) \end{array}$$

where the two vertical maps are given by $\chi \mapsto [M, \sigma \otimes \chi]$ and $\chi \mapsto [L, \tau \otimes \chi]$ respectively. Since these two arrows are finite morphisms and the first one is a quotient map by a finite group of automorphisms, it follows that BC is both regular and finite. To show the flatness of BC over its image, we will use the "miracle flatness theorem" [Hart, Exercise III.10.9] which implies that a finite surjective morphism between smooth connected varieties is automatically flat. Indeed, by [AC, Theorem 6.2(b)] the image of BC is the set of fixed points of the automorphism c of $\mathrm{Irr}(G)$ induced from the non-trivial Galois automorphism of E/F. This automorphism is easily seen to be algebraic, hence by [Iv, Proposition 1.3] the image of BC is smooth. Thus, by the second part of (i) the image by BC of a connected component of $\mathrm{Irr}(G')$ is also smooth (being the intersection of the full image with a component of $\mathrm{Irr}(G)$). Since the source is also smooth we can conclude by [Hart, Exercise III.10.9].

□

5.2 The result

For $V \in \mathcal{V}$ and $\pi \in \mathrm{Irr}(G)$ we set

$$m_V(\pi) = \dim \mathrm{Hom}_{U(V)}(\pi, \mathbb{C})$$

where $\mathrm{Hom}_{U(V)}(\pi, \mathbb{C})$ denotes the space of $U(V)$-invariant functionals on (the space of) π. We define the *degree* of the base-change map to be the function

$$\deg \mathrm{BC} : \mathrm{Irr}(G) \to \mathbb{N}$$

$$\deg \mathrm{BC}(\pi) = \dim \mathbb{C}[\mathrm{Irr}(G')]/\mathfrak{m}_\pi \, \mathbb{C}[\mathrm{Irr}(G')]$$

where $\mathfrak{m}_\pi \subset \mathbb{C}[\mathrm{Irr}(G)]$ denotes the maximal ideal corresponding to π. By Lemma 5.1.3(ii), $\deg \mathrm{BC}$ is locally constant on the image of the base-change map. Thus, to compute it we just need to consider the case where π is in general position in the image in which case we simply have $\deg \mathrm{BC}(\pi) = |\mathrm{BC}^{-1}(\pi)|$. By the description of the image and fibers of BC and its compatibility with parabolic induction (see [AC, Theorem 6.2, Proposition 6.7]), we obtain the following explicit description: if $\pi \in \mathrm{Irr}(G)$ is the Langlands quotient of an induced representation of the form

$$\sigma_1 \times \ldots \times \sigma_k$$

where for each $1 \leqslant i \leqslant k$, $\sigma_i \in \Pi_{\mathrm{ess},2}(\mathrm{GL}_{n_i}(E))$ for some positive integer n_i, then we have

$$\deg \mathrm{BC}(\pi) = \begin{cases} 2^{|\{1 \leqslant i \leqslant k | \sigma_i^c \simeq \sigma_i\}|} & \text{if } \pi \simeq \pi^c, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.1}$$

The following result is proved by Feigon-Lapid-Offen in [FLO, Theorem 0.2].

Theorem 5.2.1 (Feigon-Lapid-Offen)**.** *For every $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$ and $V \in \mathcal{V}$ we have*

$$m_V(\pi) \geqslant \begin{cases} \lceil \frac{\deg \mathrm{BC}(\pi)}{2} \rceil & \text{if } U(V) \text{ is quasi-split,} \\ \lfloor \frac{\deg \mathrm{BC}(\pi)}{2} \rfloor & \text{otherwise.} \end{cases}$$

Moreover, equality holds whenever BC *is unramified on the fiber of π.*

The goal of this chapter is to refine this result and prove the following.

Theorem 5.2.2. *For every $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$ and $V \in \mathcal{V}$ we have*

$$m_V(\pi) = \begin{cases} \lceil \frac{\deg \mathrm{BC}(\pi)}{2} \rceil & \text{if } V \text{ is quasi-split,} \\ \lfloor \frac{\deg \mathrm{BC}(\pi)}{2} \rfloor & \text{otherwise.} \end{cases}$$

5.3 First step: Reduction to the tempered case

For $\pi \in \mathrm{Irr}(G)$ we set

$$m(\pi) := \sum_{V \in \mathcal{V}} m_V(\pi).$$

Note that, since we are in the p-adic case, the above sum contains only two terms. Moreover, if n is odd every $V \in \mathcal{V}$ is quasi-split whereas, if n is even one of the Hermitian spaces in $\mathcal{V}$ is quasi-split and the other is not. Using (5.2.1), we readily check that if n is odd then $\deg \mathrm{BC}(\pi)$ is always even. Therefore, by Theorem 5.2.1, Theorem 5.2.2 is equivalent to

$$m(\pi) \leqslant \deg \mathrm{BC}(\pi) \tag{5.3.1}$$

for every $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$.

Let $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$. It can be written as

$$\pi = \tau_1|\det|^{\lambda_1} \times \ldots \times \tau_t|\det|^{\lambda_t}$$

where, for each $1 \leqslant i \leqslant t$, $\tau_i \in \mathrm{Temp}(\mathrm{GL}_{n_i}(E))$ for some positive integer n_i and $\lambda_1, \ldots, \lambda_t$ are real numbers satisfying $\lambda_1 > \lambda_2 > \ldots > \lambda_t$. For every $1 \leqslant i \leqslant t$, we define $m(\tau_i)$ and $\deg \mathrm{BC}(\tau_i)$ similarly to $m(\pi)$ and $\deg \mathrm{BC}(\pi)$ (just replacing n by n_i). The proposition below will allow to reduce the proof of Theorem 5.2.2 to the case where π is tempered.

Proposition 5.3.1. *We have*

(i) $\deg \mathrm{BC}(\pi) = \deg \mathrm{BC}(\tau_1) \ldots \deg \mathrm{BC}(\tau_t)$;

(ii) $m(\pi) = m(\tau_1) \ldots m(\tau_t)$.

Proof. (i) can be inferred directly from the description (5.2.1) of $\deg \mathrm{BC}(\pi)$. The proof of (ii) essentially follows from the analysis performed in [FLO, §6] but is not explicitely stated there. Therefore, we shall now explain carefully this deduction. Let

$$M = \mathrm{GL}_{n_1}(E) \times \ldots \times \mathrm{GL}_{n_t}(E)$$

be the standard Levi subgroup of G from which π is induced as a standard module and

$$\tau = \tau_1|\det|^{\lambda_1} \boxtimes \ldots \boxtimes \tau_t|\det|^{\lambda_t} \in \mathrm{Irr}(M)$$

so that $\pi \simeq I_P^G(\tau)$ where P is the standard parabolic subgroup with Levi M. By [FLO, Lemma 6.7], we just need to check that the "unitary periods of π are supported on open P-orbits" with the terminology of *loc. cit.* (see [FLO, Definition 6.6]). Here, the P-orbits refer to the action of P on X. Given the explicit description of P-orbits from [FLO, §6.1] and of the "unitary periods" supported on each of these P-orbit from [FLO, Lemma 6.4], we just need to show the following: if $n_i = n_{i,t} + \ldots + n_{i,1}$ are partitions of the n_i's satisfying $n_{i,j} = n_{j,i}$ for every $1 \leqslant i, j \leqslant t$ which are not all trivial (i.e. there exist $1 \leqslant i \neq j \leqslant t$ with $n_{i,j} \neq 0$), P_i stands for the standard parabolic subgroup of $\mathrm{GL}_{n_i}(E)$ associated to this partition of n_i with standard Levi

$$M_i = \mathrm{GL}_{n_{i,t}}(E) \times \ldots \times \mathrm{GL}_{n_{i,1}}(E) \tag{5.3.2}$$

and $J_{P_i}(\tau_i|\det|^{\lambda_i})$ denotes the normalized Jacquet module with respect to this parabolic, there is no irreducible subquotients

$$\rho_i = \rho_{i,t} \boxtimes \ldots \boxtimes \rho_{i,1} \in \mathrm{Irr}(M_i)$$

of the $J_{P_i}(\tau_i)$, $1 \leqslant i \leqslant t$, such that $\rho_{ij} \simeq \rho_{ji}^c$ for every $1 \leqslant i \neq j \leqslant t$. Assume, by way of contradiction, that there exist such partitions and irreducible subquotients of the Jacquet modules. Let $1 \leqslant i \leqslant t$ be the smallest index such that the partition of n_i is non-trivial and $1 \leqslant j \leqslant t$ be the largest index such that $n_{ij} \neq 0$. Note that $j > i$ as the partition of n_i is non-trivial and $n_{ik} = n_{ki} = 0$ for every $k < i$ by minimality of i. Let μ be the real exponent of the central character $\omega_{\rho_{ij}} = \omega_{\rho_{ji}}^c$. As $\mathrm{GL}_{n_{ij}}(E)$ is the first non-trivial group in the product decomposition (5.3.2) of M_i, by Casselman's criterion of temperedness [Wald1, Proposition III.2.2] we have $\lambda_i \leqslant \frac{\mu}{n_{ij}}$. Similarly, since $n_{jk} = n_{kj} = 0$ for $k < i$ (again by minimality of i), by Casselman's criterion of temperedness we have $\lambda_j \geqslant \frac{\mu}{n_{ji}} = \frac{\mu}{n_{ij}}$. But $j > i$ implies that $\lambda_i > \lambda_j$ and therefore we have a contradiction. □

5.4 Second step: relation between multiplicities and FLO functionals

For $\pi \in \mathrm{Irr}(G)$, we let $C_c^\infty(X)_\pi$ be the $\pi^\vee$-isotypic quotient of $C_c^\infty(X)$ i.e. the maximal quotient which is G-isomorphic to a direct sum of copies of $\pi^\vee$. Note that by Frobenius reciprocity, since $X = \bigsqcup_{V\in\mathcal{V}} U(V)\backslash G$ (see Section 3.2), we have

$$C_c^\infty(X)_\pi \simeq (\pi^\vee)^{\oplus m(\pi)}, \quad \pi \in \mathrm{Irr}(G).$$

Therefore, the following lemma is just a consequence of the unicity of Whittaker models.

Lemma 5.4.1. *For $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$, we have*

$$m(\pi) = \dim \mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n).$$

Recall the FLO relative character J_σ associated to each $\sigma \in \mathrm{Temp}(G')$ introduced in Section 3.4. Note that $J_\sigma \in \mathrm{Hom}_N(C_c^\infty(X), \psi_n)$ for every $\sigma \in \mathrm{Temp}(G')$. Let $\pi \in \mathrm{Temp}(G)$, $\Omega_\pi \subseteq \mathrm{Irr}(G)$ be the connected component of π and $\Omega_\pi^t = \Omega_\pi \cap \mathrm{Temp}(G)$. We equip $\mathrm{Hom}_N(C_c^\infty(X), \psi_n)$ with the weak topology (that is the topology of pointwise convergence). Set

$$\mathcal{J}(\pi) := \overline{\langle J_\sigma \mid \sigma \in \mathrm{BC}^{-1}(\Omega_\pi^t)\rangle}$$

for the closure of the subspace of $\mathrm{Hom}_N(C_c^\infty(X), \psi_n)$ generated by the FLO relative characters J_σ with $\sigma \in \mathrm{BC}^{-1}(\Omega_\pi^t)$. The main result of this section is the following proposition.

Proposition 5.4.1. *We have*

$$\mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n) \subseteq \mathcal{J}(\pi).$$

Proof. Let $J \in \mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n)$. We need to show that for every $\varphi \in C_c^\infty(X)$ such that $J_\sigma(\varphi) = 0$ for all $\sigma \in \mathrm{BC}^{-1}(\Omega_\pi^t)$ we have $J(\varphi) = 0$. By Lemma 3.6.1, J and the relative characters J_σ, for $\sigma \in \mathrm{Temp}(G')$, extend continuously to $\mathcal{C}(X)$ and we will prove that the previous property holds more generally for $\varphi \in \mathcal{C}(X)$.

The application $f \in \mathcal{C}(G) \mapsto (\pi' \in \mathrm{Temp}(G) \mapsto \pi'(f))$ is injective and its image was described by Harish-Chandra [Wald1, Théorèmes VII.2.5 et VIII.1.1]. A consequence of this description is that there exists a projector $f \in \mathcal{C}(G) \mapsto e_{\Omega_\pi^t} * f \in \mathcal{C}(G)$ which is equivariant with respect to both left and right convolutions such that for every $f \in \mathcal{C}(G)$ and $\pi' \in \mathrm{Temp}(G)$ we have [5]

$$\pi'(e_{\Omega_\pi^t} * f) = \begin{cases} \pi'(f) & \text{if } \pi' \in \Omega_\pi^t, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.1}$$

By Proposition 3.5.1(ii), we can define a similar projector $\varphi \in \mathcal{C}(X) \mapsto e_{\Omega_\pi^t} * \varphi \in \mathcal{C}(X)$: for $\varphi \in \mathcal{C}(X)$, choose any $f \in C_c^\infty(G)$ such that $\varphi = R(f)\varphi$ (e.g. $\mathrm{vol}(K_0)^{-1}\mathbf{1}_{K_0}$ for a sufficiently small compact-open subgroup K_0) and set $e_{\Omega_\pi^t} * \varphi = R(e_{\Omega_\pi^t} * f)\varphi$: the fact that $e_{\Omega_\pi^t} * .$ is equivariant with respect to right convolution ensures that the result does not depend on the choice of f.

Let $\pi' \in \mathrm{Temp}(G)$ and $T_{\pi'} : \mathcal{C}(X) \to \pi'$ be a continuous G-equivariant linear map where *continuous* here means that for every compact-open subgroup K_0 of G, the restriction $\mathcal{C}(X)^J \to$

[5] The existence of such a projector can also be deduced from the description of the *tempered Bernstein center* by Schneider and Zink [SZ]

$(\pi^\vee)^{K_0}$ is continuous. Then, $T_{\pi'}(R(f)\varphi) = \pi'(f)T_{\pi'}(\varphi)$ for every $(f,\varphi) \in \mathcal{C}(G)\times\mathcal{C}(X)$ and therefore, by (5.4.1) and the definition of $e_{\Omega_\pi^t} * \varphi$, it follows that:

$$T_{\pi'}(e_{\Omega_\pi^t} * \varphi) = \begin{cases} T_{\pi'}(\varphi) & \text{if } \pi' \in \Omega_\pi^t, \\ 0 & \text{otherwise} \end{cases} \tag{5.4.2}$$

for all $\varphi \in \mathcal{C}(X)$.

By Frobenius reciprocity, J and J_σ, for $\sigma \in \mathrm{Temp}(G')$, induce continuous G-equivariant linear maps $\mathcal{C}(X) \to \mathcal{W}(\pi^\vee, \psi_n)$ and $\mathcal{C}(X) \to \mathcal{W}(\mathrm{BC}(\sigma)^\vee, \psi_n)$ respectively. Thus, by the above, we have

$$J(e_{\Omega_\pi^t}\varphi) = J(\varphi)$$

and

$$J_\sigma(e_{\Omega_\pi^t}\varphi) = \begin{cases} J_\sigma(\varphi) & \text{if } \sigma \in \mathrm{BC}^{-1}(\Omega_\pi^t), \\ 0 & \text{otherwise,} \end{cases}$$

for every $\varphi \in \mathcal{C}(X)$ and $\sigma \in \mathrm{Temp}(G')$.

As a consequence, up to replacing φ by $e_{\Omega_\pi^t}\varphi$, we only need to show that:

(5.4.3) For every $\varphi \in \mathcal{C}(X)$ such that $J_\sigma(\varphi) = 0$ for every $\sigma \in \mathrm{Temp}(G')$, we have $J(\varphi) = 0$.

We henceforth fix a function $\varphi \in \mathcal{C}(X)$ satisfying $J_\sigma(\varphi) = 0$ for every $\sigma \in \mathrm{Temp}(G')$.

By Lemma 3.6.1, there exists $F \in \mathcal{C}^w(X)$ such that

$$J(\varphi) = \int_N^* \langle R(u)\varphi, F\rangle_X \psi_n(u)^{-1} du.$$

Let $(X_k)_{k\geqslant 1}$ be an increasing and exhausting sequence of K-invariant compact subsets of X and set $F_k = \mathbf{1}_{X_k}F$ for every $k \geqslant 1$. We can show, by the same argument as for (3.5.2), that the sequence $(F_k)_{k\geqslant 1}$ converges to F in $\mathcal{C}^w(X)$. Hence, by Proposition 3.5.1(i), we have

$$J(\varphi) = \lim_{k\to\infty} \int_N^* \langle R(u)\varphi, F_k\rangle_X \psi_n(u)^{-1} du = \lim_{k\to\infty} J(\varphi, F_k)$$

with the notation of Section 4.1. Therefore, by Theorem 4.2.3 and the hypothesis made on φ, we have

$$J(\varphi) = \lim_{k\to\infty} \int_{\mathrm{Temp}(G')} J_\sigma(\varphi)\overline{J_\sigma(F_k)} d\mu_{G'}(\sigma) = 0.$$

This shows (5.4.3) and ends the proof of the proposition. □

5.5 End of the proof of Theorem 5.2.2

For convenience, here we normalize the action of the Bernstein center $\mathcal{Z}(G)$ on $C_c^\infty(X)$ such that $z \in \mathcal{Z}(G)$ acts on the coinvariant space $C_c^\infty(X)_\pi$ by the scalar $z(\lambda(\pi))$ for every $\pi \in \mathrm{Irr}(G)$.

Let $\pi \in \mathrm{Temp}(G)$ and $\Omega_\pi \subseteq \mathrm{Irr}(G)$ be the connected component of π. Set $\Omega_\pi^t = \Omega_\pi \cap \mathrm{Temp}(G)$, $\Omega'_\pi = \mathrm{BC}^{-1}(\Omega_\pi)$, $\Omega_\pi^\lambda = \lambda(\Omega_\pi) \subseteq \mathcal{B}(G)$ and $(\Omega'_\pi)^\lambda = \lambda(\Omega'_\pi) \subseteq \mathcal{B}(G')$. Let V be the space of functions of the form

$$\sigma \in \mathrm{BC}^{-1}(\Omega_\pi^t) \mapsto J_\sigma(\varphi)$$

where $\varphi \in C_c^\infty(X)$. Then, V is a quotient of $C_c^\infty(X)$ by a $\mathcal{Z}(G)$-submodule. Moreover, by the definition of FLO functionals (Theorem 3.4.1) and the existence of the Jacquet-Ye transfer (Theorem 3.3.1), V is also the space of functions of the form

$$\sigma \in \mathrm{BC}^{-1}(\Omega_\pi^t) \mapsto I_\sigma(f')$$

where $f' \in C_c^\infty(G')$. Note that, by Lemma 5.1.1, $(\Omega'_\pi)^\lambda$ is Zariski closed in $\mathcal{B}(G')$. Therefore, by Theorem 2.4.1, V is the space of restrictions to $\mathrm{BC}^{-1}(\Omega_\pi^t)$ of the algebra of regular functions $\mathbb{C}[(\Omega'_\pi)^\lambda]$ on $(\Omega'_\pi)^\lambda$ through the map λ. As $\mathrm{BC}^{-1}(\Omega_\pi^t) = \Omega'_\pi \cap \mathrm{Temp}(G')$ is Zariski dense in Ω'_π by (5.1.1), this gives an isomorphism

$$V \simeq \mathbb{C}[(\Omega'_\pi)^\lambda] \tag{5.5.1}$$

through which the action of $\mathcal{Z}(G)$ is given by the pullback $\mathrm{BC}^* : \mathcal{Z}(G) = \mathbb{C}[\mathcal{B}(G)] \to \mathbb{C}[\mathcal{B}(G')]$.

Let $\mathfrak{m}_{\lambda(\pi)} \subseteq \mathcal{Z}(G)$ be the maximal ideal corresponding to $\lambda(\pi) \in \mathcal{B}(G)$. Then, by Proposition 5.4.1, each element of $\mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n)$ factorizes through the quotient $C_c^\infty(X) \to V$ and therefore, by the theory of the Bernstein center and the isomorphism (5.5.1), also through

$$V/\mathfrak{m}_{\lambda(\pi)}V \simeq \mathbb{C}[(\Omega'_\pi)^\lambda]/\mathfrak{m}_{\lambda(\pi)}\,\mathbb{C}[(\Omega'_\pi)^\lambda].$$

Consequently, by Lemma 5.4.1, we have

$$m(\pi) \leqslant \dim(\mathbb{C}[(\Omega'_\pi)^\lambda]/\mathfrak{m}_{\lambda(\pi)}\,\mathbb{C}[(\Omega'_\pi)^\lambda]). \tag{5.5.2}$$

Consider the following commutative diagram (coming from restriction of (5.1.8))

$$\begin{array}{ccc} \Omega'_\pi & \xrightarrow{\mathrm{BC}} & \Omega_\pi \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} \\ (\Omega'_\pi)^\lambda & \xrightarrow{\mathrm{BC}} & \Omega_\pi^\lambda. \end{array}$$

By Proposition 5.1.1 and Lemma 5.1.3 (i), the two vertical arrows are isomorphisms when restricted to suitable Zariski open neighborhood of $\lambda(\pi)$ and $\mathrm{BC}^{-1}(\lambda(\pi)) \cap (\Omega'_\pi)^\lambda = \lambda(\mathrm{BC}^{-1}(\pi))$. Therefore,

$$\mathbb{C}[(\Omega'_\pi)^\lambda]/\mathfrak{m}_{\lambda(\pi)}\,\mathbb{C}[(\Omega'_\pi)^\lambda] \simeq \mathbb{C}[\Omega'_\pi]/\mathfrak{m}_\pi\,\mathbb{C}[\Omega'_\pi].$$

Combining this with (5.5.2), we obtain

$$m(\pi) \leqslant \dim(\mathbb{C}[\Omega'_\pi]/\mathfrak{m}_\pi\,\mathbb{C}[(\Omega'_\pi)]) = \deg \mathrm{BC}(\pi).$$

We have just proven that (5.3.1) holds for every $\pi \in \mathrm{Temp}(G)$ and therefore, by Proposition 5.3.1, also for every $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$. This ends the proof of Theorem 5.2.2.

6 A Plancherel formula for X and relation to factorization of global periods

In this chapter, we keep the notation introduced in the Chapters 3 and 4 and we don't assume anymore that F is a p-adic field (i.e. we allow $F = \mathbb{R}$). The goal of this part is to establish

an explicit Plancherel formula for X. More precisely, we will prove that the L^2-inner product $\langle .,. \rangle_X$ on X decomposes as an integral of certain G-invariant semi-positive Hermitian forms $\langle .,. \rangle_{X,\sigma}$ that are indexed by $\sigma \in \mathrm{Temp}(G')$ and "living on $\mathrm{BC}(\sigma)$" in the sense that they factorize through the $\mathrm{BC}(\sigma)^\vee$-coinvariant space $C_c^\infty(X)_{\mathrm{BC}(\sigma)}$ (see Theorem 6.1.1). The Hermitian forms $\langle .,. \rangle_{X,\sigma}$ are defined through the FLO functionals α^σ of Section 3.4 and moreover the underlying spectral measure is the Plancherel measure $d\mu_{G'}$ of G'. According to Bernstein [Ber3], such a decomposition induces an isomorphism of unitary representations

$$L^2(X) \simeq \int_{\mathrm{Temp}(G')}^{\oplus} \mathrm{BC}(\sigma) d\mu_{G'}(\sigma) \tag{6.0.1}$$

and it is actually also equivalent to a certain Plancherel inversion formula expressing any test function $\varphi \in C_c^\infty(X)$ as an integral of "generalized eigenfunctions" φ_σ (see Theorem 6.1.2). The isomorphism (6.0.1) can be seen as a particular case of a general conjecture of Sakellaridis-Venkatesh on the L^2-spectrum of spherical varieties [SV, Conjecture 16.2.2]. More precisely, in [SV] a dual group is associated to any spherical variety[6] which for the case at hand is the group $\check{G}_X = \mathrm{GL}_n(\mathbb{C}) = \check{G}'$ coming with a natural "distinguished morphism" $\check{G}_X \to \check{G}$ to the dual group of G. Here, this morphism extends naturally to the base-change map between L-groups ${}^L G' \to {}^L G$ and [SV, Conjecture 16.2.2], suitably interpreted, predicts exactly a decomposition of the G-unitary representation $L^2(X)$ of the form (6.0.1). A concrete consequence of this Plancherel decomposition is a description of the so-called *relative discrete series* of X (see Corollary 6.1.1).

The precise statement of the Plancherel formula is given in the next section. The proof, which is relatively short and builds upon the local Jacquet-Ye trace formula of Chapter 4 together with the Fourier inversion formula (3.1.1), occupies Section 6.2. In the final Section 6.3, we revisit the work of Feigon-Lapid-Offen [FLO] on the factorization of unitary periods (generalizing previous work of Jacquet [Jac01]) to make the relation to the local Plancherel decomposition we have obtained more transparent. That there is such a relation is of course not surprising, since the FLO functionals we use to compute the Plancherel decomposition are also the main local input in *loc. cit.* to the global period factorization, but once properly reformulated we find this connection to be in striking accordance with general speculations of Sakellaridis-Venkatesh on the factorization of global spherical periods [SV, §17] which is why we have included such a discussion here.

6.1 The statement

Let $\sigma \in \mathrm{Temp}(G')$. Recall from Section 3.4 that to σ is associated a functional $\alpha^\sigma \in \mathcal{E}_G(X, \mathcal{W}(\pi, \psi_n)^*)$ where $\pi = \mathrm{BC}(\sigma)$. For $\varphi \in C_c^\infty(X)$, we construct as in Section 3.4 a smooth functional $\varphi \cdot \alpha^\sigma \in \mathcal{W}(\pi, \psi_n)^\vee$ that we identify with an element of $\mathcal{W}(\pi^\vee, \psi_n^{-1})$ through the invariant inner product $\langle .,. \rangle_{\mathrm{Whitt}}$ (3.1.2). For every $\varphi_1, \varphi_2 \in C_c^\infty(X)$, we set

$$\langle \varphi_1, \varphi_2 \rangle_{X,\sigma} := \langle \varphi_1 \cdot \alpha^\sigma, \varphi_2 \cdot \alpha^\sigma \rangle_{\mathrm{Whitt}}.$$

Obviously, $\langle .,. \rangle_{X,\sigma}$ is a G-invariant positive semi-definite Hermitian form that factorizes through the $\pi^\vee$-coinvariants $C_c^\infty(X) \to C_c^\infty(X)_\pi$.

Finally, recall that $\langle .,. \rangle_X$ stands for the L^2-scalar product on X and $d\mu_{G'}$ denotes the Plancherel measure on G'.

[6] This construction actually only works well under a suitable extra technical condition (namely that the spherical variety has no root of 'type N') for which we refer the reader to *loc. cit.*

Theorem 6.1.1. *For every $\varphi_1, \varphi_2 \in C_c^\infty(X)$, we have*

$$\langle \varphi_1, \varphi_2 \rangle_X = \int_{\mathrm{Temp}(G')} \langle \varphi_1, \varphi_2 \rangle_{X,\sigma} d\mu_{G'}(\sigma)$$

where the right hand side is absolutely convergent.

Note that the action of the center $Z(G) = E^\times$ on X factorizes through the quotient $E^\times \to N(E^\times)$. Let $\chi : N(E^\times) \to \mathbb{S}^1$ be a unitary character and $L^2(X, \chi)$ be the space of functions $f : X \to \mathbb{C}$ satisfying $f(xz) = \chi(z)f(x)$ for every $(x, z) \in X \times Z(G)$ and which are square-integrable on $X/Z(G)$. Let $L^2(X, \chi)_{\mathrm{disc}}$ the subspace generated by all the irreducible smooth submodules of $L^2(X, \chi)$ (the so-called *relative discrete series*) and $\Pi_{2,\chi}(G')$ be the subset of representations $\sigma \in \Pi_2(G')$ whose central character restricted to $N(E^\times) \subset Z(G')$ is equal to χ. The above decomposition of $L^2(X)$ admits the following concrete representation-theoretic corollary.

Corollary 6.1.1. *There is a G-isomorphism*

$$L^2(X, \chi)_{\mathrm{disc}} \simeq \bigoplus_{\sigma \in \Pi_{2,\chi}(G')} \mathrm{BC}(\sigma).$$

Let $x \in X$. The value of α^σ at x is a G_x-invariant functional $\alpha_x^\sigma : \mathcal{W}(\pi, \psi_n) \to \mathbb{C}$. Identifying its complex conjugate $\overline{\alpha_x^\sigma}$ with a functional on $\overline{\mathcal{W}(\pi, \psi_n)} = \mathcal{W}(\pi^\vee, \psi_n^{-1})$, for every $\varphi \in C_c^\infty(X)$ we set

$$\varphi_\sigma(x) = \langle \varphi \cdot \alpha^\sigma, \overline{\alpha_x^\sigma} \rangle.$$

Note that the function φ_σ generates (by right-translation) a representation isomorphic to $\pi^\vee = \mathrm{BC}(\sigma)^\vee$. In this sense, it is a "generalized eigenfunction". The following explicit "Plancherel inversion formula" follows from Theorem 6.1.1 by specializing it to the case where $\varphi_1 = \varphi$ and $\varphi_2 = \mathbf{1}_{xK_0}$ for K_0 a sufficiently small compact-open subgroup of G in the p-adic case. In the Archimedean case, we can argue in a similar way using the Dixmier-Malliavin theorem (details are left to the reader).

Theorem 6.1.2. *For every $\varphi \in C_c^\infty(X)$ and $x \in X$, we have*

$$\varphi(x) = \int_{\mathrm{Temp}(G')} \varphi_\sigma(x) d\mu_{G'}(\sigma)$$

where the right hand side is absolutely convergent.

6.2 Proof of Theorem 6.1.1

Note that, for every $\sigma \in \mathrm{Temp}(G')$ and $\varphi_1, \varphi_2 \in C_c^\infty(X)$ and since the scalar product $\langle ., . \rangle_{X,\sigma}$ is G-invariant and factorizes through the $\pi^\vee = \mathrm{BC}(\sigma^\vee)$-coinvariants $C_c^\infty(X)_\pi$, the function $g \in G \mapsto \langle R(g)\varphi_1, \varphi_2 \rangle_{X,\sigma}$ is a finite sum of matrix coefficients of $\pi^\vee$ hence belongs to $\mathcal{C}^w(G)$. In particular, we can apply to it the regularized integral $\int_N^* .\psi_n(u)^{-1} du$ of Section 2.1.

Lemma 6.2.1. *For every $\sigma \in \mathrm{Temp}(G')$ and $\varphi_1, \varphi_2 \in C_c^\infty(X)$, we have*

$$J_\sigma(\varphi_1)\overline{J_\sigma(\varphi_2)} = \int_N^* \langle R(u)\varphi_1, \varphi_2 \rangle_{X,\sigma} \psi_n(u)^{-1} du.$$

Proof. By (3.1.3) and the definition of $\langle .,. \rangle_{X,\sigma}$ and J_σ, we have

$$\begin{aligned}\int_N^* \langle R(u)\varphi_1, \varphi_2\rangle_{X,\sigma}\psi_n(u)^{-1}du &= \int_N^* \langle R(u^{-1})(\varphi_1\cdot\alpha^\sigma), \varphi_2\cdot\alpha^\sigma\rangle_{\mathrm{Whitt}}\psi_n(u)^{-1}du \\ &= \langle \varphi_1\cdot\alpha^\sigma, \lambda_1^\vee\rangle\overline{\langle \varphi_2\cdot\alpha^\sigma, \lambda_1^\vee\rangle} = J_\sigma(\varphi_1)\overline{J_\sigma(\varphi_2)}\end{aligned}$$

where we recall that $\lambda_1^\vee$ stands for the functional $W^\vee \in \mathcal{W}(\pi^\vee, \psi_n^{-1}) \mapsto W^\vee(1)$. □

We can now finish the proof of Theorem 6.1.1. Since both $\langle .,. \rangle_X$ and $\langle .,. \rangle_{X,\sigma}$, $\sigma \in \mathrm{Temp}(G')$, are positive semi-definite Hermitian forms, by Cauchy-Schwarz and the polarization formula, it suffices to prove the theorem when $\varphi_1 = \varphi_2 = \varphi \in C_c^\infty(X)$. By (3.2.4), (3.1.1), the definition (4.1.1) of $J(\varphi,\varphi)$ and Theorem 4.2.1, we have

$$\begin{aligned}\langle \varphi, \varphi\rangle_X &= \int_{N\backslash P} J(R(p)\varphi, R(p)\varphi)dp \\ &= \int_{N\backslash P}\int_{\mathrm{Temp}(G')} |J_\sigma(R(p)\varphi)|^2 d\mu_{G'}(\sigma)dp.\end{aligned}$$

Since the integrand in the last expression above is nonnegative, this expression is absolutely convergent. By Lemma 6.2.1 and the inversion formula (3.1.1), we have

$$\int_{N\backslash P} |J_\sigma(R(p)\varphi)|^2 dp = \langle \varphi, \varphi\rangle_{X,\sigma}$$

for every $\sigma \in \mathrm{Temp}(G')$. Hence, we get

$$\langle \varphi, \varphi\rangle_X = \int_{\mathrm{Temp}(G')}\int_{N\backslash P} |J_\sigma(R(p)\varphi)|^2 dp d\mu_{G'}(\sigma) = \int_{\mathrm{Temp}(G')} \langle \varphi, \varphi\rangle_{X,\sigma} d\mu_{G'}(\sigma)$$

showing at once the identity and the convergence of the right-and side of Theorem 6.1.1 when $\varphi_1 = \varphi_2 = \varphi$.

6.3 Relation to the factorization of global periods

In this section, we assume that n is odd.

Recall that there is a natural left $F^\times$-action on X. We denote the corresponding diagonal action by left translation of $F^\times$ on $C_c^\infty(X\times X)$ by L^Δ (that is $L^\Delta(\lambda)\Phi = \Phi(\lambda^{-1}.,\lambda^{-1}.)$ for $\Phi \in C_c^\infty(X\times X)$ and $\lambda \in F^\times$). Let $C_c^\infty(X\times X)_G$ be the G-coinvariant space of $C_c^\infty(X\times X)$ for the diagonal action by right translation of G. Then, we say that a function $\Phi \in C_c^\infty(X\times X)$ is $F^\times$-*stable* if for every $\lambda \in F^\times$, $\Phi - L^\Delta(\lambda)\Phi$ maps to 0 in $C_c^\infty(X\times X)_G$. By (3.4.2), we readily check that if $\Phi = \varphi_1\otimes\varphi_2$ is $F^\times$-stable then for every $\sigma \in \mathrm{Temp}(G')$, we have

$$\langle \varphi_1\cdot\alpha^{\sigma\otimes\eta}, \varphi_2\cdot\alpha^\sigma\rangle_{\mathrm{Whitt}} = 0. \tag{6.3.1}$$

We now move to a global setting and consider a quadratic extension k/k' of number fields. We write $\mathbb{A}$ for the adele ring of k', $\eta : \mathbb{A}^\times/(k')^\times \to \{\pm 1\}$ for the idele class character associated to the extension and for every place v of k', we denote by k'_v the corresponding completion, by $\mathcal{O}_v$ its ring of integers in case it is non-Archimedean and by k_v the tensor product $k\otimes_{k'} k'_v$. We also

change slightly notation to denote by G' the group GL_n over k', by $G = \mathrm{Res}_{k/k'}\mathrm{GL}_n$ the algebraic group obtained by restriction of scalar of GL_n from k to k' and by X the algebraic variety (over k') of non-degenerate Hermitian forms on k^n. There is a natural right action of G on X and for each place v of k' inert in k, the groups $G'_v = G'(k'_v)$, $G_v = G(k'_v)$ and the variety $X_v = X(k'_v)$ are what we have denoted G', G and X so far for $F = k'_v$ and $E = k_v$.

When v is inert in k, for every $\sigma_v \in \mathrm{Temp}(G'_v)$ we denote by $\langle .,. \rangle_{X_v,\sigma_v}$ the inner product on $C_c^\infty(X_v)$ defined in Section 6.1. When v splits in k, we define an inner product $\langle .,. \rangle_{X_v,\sigma_v}$ on $C_c^\infty(X_v)$ for every $\sigma_v \in \mathrm{Temp}(G'_v)$ as follows: choosing a place of k above v we get an identification $k_v \simeq k'_v \times k'_v$ and projection on the first component induces an isomorphism $X_v \simeq \mathrm{GL}_n(k'_v) = G'_v$, then we set

$$\langle \varphi_{1,v}, \varphi_{2,v} \rangle_{X_v,\sigma_v} = \mathrm{Trace}(\sigma_v(\varphi_{1,v} \star \varphi_{2,v}^*)), \quad \varphi_1, \varphi_2 \in C_c^\infty(X_v),$$

where $(\varphi_{1,v} \star \varphi_{2,v}^*)(x) = \displaystyle\int_{X_v} \varphi_{1,v}(xy)\overline{\varphi_{2,v}(y)}dy$ (for $x \in X_v$) and $\sigma_v(\varphi_v) = \displaystyle\int_{G'_v} \varphi_v(h)\sigma_v(h)dh$ (for $\varphi_v \in C_c^\infty(X_v) = C_c^\infty(G'_v)$). Note that for these inner products, the analog of Theorem 6.1.1 holds by Harish-Chandra Plancherel formula for G'_v.

When the place v is split, by the above definition, it is clear that the inner product $\langle .,. \rangle_{X_v,\sigma_v}$ only depends on the choice of invariant measures on X_v and G'_v. It is also true when v is inert as follows from the identity of Theorem 6.1.1 (the Plancherel measure $d\mu_{G'_v}(\sigma_v)$ is inversely proportional to the Haar measure on G'_v). This can alternativey be checked (slightly painfully) by tracing back all the constructions and normalizations of this paper (More precisely, we have made two auxilliary choices in the construction: a Haar measure on T' and a nontrivial additive character ψ').

We now normalize the local measures on X_v and G'_v so that they factorize the global invariant Tamagawa measures on $X(\mathbb{A})$ and $G'(\mathbb{A})$ and give, for almost all places v, volume 1 to the subsets of integral points $X(\mathcal{O}_v)$, $G'(\mathcal{O}_v)$.

Let $\Phi = \varphi_1 \otimes \varphi_2 \in C_c^\infty(X(\mathbb{A})) \otimes C_c^\infty(X(\mathbb{A}))$ and assume that the functions φ_1, φ_2 are products $\varphi_1 = \prod_v \varphi_{1,v}$, $\varphi_2 = \prod_v \varphi_{2,v}$ where $\varphi_{1,v}, \varphi_{2,v} \in C_c^\infty(X_v)$ for each place v of k'. Let $\sigma = \bigotimes'_v \sigma_v$ be a cuspidal automorphic representation of $G'(\mathbb{A})$ such that for each place v, the local representation σ_v is tempered. We denote by $L(s,\sigma,\mathrm{Ad})$ (resp. $L(s,\sigma,\mathrm{Ad}\otimes\eta)$) the adjoint L-function $L(s,\sigma\times\sigma^\vee)$ (resp. the twisted adjoint L-function $L(s,\sigma\eta\times\sigma^\vee)$) of σ. For any finite set S of places (resp. place v), we write $L^S(s,\sigma,\mathrm{Ad})$ and $L^S(s,\sigma,\mathrm{Ad}\otimes\eta)$ (resp. $L(s,\sigma_v,\mathrm{Ad})$ and $L(s,\sigma_v,\mathrm{Ad}\otimes\eta)$) for the corresponding partial L-functions (resp. local L-factors) and we set $L^{*,S}(1,\sigma,\mathrm{Ad}) = \mathrm{Res}_{s=1} L^S(s,\sigma,\mathrm{Ad})$. Since n is odd, $\sigma \not\simeq \sigma\otimes\eta$ and the partial L-function $L^S(s,\sigma,\mathrm{Ad}\otimes\eta)$ is regular at $s=1$ (for any S). Moreover, by the unramified computations of [FLO, Lemma 3.9] and [JS, Proposition 2.3], for almost all places v of k' we have

$$\langle \varphi_{1,v}, \varphi_{2,v} \rangle_{X_v,\sigma_v} = \frac{L(1,\sigma_v,\mathrm{Ad}\otimes\eta)}{L(1,\sigma_v,\mathrm{Ad})}.$$

(Note that when v is split, the right-hand side is simply 1). Therefore, for any sufficiently large finite set of places S of k', we can set

$$\langle \varphi_1, \varphi_2 \rangle_{X,\sigma} = \frac{L^S(1,\sigma,\mathrm{Ad}\otimes\eta)}{L^{*,S}(1,\sigma,\mathrm{Ad})} \prod_{v\in S} \langle \varphi_{1,v}, \varphi_{2,v} \rangle_{X_v,\sigma_v}.$$

Let $\varphi \in C_c^\infty(X(\mathbb{A}))$. We denote by $\Sigma\varphi$ the function on $[G] = G(k')\backslash G(\mathbb{A})$ defined by

$$(\Sigma\varphi)(g) = \sum_{x\in X(k')} \varphi(xg), \quad g \in [G].$$

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. We equip it with the Petersson inner product

$$\langle \phi, \phi \rangle_{\mathrm{Pet}} = \int_{G(k')\backslash G(\mathbb{A})^1} |\phi(g)|^2 dg$$

where $G(\mathbb{A})^1$ is the subgroup of matrices $g \in G(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_k)$ ($\mathbb{A}_k$ denoting the adele ring of k) such that $|\det(g)| = 1$ and dg is the Tamagawa measure (i.e. the one giving $G(k')\backslash G(\mathbb{A})^1$ volume 1). We then write $(\Sigma\varphi)_\pi$ for the π-projection of $\Sigma\varphi$ that is

$$(\Sigma\varphi)_\pi = \sum_\phi \langle \Sigma\varphi, \phi \rangle_{[G]} \phi$$

where the sum runs over an orthonormal basis of π and $\langle ., . \rangle_{[G]}$ stands for the L^2-inner product on $[G]$ (again with respect to the Tamagawa measure).

For any cuspidal automorphic representation σ of $G'(\mathbb{A})$, we let $\mathrm{BC}(\sigma)$ be the automorphic base-change of σ to $G(\mathbb{A})$ [AC].

The following result is simply a reformulation of a theorem of Feigon-Lapid-Offen [FLO, Theorem 10.2] on the factorization of unitary periods of cuspidal automorphic representations of G (following an approach of Jacquet who has established a similar result when $n = 3$ for quasi-split unitary groups [Jac01]). The main reason to restate the result in the form below, is to make the relation to the explicit local Plancherel decomposition of Theorem 6.1.1 more transparent. In particular, we find this formulation to be pleasantly aligned with certain speculations of Sakellaridis-Venkatesh on the factorization of general spherical periods [SV, §17].

Theorem 6.3.1 (Feigon-Lapid-Offen, Jacquet (n=3))**.** *Assume that n is odd. Let $\Phi = \varphi_1 \otimes \varphi_2 \in C_c^\infty(X(\mathbb{A})) \otimes C_c^\infty(X(\mathbb{A}))$ be a factorizable test function $\Phi = \prod_v \Phi_v$ and let π be a cuspidal automorphic representation of $G(\mathbb{A})$. Assume that for at least one inert place v, the function Φ_v is ${k'_v}^\times$-stable and that for every place v, the representation π_v is tempered. Then, we have*

$$\langle (\Sigma\varphi_1)_\pi, (\Sigma\varphi_2)_\pi \rangle_{\mathrm{Pet}} = \sum_{\mathrm{BC}(\sigma)=\pi} \langle \varphi_1, \varphi_2 \rangle_{X,\sigma} \tag{6.3.2}$$

where the sum runs over cuspidal automorphic representations σ of $G'(\mathbb{A})$ such that $\mathrm{BC}(\sigma) = \pi$.

Proof. Unfolding all the definitions, we arrive at

$$\langle (\Sigma\varphi_1)_\pi, (\Sigma\varphi_2)_\pi \rangle_{\mathrm{Pet}} = \sum_\phi \frac{\langle \Sigma\varphi_1, \phi \rangle_{[G]} \langle \phi, \Sigma\varphi_2 \rangle_{[G]}}{\langle \phi, \phi \rangle_{\mathrm{Pet}}} \tag{6.3.3}$$

the sum being over an orthogonal basis of π and

$$\langle \Sigma\varphi_i, \phi \rangle_{[G]} = \sum_{x \in X(k')/G(k')} \int_{G_x(\mathbb{A})\backslash G(\mathbb{A})} \varphi_i(xg) \overline{P_{G_x}(R(g)\phi)} dg \tag{6.3.4}$$

for $i = 1, 2$, where $P_{G_x} : \phi \mapsto \int_{[G_x]} \phi(h) dh$ denotes the period integral over G_x and the measure on $G_x(\mathbb{A})$ is again the Tamagawa measure.

We now fix a global nontrivial additive character $\psi' : \mathbb{A}/k' \to \mathbb{C}^\times$ and we set $\psi = \psi' \circ \mathrm{Tr}_{k/k'} : \mathbb{A}_k/k \to \mathbb{C}^\times$. For each place v of k, we normalize the right Haar measures on the mirabolic subgroups

$P'_v = P_n(k'_v)$ and $P_v = P_n(k_v)$ so that the Fourier inversion formulas (3.1.1) are satisfied for the local additive characters ψ'_v and ψ_v. We also set $N' = N_{n,k'}$, $N = \mathrm{Res}_{k/k'} N_{n,k}$ and we equip $N'(\mathbb{A})$, $N(\mathbb{A})$ with the Haar measures giving $N'(k')\backslash N'(\mathbb{A})$, $N(k')\backslash N(\mathbb{A})$ volume 1. With these normalizations, we can define local FLO functionals as in Section 3.4 by using Haar measures on the local groups $N'_v = N'(k'_v)$, $N_v = N(k'_v)$ that factorize the global ones. Finally, we define a generic character ψ_n of $N(\mathbb{A})$ using the character ψ as in the local case (see Section 3.1).

Let $x \in X(k')$. By [FLO, Theorem 10.2], P_{G_x} vanishes on π unless it is the base-change of some cuspidal automorphic representation σ of $G'(\mathbb{A})$ in which case for any factorizable vector $\phi \in \pi$, we have

$$P_{G_x}(\phi) = 2\alpha_x^\sigma(W_\phi) \tag{6.3.5}$$

where $W_\phi(g) = \int_{[N]} \phi(ug)\psi_n(u)^{-1}du = \prod_v W_{\phi,v}$ is the Whittaker function associated to ϕ and $\alpha_x^\sigma(W_\phi)$ is defined by

$$\alpha_x^\sigma(W_\phi) = L(1, \sigma, \mathrm{Ad}\otimes\eta) \prod_v L(1, \sigma_v, \mathrm{Ad}\otimes\eta)^{-1} \alpha_x^{\sigma_v}(W_{\phi,v}).$$

From now on we assume that $\pi = \mathrm{BC}(\sigma)$ for some cuspidal automorphic representation σ of $G'(\mathbb{A})$ (as otherwise the just quoted result of Feigon-Lapid-Offen implies that both sides of (6.3.2) are zero). Plugging this into (6.3.4), we obtain

$$\langle \Sigma\varphi_i, \phi\rangle_{[G]} = 2 \sum_{x \in X(k')/G(k')} (\varphi_{i,x} \cdot \alpha^\sigma)(W_\phi)$$

for $i = 1, 2$ where $\varphi_{i,x}$ denotes the restriction of φ_i to the $G(\mathbb{A})$-orbit of x and we have set

$$(\varphi \cdot \alpha^\sigma)(W_\phi) = \int_{X(\mathbb{A})} \varphi(x)\alpha_x^\sigma(W_\phi)dx$$

for every $\varphi \in C_c^\infty(X(\mathbb{A}))$ and $\phi \in \pi$. Together with (6.3.3), this gives

$$\langle (\Sigma\varphi_1)_\pi, (\Sigma\varphi_2)_\pi\rangle_{\mathrm{Pet}} = 4 \sum_{x \in X(k')/G(k')} \sum_\phi \frac{(\varphi_{1,x} \cdot \alpha^\sigma)(W_\phi)\overline{(\varphi_{2,x} \cdot \alpha^\sigma)(W_\phi)}}{\langle \phi, \phi\rangle_{\mathrm{Pet}}}. \tag{6.3.6}$$

For any factorizable vector $\phi \in \pi$, we set

$$\langle W_\phi, W_\phi\rangle_{\mathrm{Whitt}} = L^*(1, \pi, \mathrm{Ad}) \prod_v L(1, \pi_v, \mathrm{Ad})^{-1} \langle W_{\phi,v}, W_{\phi,v}\rangle_{\mathrm{Whitt}}.$$

Then, by [JS, §4] (see also [FLO, Eq. (10.1) p.265] or [Zha, Proposition 3.1][7]), we have $\langle \phi, \phi\rangle_{\mathrm{Pet}} = \langle W_\phi, W_\phi\rangle_{\mathrm{Whitt}}$ so that (6.3.6) can be rewritten as

$$\langle (\Sigma\varphi_1)_\pi, (\Sigma\varphi_2)_\pi\rangle_{\mathrm{Pet}} = 4 \sum_{x \in X(k')/G(k')} \sum_\phi \frac{(\varphi_{1,x} \cdot \alpha^\sigma)(W_\phi)\overline{(\varphi_{2,x} \cdot \alpha^\sigma)(W_\phi)}}{\langle W_\phi, W_\phi\rangle_{\mathrm{Whitt}}}. \tag{6.3.7}$$

[7]Note that the normalization of the Petterson inner product in *loc. cit.* is different from ours. Namely, there it is normalized as the L^2-inner product on $[\mathrm{PGL}_n]$ for the Tamagawa measure (thus giving $[\mathrm{PGL}_n]$ volume n).

Let $\mathrm{disc} : X \to \mathbb{G}_m$ be the regular map that sends $x \in X$ to its discriminant in the standard basis of k^n. Then, by global class field theory, the natural map $X(k')/G(k') \to X(\mathbb{A})/G(\mathbb{A})$ is injective with image the set of orbits $x \in X(\mathbb{A})/G(\mathbb{A})$ such that $\eta(\mathrm{disc}(x)) = 1$. On the other hand, by [FLO, Lemma 3.5], we have $\varphi_x \cdot \alpha^{\sigma\otimes\eta} = \eta(\mathrm{disc}(x))\varphi_x \cdot \alpha^{\sigma}$ for $\varphi \in C_c^\infty(X(\mathbb{A}))$ and $x \in X(\mathbb{A})$. This allows to rewrite the identity (6.3.7) as

(6.3.8)

$$\langle(\Sigma\varphi_1)_\pi, (\Sigma\varphi_2)_\pi\rangle_{\mathrm{Pet}} = \sum_{x\in X(\mathbb{A})/G(\mathbb{A})} \sum_{\phi} \frac{(\varphi_{1,x}\cdot\alpha^\sigma + \varphi_{1,x}\cdot\alpha^{\sigma\otimes\eta})(W_\phi)\overline{(\varphi_{2,x}\cdot\alpha^\sigma + \varphi_{2,x}\cdot\alpha^{\sigma\otimes\eta})(W_\phi)}}{\langle W_\phi, W_\phi\rangle_{\mathrm{Whitt}}}$$

$$= \langle\varphi_1\cdot\alpha^\sigma, \varphi_2\cdot\alpha^\sigma\rangle_{\mathrm{Whitt}} + \langle\varphi_1\cdot\alpha^{\sigma\otimes\eta}, \varphi_2\cdot\alpha^{\sigma\otimes\eta}\rangle_{\mathrm{Whitt}} + \langle\varphi_1\cdot\alpha^{\sigma}, \varphi_2\cdot\alpha^{\sigma\otimes\eta}\rangle_{\mathrm{Whitt}} + \langle\varphi_1\cdot\alpha^{\sigma\otimes\eta}, \varphi_2\cdot\alpha^{\sigma}\rangle_{\mathrm{Whitt}}$$

where as in the local case for every $\varphi \in C_c^\infty(X(\mathbb{A}))$ we have identified $\varphi\cdot\alpha^\sigma$ and $\varphi\cdot\alpha^{\sigma\otimes\eta}$ with elements of the global Whittaker model $\mathcal{W}(\pi^\vee, \psi_n^{-1})$ through the inner product $\langle .,.\rangle_{\mathrm{Whitt}}$. From the definitions it is clear that

$$\langle\varphi_1\cdot\alpha^\sigma, \varphi_2\cdot\alpha^\sigma\rangle_{\mathrm{Whitt}} = \langle\varphi_1,\varphi_2\rangle_{X,\sigma} \text{ and } \langle\varphi_1\cdot\alpha^{\sigma\otimes\eta}, \varphi_2\cdot\alpha^{\sigma\otimes\eta}\rangle_{\mathrm{Whitt}} = \langle\varphi_1,\varphi_2\rangle_{X,\sigma\otimes\eta}$$

whereas the hypothesis that Φ_v is ${k'_v}^\times$-stable for at least one inert place v implies (by (6.3.1)) that

$$\langle\varphi_1\cdot\alpha^\sigma, \varphi_2\cdot\alpha^{\sigma\otimes\eta}\rangle_{\mathrm{Whitt}} = \langle\varphi_1\cdot\alpha^{\sigma\otimes\eta}, \varphi_2\cdot\alpha^{\sigma}\rangle_{\mathrm{Whitt}} = 0.$$

Together with (6.3.8) and the fact that the only cuspidal automorphic representations of $G'(\mathbb{A})$ with base-change π are σ and $\sigma\otimes\eta$ [AC, Theorem 4.2], this gives identity (6.3.2). □

Final remark. *To finish this paper, we would like to offer a word of explanation on the assumption in the theorem above and its relation to the (author's interpretation of) speculations made by Sakellaridis-Venkatesh in [SV, §17][8]. Namely, we can see the formal (non-convergent) expression* $\mathrm{RTF}_{X\times X/G}(\Phi) = \langle\Sigma\varphi_1, \Sigma\varphi_2\rangle_{[G]}$ *as a version of Jacquet's relative formula for the variety* X*. This expression decomposes (again formally) as a sum of orbital integrals of* Φ *for the diagonal action of* G *on* $X\times X$*. Note that, in the case at hand, there is a stability issue: different rational orbits for this action may become the same over the algebraic closure. Therefore, a natural expectation would be that a stabilization process, similar to the one for the Arthur-Selberg trace formula, can lead to a stable version* $\mathrm{STF}_{X\times X/G}(\Phi)$ *of this trace formula. Now, we interpret[9] the speculations in [SV, §17] as saying that* $\mathrm{STF}_{X\times X/G}(\Phi)$ *should decompose as an integral over the* L^2*-automorphic spectrum of* G' *(for a suitable canonical spectral measure) of the scalar product* $\langle\varphi_1,\varphi_2\rangle_{X,\sigma}$*. Of course, all of this is based on many formal statements that the author cannot make precise here (In particular, the scalar products* $\langle .,.\rangle_{X,\sigma}$ *have only been defined when* σ *is tempered. The definition naturally extends to generic* σ *but e.g. it is not obvious how to make sense of them for the residual representations.) but this at least can be used as a rationale for the statement of Theorem 6.3.1: the assumption of being* ${k'_v}^\times$*-stable should be seen as a weak version of stability in this context and the result roughly says that (when* n *is odd) it is nevertheless enough to get the correct stable cuspidal contributions.*

[8] Strictly speaking, the situation considered here is not even covered in *loc. cit.* since they assume local multiplicity one. Therefore, our discussion should be seen as a kind of "speculation over a speculation".

[9] We of course try to follow the general spirit of Sakellaridis-Venkatesh's vision but any error or misinterpretation is the author's responsability only.

References

[AG] A. Aizenbud, D. Gourevitch: *Smooth transfer of Kloosterman integrals (the Archimedean case)*, Amer. J. Math. 135 (2013), no. 1, 143-182.

[AC] J. Arthur, L. Clozel: *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, 120. Princeton University Press, Princeton, NJ, 1989. xiv+230 pp.

[Bar] E. M. Baruch: *A proof of Kirillov's conjecture*, Ann. Math. (2), 158 (2003), 207-252.

[BO] Y. Benoist, H. Oh, *Polar decomposition for p-adic symmetric spaces*, Int. Math. Res. Not. IMRN 2007, no. 24, Art. ID rnm121, 20 pp.

[Ber1] J. N. Bernstein: *P-invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-Archimedean case)*, in "Lie group representations, II." Lecture Notes in Math., vol. 1041, pp. 50-102, Springer, Berlin, 1984.

[Ber2] J. N. Bernstein: *Le "centre" de Bernstein*, Notes by P. Deligne. Travaux en Cours, Representations of reductive groups over a local field, 1-32, Hermann, Paris, 1984.

[Ber3] J. N. Bernstein: *On the support of Plancherel measure*, J. Geom. Phys. 5 (1988), no. 4, 663-710.

[BDK] J. Bernstein, P. Deligne, D. Kazhdan: *Trace Paley-Wiener theorem for reductive p-adic groups*, J. Analyse Math. 47 (1986), 180-192.

[BZ] I. N. Bernstein, A. V. Zelevinsky: *Induced representations of reductive $\mathfrak{p}$-adic groups. I*, Ann. Sci. École Norm. Sup. (4) 10 (1977), no. 4, 441-472.

[Beu1] R. Beuzart-Plessis: *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the Archimedean case*, to appear in Astérisque, arXiv:1506.01452

[Beu2] R. Beuzart-Plessis, *On distinguished square-integrable representations for Galois pairs and a conjecture of Prasad*, Invent. Math. 214 (2018), no. 1, 437-521.

[Beu3] R. Beuzart-Plessis: *Plancherel formula for $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(E)$ and applications to the Ichino-Ikeda and formal degree conjectures for unitary groups*, arXiv:1812.00047

[BW] A. Borel, N. Wallach: *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, 2000. xviii+260 pp.

[Cas1] W. Casselman: *Introduction to the theory of admissible representations of p-adic reductive groups*, 1975.

[Cas2] W. Casselman: *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. 41 (1989), no. 3, 385-438.

[CHH] M. Cowling, U. Haagerup, R. Howe: *Almost L^2 matrix coefficients*, J. für die reine und angewandte Math., vol.387 (1988), pp.97-110

[Del] P. Delorme, *Constant term of smooth H_ψ-spherical functions on a reductive p-adic group*, Trans. Amer. Math. Soc. 362 (2010), no. 2, 933-955.

[DS] P. Delorme, V. Sécherre, *An analogue of the Cartan decomposition for p-adic reductive symmetric spaces of split p-adic reductive groups*, Pacific J. Math. 251 (2011), no. 1, 1-21.

[DM] J. Dixmier P. Malliavin: *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) 102 (1978), no. 4, 307-330.

[Fe] B. Feigon: *A relative trace formula for $PGL(2)$ in the local setting*, Pacific J. Math. 260 (2012), no. 2, 395-432.

[FLO] B. Feigon, E. Lapid, O. Offen: *On representations distinguished by unitary groups*, Publ. Math. Inst. Hautes Études Sci. 115 (2012), 185-323.

[Gro] A. Grothendieck: *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 1955 (1955), no. 16.

[GO] M. Gurevich, O. Offen, *A criterion for integrability of matrix coefficients with respect to a symmetric pair*, J. Funct. Anal. 270 (2016), no. 12, 4478-4512.

[H-C] Harish-Chandra: *Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula*, Ann. of Math. 104 (1976), 117-201.

[Hart] R. Hartshorne: *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[Hump] J. Humphreys: *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21 (Springer, 1975).

[Iv] B. Iversen: *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. 16 (1972), 229-236.

[Jac01] H. Jacquet: *Factorization of period integrals*, J. Number Theory, 87 (2001), 109-143

[Jac03] H. Jacquet: *Smooth transfer of Kloosterman integrals*, Duke Math. J., 120 (2003), 121-152.

[Jac04] H. Jacquet: *Kloosterman identities over a quadratic extension*, Ann. Math. (2), 160 (2004), 755-779.

[Jac05] H. Jacquet: *Kloosterman identities over a quadratic extension. II*, Ann. Sci. Éc. Norm. Super. (4), 38 (2005), 609-669.

[Jac10] H. Jacquet: *Distinction by the quasi-split unitary group*, Isr. J. Math., 178 (2010), 269-324.

[JS] H. Jacquet, J. A. Shalika: *On Euler products and the classification of automorphic representations I*, Amer. J. Math. 103 (1981),449-558.

[JY90] H. Jacquet, Y. Ye: *Une remarque sur le changement de base quadratique*, C. R. Acad. Sci. Paris Sér. I Math., 311 (1990), 671-676.

[JY92] H. Jacquet, Y. Ye: *Kloosterman integrals for $GL(3)$*, Bull. Soc. Math. Fr., 120 (1992), 263-295.

[JY96] H. Jacquet, Y. Ye: *Distinguished representations and quadratic base change for GL(3)*, Trans. Amer. Math. Soc. 348 (1996), no. 3, 913-939.

[KT] S. Kato, K. Takano: *Subrepresentation theorem for p-adic symmetric spaces*, Int. Math. Res. Not. IMRN, (11):Art. ID rnn028, 40, 2008.

[Lag] N. Lagier: *Terme constant de fonctions sur un espace symétrique réductif p-adique*, J. Funct. Anal., 254(4):1088-1145, 2008.

[LM] E. Lapid, Z. Mao: *A conjecture on Whittaker-Fourier coefficients of cusp forms*, J. Number Theory 146 (2015), 448-505.

[Pras] D. Prasad, *A 'relative' local Langlands correspondence*, arXiv preprint 2015, arXiv:1512.04347

[Ren] D. Renard: *Représentations des groupes réductifs p-adiques*, Cours Spécialisés, 17. Société Mathématique de France, Paris, 2010. vi+332 pp.

[Rod] F. Rodier: *Modèles de Whittaker des représentations admissibles des groupes réductifs p-adiques quasi-déployés*, unpublished manuscript available at `http://iml.univ-mrs.fr/~rodier/archives.html`.

[SV] Y. Sakellaridis, A. Venkatesh: *Periods and harmonic analysis on spherical varieties*, Astérisque No. 396 (2017), viii+360 pp.

[SZ] P. Schneider, E.-W. Zink: *The algebraic theory of tempered representations of p-adic groups. II. Projective generators*, Geom. Funct. Anal. 17 (2008), no. 6, 2018-2065.

[Smith] J. M. Smith: *Local unitary periods and relative discrete series*, Pacific J. Math. 297 (2018), no. 1, 225-256.

[Stacks] The Stacks project authors: *The Stacks project*, `https://stacks.math.columbia.edu`, 2019.

[Sun] B. Sun: *Bounding Matrix Coefficients for Smooth Vectors of Tempered Representations*, Proc. of the AMS, Vol. 137, No. 1 (Jan 2009), pp. 353-357.

[Tr] F. Trèves: *Topological vector spaces, distributions and kernels*, Academic Press, New York-London 1967 xvi+624 pp.

[Var] V. S. Varadarajan: *Harmonic analysis on real reductive groups*, Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin-New York, 1977. v+521 pp.

[Wald1] J.-L. Waldspurger: *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu 2 (2003), no. 2, 235-333.

[WallI] N. R. Wallach: *Real reductive groups I*, Pure and Applied Mathematics, 132. Academic Press, Inc., Boston, MA, 1988. xx+412 pp.

[WallII] N. R. Wallach: *Real reductive groups II*, Academic Press 1992.

[Ye88] Y. Ye: *Kloosterman integrals and base change*, in "Number Theory and Its Applications in China" Contemp.Math., vol. 77, pp. 163-170, Amer. Math. Soc., Providence, 1988.

[Ye89] Y. Ye: *Kloosterman integrals and base change for $GL(2)$*, J. Reine Angew. Math., 400 (1989), 57-121.

[Ze] A. V. Zelevinsky: *Induced representations of reductive p-adic groups. II. On irreducible representations of GL(n)*, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 2, 165-210.

[Zha] W. Zhang: *Automorphic period and the central value of Rankin-Selberg L-function*, J. Amer. Math. Soc. 27 (2014), no. 2, 541-612.

Raphaël Beuzart-Plessis
Aix Marseille Univ, CNRS
Centrale Marseille
I2M
Marseille
France

email:
raphael.beuzart-plessis@univ-amu.fr