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ABOUT THE STEIN EQUATION FOR THE GENERALIZED INVERSE GAUSSIAN AND KUMMER DISTRIBUTIONS

Essomanda Konzou[1,2] and Angelo Efoevi Koudou[1,*]

Abstract. We observe that the density of the Kummer distribution satisfies a certain differential equation, leading to a Stein characterization of this distribution and to a solution of the related Stein equation. A bound is derived for the solution and for its first and second derivatives. To provide a bound for the solution we partly use the same framework as in Gaunt 2017 [Stein, *ESAIM: PS* **21** (2017) 303–316] in the case of the generalized inverse Gaussian distribution, which we revisit by correcting a minor error. We also bound the first and second derivatives of the Stein equation in the latter case.

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1. Introduction

For $a > 0$, $b \in \mathbb{R}$, $c > 0$, the Kummer distribution with parameters a, b, c has density

$$k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^{a-1}(1+x)^{-a-b}e^{-cx}, \ (x > 0)$$

where ψ is the confluent hypergeometric function of the second kind.

The generalized inverse Gaussian (hereafter *GIG*) distribution with parameters $p \in \mathbb{R}$, $a > 0$, $b > 0$ has density

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1}e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0,$$

where K_p is the modified Bessel function of the third kind.

For details on GIG and Kummer distributions see [6, 7, 10] and references therein, where one can see for instance that these distributions are involved in some characterization problems related to the so-called Matsumoto-Yor property.

In this paper, these two distributions are considered in the context of Stein's method. This method introduced in [14] is a technique used to bound the error in the approximation of the distribution of a random variable of

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interest by another probability (for instance the normal) distribution. For an overview of Stein's method see [1, 11]. The first steps of this method consist in finding an operator called Stein operator characterizing the targeted distribution, then solving the corresponding so-called *Stein equation*.

One finds in [14] a seminal instance of the method, where Stein showed that a random variable X has a standard normal distribution if and only if for all real-valued absolutely continuous function f such that $\mathbb{E}\,|f'(Z)| < \infty$ for $Z \sim N(0,1)$,

$$\mathbb{E}\,[f'(X) - Xf(X)] = 0.$$

The corresponding *Stein equation* is

$$f'(x) - xf(x) = h(x) - \mathbb{E}h(Z)$$

where h is a bounded function and Z a random variable following the standard normal distribution. The operator $f \mapsto T_f$ defined by $(T_f)(x) = f'(x) - xf(x)$ is the corresponding *Stein operator*.

If a function f_h is a solution of the previous equation, then for any random variable U we have

$$|\mathbb{E}(f_h'(U) - Uf_h(U))| = |\mathbb{E}(h(U)) - \mathbb{E}(h(Z))|.$$

Thus, in order to bound $|\mathbb{E}(h(U)) - \mathbb{E}(h(Z))|$ given h, its enough to find a solution f_h of the Stein equation and to bound the left-hand side of the previous equation. The problem of solving the Stein equation for other distributions than the standard normal distribution and bounding the solution and its derivatives has been widely studied in the literature (see [4] among many others).

The aim of this paper is to solve the Stein equation and derive a bound of the solution and of its first and second derivatives for the Kummer distribution and for the generalized inverse Gaussian distribution. The idea of this paper emerged by reading the remarkable work by [2] about a Stein characterization of the generalized hyperbolic distribution of which the generalized inverse Gaussian distribution (GIG) is a limiting case. Among many other results, [2] solved the GIG Stein equation and bounded the solution by using a general result obtained in [12] when the targeted distribution has a density g satisfying

$$(s(x)g(x))' = \tau(x)g(x) \tag{1.1}$$

for some polynomial functions s and τ. Also a bound was obtained for the solution under the condition that the function τ be a decreasing linear function. But since this linearity condition does not hold in the GIG case, the bound given by [2] has to be slightly corrected. This is done in Theorem 3.2 after recalling the general framework of Schoutens [12] and adapting it to the cases where τ is decreasing but not necessarily linear. Indeed, we realized that the procedure adopted in [12] still works, via a slight change, even if τ is not linear.

Observing that the Kummer density also satisfies (1.1), we can use the same methodology (Thm. 4.2) for this distribution. We begin by putting the restrictions $p \leq -1$ for the GIG density and $1 - b - c \leq 0$ for the Kummer density in order for the corresponding function τ to be decreasing on $(0, \infty)$. But we finally derived bounds without these restrictions, due to the form of the function τ in the GIG and Kummer cases.

In Section 2 we recall the general framework established by [12] for densities g satisfying (1.1) without the assumption of linearity of τ, but assuming that τ is decreasing. We retrieve the Stein operator given in [12] by using the density approach initiated in [15] and further developed in [9]. In the same section we bound the solution in a setting where the function τ is not decreasing but has a particular behaviour shared by the GIG and Kummer cases. The results are applied to the GIG distribution in Section 3 (the proof of the bound of the second derivative is postponed to the Appendix A), and to the Kummer one in Section 4. In Section 5 we give some examples of possible applications that will be investigated in future work.

2. Stein characterization, solution of the Stein equation and bound for the solution in the Schoutens framework

Theorem 1 in [12] addressed the problem of establishing a Stein characterization for probability distributions with density g satisfying (1.1) for some polynomial functions s et τ, and proved that a Stein operator in this case is $f \mapsto sf' + \tau f$. We realized (see the following theorem) that the same Stein operator can be arrived at by using the density approach designed in [15] and [9]. The support of the density may be any interval, but here we take this support to be $(0, \infty)$ in the purpose of the application to the GIG and Kummer distributions.

Theorem 2.1. *Consider a density g on $(0, \infty)$ such that (1.1) holds for some polynomial functions s and τ. Then a positive random variable X has density g if and only if for any differentiable function f such that $\lim_{x\to 0} s(x)g(x)f(x) = \lim_{x\to\infty} s(x)g(x)f(x) = 0$,*

$$\mathbb{E}\left[s(X)f'(X) + \tau(X)f(X)\right] = 0.$$

Proof. We use Corollary 2.1 of [9]. According to this corollary, a Stein operator related to the density function g is

$$T_g f(x) = \frac{1}{g(x)}(fg)'(x).$$

Applyng this operator to sf, we have

$$\begin{aligned} T_g(sf)(x) &= \frac{1}{g(x)}(sfg)'(x) \\ &= \frac{1}{g(x)}(f'(x)s(x)g(x) + f(x)(sg)'(x)) \end{aligned}$$

which, by (1.1), reads

$$\begin{aligned} T_g(sf)(x) &= \frac{1}{g(x)}(f'(x)s(x)g(x) + f(x)\tau(x)g(x)) \\ &= f'(x)s(x) + f(x)\tau(x). \end{aligned}$$

□

Theorem 2.1 shows that the Stein equation related to any density g satisfying (1.1) enjoys the tractable form

$$s(x)f'(x) + \tau(x)f(x) = h(x) - \mathbb{E}h(W) \tag{2.1}$$

where W is random variable with density g. Schoutens [12] found a solution to the Stein equation (2.1) and established a bound for the solution, under the condition that the function τ be a decreasing linear function (which is the case for the so-called Pearson and Ord classes of distributions considered in [12]).

The following result comes from Proposition 1 in [12]. We again take the support of the density function to be $(0, \infty)$.

Proposition 2.2. *Consider a density function $g > 0$ on $(0, \infty)$ satisfying equation (1.1), for some polynomial functions s and τ. Then a solution of the Stein equation (2.1) is*

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)g(x)} \int_0^x g(t)\,[h(t) - \mathbb{E}h(W)]\,\mathrm{d}t \\ =& \frac{-1}{s(x)g(x)} \int_x^{+\infty} g(t)\,[h(t) - \mathbb{E}h(W)]\,\mathrm{d}t. \end{aligned} \tag{2.2}$$

Remark 2.3. The proof of this proposition is given in [12] just by calculating the derivative of the function f_h defined by (2.2) and checking directly that f_h satisfies (2.1). Our following proposition complements this result.

Proposition 2.4. *Under the notation and assumptions of Proposition 2.2,*

- *The solutions of the Stein equation (2.1) are of the form*

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)g(x)} \int_0^x g(t)\,[h(t) - \mathbb{E}h(W)]\,\mathrm{d}t + \frac{C}{s(x)g(x)} \\ =& \frac{-1}{s(x)g(x)} \int_x^{+\infty} g(t)\,[h(t) - \mathbb{E}h(W)]\,\mathrm{d}t + \frac{C}{s(x)g(x)} \end{aligned} \tag{2.3}$$

  where C is constant.
- *Suppose $\lim_{x\to 0} s(x)g(x) = 0$. For the solution to be bounded, it is necessary that $C = 0$ in (2.3).*

Proof. Multiplying both sides of (2.1) by $g(x)$ we have

$$s(x)g(x)f'(x) + \tau(x)g(x)f(x) = g(x)(h(x) - \mathbb{E}h(W))$$

which, by (1.1), can be written

$$s(x)g(x)f'(x) + (sg)'(x)f(x) = g(x)(h(x) - \mathbb{E}h(W)),$$

i.e.

$$(sgf)'(x) = g(x)(h(x) - \mathbb{E}h(W)).$$

As a consequence, there exists a constant C such that

$$s(x)g(x)f(x) = \int_0^x g(t)\,[h(t) - \mathbb{E}h(W)]\,\mathrm{d}t + C \tag{2.4}$$

which implies (2.3).

Suppose f is bounded. Since $\lim_{x\to 0} s(x)g(x) = 0$, letting x tend to 0 in (2.4) yields $C = 0$.

The second expression for f_h follows from the fact that, since W has density g,

$$\int_0^{+\infty} g(t)\,[h(t) - \mathbb{E}h(W)]\,\mathrm{d}t = 0.$$

□

The following proposition proves that the solution given by (2.2) is bounded indeed if h is bounded, and thus is the unique bounded solution to the Stein equation associated to the density g. A bound of this solution is provided.

Proposition 2.5. *Consider a density function $g > 0$ on $(0, \infty)$ satisfying equation (1.1), where s and τ are polynomial functions such that $s > 0$ on $(0, \infty)$. Assume that $\lim_{x\to 0} s(x)g(x) = \lim_{x\to\infty} s(x)g(x) = 0$.*

If h is a bounded continuous function, then

$$||f_h|| \leq M\, ||h(\cdot) - \mathbb{E}h(W)|| \tag{2.5}$$

where W has density g, $||f_h|| = \sup_{x>0} |f_h(x)|$ and the constant M is defined as follows:

1. *Suppose τ is decreasing and has a unique zero α on $(0, \infty)$. Then*

$$M = \max\left(\frac{1}{s(\alpha)g(\alpha)}\int_0^\alpha g(t)\mathrm{d}t; \frac{1}{s(\alpha)g(\alpha)}\int_\alpha^{+\infty} g(t)\mathrm{d}t\right).$$

2. *Assume that $\tau(0) > 0$ and, for some $\beta > 0$, τ is increasing on $(0, \beta]$, decreasing on $[\beta, \infty)$ and has a unique zero α on $[\beta, \infty)$. Then*

$$M = \max\left(\frac{1}{\tau(0)};\ \frac{1}{s(\beta)g(\beta)}\right).$$

Remark 2.6. The first item of Proposition 2.5 is a reformulation of Lemma 1 in [12] without the assumption that τ is linear. With this assumption, [12] established the same bound with $\alpha = \mathbb{E}(W)$ (for a random variable W with density g), which is not true if τ is not linear. The proof given below follows the lines of that of [12] where we observed that the assumption of linearity of τ was used nowhere except to state that its only zero is $\alpha = \mathbb{E}(W)$.

The proof of item 1 of Proposition 2.5 uses the following lemma:

Lemma 2.7. *Under the assumptions of Proposition 2.5,*

$$\int_0^x g(t)\mathrm{d}t \leq \frac{s(x)g(x)}{\tau(x)} \quad \textit{for} \quad x < \alpha$$

and

$$\int_x^{+\infty} g(t)\mathrm{d}t \leq \frac{-s(x)g(x)}{\tau(x)} \quad \textit{for} \quad x > \alpha.$$

Proof. Suppose $x < \alpha$. Since τ is positive and decreasing on $(0, \alpha)$, we have $\dfrac{\tau(t)}{\tau(x)} \geq 1$ for all $t \leq x$. Therefore

$$\begin{aligned}\int_0^x g(t)\mathrm{d}t \leq & \int_0^x g(t)\frac{\tau(t)}{\tau(x)}\mathrm{d}t \\ = & \frac{1}{\tau(x)}\int_0^x \tau(t)g(t)\mathrm{d}t \\ = & \frac{s(x)g(x)}{\tau(x)}\end{aligned}$$

because of (1.1) and as $\lim\limits_{t\to 0} s(t)g(t) = 0$.

For $x > \alpha$, since τ is negative and decreasing on (α, ∞), we have $\dfrac{\tau(t)}{\tau(x)} \geq 1$ for all $t \geq x$. As a consequence,

$$\begin{aligned}\int_x^{+\infty} g(t)\mathrm{d}t \leq & \int_x^{+\infty} g(t)\frac{\tau(t)}{\tau(x)}\mathrm{d}t \\ = & \frac{-s(x)g(x)}{\tau(x)}\end{aligned}$$

since $\lim\limits_{t\to\infty} s(t)g(t) = 0$. □

Now, let us prove the item 1 of Proposition 2.5.

Proof. For $x < \alpha$,

$$\begin{aligned}|f_h(x)| = & \left|\frac{1}{s(x)g(x)}\int_0^x g(t)\left[h(t) - \mathbb{E}h(W)\right]\mathrm{d}t\right| \\ \leq & \frac{1}{s(x)g(x)}\int_0^x g(t)\left|h(t) - \mathbb{E}h(W)\right|\mathrm{d}t \\ \leq & \left|\left|h(\cdot) - \mathbb{E}h(W)\right|\right|\frac{1}{s(x)g(x)}\int_0^x g(t)\mathrm{d}t.\end{aligned}$$

Let $l(x) = \dfrac{1}{s(x)g(x)}\displaystyle\int_0^x g(t)\mathrm{d}t$. Then l is differentiable on $(0, \infty)$ and

$$\begin{aligned}l'(x) = & \frac{-\left(s(x)g(x)\right)'}{\left(s(x)g(x)\right)^2}\int_0^x g(t)\mathrm{d}t + \frac{1}{s(x)} \\ = & \frac{-\tau(x)}{s^2(x)g(x)}\int_0^x g(t)\mathrm{d}t + \frac{1}{s(x)}.\end{aligned}$$

Using Lemma 2.7, we conclude that $l'(x) \geq 0$. Then $l(x) \leq l(\alpha)$.

For $x > \alpha$,

$$|f_h(x)| \leq \left|\left|h(x) - \mathbb{E}h(W)\right|\right|\frac{1}{s(x)g(x)}\int_x^{+\infty} g(t)\mathrm{d}t.$$

Let $u(x) = \dfrac{1}{s(x)g(x)} \displaystyle\int_x^{+\infty} g(t)\mathrm{d}t$. The function u is differentiable on $(0,\infty)$ and

$$u'(x) = \frac{-\tau(x)}{s^2(x)g(x)} \int_x^{+\infty} g(t)\mathrm{d}t - \frac{1}{s(x)}.$$

By Lemma 2.7, we conclude that $u'(x) \leq 0$. Then $u(x) \leq u(\alpha)$. □

Proof of item 2 of Proposition 2.5.

τ is increasing and positive on $[0,\beta]$, so that $\dfrac{\tau(t)}{\tau(0)} \geq 1$ for $0 \leq t \leq \beta$. Thus, for $x \leq \beta$,

$$\begin{aligned}
|f_h(x)| &= \left| \frac{1}{s(x)g(x)} \int_0^x g(t)\left[h(t) - \mathbb{E}h(W)\right]\mathrm{d}t \right| \\
&\leq \frac{1}{s(x)g(x)} \int_0^x g(t)\left|h(t) - \mathbb{E}h(W)\right|\mathrm{d}t \\
&\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(x)g(x)} \int_0^x g(t)\mathrm{d}t \\
&\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(x)g(x)} \int_0^x \frac{\tau(t)}{\tau(0)} g(t)\mathrm{d}t \\
&= \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{\tau(0)s(x)g(x)} \int_0^x \left(s(t)g(t)\right)' \mathrm{d}t \\
&= \frac{1}{\tau(0)} \left\|h(\cdot) - \mathbb{E}h(W)\right\|.
\end{aligned}$$

For $\beta \leq x \leq \alpha$, we have again

$$\begin{aligned}
|f_h(x)| &\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(x)g(x)} \int_0^x g(t)\mathrm{d}t \\
&\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(x)g(x)} \\
&\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(\beta)g(\beta)}
\end{aligned}$$

because $x \mapsto s(x)g(x)$ is increasing on $[\beta,\alpha]$ (its derivative is $\tau(x)g(x)$ which is positive on $[\beta,\alpha]$).

For $x > \alpha$, an argument similar to the one used in the proof of Proposition 2.5 yields

$$\begin{aligned}
|f_h(x)| &\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(\alpha)g(\alpha)} \int_x^{+\infty} g(t)\mathrm{d}t \\
&\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(\alpha)g(\alpha)} \\
&\leq \left\|h(\cdot) - \mathbb{E}h(W)\right\| \frac{1}{s(\beta)g(\beta)}.
\end{aligned}$$

□

Remark 2.8. Note that [13] obtained bounds for the solution of the Stein equation and its first derivative in another general context where τ is not decreasing, under other assumptions that are not needed in the GIG and Kummer cases, where we obtain more explicit bounds, as shown in the two next sections.

3. About the Stein equation of the generalized inverse Gaussian distribution

Recall that the density of the GIG distribution with parameters $p \in \mathbb{R}$, $a > 0$, $b > 0$ is

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0, \tag{3.1}$$

where K_p is the modified Bessel function of the third kind.

Let

$$s(x) = x^2 \quad \text{and} \quad \tau_{p,a,b}(x) = \frac{b}{2} + (p+1)x - \frac{a}{2}x^2. \tag{3.2}$$

Then, as observed by [2], the GIG density $g_{p,a,b}$ satisfies

$$(s(x)g_{p,a,b}(x))' = \tau_{p,a,b}(x)g_{p,a,b}(x). \tag{3.3}$$

This enables us to apply Theorem 2.1 to retrieve the following Stein characterization of the GIG distribution given in [7] and [2]:

Proposition 3.1. *A random variable X follows the GIG distribution with density $g_{p,a,b}$ if and only if, for all real-valued and differentiable function f such that $\lim_{x\to\infty} g_{p,a,b}(x)f(x) = \lim_{x\to 0} g_{p,a,b}(x)f(x) = 0$, and such that the following expectation exists, we have:*

$$\mathbb{E}\left[X^2 f'(X) + \left(\frac{b}{2} + (p+1)X - \frac{a}{2}X^2\right) f(X)\right] = 0.$$

The corresponding Stein equation is

$$x^2 f'(x) + \left(\frac{b}{2} + (p+1)x - \frac{a}{2}x^2\right) f(x) = h(x) - \mathbb{E}h(W) \tag{3.4}$$

where h is a bounded function and W a random variable following the *GIG* distribution with parameters p, a, b.

3.1. Bound of the solution for $p \leq -1$

We apply Propositions 2.4 and 2.5 to solve equation (3.4) and bound the solution. Let us check that the assumptions of these propositions are true in the GIG case.

Firstly, we note that, by (3.1),

$$s(x)g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p+1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0,$$

which shows that $\lim_{x\to\infty} s(x)g_{p,a,b}(x) = \lim_{x\to 0} s(x)g_{p,a,b}(x) = 0$.

Secondly, observe that if $p \leq -1$, then the function $\tau_{p,a,b}$ defined by (3.2) is decreasing on $(0,\infty)$ and that its only zero on $(0,\infty)$ is $\alpha = \dfrac{p+1+\sqrt{(p+1)^2+ab}}{a}$.

Thus, by using Propositions 2.4 and 2.5, we obtain the following theorem.

Theorem 3.2. *The GIG Stein equation (3.4) has solution*

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)g_{p,a,b}(x)} \int_0^x g_{p,a,b}(t)\,[h(t)-\mathbb{E}h(W)]\,\mathrm{d}t \\ =& \frac{-1}{s(x)g_{p,a,b}(x)} \int_x^{+\infty} g_{p,a,b}(t)\,[h(t)-\mathbb{E}h(W)]\,\mathrm{d}t \end{aligned} \tag{3.5}$$

where W follows the GIG distribution with parameters p,a,b.

If h is a bounded continuous function and $p \leq -1$, then the function defined by (3.5) is the unique bounded solution of (3.4) and

$$||f_h|| \leq M\,||h(\cdot)-\mathbb{E}h(W)||$$

where

$$\alpha = \frac{p+1+\sqrt{(p+1)^2+ab}}{a}, \tag{3.6}$$

$$M = \max\left(\frac{1}{\alpha^2 g_{p,a,b}(\alpha)}\int_0^\alpha g_{p,a,b}(t)\mathrm{d}t; \frac{1}{\alpha^2 g_{p,a,b}(\alpha)}\int_\alpha^{+\infty} g_{p,a,b}(t)\mathrm{d}t\right). \tag{3.7}$$

Remark 3.3. This result was claimed by Gaunt (see [2]) with $\alpha = \mathbb{E}(W)$ by applying Proposition 1 of [12]. The only slight mistake is that $\tau_{p,a,b}$ is not a polynomial function of degree one as in [12].

Remark 3.4. By bounding by 1 the integrals in equation (3.7), one obtains

$$\begin{aligned} ||f_h|| &\leq \frac{1}{\alpha^2 g_{p,a,b}(\alpha)}\,||h(\cdot)-\mathbb{E}h(W)|| \\ &= \frac{a^2}{\left(ab+2(p+1)\left(p+1+\sqrt{(p+1)^2+ab}\right)\right) g_{p,a,b}\left(\dfrac{p+1+\sqrt{(p+1)^2+ab}}{a}\right)}\,||h(\cdot)-\mathbb{E}h(W)||\,. \end{aligned}$$

3.2. Bound of the solution for $p > -1$

Theorem 3.5. *Consider $p > -1$, $a > 0$, $b > 0$ and α defined by (3.6). Let $h : (0,+\infty) \to \mathbb{R}$ be a continuous bounded function. The solution f_h given by (3.5) of the Stein equation for the GIG(p,a,b) distribution with density $g_{p,a,b}$ is such that*

$$\|f_h\| \leq \max\left(\frac{2}{b}, \frac{a^2}{(p+1)^2 g_{p,a,b}(\frac{p+1}{a})}\right)\|h(\cdot)-\mathbb{E}h(W)\|\,. \tag{3.8}$$

Proof. We use the item 2 of Proposition 2.5 with $\tau(x) = \tau_{p,a,b}(x) = \frac{b}{2} + (p+1)x - \frac{a}{2}x^2$, $\tau(0) = b/2$ and $\beta = (p+1)/a$. □

3.3. Bound of the first derivative

Theorem 3.6. *Let $W \sim GIG(p,a,b)$. Consider again α defined by (3.6), let*

$$\alpha_1 = \frac{p+3+\sqrt{(p+3)^2+ab}}{a}$$

and let M be given by

$$M = \begin{cases} \max\left(\dfrac{1}{\alpha^2 g_{p,a,b}(\alpha)}\displaystyle\int_0^{\alpha} g_{p,a,b}(t)\mathrm{d}t; \dfrac{1}{\alpha^2 g_{p,a,b}(\alpha)}\displaystyle\int_{\alpha}^{+\infty} g_{p,a,b}(t)\mathrm{d}t\right) & \text{if } p \leq -1 \\ \max\left(\frac{2}{b}, \dfrac{a^2}{(p+1)^2 g_{p,a,b}(\frac{p+1}{a})}\right) & \text{if } p > -1. \end{cases} \tag{3.9}$$

Let $h : (0,+\infty) \to \mathbb{R}$ be a bounded, differentiable function with bounded derivative h'. Consider

$$C_1 = \frac{2}{b}\left(\|h'\| + \max(2,|p+1|)M\,\|h(\cdot) - \mathbb{E}h(W)\|\right),$$

$$C_2 = \left(\|h'\| + (2+\sqrt{(p+3)^2+ab})M\,\|h(\cdot) - \mathbb{E}h(W)\|\right)\frac{1}{\alpha_1^2 g_{p+2,a,b}(\alpha_1)}\int_0^{\alpha_1} g_{p+2,a,b}(t)\mathrm{d}t,$$

$$C_3 = \frac{2}{\alpha_1^2}\,\|h(\cdot) - \mathbb{E}h(W)\|,$$

$$C_4 = \left(\|h'\| + (2+\sqrt{(p+3)^2+ab})M\,\|h(\cdot) - \mathbb{E}h(W)\|\right)\frac{a^2}{(p+3)^2 g_{p+2,a,b}(\frac{p+3}{a})}.$$

The derivative f_h' of the solution f_h of equation (3.4) is such that

$$\|f_h'\| \leq M' = \begin{cases} \max(C_2, C_3) & \text{if } p \leq -3 \\ \max(C_1, C_3, C_4) & \text{if } p > -3. \end{cases} \tag{3.10}$$

Proof. First, assume that $p \leq -3$.

f_h satisfies the differential equation

$$x^2 f''(x) + \left(\frac{b}{2} + (p+3)x - \frac{a}{2}x^2\right) f'(x) = h'(x) - (p+1-ax)f(x). \tag{3.11}$$

Let $h_1(x) = h'(x) - (p+1-ax)f_h(x)$. Then, by equation (3.11), f_h' solves the differential equation

$$x^2 f'(x) + \left(\frac{b}{2} + (p+3)x - \frac{a}{2}x^2\right) f(x) = h_1(x). \tag{3.12}$$

Since $\mathbb{E}h_1(Z) = \mathbb{E}(Z^2 f'(Z) + \left(\frac{b}{2} + (p+3)Z - \frac{a}{2}Z^2\right) f(Z) = 0$ if $Z \sim GIG(p+2,a,b)$ by the Stein characterization of $GIG(p+2,a,b)$, then equation (3.12) is the $GIG(p+2,a,b)$ Stein equation. An instance of such a strategy can be found in [3] when bounding the derivatives of the Gamma Stein equation.

Thus, for $p \leq -3$, *i.e.* $p+2 \leq -1$, we can apply the same argument as in Theorem 3.2 to state that if $x < \alpha_1$ and h_1 is bounded, then

$$|f_h'(x)| \leq \|h_1\| \frac{1}{s(\alpha_1)g_1(\alpha_1)} \int_0^{\alpha_1} g_1(t)\mathrm{d}t,$$

where $g_1 := g_{p+2,a,b}$ is the density of the $GIG(p+2,a,b)$ distribution and $\alpha_1 = \dfrac{p+3+\sqrt{(p+3)^2+ab}}{a}$.

We have

$$\begin{aligned}
|h_1(x)| &= |h'(x) - (p+1-ax)f(x)| \\
&\leq \|h'\| + |p+1-ax|\,\|f_h\| \\
&= \|h'\| + (-p-1+ax)\,\|f_h\| \quad (p+1-ax<0 \ \ \forall p \leq -3, x>0) \\
&\leq \|h'\| + (-p-1+a\alpha_1)\,\|f_h\| \quad (x<\alpha_1) \\
&= \|h'\| + (2+\sqrt{(p+3)^2+ab})\,\|f_h\| \\
&\leq \|h'\| + (2+\sqrt{(p+3)^2+ab})M\,\|h(\cdot) - \mathbb{E}h(W)\|\,.
\end{aligned}$$

Hence, for $p \leq -3$ and $x \leq \alpha_1$,

$$|f_h'(x)| \leq \left(\|h'\| + (2+\sqrt{(p+3)^2+ab})M\,\|h(\cdot) - \mathbb{E}h(W)\|\right) \frac{1}{s(\alpha_1)g_1(\alpha_1)} \int_0^{\alpha_1} g_1(t)\mathrm{d}t.$$

If $p \leq -3$ and $x \geq \alpha_1$, the function $\tau := \tau_{p,a,b}$ (given by (3.2)) is decreasing and negative. By (3.5) and (3.3),

$$\begin{aligned}
f_h'(x) &= \frac{h(x) - \mathbb{E}h(W)}{s(x)} - \frac{\tau(x)f_h(x)}{s(x)} \\
|f_h'(x)| &\leq \frac{\|h(\cdot) - \mathbb{E}h(W)\|}{s(x)} + \frac{|\tau(x)f_h(x)|}{s(x)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|\tau(x)f_h(x)| &= \left|\frac{-\tau(x)}{s(x)g(x)} \int_x^{+\infty} g(t)[h(t) - \mathbb{E}h(W)]\mathrm{d}t\right| \\
&\leq \|h(\cdot) - \mathbb{E}h(W)\|\,\frac{-\tau(x)}{s(x)g(x)} \int_x^{+\infty} g(t)\mathrm{d}t \quad (\tau \ \text{ is negative}) \\
&\leq \|h(\cdot) - \mathbb{E}h(W)\|\,\frac{-\tau(x)}{s(x)g(x)} \int_x^{+\infty} \frac{\tau(t)}{\tau(x)} g(t)\mathrm{d}t \quad \left(\frac{\tau(t)}{\tau(x)} \geq 1 \ \ \forall x \leq t\right) \\
&= \|h(\cdot) - \mathbb{E}h(W)\|\,.
\end{aligned}$$

Hence $|f_h'(x)| \leq \dfrac{2}{s(x)}\,\|h(\cdot) - \mathbb{E}h(W)\| \leq \dfrac{2}{\alpha_1^2}\,\|h(\cdot) - \mathbb{E}h(W)\|\,.$

Now assume that $p > -3$.

On $\left(0, \frac{p+3}{a}\right)$, $\tau_1 := \tau_{p+2,a,b}$ is such that $\frac{\tau_1(t)}{\tau_1(0)} \geq 1 \ \forall 0 \leq t \leq \frac{p+3}{a}$ so that

$$\begin{aligned} |h_1(x)| &\leq \|h'\| + |p+1-ax| \, \|f_h\| \\ &\leq \|h'\| + \max(2, |p+1|) \, \|f_h\| \\ &\leq \|h'\| + \max(2, |p+1|) M \, \|h(\cdot) - \mathbb{E}h(W)\| , \end{aligned}$$

$$\begin{aligned} |f_h'(x)| &\leq \|h_1\| \, \frac{1}{s(x)g_1(x)} \int_0^x g_1(t)\mathrm{d}t \\ &\leq \|h_1\| \, \frac{1}{\tau_1(0)s(x)g_1(x)} \int_0^x \tau_1(t)g_1(t)\mathrm{d}t \\ &= \frac{1}{\tau_1(0)} \, \|h_1\| \\ &= \frac{2}{b} \, \|h_1\| \end{aligned}$$

and

$$\|f_h'\| \leq \frac{2}{b} \left(||h'|| + \max(2, |p+1|) M \, \|h(\cdot) - \mathbb{E}h(W)\| \right).$$

On $\left(\frac{p+3}{a}, \infty\right)$, the function τ_1 is decreasing and we use the same arguments as in the case $p \leq -3$. □

3.4. Bound of the second derivative

Theorem 3.7. *Let $W \sim GIG(p, a, b)$. Define $\alpha_2 = \dfrac{p+5+\sqrt{(p+5)^2+ab}}{a}$ and denote by $g_2 := g_{p+4,a,b}$ the density of the $GIG(p+4, a, b)$ distribution.*

Let $h : (0, \infty) \to \mathbb{R}$ be a bounded, twice differentiable function such that h' and h'' are bounded. The second derivative f_h'' of the solution f_h of equation (3.4) is such that

$$\|f_h''\| \leq M'' = \begin{cases} \max\left(C_5, C_6\right) & \text{if } \ p \leq -5 \\ \max\left(C_4, C_6, C_7\right) & \text{if } \ p > -5 \end{cases} \tag{3.13}$$

where

$$C_4 = \frac{2}{b} \left(\|h''\| + \max(6, |2p+4|) M' + aM \, \|h(\cdot) - \mathbb{E}h(W)\| \right),$$

$$C_5 = \left(\|h''\| + (6 + 2\sqrt{(p+5)^2+ab}) M' + aM \, \|h(\cdot) - \mathbb{E}h(W)\| \right) \frac{1}{\alpha_2^2 g_2(\alpha_2)} \int_0^{\alpha_2} g_2(t)\mathrm{d}t,$$

$$C_6 = \frac{2}{\alpha_2^2} \left(\|h'\| + \frac{4a + a\sqrt{(p+5)^2+ab}}{4p+20+4\sqrt{(p+5)^2+ab}} \, \|h(\cdot) - \mathbb{E}h(W)\| \right),$$

$$C_7 = \left(\|h''\| + (6 + 2\sqrt{(p+5)^2 + ab})M' + aM \|h(\cdot) - \mathbb{E}h(W)\| \right) \frac{a^2}{(p+5)^2 g_2(\frac{p+5}{a})}$$

and M is given by (3.9) *and M' by* (3.10).

The proof, similar to that of the bound of the first derivative, is given in appendix.

4. About the Stein equation related to the Kummer distribution

Recall that for $a > 0$, $b \in \mathbb{R}$, $c > 0$, the Kummer distribution $K(a, b, c)$ has density

$$k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^{a-1}(1+x)^{-a-b}e^{-cx}, \ (x > 0)$$

where ψ is the confluent hypergeometric function of the second kind. Let

$$s(x) = x(1+x) \quad \text{and} \quad \tau(x) = (1-b)x - cx(1+x) + a. \tag{4.1}$$

Then

$$(s(x)k_{a,b,c}(x))' = \tau(x)k_{a,b,c}(x).$$

Then we can use Theorem 2.1 to obtain the following Stein characterization of the Kummer distribution:

Theorem 4.1. *A random variable X follows the $K(p, a, b)$ distribution if and only if, for all differentiable functions f such that the expectation exists,*

$$\mathbb{E}\left[X(X+1)(f'(X) + ((1-b)X - cX(1+X) + a)\, f(X)\right] = 0.$$

The corresponding Stein equation is

$$x(x+1)f'(x) + [(1-b)x - cx(1+x) + a]\, f(x) = h(x) - \mathbb{E}h(W) \tag{4.2}$$

where W has density $k_{a,b,c}$.

We have

$$s(x)k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^a(1+x)^{1-a-b}e^{-cx}, \quad x > 0$$

and we see that $\lim_{x\to\infty} s(x)k_{a,b,c}(x) = \lim_{x\to 0} s(x)k_{a,b,c}(x) = 0$.

4.1. Bound of the solution in the case $1 - b - c \leq 0$

Note that if $1 - b - c \leq 0$, then the function τ defined by (4.1) is decreasing on $(0, \infty)$ and its only zero on this interval is

$$\alpha = \frac{1 - b - c + \sqrt{(1-b-c)^2 + 4ac}}{2c}. \tag{4.3}$$

Then we use again Proposition 2.4 and Proposition 2.5 to obtain the following result:

Theorem 4.2. *The Kummer Stein equation (4.2) has solution*

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)k_{a,b,c}(x)} \int_0^x k_{a,b,c}(t)\left[h(t) - \mathbb{E}h(W)\right] \mathrm{d}t \\ =& \frac{-1}{s(x)k_{a,b,c}(x)} \int_x^{+\infty} k_{a,b,c}(t)\left[h(t) - \mathbb{E}h(W)\right] \mathrm{d}t \end{aligned} \tag{4.4}$$

where $W \sim K(a,b,c)$.

If h *is a bounded continuous function and* $1-b-c \leq 0$, *then* f_h *defined by (4.4) is the unique bounded solution of (4.2) and*

$$||f_h|| \leq M\, ||h(\cdot) - \mathbb{E}h(W)||$$

where

$$M = \max\left(\frac{1}{s(\alpha)k_{a,b,c}(\alpha)} \int_0^{\alpha} k_{a,b,c}(t)\mathrm{d}t; \frac{1}{s(\alpha)k_{a,b,c}(\alpha)} \int_{\alpha}^{+\infty} k_{a,b,c}(t)\mathrm{d}t\right). \tag{4.5}$$

Remark 4.3. By bounding by 1 the integrals in equation (4.5), one obtains

$$\begin{aligned} \|f_h\| &\leq \frac{1}{\alpha(\alpha+1)k_{a,b,c}(\alpha)} \|h(\cdot) - \mathbb{E}h(W)\| \\ &= \frac{2c^2}{\left(2ac + (1-b)\left(1-b-c+\sqrt{(1-b-c)^2+4ac}\right)\right) k_{a,b,c}\left(\frac{1-b-c+\sqrt{(1-b-c)^2+4ac}}{2c}\right)} \|h(\cdot) - \mathbb{E}h(W)\|. \end{aligned}$$

4.2. Bound of the solution in the case $1-b-c>0$

Theorem 4.4. *Let* $h:(0,\infty)\to\mathbb{R}$ *be a continuous, bounded function. If* $1-b-c>0$, *then the solution* f_h *of the Stein equation (4.2) for the Kummer distribution with parameters* $a>0$, $b\in\mathbb{R}$, $c>0$, *is such that*

$$\|f_h\| \leq K\, \|h(\cdot) - \mathbb{E}h(W)\|$$

where

$$K = \max\left(\frac{1}{a}, \frac{c^2}{(1-b)(1-b-c)k_{a,b,c}(\frac{1-b-c}{c})}\right).$$

Proof. We apply Proposition 2.5. □

In the sequel we give the results bounding the first and second derivatives of the solution of the Kummer Stein equation. We do not give the proofs because the arguments are similar to those used for the GIG Stein equation.

4.3. Bound of the derivative

Theorem 4.5. *Let* h *be a continuous bounded function such that the derivative* h' *of* h *is bounded. Consider the solution* f_h *of the Stein equation (4.2) related to the Kummer distribution with parameters* a,b,c.

Let $\alpha_1 = \dfrac{3-b-c+\sqrt{(3-b-c)^2+4c(a+1)}}{2c}$. *Then*

$$\|f_h'\| \leq K' = \begin{cases} \max(K_2, K_3) & \text{if } 3-b-c \leq 0 \\ \max(K_1, K_3, K_4) & \text{if } 3-b-c > 0 \end{cases} \tag{4.6}$$

where

$$K_1 = \frac{1}{a+1}\left(\|h'\| + \max(2, |1-b-c|)K\,\|h(\cdot) - \mathbb{E}h(W)\|\right),$$

$$K_2 = \left(\|h'\| + (2+\sqrt{(3-b-c)^2+4c(a+1)})K\,\|h(\cdot) - \mathbb{E}h(W)\|\right) \frac{1}{\alpha_1(1+\alpha_1)k_{a+1,b-2,c}(\alpha_1)} \int_0^{\alpha_1} k_{a+1,b-2,c}(t)\mathrm{d}t,$$

$$K_3 = \frac{2}{\alpha_1(1+\alpha_1)}\,\|h(\cdot) - \mathbb{E}h(W)\|,$$

$$K_4 = \left(\|h'\| + (2+\sqrt{(3-b-c)^2+4c(a+1)})K\,\|h(\cdot) - \mathbb{E}h(W)\|\right) \frac{c^2}{(3-b)(3-b-c)k_{a+1,b-2,c}(\frac{3-b-c}{c})}$$

and K *is given by*

$$K = \begin{cases} \max\left(\dfrac{1}{s(\alpha)k_{a,b,c}(\alpha)}\displaystyle\int_0^{\alpha} k_{a,b,c}(t)\mathrm{d}t; \dfrac{1}{s(\alpha)k_{a,b,c}(\alpha)}\displaystyle\int_{\alpha}^{+\infty} k_{a,b,c}(t)\mathrm{d}t\right) & \text{if } 1-b-c \leq 0 \\ \max\left(\frac{1}{a}, \dfrac{c^2}{(1-b)(1-b-c)k_{a,b,c}\left(\frac{1-b-c}{c}\right)}\right) & \text{if } 1-b-c > 0. \end{cases} \tag{4.7}$$

4.4. Bound of the second derivative

Theorem 4.6. *Let* h *be a bounded, twice differentiable function such that* h' *and* h'' *are bounded. Let*

$$\alpha_2 = \frac{5-b-c+\sqrt{(5-b-c)^2+4c(a+2)}}{2c}.$$

The second derivative f_h'' *of the solution* f_h *of equation* (4.2) *is such that*

$$\|f_h''\| \leq K'' = \begin{cases} \max(K_5, K_6) & \text{if } 5-b-c \leq 0 \\ \max(K_6, K_7, K_8) & \text{if } 5-b-c > 0 \end{cases}$$

where

$$K_5 = \left(\|h''\| + (6+2\sqrt{(5-b-c)^2+4c(a+2)})K' + aK\,\|h(\cdot) - \mathbb{E}h(W)\|\right) \times \frac{1}{\alpha_2(\alpha_2+1)k_{a+2,b-4,c}(\alpha_2)} \int_0^{\alpha_2} k_{a+2,b-4,c}(t)\mathrm{d}t,$$

$$K_6 = \frac{2}{\alpha_2(1+\alpha_2)}\left(||h'|| + \frac{4c + c\sqrt{(5-b-c)^2+4c(a+2)}}{10-2b+2\sqrt{(5-b-c)^2+4c(a+2)}}\,\|h(\cdot)-\mathbb{E}h(W)\|\right),$$

$$K_7 = \frac{1}{a+2}\left(\|h''\| + \max(6,|4-2b-2c|)K' + 2cK\,\|h(\cdot)-\mathbb{E}h(W)\|\right),$$

$$K_8 = \left(\|h''\| + (6+2\sqrt{(5-b-c)^2+4c(a+2)})K' + aK\,\|h(\cdot)-\mathbb{E}h(W)\|\right) \times \frac{c^2}{(5-b)(5-b-c)k_{a+2,b-4,c}(\frac{5-b-c}{c})},$$

K is given by (4.7) and K' by (4.6).

5. Potential applications

These results could be used to provide rates of convergence in limit problems related to the GIG and Kummer distributions. Here are two instances of applications that will be considered in future work.

5.1. The GIG distribution as the law of a continued fraction

Theorem 5.1. [8]

- *Let X and Y be two independent random variables such that $X > 0$ and $Y \sim \gamma(p, a/2)$ for $p, a > 0$, where $\gamma(p, c)$ is the gamma distribution with density proportional to $x^{p-1}e^{-cx}$.*
  Then $X =_d \frac{1}{Y+X}$ if and only if $X \sim \mathrm{GIG}(-p, a, a)$.
- *Let X, Y_1 and Y_2 be three independent random variables such that $X > 0$, $Y_1 \sim \gamma(p, b/2)$ and $Y_2 \sim \gamma(p, a/2)$ for $p, a, b > 0$. Then $X =_d \frac{1}{Y_1+\frac{1}{Y_2+X}}$ if and only if $X \sim \mathrm{GIG}(-p, a, b)$.*
- *If $(Y_i)_{i\geq 1}$ is a sequence of independent random variables such that*

$$\mathcal{L}(Y_{2i-1}) = \mathcal{L}(Y_1) = \gamma(\lambda, b/2) \quad \text{and} \quad \mathcal{L}(Y_{2i}) = \mathcal{L}(Y_2) = \gamma(\lambda, a/2); \quad i \geq 1,$$

then

$$\mathcal{L}\left(\cfrac{1}{Y_1 + \cfrac{1}{Y_2 + \cfrac{1}{Y_3 + \ddots}}}\right) = GIG(-\lambda, a, b).$$

A natural continuation of this work would be to derive bounds for the distance between

$$\mathcal{L}\left(\frac{1}{Y_1 + \frac{1}{Y_2 + \frac{1}{\cdots + \frac{1}{Y_n}}}}\right)$$

and $\mathrm{GIG}(p, a, b)$.

5.2. The Kummer distribution as the law of a continued fraction

Theorem 5.2. [5]

- *Let X, Y_1 and Y_2 be three independent random variables such that $Y_1 \sim \gamma(a,c)$ and $Y_2 \sim \gamma(a+b,c)$ for $a,c>0$ and $b>-a$. Then $X =_d \frac{Y_1}{1+\frac{Y_2}{1+X}}$ if and only if $X \sim \mathrm{K}(a,b,c)$.*
- *If $(Y_i)_{i\geq 1}$ is a sequence of independent random variables such that*

$$\mathcal{L}(Y_{2i-1}) = \mathcal{L}(Y_1) = \gamma(a,c) \quad \text{and} \quad \mathcal{L}(Y_{2i}) = \mathcal{L}(Y_2) = \gamma(a+b,c); \quad i \geq 1,$$

then

$$\mathcal{L}\left(\cfrac{Y_1}{1+\cfrac{Y_2}{1+\cfrac{Y_3}{1++\ddots}}}\right) = K(a,b,c).$$

Finding bounds for the distance between

$$\mathcal{L}\left(\cfrac{Y_1}{1+\cfrac{Y_2}{1+\cfrac{Y_3}{1+\cdots+\frac{Y_{n-1}}{1+Y_n}}}}\right)$$

and $\mathrm{K}(a,b,c)$ may be another question of interest.

Appendix A. Proof of Theorem 3.7

First, assume that $p \leq -5$. f_h'' satisfies the differential equation

$$x^2 f^{(3)}(x) + \left(\frac{b}{2} + (p+5)x - \frac{a}{2}x^2\right) f''(x) = h''(x) - 2(p+2-ax)f'(x) + af(x). \tag{A.1}$$

Let $h_2(x) = h''(x) - 2(p+2-ax)f_h'(x) + af_h(x)$. Equation (A.1) shows that f_h'' solves the Stein equation for $GIG(p+4,a,b)$ distribution with right-hand side $h_2(x) = h''(x) - 2(p+2-ax)f_h'(x) + af_h(x)$. Let

$$s(x) = x^2 \quad \text{and} \quad \tau_2(x) = \frac{b}{2} + (p+5)x - \frac{a}{2}x^2,$$

we have

$$(s(x)g_2(x))' = \tau_2(x)g_2(x)$$

where g_2 is the density of $GIG(p+4,a,b)$ distribution.

For $p \leq -5$, τ_2 is decreasing on $(0,\infty)$ and its only zero is $\alpha_2 = \dfrac{p+5+\sqrt{(p+5)^2+ab}}{a}$. If $x < \alpha_2$,

Theorem 3.5 shows that, if h_2 is bounded, the solution f_h'' of equation (A.1) is such that

$$\|f_h''\| \leq \|h_2\| \frac{1}{s(\alpha_2)g_2(\alpha_2)} \int_0^{\alpha_2} g_2(t)\mathrm{d}t.$$

We have

$$\begin{aligned}
|h_2(x)| &= |h''(x) - 2(p+2-ax)f_h'(x) + af_h(x)| \\
&\leq \|h''\| + 2|p+2-ax|\,\|f_h'\| + a\,\|f_h\| \\
&= \|h''\| + 2(-p-2+ax)\,\|f_h'\| + a\,\|f_h\| \quad (p+2-ax<0 \ \ \forall p\leq -5, x>0) \\
&\leq \|h''\| + 2(-p-2+a\alpha_2)\,\|f_h'\| + a\,\|f_h\| \quad (x<\alpha_2) \\
&= \|h''\| + (6+2\sqrt{(p+5)^2+ab})\,\|f_h'\| + a\,\|f_h\| \\
&\leq \|h''\| + (6+2\sqrt{(p+5)^2+ab})M' + aM\,\|h(\cdot)-\mathbb{E}h(W)\|\,.
\end{aligned}$$

Hence for $p\leq -5$ and $x\leq \alpha_2$,

$$\begin{aligned}
|f_h''(x)| \leq &\left(\|h''\| + (6+2\sqrt{(p+5)^2+ab})M' + aM\,\|h(\cdot)-\mathbb{E}h(W)\|\right) \\
&\times \frac{1}{s(\alpha_2)g_2(\alpha_2)} \int_0^{\alpha_2} g_2(t)\mathrm{d}t.
\end{aligned}$$

If $p\leq -5$ and $x\geq \alpha_2$, the function τ_1 defined by $\tau_1(x) = \frac{b}{2} + (p+3)x - \frac{a}{2}x^2$ is decreasing and negative. By equation (3.11),

$$\begin{aligned}
f_h''(x) &= \frac{h_1(x)}{s(x)} - \frac{\tau_1(x)f_h'(x)}{s(x)} \\
|f_h''(x)| &\leq \frac{\|h_1\|}{s(x)} + \frac{|\tau_1(x)f_h'(x)|}{s(x)} \\
|\tau_1(x)f_h'(x)| &= \left|\frac{-\tau_1(x)}{s(x)g_1(x)}\int_x^{+\infty} g_1(t)h_1(t)\mathrm{d}t\right| \\
&\leq \|h_1\|\frac{-\tau_1(x)}{s(x)g_1(x)}\int_x^{+\infty} g_1(t)\mathrm{d}t \quad (\tau_1 \text{ is negative}) \\
&\leq \|h_1\|\frac{-\tau_1(x)}{s(x)g_1(x)}\int_x^{+\infty} \frac{\tau_1(t)}{\tau_1(x)}g_1(t)\mathrm{d}t \quad \left(\frac{\tau_1(t)}{\tau_1(x)}\geq 1 \ \ \forall x\leq t\right) \\
&= \|h_1\|\,.
\end{aligned}$$

$$h_1(x) = h'(x) - (p+1-ax)f_h(x) \implies |h_1(x)| \leq \|h'\| + |(p+1-ax)f_h(x)|\,.$$

If $p\leq -5$ and $x\geq \alpha_2$, the functions $\tau := \tau_{p,a,b}$ (given by (3.2)) and $x\mapsto p+1-ax$ are decreasing and negative. Then

$$\begin{aligned}
|(p+1-ax)f_h(x)| &= \left|-\frac{p+1-ax}{s(x)g_{p,a,b}(x)}\int_x^{+\infty} g_{p,a,b}(t)\left[h(t)-\mathbb{E}h(W)\right]\mathrm{d}t\right| \\
&\leq \|h(\cdot)-\mathbb{E}h(W)\|\frac{-p-1+ax}{s(x)g_{p,a,b}(x)\tau(x)}\int_x^{+\infty}\tau(t)g_{p,a,b}(t)\mathrm{d}t \quad \left(\frac{\tau(t)}{\tau(x)}\geq 1 \ \ \forall x\leq t\right) \\
&= \|h(\cdot)-\mathbb{E}h(W)\|\frac{p+1-ax}{\tau(x)}.
\end{aligned}$$

Let $\kappa(x) = \dfrac{p+1-ax}{\tau(x)}$. κ is differentiable on (α_2, ∞) and $\kappa'(x) = -\dfrac{ab + (p+1)^2 + (p+1-ax)^2}{2\tau^2(x)} \leq 0$. Hence for $p \leq -5$ and $x \geq \alpha_2$, $\kappa(x) \leq \kappa(\alpha_2) = \dfrac{p+1-a\alpha_2}{\tau(\alpha_2)} = \dfrac{4a + a\sqrt{(p+5)^2+ab}}{4(p+5) + 4\sqrt{(p+5)^2+ab}}$ and

$$\|f_h''\| \leq \frac{2}{\alpha_2^2}\left(\|h'\| + \frac{4a + a\sqrt{(p+5)^2+ab}}{4p+20+4\sqrt{(p+5)^2+ab}}\|h(\cdot) - \mathbb{E}h(W)\|\right).$$

Hence $\|f_h''\| \leq \dfrac{2}{\alpha_2^2}\left(\|h'\| + (4 + \sqrt{(p+5)^2+ab})M\|h(\cdot) - \mathbb{E}h(W)\|\right)$.

Now assume that $p > -5$.

On $\left(0, \frac{p+5}{a}\right)$, τ_2 is such that $\frac{\tau_2(t)}{\tau_2(0)} \geq 1 \ \forall 0 \leq t \leq \frac{p+5}{a}$, so that

$$\begin{aligned}
|h_2(x)| &\leq \|h''\| + 2|p+2-ax|\,\|f_h'\| + a\|f_h\| \\
&\leq \|h''\| + \max(6, 2p+4)\,\|f_h'\| + a\|f_h\| \\
&\leq \|h''\| + \max(6, 2p+4)M' + aM\|h(\cdot) - \mathbb{E}h(W)\|.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
|f_h''(x)| &\leq \|h_2\|\frac{1}{s(x)g_2(x)}\int_0^x g_2(t)\mathrm{d}t \\
&\leq \|h_2\|\frac{1}{\tau_2(0)s(x)g_2(x)}\int_0^x \tau_2(t)g_2(t)\mathrm{d}t \\
&= \frac{1}{\tau_2(0)}\|h_2\| \\
&= \frac{2}{b}\|h_2\|
\end{aligned}$$

and

$$\|f_h''\| \leq \frac{2}{b}\left(\|h''\| + \max(6, 2p+4)M' + aM\|h(\cdot) - \mathbb{E}h(W)\|\right).$$

On $\left(\frac{p+5}{a}, \infty\right)$, the function τ_2 is decreasing and similar arguments as in the case $p \leq -5$ lead to the result.

□

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