HAL open science

# SIMPLE FOLIATED FLOWS

Eric Leichtnam, Jesús a Álvarez López, Yuri A Kordyukov

**To cite this version:**

Eric Leichtnam, Jesús a Álvarez López, Yuri A Kordyukov. SIMPLE FOLIATED FLOWS. Tohoku mathematical journal, 2020. hal-02988733

**HAL Id: hal-02988733**

**https://hal.science/hal-02988733**

Submitted on 4 Nov 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SIMPLE FOLIATED FLOWS

JESÚS A. ÁLVAREZ LÓPEZ,* YURI A. KORDYUKOV* AND ERIC LEICHTNAM*

Abstract. We describe transversely oriented foliations of codimension one on closed manifolds that admit simple foliated flows.

## Contents

Introduction 1
1. Preliminaries 3
2. Some classes of foliations of codimension one 9
3. Transversely simple foliated flows 18
4. Case of a suspension foliation 21
5. Global structure 24
6. Existence and description of simple foliated flows 26
References 29

## Introduction

In this paper, we describe transversely oriented foliations of codimension one on closed manifolds that admit simple foliated flows. Our motivation to study simple foliated flows comes from the role that they play in Deninger's program [10, 11, 12, 13, 14]. These are exactly those foliated flows for which a dynamical Lefschetz trace formula conjectured by Deninger holds. For the study of the associated Lefschetz trace formula, we refer to [1, 2, 3, 4, 5]. A related classification of foliated dynamical systems was given in [29].

Let $\mathcal{F}$ be a smooth foliation of codimension one on a closed manifold $M$. Flows on $M$ are foliated when they map leaves to leaves. This means that their infinitesimal generators are infinitesimal transformations of $(M, \mathcal{F})$. These infinitesimal transformations form the normalizer $\mathfrak{X}(M, \mathcal{F})$ of the Lie subalgebra $\mathfrak{X}(\mathcal{F}) \subset \mathfrak{X}(M)$ of vector fields tangent to the leaves, obtaining the quotient Lie algebra $\overline{\mathfrak{X}}(M, \mathcal{F}) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$. The elements of

2010 *Mathematics Subject Classification.* Primary 57R30.

*Key words and phrases.* Foliation almost without holonomy, transversely affine foliation, transversely projective foliation, simple flow, foliated flow.

*The authors are partially supported by MTM2017-89686-P (AEI/FEDER, UE) and ED431C 2019/10 (Xunta de Galicia, FEDER).

$\overline{\mathfrak{X}}(M,\mathcal{F})$, called transverse vector fields, can be considered as leafwise invariant sections of the normal bundle of $\mathcal{F}$.

Let $(\Sigma,\mathcal{H})$ be the holonomy pseudogroup of $\mathcal{F}$. The infinitesimal generators of $\mathcal{H}$-equivariant local flows on $\Sigma$ are the $\mathcal{H}$-invariant vector fields. These invariant vector fields form a Lie subalgebra $\mathfrak{X}(\Sigma,\mathcal{H})\subset\mathfrak{X}(\Sigma)$. There is a canonical identity $\overline{\mathfrak{X}}(M,\mathcal{F})\equiv\mathfrak{X}(\Sigma,\mathcal{H})$, and every foliated flow $\phi$ induces an $\mathcal{H}$-equivariant local flow $\bar\phi$ on $\Sigma$.

Simple fixed points and simple closed orbits of a flow $\phi$ can be defined by using a transversality condition between the graph of $\phi$ and the diagonal. The flow is simple when all of its fixed points and closed orbits are simple. Using the canonical identity between leaf and orbit spaces, $M/\mathcal{F}\equiv\Sigma/\mathcal{H}$, the leaves preserved by a foliated flow $\phi$, which will be shortly called preserved leaves in the sequel, correspond to $\mathcal{H}$-orbits consisting of fixed points of $\bar\phi$. A preserved leaf $L$ is called transversely simple if the corresponding fixed points $\bar p$ of $\bar\phi$ are simple. In this case, $\bar\phi^t_*=e^{\varkappa t}$ on $T_{\bar p}\Sigma\equiv\mathbb{R}$ for some $\varkappa=\varkappa_L\in\mathbb{R}^\times:=\mathbb{R}\smallsetminus\{0\}$, which depends only on $L$. It is said that $\phi$ is transversely simple when all of its preserved leaves are transversely simple. Clearly, every simple flow is transversely simple.

Let $L$ be any compact leaf whose holonomy group $\operatorname{Hol}L$ can be described by germs of homotheties at 0. This description of $\operatorname{Hol}L$ can be achieved with a foliated chart $(U,(x,y))$ around any point of $L$, where $x$ is the transverse coordinate. The same kind of description of $\operatorname{Hol}L$ is given by the foliated chart $(U,(u,y))$, with $u=x\,|x|^{\alpha-1}$ $(0<\alpha\neq1)$, which is not smooth at $U\cap L$. A transverse power change of the differentiable structure around $L$ is defined by requiring all of these new charts to be smooth. In Sections 4.4 and 5.1, we give a description of this new differential structure in terms of a defining form of $\mathcal{F}$ and a defining function of $L$ on some tubular neighborhood.

The following is our main result, which is part of Theorem 6.9.

Theorem 0.1. *Let $\mathcal{F}$ be a transversely oriented smooth foliation of codimension one on a closed manifold $M$. Then $\mathcal{F}$ admits a (transversely) simple foliated flow in the following cases and uniquely in these ones:*

- $(i)$ *$\mathcal{F}$ is a fiber bundle over $S^1$ with connected fibers.*
- $(ii)$ *$\mathcal{F}$ is a minimal $\mathbb{R}$-Lie foliation.*
- $(iii)$ *$\mathcal{F}$ is an elementary transversely affine foliation whose developing map is surjective over $\mathbb{R}$, and whose global holonomy group is a non-trivial group of homotheties.*
- $(iv)$ *$\mathcal{F}$ is a transversely projective foliation whose developing map is surjective over the real projective line $S^1_\infty=\mathbb{R}\cup\{\infty\}$, and whose global holonomy group consists of the identity and hyperbolic elements with a common fixed point set.*
- $(v)$ *$\mathcal{F}$ is obtained from $(iii)$ or $(iv)$ using transverse power changes of the differentiable structure of $M$ around the compact leaves.*

In all cases of Theorem 0.1, $\mathcal{F}$ is almost without holonomy.

In the cases (i) and (ii), $\mathcal{F}$ is defined by a non-vanishing closed form $\omega$ of degree one, and therefore it is indeed without holonomy. The group of periods of $[\omega] \in H^1(M)$ has rank 1 in (i), and rank $> 1$ in (ii).

In the case (i), all leaves are compact and we have $\overline{\mathfrak{X}}(M, \mathcal{F}) \equiv \mathfrak{X}(S^1)$. Moreover, for any even number of points, $x_1, \dots, x_{2m} \in S^1$ ($m \geq 0$), in cyclic order, and numbers $\varkappa_1, \dots, \varkappa_{2m} \in \mathbb{R}^\times$, with alternate sign, there is some (transversely) simple foliated flow $\phi$ whose preserved leaves are the fibers $L_i$ over the points $x_i$, with $\varkappa_{L_i} = \varkappa_i$. If $m > 0$, then $\phi$ has no closed orbits transverse to the leaves. If $m = 0$, then $\phi$ has no preserved leaves, and therefore no fixed points. Every transversely simple foliated flow is of this form.

In the cases (ii)–(iv), $\overline{\mathfrak{X}}(M, \mathcal{F})$ is of dimension one.

In the case (ii), $\overline{\mathfrak{X}}(M, \mathcal{F})$ is generated by a non-vanishing transverse vector field, and the transversely simple foliated flows have no preserved leaves.

In the cases (iii)–(v), there is a finite number of compact leaves, which are the preserved leaves of every transversely simple foliated flow.

In the case (iii) or (iv), for every transversely simple flow $\phi$, there is some $\varkappa \in \mathbb{R}^\times$ such that the set of numbers $\varkappa_L$ is $\{\varkappa\}$ or $\{\pm\varkappa\}$, respectively.

In the cases (iii) and (iv), the holonomy groups of the compact leaves can be described by germs of homotheties at 0. Thus transverse power changes of the differentiable structure can be considered around them to get the case (v). $\overline{\mathfrak{X}}(M, \mathcal{F})$ and the (transversely) simple foliated flows are independent of these changes of the differentiable structure. But every $|\varkappa_L|$ can be modified arbitrarily by performing such changes, keeping $\operatorname{sign}(\varkappa_L)$ invariant.

**Acknowledgment.** We thank Hiraku Nozawa for helpful discussions about the contents of this paper.

## 1. Preliminaries

Let $M$ be a (smooth) manifold of dimension $n$.

1.1. **Simple flows.** Let $Z \in \mathfrak{X}(M)$ with local flow $\phi : \Omega \to M$, where $\Omega$ is an open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$. For $p \in M$ and $t \in \mathbb{R}$, let

$$\Omega_p = \{\, \tau \in \mathbb{R} \mid (p, \tau) \in \Omega \,\}\,, \quad \Omega^t = \{\, q \in M \mid (q, t) \in \Omega \,\}\,,$$

and let $\phi^t = \phi(\cdot, t) : \Omega^t \to M$. It is said that $p \in M$ is a *fixed point* of $\phi$ if it is a fixed point of $\phi^t$ for all $t$ in some neighborhood of 0 in $\Omega_p$; in other words, if $Z(p) = 0$. The fixed point set is denoted by $\operatorname{Fix}(\phi)$. For every $p \in \operatorname{Fix}(\phi)$, there is an endomorphism $H_p$ of $T_pM$ so that $\phi^t_* = e^{tH_p}$ on $T_pM$. Then $p$ is called *simple*[1] (respectively, *generic*) if $H_p$ is an automorphism (respectively, no eigenvalue of $H_p$ has zero real part).

Now assume that $Z$ is complete with flow $\phi : M \times \mathbb{R} \to M$, which may considered as a one-parameter subgroup of diffeomorphisms, $\phi = \{\phi^t\} \subset \operatorname{Diffeo}(M)$. On $M \smallsetminus \operatorname{Fix}(\phi)$,

[1]The terms *transverse*/*elementary* are also used instead of simple/generic.

let $N\phi$ denote the normal bundle to the orbits of $\phi$; i.e., $N_p\phi = T_pM/\mathbb{R}\,Z(p)$ for all $p \in M \smallsetminus \mathrm{Fix}(\phi)$. For every closed orbit $c$ of $\phi$ (without including fixed points), let $\ell(c)$ denote its smallest positive period. Recall that $c$ is called *simple* (respectively, *generic*) if the eigenvalues of the isomorphism of $N_p\phi$ induced by $\phi_*^{\ell(c)}$ are different from 1 (respectively, have modulo different from 1) for all $p \in c$.

It is said that $\phi$ (or $Z$) is *simple* if all of its fixed points and closed orbits are simple. This means that the maps $M \times \mathbb{R}^{\pm} \to M^2 \times \mathbb{R}^{\pm}$, $(p,t) \mapsto (p, \phi^t(p), t)$ and $(p,t) \mapsto (p,p,t)$, are transverse [20, Lecture 2, Lemma 7]. Thus fixed points and closed orbits are isolated in this case; there are finitely many of them if $M$ is compact.

On the other hand, $\phi$ (or $Z$) is called *generic* if all of its fixed points and closed orbits are generic, and their stable and unstable manifolds are transverse—the definition of the stable and unstable manifolds is omitted because we will not use them. A theorem of Kupka [31, 32] and Smale [39] states that, for any closed manifold $M$, the set of generic smooth vector fields on $M$ is residual in $\mathfrak{X}(M)$ with the $C^\infty$ topology (see also [34] for the case of closed surfaces). This was generalized to open manifolds by Peixoto [35], using the strong $C^\infty$ topology.

Remark 1.1. Suppose that $M$ is closed. For $0 < f \in C^\infty(M)$, let $Z' = fZ \in \mathfrak{X}(M)$. The flow $\phi'$ of $Z'$ has the same orbits as $\phi$, considered as sets, but with possibly different time parameterizations; precisely, there is a smooth function $t' : M \times \mathbb{R} \to \mathbb{R}$ such that $\phi(p,t) = \phi'(p, t'(p,t))$ for all $(p,t)$. It easily follows that $\phi'$ is simple if and only if $\phi$ is simple.

Example 1.2. Suppose that $M$ is closed, and let $f$ be a Morse function on $M$. For any Riemannian metric on $M$, the flow $\phi$ of $\nabla f$ has no closed orbits because $f$ is strictly increasing on every orbit in $M \smallsetminus \mathrm{Fix}(\phi)$. Moreover every $p \in \mathrm{Fix}(\phi)$ is generic because $H_p$ is given by $\mathrm{Hess}\, f(p)$, whose eigenvalues are in $\mathbb{R}^\times$. The transversality of the stable and unstable manifolds of all fixed points holds for an open dense set of Riemannian metrics in the $C^2$ topology [36, Section 2.3] (see also [38]). In this case, $\nabla f$ is generic without closed orbits.

1.2. **Collar and tubular neighborhoods.** Suppose that $M$ is compact with boundary, and let $\mathring{M}$ denote its interior. There exists a *boundary defining function* $x \in C^\infty(M)$, in the sense that $x \geq 0$, $x^{-1}(0) = \partial M$, and $dx \neq 0$ on $\partial M$. Then an (open) *collar neighborhood* of the boundary, $\varpi : T \to \partial M$, can be chosen of the form[2] $T \equiv [0,\epsilon)_x \times \partial M_\varpi$ for some $\epsilon > 0$. For any chart $(V, y)$ of $\partial M$, we get a chart $(U \equiv [0,\epsilon)_x \times V, (x,y))$ of $M$ adapted to $\partial M$.

Now assume that $M$ is closed. Let $M^0 \subset M$ be a (possibly disconnected) regular and transversely oriented submanifold of codimension one, and let $M^1 = M \smallsetminus M^0$. Since $M^0$

[2]In a product, the projections may be indicated as subindexes of the factors.

is transversely oriented, there is a *defining function* $x$ of $M^0$ in some open $W \subset M$, in the sense that $x \in C^\infty(W)$, $M^0 = x^{-1}(0) \subset W$, and $dx \neq 0$ on $M^0$. Then there is an (open) *tubular neighborhood* of $M^0$ in $W$, $\varpi : T \to M^0$, of the form $T \equiv (-\epsilon, \epsilon)_x \times M^0_\varpi$ for some $\epsilon > 0$. For any chart $(V, y)$ of $M^0$, we get a chart $(U \equiv (-\epsilon, \epsilon)_x \times V, (x, y))$ of $M$ adapted to $M^0$. Let $\mathbf{M}$ be the manifold with boundary defined by "cutting" $M$ along $M^0$; i.e., modifying $M$ only on the tubular neighborhood $T \equiv (-\epsilon, \epsilon) \times M^0$, which is replaced with $\mathbf{T} = ((-\epsilon, 0] \sqcup [0, \epsilon)) \times M^0$ in the obvious sense. Thus $\partial\mathbf{M} \equiv M^0 \sqcup M^0$, and $\mathring{\mathbf{M}} \equiv M^1$. There is a canonical projection $\boldsymbol{\pi} : \mathbf{M} \to M$, which is the combination of the identity on $\mathring{\mathbf{M}} \equiv M^1$ and the map $\mathbf{T} \to T$ induced by the canonical projection $(-\epsilon, 0] \sqcup [0, \epsilon) \to (-\epsilon, \epsilon)$. This projection realizes $M$ as a quotient space of $\mathbf{M}$ by "gluing" the two copies of $M^0$ in the boundary.

The connected components of $\mathbf{M}$ can be also described as the metric completion of the connected components of $M^1$ with respect to the restriction of any Riemannian metric on $M$, and then $\boldsymbol{\pi}$ is given by taking limits of Cauchy sequences.

1.3. **Foliations.** The concepts used here are explained in standard references on foliations, like [21, 25, 26, 7, 18, 40, 8, 9, 41]. Let $\mathcal{F}$ be a (smooth) *foliation*[3] on $M$ of codimension $n'$ and dimension $n''$. Locally, $\mathcal{F}$ can be described by a (smooth) *foliated chart* $(U, x)$, where $x = (x', x'') : U \to x(U) = \Sigma \times B''$ for open balls, $\Sigma$ in $\mathbb{R}^{n'}$ and $B''$ in $\mathbb{R}^{n''}$. In the case of codimension one, we may use the notation $(x, y)$ instead of $(x', x'')$. The fibers of $x'$ are the *plaques*. The intersections of plaques of different foliated charts are open in the plaques. Thus all plaques of all foliated charts form a base of a finer topology on $M$ whose path-connected components are the *leaves*, which are injectively immersed $n''$-submanifolds. The leaf through any $p \in M$ may be denoted by $L_p$. The submanifolds transverse to the leaves are called *transversals*; for example, the fibers of the maps $x''$ are local transversals. A transversal is called *complete* when it meets all leaves. A *foliated atlas* is a covering of $M$ by foliated charts.

If a smooth map $\phi : M' \to M$ transverse to (the leaves of) $\mathcal{F}$, then the connected components of the inverse images of the leaves of $\mathcal{F}$ are the leaves of the *pull-back* $\phi^*\mathcal{F}$, which is a smooth foliation on $M'$ of codimension $n'$. For the inclusion map of any open $U \subset M$, this defines the *restriction* $\mathcal{F}|_U$.

Foliations on manifolds with boundary can be similarly defined, with leaves tangent or transverse to the boundary. The concepts and properties of foliations considered here have obvious versions with boundary.

1.4. **Holonomy.** Let $\{U_k, x_k\}$ be a foliated atlas of $\mathcal{F}$ with $x_k = (x'_k, x''_k)$ and $x_k(U_k) = \Sigma_k \times B''_k$. Assume that it is *regular* in the following sense: $\{U_k\}$ is locally finite, there are foliated charts $(V_k, y_k)$ with $\overline{U_k} \subset V_k$ and $y_k|_{U_k} = x_k$, and $U_k \cup U_l$ is in the domain of some foliated chart if $U_k \cap U_l \neq \emptyset$. Then, with the notation $\Sigma_{kl} = x'_k(U_k \cap U_l)$, the *elementary*

[3]It is also said that $(M, \mathcal{F})$ is a *foliated manifold*.

*holonomy transformations* $h_{kl} : \Sigma_{lk} \to \Sigma_{kl}$ are defined by $h_{kl}x'_l = x'_k$ on $U_k \cap U_l$. Let $\mathcal{H}$ denote the representative of the *holonomy pseudogroup* on $\Sigma := \bigsqcup_k \Sigma_k$ generated by the local transformations $h_{kl}$. The $\mathcal{H}$-orbit of every $\bar{p} \in \Sigma$ is denoted by $\mathcal{H}(\bar{p})$. The maps $x'_k$ define a homeomorphism between the leaf space $M/\mathcal{F}$ and the orbit space $\Sigma/\mathcal{H}$.

Let $c : I := [0,1] \to L$ be a path in a leaf from $p \in L \cap U_k$ to $q \in L \cap U_l$, and let $\bar{p} = x'_k(p) \in \Sigma_k$ and $\bar{q} = x'_l(q) \in \Sigma_l$. Take a partition of $I$, $0 = t_0 < t_1 < \cdots < t_m = 1$, and a sequence of indices, $k = k_1, k_2, \dots, k_m = l$, such that $c([t_{i-1}, t_i]) \subset U_{k_i}$ for $i = 1, \dots, m$. Let $h_c = h_{k_m k_{m-1}} \cdots h_{k_2 k_1}$. We have $\bar{p} \in \operatorname{dom} h_c \subset \Sigma_k$ and $\bar{q} = h_c(\bar{p}) \in \operatorname{im} h_c \subset \Sigma_l$. The germ $\mathbf{h}_c$ of $h_c$ at $\bar{p}$ is the (*germinal*) *holonomy* of $c$, and the tangent map $h_{c*} : T_{\bar{p}}\Sigma_k \to T_{\bar{q}}\Sigma_l$ is its *infinitesimal holonomy*. End-point homotopic paths in $L$ define the same holonomy. Thus, taking $p = q$ and $k = l$, we get the *holonomy homomorphism* onto the *holonomy group*, $\mathbf{h} = \mathbf{h}_L : \pi_1 L = \pi_1(L, p) \to \operatorname{Hol} L = \operatorname{Hol}(L, p)$, $[c] \mapsto \mathbf{h}_c$, which is independent of the foliated chart containing $p$ up to conjugation. The *holonomy cover* $\widetilde{L} = \widetilde{L}^{\mathrm{hol}}$ of $L$ is defined by $\pi_1 \widetilde{L} = \ker \mathbf{h}_L$. If[4] $\operatorname{Hol} L = \{e\}$, it is said that $L$ has *no holonomy*. The union of leaves without holonomy is a dense $G_\delta$ subset [23, 15]. If all leaves have no holonomy, then $\mathcal{F}$ is said to be *without holonomy*. According to Reeb's local stability, if $L$ is compact, then the germ of $\mathcal{F}$ at $L$ is determined by $\mathbf{h}_L$ using a construction called *suspension* [21, Section 2.7] (see also [25, Theorem 2.1.7], [7, Theorem IV.2], [18, Theorem II.2.29], [8, Theorem 2.3.9]). Similarly, we have the concepts of *infinitesimal holonomy groups* of the leaves, and leaves/foliations *without infinitesimal holonomy*.

With the above notation, an element of $\operatorname{Hol} L$ is called *quasi-analytic* if, either it is the identity, or it is represented by some local transformation $h$ such that $h|_V \neq \mathrm{id}_V$ for all open $V \subset \operatorname{dom} h$ with $\bar{p} \in \overline{V}$. $\operatorname{Hol} L$ is called *quasi-analytic* when all of its elements are quasi-analytic.

In the case of codimension one, $\operatorname{Hol} L$ can be described by germs at 0 of local transformations of $\mathbb{R}$. Then $\mathcal{F}$ is said to be *infinitesimally $C^\infty$-trivial* at $L$ if $h'(0) = 1$ and $h^{(k)}(0) = 0$ $(k > 1)$ for all local transformation $h$ representing an element of $\operatorname{Hol} L$. For instance, this property is satisfied if $\operatorname{Hol} L$ is generated by non-quasi-analytic elements.

1.5. **Infinitesimal transformations and transverse vector fields.** Let $T\mathcal{F} \subset TM$ denote the subbundle of vectors tangent to the leaves, and let $N\mathcal{F} = TM/T\mathcal{F}$. The terms *leafwise*[5]/*normal* are used for these vector bundles, their elements and smooth sections (vector fields). The leafwise vector fields form a Lie subalgebra and $C^\infty(M)$-submodule, $\mathfrak{X}(\mathcal{F}) \subset \mathfrak{X}(M)$. Its normalizer is the Lie algebra $\mathfrak{X}(M, \mathcal{F})$ of *infinitesimal transformations* of $(M, \mathcal{F})$, and $\overline{\mathfrak{X}}(M, \mathcal{F}) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$ is the Lie algebra of *transverse vector fields*. An *orientation* (respectively, *transverse orientation*) of $\mathcal{F}$ is an orientation of the vector bundle $T\mathcal{F}$ (respectively, $N\mathcal{F}$).

[4]In abstract groups, the identity element is denoted by $e$.

[5]The terms "*tangent*" or "*vertical*" are also used instead of "leafwise".

For any $X$ in $TM$ (respectively, $\mathfrak{X}(M)$ or $\mathfrak{X}(M,\mathcal{F})$), let $\overline{X}$ denote the induced element of[6] $N\mathcal{F}$ (respectively, $C^\infty(M;N\mathcal{F})$ or $\overline{\mathfrak{X}}(M,\mathcal{F})$). $N\mathcal{F}$ becomes a *leafwise flat* vector bundle with the canonical flat $T\mathcal{F}$-partial connection $\nabla^{\mathcal{F}}$ given by $\nabla^{\mathcal{F}}_V\overline{X} = \overline{[V,X]}$ for $V \in \mathfrak{X}(\mathcal{F})$ and $X \in \mathfrak{X}(M)$. The leafwise parallel transport along any piecewise smooth path $c$ is the infinitesimal holonomy $h_{c*} : T_{\bar{p}}\Sigma_k \equiv N_p\mathcal{F} \to T_{\bar{q}}\Sigma_l \equiv N_q\mathcal{F}$.

$\overline{\mathfrak{X}}(M,\mathcal{F})$ can be realized as the linear subspace of $C^\infty(M;N\mathcal{F})$ consisting of leafwise flat normal vector fields. The local projections $x'_k$ induce a canonical isomorphism of $\overline{\mathfrak{X}}(M,\mathcal{F})$ to the Lie algebra $\mathfrak{X}(\Sigma,\mathcal{H})$ of $\mathcal{H}$-invariant tangent vector fields on $\Sigma$. The notation $\overline{X}$ is also used for the element of $\mathfrak{X}(\Sigma,\mathcal{H})$ that corresponds to $X \in \overline{\mathfrak{X}}(M,\mathcal{F})$.

When $M$ is not closed, let $\mathfrak{X}_{\mathrm{com}}(\mathcal{F}) \subset \mathfrak{X}(\mathcal{F})$ and $\mathfrak{X}_{\mathrm{com}}(M,\mathcal{F}) \subset \mathfrak{X}(M,\mathcal{F})$ denote the subsets of complete vector fields, and $\overline{\mathfrak{X}}_{\mathrm{com}}(M,\mathcal{F}) \subset \overline{\mathfrak{X}}(M,\mathcal{F})$ the projection of $\mathfrak{X}_{\mathrm{com}}(M,\mathcal{F})$.

1.6. **Foliated maps and foliated flows.** A (smooth) map between foliated manifolds, $\phi : (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$, is called *foliated* if it maps leaves to leaves. Then its tangent map defines morphisms, $\phi_* : T\mathcal{F}_1 \to T\mathcal{F}_2$ and $\phi_* : N\mathcal{F}_1 \to N\mathcal{F}_2$, the second one being compatible with the leafwise flat structures.

Let $\mathrm{Diffeo}(M,\mathcal{F}) \subset \mathrm{Diffeo}(M)$ be the subgroup of foliated diffeomorphisms. A smooth flow $\phi$ on $M$ is called *foliated* if $\phi^t \in \mathrm{Diffeo}(M,\mathcal{F})$ for all $t$. This concept can be extended to a local flow $\phi : \Omega \to M$ by considering the restriction to $\Omega$ of the foliation on $M \times \mathbb{R}$ with leaves $L \times \{t\}$, for leaves $L$ of $\mathcal{F}$ and points $t \in \mathbb{R}$. For $X \in \mathfrak{X}(M)$ (respectively, $X \in \mathfrak{X}_{\mathrm{com}}(M)$), we have $X \in \mathfrak{X}(M,\mathcal{F})$ (respectively, $X \in \mathfrak{X}_{\mathrm{com}}(M,\mathcal{F})$) if and only if its local flow (respectively, flow) is foliated.

For $X \in \mathfrak{X}_{\mathrm{com}}(M,\mathcal{F})$ with foliated flow $\phi$, let $\bar{\phi}$ be the local flow on $\Sigma$ generated by $\overline{X} \in \mathfrak{X}(\Sigma,\mathcal{H})$, which corresponds to $\phi$ via the maps $x'_k$. In an obvious sense, $\bar{\phi}$ is $\mathcal{H}$-equivariant, and therefore it defines an $\mathcal{H}$-equivariant local flow $\bar{\phi}$ on any other representative of the holonomy pseudogroup.

1.7. **Riemannian foliations.** The $\mathcal{H}$-invariant structures on $\Sigma$ are called (*invariant*) *transverse structures*. A transverse orientation has this interpretation. Other examples are *transverse Riemannian metrics* and *transverse parallelisms*. Their existence defines the classes of (*transversely*) *Riemannian* and *transversely parallelizable* (*TP*) foliations. A Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}(\Sigma,\mathcal{H})$ generated by a transverse parallelism is called a *transverse Lie structure*, giving rise to the concept of ($\mathfrak{g}$-)*Lie foliation*.

Let $G$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. $\mathcal{F}$ is a $\mathfrak{g}$-Lie foliation just when $\{U_k, x_k\}$ can be chosen so that every $\Sigma_k$ is realized as an open subset of $G$ and the maps $h_{kl}$ are restrictions of left translations.

Using the canonical isomorphism $\overline{\mathfrak{X}}(M,\mathcal{F}) \cong \mathfrak{X}(\Sigma,\mathcal{H})$, a transverse parallelism can be given by a global frame of $N\mathcal{F}$ consisting of transverse vector fields $\overline{X_1},\dots,\overline{X_{n'}}$. This

[6]The space of smooth sections of a vector bundle $E$ is denoted by $C^\infty(M;E)$.

frame defines a transverse Lie structure when it is a base of a Lie subalgebra $\mathfrak{g} \subset \overline{\mathfrak{X}}(M, \mathcal{F})$. If moreover $\overline{X_1}, \dots, \overline{X_{n'}} \in \overline{\mathfrak{X}}_{\mathrm{com}}(M, \mathcal{F})$, the TP or Lie foliation $\mathcal{F}$ is called *complete*.

Similarly, a transverse Riemannian metric can be described as a leafwise flat Euclidean structure on $N\mathcal{F}$. It is induced by a *bundle-like metric* on $M$, in the sense that the maps $x'_k$ are Riemannian submersions.

It is said that $\mathcal{F}$ is *transitive at* $p \in M$ when the evaluation map $\mathrm{ev}_p : \mathfrak{X}(M, \mathcal{F}) \to T_pM$ is surjective, or, equivalently, the evaluation map $\overline{\mathrm{ev}}_p : \overline{\mathfrak{X}}(M, \mathcal{F}) \subset C^\infty(M; N\mathcal{F}) \to N_p\mathcal{F}$ is surjective. Similarly, $\mathcal{F}$ is called *transversely complete* (*TC*) *at* $p$ if $\mathrm{ev}_p(\mathfrak{X}_{\mathrm{com}}(M, \mathcal{F}))$ generates $T_pM$, or, equivalently, $\overline{\mathrm{ev}}_p(\overline{\mathfrak{X}}_{\mathrm{com}}(M, \mathcal{F}))$ generates $N_p\mathcal{F}$. The transitive/TC point set is open and saturated. $\mathcal{F}$ is called *transitive*/*TC* if it is transitive/TC at every point [33, Section 4.5].

TP foliations are transitive, and transitive foliations are Riemannian. In turn, Molino's theory describes Riemannian foliations in terms of TP foliations [33]. A Riemannian foliation is called *complete* if, using Molino's theory, the corresponding TP foliation is TC. Furthermore Molino's theory describes TC foliations in terms of complete Lie foliations with dense leaves. On the other hand, complete Lie foliations have the following description due to Fedida [16, 17] (see also [33, Theorem 4.1 and Lemma 4.5]). Assume that $M$ is connected and $\mathcal{F}$ a complete $\mathfrak{g}$-Lie foliation. Let $G$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then there is a regular covering $\pi : \widetilde{M} \to M$, a fiber bundle $D : \widetilde{M} \to G$ (the *developing* map) and a monomorphism[7] $h : \Gamma := \mathrm{Aut}(\pi) \equiv \pi_1 M / \pi_1 \widetilde{M} \to G$ (the *holonomy* homomorphism) such that the leaves of $\widetilde{\mathcal{F}} := \pi^* \mathcal{F}$ are the fibers of $D$, and $D$ is $h$-equivariant with respect to the left action of $G$ on itself by left translations. As a consequence, $\pi$ restricts to diffeomorphisms between the leaves of $\widetilde{\mathcal{F}}$ and $\mathcal{F}$. The subgroup $\mathrm{Hol}\,\mathcal{F} := \mathrm{im}\, h \subset G$, isomorphic to $\Gamma$, is called the *global holonomy group*. Since $D$ induces an identity $\widetilde{M}/\widetilde{\mathcal{F}} \equiv G$, the $\pi$-lift and $D$-projection of vector fields define identities

$$\overline{\mathfrak{X}}(M, \mathcal{F}) \equiv \overline{\mathfrak{X}}(\widetilde{M}, \widetilde{\mathcal{F}}, \Gamma) \equiv \mathfrak{X}(G, \mathrm{Hol}\,\mathcal{F}) \ , \tag{1.1}$$

where a group within the parentheses to denote subspaces of invariant sections[8]. These identities give a precise realization of $\mathfrak{g} \subset \overline{\mathfrak{X}}(M, \mathcal{F})$ as the Lie algebra of left invariant vector fields on $G$. The holonomy pseudogroup of $\mathcal{F}$ is equivalent to the pseudogroup on $G$ generated by the action of $\mathrm{Hol}\,\mathcal{F}$ by left translations. Thus the leaves are dense if and only if $\mathrm{Hol}\,\mathcal{F}$ is dense in $G$, which means $\mathfrak{g} = \overline{\mathfrak{X}}(M, \mathcal{F})$.

1.8. **Homogeneous foliations.** More generally, consider the homogeneous space $S = G/H$, defined by a closed subgroup of a connected Lie group, $H \subset G$. It is said that $\mathcal{F}$ is a (*transversely*) *homogeneous* ($(G, S)$-) *foliation* if $\{U_k, x_k\}$ can be chosen so that every $\Sigma_k$ is realized as an open subset of $S$ and the maps $h_{kl}$ are restrictions of the action of elements of $G$. In this case, there is a regular covering $\pi : \widetilde{M} \to M$, a smooth submersion

[7] $\mathrm{Aut}(\pi)$ denotes the group of deck transformations of the covering $\pi : \widetilde{M} \to M$.

[8] This is preferred rather than the usual subindex to agree with $\mathfrak{X}(\Sigma, \mathcal{H})$ and $\mathfrak{X}(M, \mathcal{F})$.

$D : \widetilde{M} \to S$ and a monomorphism $h : \Gamma := \mathrm{Aut}(\pi) \equiv \pi_1 L/\pi_1 \widetilde{L} \to G$ such that the leaves of $\widetilde{\mathcal{F}} := \pi^* \mathcal{F}$ are the connected components of the fibers of $D$, and $D$ is $h$-equivariant [6] (see also [18, Section III.3]). The terms of Fedida's description are also used in this case, as well as the notation $\mathrm{Hol}\,\mathcal{F} = \mathrm{im}\, h$. This description is determined up to conjugation in $G$ in an obvious sense. Now $\widetilde{M}/\widetilde{\mathcal{F}}$ is a possibly non-Hausdorff smooth manifold, and $D$ induces a local diffeomorphism $\overline{D} : \widetilde{M}/\widetilde{\mathcal{F}} \to S$, which is $h$-equivariant with respect to the induced $\Gamma$-action on $\widetilde{M}/\widetilde{\mathcal{F}}$. Like in (1.1), we get

$$\overline{\mathfrak{X}}(M, \mathcal{F}) \equiv \overline{\mathfrak{X}}(\widetilde{M}, \widetilde{\mathcal{F}}, \Gamma) \equiv \mathfrak{X}(\widetilde{M}/\widetilde{\mathcal{F}}, \Gamma) \supset \mathfrak{X}(\mathrm{im}\, D, \mathrm{Hol}\,\mathcal{F}) \ . \tag{1.2}$$

The holonomy pseudogroup of $\mathcal{F}$ is equivalent to the pseudogroup generated by the action of $\Gamma$ on $\widetilde{M}/\widetilde{\mathcal{F}}$. In particular, for leaves, $L$ of $\mathcal{F}$ and $\widetilde{L}$ of $\widetilde{\mathcal{F}}$ with $\pi(\widetilde{L}) = L$ and $D(\widetilde{L}) = x \in S$, we have

$$\mathrm{Hol}\, L \equiv \{\, \gamma \in \Gamma \mid \gamma \cdot \widetilde{L} = \widetilde{L} \,\} \cong h(\{\, \gamma \in \Gamma \mid \gamma \cdot \widetilde{L} = \widetilde{L} \,\}) \subset \mathrm{Hol}_x\, \mathcal{F} \ , \tag{1.3}$$

where $\mathrm{Hol}_x\, \mathcal{F} \subset \mathrm{Hol}\,\mathcal{F}$ is the isotropy subgroup at $x$.

## 2. Some classes of foliations of codimension one

2.1. **Preliminary considerations.** Let $\mathcal{F}$ be a smooth foliation of codimension one on a closed $n$-manifold $M$. Suppose that $\mathcal{F}$ is transversely oriented, obtaining[9] $\omega, \theta \in C^\infty(M; \Lambda^1)$ such that $\omega$ defines[10] $\mathcal{F}$ (with its transverse orientation) and $d\omega = \theta \wedge \omega$. There is some $X \in \mathfrak{X}(M)$ with $\omega(X) = 1$; in fact, $\overline{X} \in C^\infty(M; N\mathcal{F})$ and $\omega$ determine each other. Note that $\mathcal{F}$ is Riemannian just when $\omega$ can be chosen so that $d\omega = 0$ ($\theta = 0$); i.e., $X \in \mathfrak{X}(M, \mathcal{F})$. Actually, $\mathcal{F}$ is an $\mathbb{R}$-Lie foliation in this case because $\mathbb{R}\,\overline{X}$ is a Lie subalgebra of $\overline{\mathfrak{X}}(M, \mathcal{F})$.

Take any leaf $L$ and $p \in L$, and a local transversal $\Sigma \equiv (-\epsilon, \epsilon)$ through $p \equiv 0$ so that the transverse orientation corresponds to the standard orientation of $(-\epsilon, \epsilon)$. Since the holonomy maps defining the elements of $\mathrm{Hol}(L, p)$ preserve the orientation of $(-\epsilon, \epsilon)$, they can be restricted to $(-\epsilon, 0]$ and $[0, \epsilon)$, defining the *lateral holonomy groups* $\mathrm{Hol}_\pm(L, p) = \mathrm{Hol}_\pm\, L$.

Recall that $L$ is said to be *locally dense* if it is dense in some open saturated set. On the other hand, $L$ is said to be *resilient* if there is some element of $\mathrm{Hol}(L, p)$, represented by some local diffeomorphism $f$ defined around $p$ in $\Sigma$, and there is some $q \neq p$ in $L \cap \mathrm{dom}\, f$ such that the sequence $f^k(q)$ is defined and converges to $p$.

Now a smooth connected closed transversal of $\mathcal{F}$ is a smooth embedding $c : S^1 \to M$ transverse to the leaves. It always has a (closed) tubular neighborhood $\varpi : T \to c(S^1) \equiv S^1$ in $M$, which can be chosen to be *foliated* in the sense that its fibers are $(n-1)$ disks

[9] We use the notation $\Lambda = \Lambda M = \bigwedge T^* M$.

[10] This means that $T\mathcal{F} = \ker \omega$ and the transverse orientation is induced by $\omega$ on $N\mathcal{F}$.

in the leaves. If $\mathcal{F}$ is also oriented, then $\varpi$ trivial, $T \equiv S^1_\varpi \times D^{n-1}$, where $D^{n-1}$ is the standard disk in $\mathbb{R}^{n-1}$.

2.2. **$\mathbb{R}$-Lie foliations.** Suppose that $\mathcal{F}$ is a transversely complete $\mathbb{R}$-Lie foliation. This means that there is some $Z \in \mathfrak{X}_{\mathrm{com}}(M, \mathcal{F})$ such that $\overline{Z} \neq 0$ everywhere. Equivalently, the orbits of the foliated flow $\phi$ of $Z$ are transverse to $\mathcal{F}$. The Fedida's description of $\mathcal{F}$ is given by a regular covering map $\pi : \widetilde{M} \to M$, a holonomy homomorphism $h : \Gamma := \mathrm{Aut}(\pi) \to \mathbb{R}$, and the developing map $D : \widetilde{M} \to \mathbb{R}$ (Section 1.7). Thus $\Gamma \cong \mathrm{im}\, h \subset \mathbb{R}$ is abelian and torsion free. Let $\widetilde{Z}$ and $\tilde{\phi}$ be the lifts of $Z$ and $\phi$ to $\widetilde{M}$. Then $\widetilde{Z}$ is $\Gamma$-invariant and $D$-projectable. Without lost of generality, we can assume $D_*\widetilde{Z} = \partial_x \in \mathfrak{X}(\mathbb{R})$, where $x$ denotes the standard global coordinate of $\mathbb{R}$. Thus $\tilde{\phi}$ is $\Gamma$-equivariant and induces via $D$ the flow $\bar{\phi}$ on $\mathbb{R}$ defined by $\bar{\phi}^t(x) = t + x$. This is the equivariant local flow induced by $\phi$ on this representative of the holonomy pseudogroup (Section 1.7). It is easy to check that $\phi^t$ preserves every leaf of $\mathcal{F}$ if and only if $t \in \mathrm{Hol}\,\mathcal{F}$.

Example 2.1. The simplest example of minimal $\mathbb{R}$-Lie foliation on a closed manifold is the Kronecker's flow on the torus $T^2 \equiv \mathbb{R}^2/\mathbb{Z}^2$ [8, Example 1.1.5]. It is induced by a foliation on $\mathbb{R}^2$ by parallel lines with irrational slope. This construction has an obvious generalization to higher dimensions, obtaining minimal $\mathbb{R}$-Lie foliations on every torus $T^n \equiv \mathbb{R}^n/\mathbb{Z}^n$ induced by foliations on $\mathbb{R}^n$ by appropriate parallel hyperplanes [8, Example 1.1.8].

2.3. **Foliations almost without holonomy.** Recall that $\mathcal{F}$ is said to be *almost without holonomy* when all non-compact leaves have no holonomy. The structure of such a foliation was described by Hector using the following *model foliations* $\mathcal{G}$ on compact manifolds $N$ (possibly with boundary) [22, Structure Theorem], [24, Theorem 1]:

(0) $\mathcal{G}$ is given by a trivial bundle over $[0, 1]$,
(1) $\mathring{\mathcal{G}} := \mathcal{G}|_{\mathring{N}}$ is given by a fiber bundle over $S^1$, or
(2) all leaves of $\mathring{\mathcal{G}}$ are dense in $\mathring{N}$.

In the case where $\mathcal{F}$ has finitely many leaves with holonomy, Hector's description is as follows. Let $M^0$ be the finite union of compact leaves with holonomy. Let $M^1 = M \smallsetminus M^0$, whose connected components are denoted by $M^1_l$, with $l$ running in a finite index set, and let $\mathcal{F}^1_l = \mathcal{F}|_{M^1_l}$. For every $l$, there is a connected compact manifold[11] $M_l$, possibly with boundary, endowed with a smooth transversely oriented foliation $\mathcal{F}_l$ tangent to the boundary, satisfying the following. Equipping $\mathbf{M} := \bigsqcup_l M_l$ with the combination $\boldsymbol{\mathcal{F}}$ of the foliations $\mathcal{F}_l$, there is a foliated smooth local embedding $\boldsymbol{\pi} : (\mathbf{M}, \boldsymbol{\mathcal{F}}) \to (M, \mathcal{F})$, preserving the transverse orientations, so that $\boldsymbol{\pi} : \mathring{M}_l \to M^1_l$ is a diffeomorphism for all $l$ (we may write $\mathring{M}_l \equiv M^1_l$), $\boldsymbol{\pi} : \partial\mathbf{M} \to M^0$ is a 2-fold covering map, and every $\mathcal{F}_l$ is a model foliation.

[11]Since $M_l$ is the metric completion of $M^1_l$, the notation $\widehat{M}^1_l$ and $\widehat{\mathcal{F}}^1_l$ would be more standard. But the notation $M_l$ is more appropriate for our use in [5] involving b-calculus.

$M$ can be described by gluing the manifolds $M_l$ along corresponding pairs of boundary components. Equivalently, $\mathbf{M}$ can be described by cutting $M$ along $M^0$ (Section 1.2). Thus $\partial\mathbf{M} \equiv M^0 \sqcup M^0$, and $\boldsymbol{\pi}$ defines diffeomorphisms between corresponding connected components of $\partial\mathbf{M}$ and $M^0$.

Remark 2.2. (i) (See [24, Lemma 7] and its proof.) For indices $l_\pm$, and boundary leaves $L_\pm$ of $\mathcal{F}_{l_\pm}$ with $L := \boldsymbol{\pi}(L_+) = \boldsymbol{\pi}(L_-)$, we have $\mathrm{Hol}(L_\pm) \equiv \mathrm{Hol}_\pm L$. $\mathrm{Hol}_\pm L$ is the germ group at 0 of a pseudogroup $\mathcal{H}_{L,\pm}$ of local transformations of $\mathbb{R}^\pm \cup \{0\}$, generated by a (possibly empty) set of contractions and dilatations defined around 0. It follows that $\mathrm{Hol}_\pm L$ is an Archimedean totally ordered group, and therefore it is isomorphic to a subgroup of $(\mathbb{R}, +)$, obtaining that $\mathrm{Hol}_\pm L$ is abelian and torsion free. It is easy to see that the orbits of $\mathcal{H}_{L,\pm}$ on $\mathbb{R}^\pm$ are singletons (respectively, monotone sequences with limit 0, or dense) just when the rank of $\mathrm{Hol}_\pm L$ is 0 (respectively, 1, or $> 1$).

(ii) If $\mathcal{F}_l$ is a model (0), or a model (1) with $\partial M_l = \emptyset$ ($M_l = M$ and $M^0 = \emptyset$), then the leaves of $\mathcal{F}_l$ are compact.

(iii) If $\mathcal{F}_l$ is a model (1) with $\partial M_l \neq \emptyset$, or a model (2), then the leaves of $\mathring{\mathcal{F}}_l$ are not compact. In fact, the whole of $\partial M_l$ is contained in the closure of every leaf of $\mathring{\mathcal{F}}_l$. Hence, according to (i), the holonomy groups of the boundary leaves of $\mathcal{F}_l$ are of rank 1 (respectively, $> 1$) if and only if $\mathcal{F}_l$ is a model (1) with $\partial M_l \neq \emptyset$ (respectively, a model (2)).

(iv) If $\mathcal{F}_l$ is a model (2), then $\mathring{\mathcal{F}}_l$ becomes a complete $\mathbb{R}$-Lie foliation after a possible change of the differentiable structure of $\mathring{M}_l$, keeping the same differentiable structure on the leaves [24, Theorem 2]. If moreover $\partial M_l = \emptyset$, then $\mathcal{F}$ is homeomorphic to a minimal $\mathbb{R}$-Lie foliation.

(v) $\mathcal{F}^1$ has no holonomy, and therefore $\mathcal{F}$ has no resilient leaves. This holds because $\mathring{\mathcal{F}}_l$ is given by a fiber bundle in the models (0) and (1), and is homeomorphic to a Lie foliation in the model (2) by (iv).

(vi) According to (ii) and (iii), the description holds as well if $M^0$ is any finite union of compact leaves, including all leaves with holonomy. Thus, if $\mathcal{F}_l$ is a model (1) with $\partial M_l = \emptyset$, then $M_l = M$ can be cut into models (0) by adding compact leaves to $M^0$. Conversely, if all foliations $\mathcal{F}_l$ are models (0), then $\mathcal{F}$ is a model (1) with $\partial M = \emptyset$.

(vii) In the models (1) and (2), $\mathring{\mathcal{F}}_l$ has smooth complete closed transversals (see [8, Lemma 3.3.7]).

Proposition 2.3. *If* $\mathrm{Hol}\, L$ *is quasi-analytic for all leaf* $L \subset M^0$*, then all foliations* $\mathcal{F}_l$ *have the same model.*

Proof. For all leaves $L \subset M^0$, we have $\mathrm{Hol}_+ L \cong \mathrm{Hol}_- L \cong \mathrm{Hol}\, L$ by the hypothesis on $\mathrm{Hol}\, L$. Then, by Remark 2.2 (i)–(iii) and since $M$ is connected, the rank of the holonomy

groups of all boundary leaves of all foliations $\mathcal{F}_l$ is simultaneously 0, 1 or $> 1$, and all foliations $\mathcal{F}_l$ have the same model. $\square$

Example 2.4. A Reeb component on $D^{n-1} \times S^1$ is a model (1) [8, Examples 1.1.12 and 3.3.11], [18, Example I.3.14 (i)], [25, Section II.1.4.4]. All of the Reeb components on $D^{n-1} \times S^1$ are homeomorphic, but they may not be diffeomorphic.

The Reeb components on $D^1 \times S^1 = [-1,1] \times S^1$ can be described as follows. Let $f : (-1,1) \to \mathbb{R}$ be a smooth function such that $|f^{(k)}(x)| \to \infty$ as $x \to \pm 1$ for all order $k$. Then the graphs of the functions $f + c$ ($c \in \mathbb{R}$) are the interior leaves of a smooth foliation tangent to the boundary on the strip $[-1,1] \times \mathbb{R}$, which induces a smooth foliation $\mathcal{G}$ on $[-1,1] \times S^1 \equiv [-1,1] \times \mathbb{R}/\mathbb{Z}$. Its boundary leaves are $L_\pm = \{\pm 1\} \times S^1$. The following examples of $f$ produce non-diffeomorphic foliations:

(i) If $f(x) = \exp \frac{1}{1-x^2}$, then $\mathcal{G}$ is infinitesimally $C^\infty$-trivial at $L_\pm$.
(ii) If $f(x) = \frac{x^2}{1-x^2}$, then $\mathcal{G}$ is not infinitesimally $C^\infty$-trivial at $L_\pm$, but $L_\pm$ is without infinitesimal holonomy.
(iii) If $|f(x)| = \ln(1-|x|)^\mu$ ($\mu > 0$) for $1 - |x|$ small enough, then $\operatorname{Hol} L_\pm$ is generated by the germ of $u \mapsto e^{1/\mu} u$ at 0 in $[0, \infty)$.

Example 2.5. Let $\mathcal{G}_a$ ($a = 1, 2$) be transversely oriented models (1) or (2) of dimension $> 1$ on manifolds $N_a$. If there is a diffeomorphism $\phi$ between boundary leaves, $L_a$ of $\mathcal{G}_a$, then a tangential gluing via $\phi$ can be made, obtaining a foliation $\mathcal{G}$ on $N := N_1 \cup_\phi N_2$, with the compact leaf $L := L_1 \cup_\phi L_2 \subset \mathring{N}$ [8, Section 3.4], [18, Example I.3.14 (i)], [26, Theorem IV.4.2.2]. $\mathcal{G}$ may not be smooth. It is smooth only when, for all $\sigma \in \pi_1 L_1$, the combination of representatives of $\mathbf{h}_\sigma$ and $\mathbf{h}_{\phi_* \sigma}$ are smooth maps (considering the elements of $\operatorname{Hol} L_1$ and $\operatorname{Hol} L_2$ as germs at 0 of local transformations of $(-\infty, 0]$ and $[0, \infty)$, respectively). For example, this is true if every $\mathbf{h}_\sigma$ and $\mathbf{h}_{\phi_* \sigma}$ are germs of homotheties at 0 with the same ratio. This property is also guaranteed when every $\mathcal{G}_a$ is infinitesimally $C^\infty$-trivial at $L_a$ [8, Proposition 3.4.2].

We can continue making tangential gluing of models to produce a foliation $\mathcal{F}$ on a closed manifold $M$. If every tangential gluing preserves smoothness, then $\mathcal{F}$ is almost without holonomy with finitely many leaves with holonomy. The following are some examples of foliations obtained in this way:

(i) The Reeb foliation $\mathcal{F}$ on $S^3$ is almost without holonomy and has one compact leaf $L$. It is obtained by tangential gluing of two Reeb components on $D^2 \times S^1$, so that the gluing map interchanges meridian and longitude in the boundary leaves $S^1 \times S^1$ [8, Example 3.4.3 and Exercise 3.4.4], [18, Examples I.3.14]. Since $\operatorname{Hol} L$ has non-quasi-analytic generators, the Reeb components must be infinitesimally $C^\infty$-trivial at the boundary leaves to get smoothness of $\mathcal{F}$.
(ii) Let $\mathcal{F}$ be a foliation on $S^{n-1} \times S^1$ obtained by tangential gluing of two Reeb components on $D^{n-1} \times S^1$ using the identity map on the boundary leaves $S^{n-2} \times S^1$.

$\mathcal{F}$ becomes smooth if the Reeb components are infinitesimally $C^\infty$-trivial at the boundary leaves, but now this condition is not necessary to get smoothness (see Example 2.13 below).

(iii) A smooth foliation $\mathcal{F}$ on the 2-torus or on the Klein bottle can be constructed by tangential gluing of Reeb components on $[-1,1]\times S^1$ of the type in Example 2.4 (iii), all of them constructed with the same constant $\mu$. The holonomy groups of the leaves with holonomy are generated by the germ of $u \mapsto e^{1/\mu}u$ at 0 in $\mathbb{R}$.

Example 2.6. Let $\mathcal{F}$ and $\mathcal{G}$ be oriented and transversely orientable foliations of codimension one on closed $n$-manifolds $M$ and $N$ ($n \geq 2$). Suppose that both of them are almost without holonomy, and that they have finitely many leaves with holonomy. Take smooth closed transversals, $c : S^1 \to M^1$ of $\mathcal{F}^1$ and $d : S^1 \to N^1$ of $\mathcal{G}^1$ (Remark 2.2 (vii)), and let $\mathcal{F}'$ be the connected sum of $\mathcal{F}$ and $\mathcal{G}$ along $c$ and $d$ [18, Example I.2.20 (i)]. $\mathcal{F}'$ is another transversely orientable foliation almost without holonomy on a closed manifold, and it has finitely many leaves with holonomy.

For models (1) or (2), we can also consider their connected sum along smooth closed transversals in their interior. The result is a model (1) if both foliations are models (1), and a model (2) otherwise.

Example 2.7. Let $\mathcal{F}$ be an oriented and transversely orientable foliation of codimension one on a closed $n$-manifold $M$. Suppose that $\mathcal{F}$ is almost without holonomy, and that it has finitely many leaves with holonomy. Let $(M', \mathcal{F}')$ be the turbulization of $(M, \mathcal{F})$ along a smooth closed transversal $c : S^1 \to M^1$ of $\mathcal{F}^1$ [8, Example 3.3.11], [18, Section I.2.18]. $\mathcal{F}'$ is another transversely orientable foliation almost without holonomy, and it has finitely many leaves with holonomy. Actually, $\mathcal{F}'$ can be considered as a connected sum along $c$ of $\mathcal{F}$ and the foliation of Example 2.5 (ii).

The turbulization can be also applied to a model (1) or (2) along a smooth closed transversal in its interior. After removing the interior of the resulting Reeb component, we get a model of the same type.

2.4. **Transversely affine foliations.** Consider $\mathbb{R}$ as the homogeneous space defined by the canonical action of $\mathrm{Aff}^+(\mathbb{R})$, the Lie group of its orientation preserving affine transformations. It is said that $\mathcal{F}$ is *transversely affine* if it is a transversely homogeneus $(\mathrm{Aff}^+(\mathbb{R}), \mathbb{R})$-foliation[12]. This means that, according to Section 2.1, $\omega$ and $\theta$ can be chosen so that $d\theta = 0$ [37]; it will be said that the transversely affine foliation $\mathcal{F}$ is defined by $(\omega, \theta)$. In this case, the description of Section 1.8 is given by $\pi : \widetilde{M} \to M$, $D : \widetilde{M} \to \mathbb{R}$, $h : \Gamma \to \mathrm{Aff}^+(\mathbb{R})$ and $\mathrm{Hol}\,\mathcal{F} \subset \mathrm{Aff}^+(\mathbb{R})$.

Assume that $\mathcal{F}$ is transversely affine. Then $\Gamma \neq \{e\}$ because $D(\widetilde{M})$ is open in $\mathbb{R}$. Furthermore $\mathcal{F}$ has a finite number of compact leaves with holonomy [18, Proposition III.3.10],

[12]We only consider transversely affine foliations that are transversely oriented. The group $\mathrm{Aff}(\mathbb{R})$ of affine transformations would define transversely affine foliations that may not be transversely oriented.

but non-compact leaves may also have holonomy. A theorem of Inaba [27, Theorem 1.2] states that, either $\mathcal{F}$ is almost without holonomy and $\operatorname{Hol}\mathcal{F}$ is abelian (the *elementary* case), or $\mathcal{F}$ has a locally dense resilient leaf and $\operatorname{Hol}\mathcal{F}$ is non-abelian.

From now on, consider only the elementary case. Then:

(a) either $\operatorname{Hol}\mathcal{F}$ is a group of translations; or

(b) $\operatorname{Hol}\mathcal{F}$ is conjugate by some translation to a group of homotheties.

In the case (a), $\mathcal{F}$ is an $\mathbb{R}$-Lie foliation on a closed manifold, whose Fedida's description is given by $\pi$, $D$ and $h$; in particular, $\operatorname{im} D = \mathbb{R}$.

In the case (b), after conjugation, we can assume that $\operatorname{Hol}\mathcal{F}$ is indeed a group of homotheties. Since $\operatorname{im} D$ is $\operatorname{Hol}\mathcal{F}$-invariant and $\operatorname{Hol}\mathcal{F} \neq \{\mathrm{id}_{\mathbb{R}}\}$, either $\operatorname{im} D = \mathbb{R}^{\pm}$, or $\operatorname{im} D = \mathbb{R}$. If $\operatorname{im} D = \mathbb{R}^{\pm}$, we can pass to a group of translations by using $\ln|D|$ instead of $D$. Thus, if $\mathcal{F}$ is not an $\mathbb{R}$-Lie foliation, we can assume that $\operatorname{Hol}\mathcal{F}$ is a non-trivial group of homotheties and $\operatorname{im} D = \mathbb{R}$. Let us analyze this case using the notation of Section 2.3.

Lemma 2.8. *(i)* $M^0 = \pi(D^{-1}(0))$.

*(ii) The holonomy groups of leaves in* $M^0$ *are isomorphic to non-trivial subgroups of* $\operatorname{Hol}_0\mathcal{F}$.

*(iii) All foliations* $\mathcal{F}_l$ *have the same model, either* (1) *with* $\partial M_l \neq \emptyset$*, or (2).*

Proof. By Proposition 2.3, all foliations $\mathcal{F}_l$ have the same model, which is neither (0), nor (1) with $\partial M_l = \emptyset$, otherwise $\mathcal{F}$ would be an $\mathbb{R}$-Lie foliation. Thus (iii) holds. It also follows that the holonomy groups of the leaves in $M^0$ cannot be trivial, obtaining "$\subset$" in (i) because $\operatorname{Hol}_0\mathcal{F}$ is the only non-trivial isotropy group. Hence (ii) is true by (1.3).

There is a regular foliated atlas $\{U_k, x_k\}$ of $\mathcal{F}$ such that, for every $k$, there is foliated chart $(\widetilde{U}_k, \tilde{x}_k)$ of $\widetilde{\mathcal{F}}$ so that $\pi : \widetilde{U}_k \to U_k$ is a diffeomorphism, $\tilde{x}_k = x_k\pi$ and $\tilde{x}_k' = D|_{\widetilde{U}_k}$. Hence $D^{-1}(0)$ contains just one plaque of every $(\widetilde{U}_k, \tilde{x}_k)$. Since $\{U_k, x_k\}$ is finite, and $D^{-1}(0)$ is $\Gamma$-invariant because 0 is fixed by $\operatorname{Hol}\mathcal{F}$, it follows that $\pi(D^{-1}(0))$ contains a finite number of plaques of the foliated atlas $\{U_k, x_k\}$. So $\pi(D^{-1}(0))$ is a finite union of compact leaves because $\{U_k, x_k\}$ is regular. This shows "$\supset$" in (i) by (iii) and Remark 2.2 (iii). $\square$

Note that $x\partial_x \in \mathfrak{X}(\mathbb{R})$ is invariant by homotheties. Let $\operatorname{Diffeo}(\mathbb{R}, 0) \subset \operatorname{Diffeo}(\mathbb{R})$ denote the subgroup of diffeomorphisms that fix 0.

Lemma 2.9. *(i) If* $Z \in \mathfrak{X}(\mathbb{R})$ *is invariant by some homothety* $h \neq \mathrm{id}_{\mathbb{R}}$*, then* $Z = \varkappa x\partial_x$ *for some* $\varkappa \in \mathbb{R}$.

*(ii) If* $h \in \operatorname{Diffeo}(\mathbb{R}, 0)$ *preserves* $x\partial_x$*, then* $h$ *is a homothety.*

Proof. Let us prove (i). We can assume $h(x) = \lambda x$ ($x \in \mathbb{R}$) for some $\lambda > 1$; otherwise consider $h^{-1}$. Any $h$-invariant $Z \in \mathfrak{X}(\mathbb{R})$ vanishes at 0 because this is the only fixed point of $h$. Thus $Z = xf\partial_x$ for some $f \in C^\infty(\mathbb{R})$. From the $h$-invariance of both $Z$ and $x\partial_x$, and since $x\partial_x$ only vanishes at $x = 0$, we get that $f$ is $h$-invariant. So $f(0) = \lim_{m\to\infty} f(x/\lambda^m) = f(x)$ for all $x \in \mathbb{R}$; i.e., $f$ is constant.

Let us prove (ii). Since $h$ preserves $x\partial_x$, it commutes with the flow of $x\partial_x$; i.e., $h(e^t x) = e^t h(x)$ for all $x, t \in \mathbb{R}$. Therefore $x \mapsto h(x)/x$ is constant on $\mathbb{R}^{\pm}$. Since $h$ is smooth at zero, it follows that $h$ is a homothety. □

Remark 2.10. The same arguments can be used to show versions of Lemma 2.9 on intervals $J$ of the form $(-\epsilon, \epsilon)$, $[0, \epsilon)$ or $(-\epsilon, 0]$ $(0 < \epsilon \leq \infty)$:

(i) If $Z \in \mathfrak{X}(J)$ is invariant by the restriction to $J$ of the pseudogroup generated by some homothety $\neq \mathrm{id}_{\mathbb{R}}$, then $Z = \varkappa x\partial_x$ for some $\varkappa \in \mathbb{R}$.

(ii) If a smooth pointed embedding $h : (J, 0) \to (\mathbb{R}, 0)$ preserves $x\partial_x$, then $h$ is the restriction of a homothety.

By Lemma 2.9 (i), $\mathfrak{X}(\mathbb{R}, \mathrm{Hol}\,\mathcal{F}) = \mathbb{R}\, x\partial_x$. Let $\overline{Z} \in \overline{\mathfrak{X}}(M, \mathcal{F})$ be defined by $x\partial_x \in \mathfrak{X}(\mathbb{R}, \mathrm{Hol}\,\mathcal{F})$ according to (1.2). By Lemma 2.8 (i), the zero set of $\overline{Z}$ is $M^0$. Thus $\mathcal{F}^1_l \equiv \mathring{\mathcal{F}}_l$ becomes a complete $\mathbb{R}$-Lie foliation with the restriction of $\overline{Z}$ to every $M^1_l \equiv \mathring{M}_l$, without having to change the differentiable structure (cf. Remark 2.2 (iv)).

Lemma 2.11. *For any neighborhood $V$ in $M$ of a leaf $L \subset M^0$, every $\overline{Z} \in \overline{\mathfrak{X}}(M, \mathcal{F})$ is determined by $\overline{Z}|_V$.*

Proof. With the notation of Remark 2.2 (i) for this particular $L$, any leaf of $\mathcal{F}^1_{l_\pm}$ meets $V$ by Remark 2.2 (iii). So the restriction $\overline{Z}$ to $\overline{M^1_{l_+} \cup M^1_{l_-}}$ is determined by $\overline{Z}|_V$. By Lemma 2.9 (i) and Remark 2.10 (i), and using the Reeb's local stability, it follows that the restriction $\overline{Z}$ to some neighborhood of $\overline{M^1_{l_+} \cup M^1_{l_-}}$ is also determined by $\overline{Z}|_V$. Then we can apply the same argument to all closures $\overline{M^1_l}$ that meet $\overline{M^1_{l_+} \cup M^1_{l_-}}$. Continuing in this way, the result follows because $M$ is connected. □

Proposition 2.12. $\overline{\mathfrak{X}}(M, \mathcal{F}) \equiv \mathfrak{X}(\mathbb{R}, \mathrm{Hol}\,\mathcal{F})$ *via* (1.2).

Proof. We have to prove that the injection of (1.2) is surjective in this case. Let $Z \in \mathfrak{X}(\widetilde{M}/\widetilde{\mathcal{F}}, \Gamma)$. Take leaves, $L \subset M^0$ of $\mathcal{F}$ and $\widetilde{L}$ of $\widetilde{\mathcal{F}}$ with $\pi(\widetilde{L}) = L$. There are open neighborhoods, $V$ of $\widetilde{L}$ in $\widetilde{M}/\widetilde{\mathcal{F}}$ and $W$ of 0 in $\mathbb{R}$, so that $\overline{D} : V \to W$ is a diffeomorphism. Consider $\{e\} \neq \mathrm{Hol}\, L \subset \mathrm{Hol}_0\, \mathcal{F}$ according to (1.3). By Lemma 2.9 (i) and Remark 2.10 (i), $D_*(Z|_V) = \varkappa x\partial_x|_V$ for some $\varkappa \in \mathbb{R}$ if $V$ and $W$ are small enough. So, by Lemma 2.11, $Z$ corresponds to $\varkappa x\partial_x \in \mathfrak{X}(\mathbb{R}, \mathrm{Hol}\,\mathcal{F})$ via (1.2). □

The transverse orientation of every $\mathcal{F}_l$ is directed, either outward on all boundary leaves of $M_l$, or inward on all of them [27, Lemma 3.4]. Thus no pair of boundary components of the same $M_l$ is glued to get $M$. So, not only $\mathring{M}_l \equiv M^1_l$, but also $M_l \equiv \overline{M^1_l}$ via $\boldsymbol{\pi}$. In particular, there have to be at least two manifolds $M_l$, and $M^0$ contains at least two leaves.

Example 2.13. Let $\widetilde{\mathcal{F}}$ denote the foliation on $\widetilde{M} := \mathbb{R}^n \smallsetminus \{0\}$ $(n > 1)$ whose leaves are the connected components of the last coordinate projection $D : \widetilde{M} \to \mathbb{R}$. Multiplication by

any $\lambda > 1$ defines an action of $\mathbb{Z}$ on $\widetilde{M}$, giving rise to a covering $\pi_\lambda : \widetilde{M} \to M_\lambda$, where $M_\lambda$ is diffeomorphic to $S^{n-1} \times S^1$. Since $\widetilde{\mathcal{F}}$ is $\mathbb{Z}$-invariant, it induces an elementary transversely affine foliation $\mathcal{F}_\lambda$ on $M_\lambda$, being $\pi_\lambda$ and $D$ the maps of its description of Section 1.8. $M^0_\lambda = \pi_\lambda(D^{-1}(0))$ is diffeomorphic to $S^{n-2} \times S^1$. Thus there are two compact leaves if $n = 2$, and one compact leaf if $n > 2$. $M^1_\lambda$ has two components, $M^1_{\lambda,\pm} = \pi_\lambda(\mathbb{R}^\pm)$. The corresponding foliated manifolds with boundary, $(M_{\lambda,\pm}, \mathcal{F}_{\lambda,\pm})$, are transversely affine Reeb components on $D^{n-1} \times S^1$ [25, Section 1.4.4], using the obvious extension of this property to foliations on manifolds with boundary. A different description of these transversely affine Reeb components is given in [8, Example 1.1.12].

Example 2.14. Consider the standard affine structure on $\mathbb{R}$, and its restriction to $\mathbb{R}^+$. The affine circles are [30], [19, Appendix to Section 2]:

(i) the quotient of $\mathbb{R}$ by the additive action of $\mathbb{Z}$; and,
(ii) for every $\lambda > 1$, the quotient of $\mathbb{R}^+$ by the multiplicative action of $\lambda\mathbb{Z}$.

After fixing an orientation, affine structures on $S^1$ are the transversely affine structures $(\omega, \theta)$ of the foliation by points. Then the affine structure defined by $(\omega, \theta)$ is isomorphic to (i) if $\int_{S^1} \theta = 0$, and isomorphic to (ii) for some $\lambda > 1$ if $|\int_{S^1} \theta| = \ln \lambda$. Thus $|\int_{S^1} \theta|$ classifies these structures on $S^1$; indeed, $\int_{S^1} \theta$ classifies these structures up to orientation preserving isomorphisms [18, Section III.3.3], [37, Section 4.1].

Now let $\mathcal{F}$ be a transversely affine foliation on a closed manifold $M$ defined by $(\omega, \theta)$. Any smooth closed transversal $c : S^1 \to M$ of $\mathcal{F}$ induces the orientation and affine structure on $S^1$ given by $(c^*\omega, c^*\theta)$.

In Example 2.6, suppose $\mathcal{F}$ and $\mathcal{G}$ are transversely affine, defined by $(\omega, \theta)$ and $(\alpha, \beta)$, respectively. If they induce the same orientation and affine structure on $S^1$ via $c$ and $d$ ($c^*\omega = f\, d^*\alpha$ for some $0 < f \in C^\infty(S^1)$ and $\int_{S^1} c^*\theta = \int_{S^1} c^*\beta$), then $\mathcal{F}'$ clearly becomes transversely affine.

In Example 2.13, let $c_{\lambda,\pm} : S^1 \to M_\lambda$ be a smooth closed transversal of $\mathcal{F}_\lambda$ that cuts every leaf of $\mathcal{F}^1_{\lambda,\pm}$ once, and induces the standard orientation of $S^1$. Via $c_{\lambda,\pm}$, we get the affine structure (ii) on $S^1$ defined with $\lambda$.

In Example 2.7, if $\mathcal{F}$ is also transversely affine, inducing the standard orientation on $S^1$ via $c$, then there is a transversely affine turbulization along $c$ if and only if $\ln \lambda := \int_{S^1} c^*\theta \neq 0$ (taking the connected sum with $\mathcal{F}_\lambda$ along $c$ and $c_{\lambda,\pm}$) [37, Section 2].

2.5. **Transversely projective foliations.** Recall that $\mathrm{SL}(2,\mathbb{R})$ is the Lie group of $2 \times 2$ matrices of determinant one, and $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$, where $I$ denotes the identity matrix. $\mathrm{PSL}(2,\mathbb{R})$ acts on the projective line $S^1_\infty = \mathbb{R} \cup \{\infty\}$ by projective transformations, the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ being $x \mapsto (ax+b)/(cx+d)$. The stabilizer of $\infty$ consists of the upper triangular matrices ($c = 0$), whose restriction to $\mathbb{R}$ gives $\mathrm{Aff}^+(\mathbb{R})$. An element $A \in \mathrm{PSL}(2,\mathbb{R})$ is called *hyperbolic*, *parabolic* or *elliptic* if it has 2, 1 or 0 fixed points in $S^1_\infty$, respectively. Elliptic elements are conjugate to rotations (elements

of $\mathrm{PSO}(2) = \mathrm{SO}(2)/\{\pm I\}$) different from the identity. The hyperbolic and parabolic elements are conjugate to transformations of the form $x \mapsto \lambda x$ ($\lambda > 0$) and $x \mapsto x + \lambda$ ($\lambda \neq 0$), respectively.

It is said that $\mathcal{F}$ is *transversely projective* if it is a transversely homogeneus $(\mathrm{PSL}(2,\mathbb{R}), S^1_\infty)$-foliation. This means that, according to Section 2.1, $\omega$ and $\theta$ can be chosen so that $d\theta = \eta \wedge \omega$ and $d\eta = \eta \wedge \theta$ for some $\eta \in C^\infty(M; \Lambda^1)$ [6]. In this case, the corresponding description of Section 1.8 is given by $\pi : \widetilde{M} \to M$, $D : \widetilde{M} \to S^1_\infty$, $h : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{Hol}\,\mathcal{F} \subset \mathrm{PSL}(2,\mathbb{R})$.

Assume that $\mathcal{F}$ is transversely projective and almost without holonomy. Then Inaba and Matsumoto proved that either of the following holds [28, Proposition 2.1, the proof of Proposition 3.4 and its remark]:

(a) $\mathrm{Hol}\,\mathcal{F}$ is conjugate to an abelian subgroup of $\mathrm{PSO}(2)$.
(b) $\mathrm{Hol}\,\mathcal{F}$ consists of the identity, hyperbolic elements with a common fixed point set and possible elliptic elements which keep the fixed point set invariant.
(c) $\mathrm{Hol}\,\mathcal{F}$ is conjugate to a subgroup of the stabilizer of $\infty$.

In the case (a), $\mathcal{F}$ is an $\mathbb{R}$-Lie foliation.

In the case (c), we can assume that $\mathrm{Hol}\,\mathcal{F}$ is a subgroup of the stabilizer of $\infty$ after conjugation. If $\infty \notin \mathrm{im}\,D$, then $\mathcal{F}$ is transversely affine. If $\infty \in \mathrm{im}\,D$ and $\mathrm{Hol}\,\mathcal{F}$ does not contain parabolic elements, then $\mathcal{F}$ satisfies (b). If $\infty \in \mathrm{im}\,D$ and $\mathrm{Hol}\,\mathcal{F}$ has some parabolic element $h$, then the fixed point of $h$ is $\infty$, and $\pi(D^{-1}(\infty))$ consists of some compact leaves whose holonomy group cannot be given by germs of homotheties.

In the case (b), $\mathrm{Hol}\,\mathcal{F}$ is virtually abelian, and it is abelian just when there are no elliptic elements. After conjugation, we can assume that the fixed point set of the hyperbolic elements is $\{0,\infty\}$. Since $\mathrm{im}\,D$ is $\mathrm{Hol}\,\mathcal{F}$-invariant and $\widetilde{M}$ is connected, it follows that $\mathrm{im}\,D$ is $\mathbb{R}^\pm$, $\mathbb{R}$, $S^1_\infty \smallsetminus \{0\}$ or $S^1_\infty$. If $\mathrm{im}\,D = \mathbb{R}^\pm$ or $\mathrm{im}\,D = \mathbb{R}$, then $\mathcal{F}$ is transversely affine. If $\mathrm{im}\,D = S^1_\infty \smallsetminus \{0\}$, then we pass to the case $\mathrm{im}\,D = \mathbb{R}$ using conjugation by the rotation $x \mapsto -1/x$ of $S^1_\infty$. Thus, if $\mathcal{F}$ is not transversely affine, then $\mathrm{im}\,D = S^1_\infty$. Let us analyze the last case from now on.

Now an obvious version of Lemma 2.8 follows with a similar proof, where $D^{-1}(\{0,\infty\})$ is used in (i) instead of $D^{-1}(0)$, and subgroups of $\mathrm{Hol}_0\,\mathcal{F}$ or $\mathrm{Hol}_\infty\,\mathcal{F}$ are used in (ii) instead of just subgroups of $\mathrm{Hol}_0\,\mathcal{F}$.

Note that $x\partial_x \in \mathfrak{X}(\mathbb{R})$ extends to a smooth vector field on $S^1_\infty$, also denoted by $x\partial_x$, which is invariant by all hyperbolic elements with fixed point set $\{0,\infty\}$. In fact, $x\partial_x$ on $S^1_\infty \smallsetminus \{0\}$ corresponds to $-y\partial_y$ on $\mathbb{R}$ by the rotation $x \mapsto y = -1/x$ of $S^1_\infty$.

Lemma 2.15. *If $Z \in \mathfrak{X}(S^1_\infty)$ is invariant by some hyperbolic element whose fixed point set is $\{0,\infty\}$, then $Z = \varkappa x\partial_x$ for some $\varkappa \in \mathbb{R}$. In particular, $\mathfrak{X}(S^1_\infty, \mathrm{Hol}\,\mathcal{F}) = \mathbb{R}\,x\partial_x$ if $\mathrm{Hol}\,\mathcal{F}$ has no elliptic element, otherwise $\mathfrak{X}(S^1_\infty, \mathrm{Hol}\,\mathcal{F}) = 0$.*

Proof. By Lemma 2.9 (i), $Z|_{\mathbb{R}} = \varkappa x\partial_x$ for some $\varkappa \in \mathbb{R}$ because the restriction to $\mathbb{R}$ of any hyperbolic element with fixed point set $\{0,\infty\}$ is a homothety different from the identity. So $Z = \varkappa x\partial_x$ on $S^1_\infty$.

The last assertion is true because any elliptic element $A$ preserving $\{0,\infty\}$ is conjugated to the rotation $x \mapsto -1/x$ by some hyperbolic element with fixed point set $\{0,\infty\}$, and therefore $A_*(x\partial_x) = -x\partial_x$. □

Like in Section 2.4, every $\mathcal{F}^1_l \equiv \mathring{\mathcal{F}}_l$ becomes a complete $\mathbb{R}$-Lie foliation with the restriction to $M^1_l \equiv \mathring{M}_l$ of the element of $\overline{\mathfrak{X}}(M,\mathcal{F})$ defined by $x\partial_x \in \mathfrak{X}(\mathbb{R}, \operatorname{Hol}\mathcal{F})$ via (1.2). Moreover the statements of Lemma 2.11 and Proposition 2.12 hold as well, with the obvious adaptations of the proofs.

Now the transverse orientation of every $\mathcal{F}_l$ may be directed outward and inward on different boundary leaves of $M_l$. Anyway, $M^0$ contains at least two leaves because $\emptyset \neq \pi(D^{-1}(0)), \pi(D^{-1}(\infty)) \subset M^0$.

Example 2.16. The identity and the hyperbolic elements with common fixed point set $\{0,\infty\}$ form an abelian and torsion free subgroup $H \subset \mathrm{PSL}(2,\mathbb{R})$ (its restriction to $\mathbb{R}$ is the group of orientation preserving homotheties). Let $\Gamma \subset H$ be a subgroup of finite rank, and let $\widetilde{L}$ be a $\Gamma$-covering of the closed oriented surface $L$ of genus two. Let $\widetilde{M} = S^1_\infty \times \widetilde{L}$ with the foliation $\widetilde{\mathcal{F}}$ by the fibers of the first factor projection $D : \widetilde{M} \to S^1_\infty$. The diagonal action of $\Gamma$ on $\widetilde{M}$, given by $\gamma \cdot (x,\tilde{y}) = (\gamma(x), \gamma\cdot\tilde{y})$, preserves $\widetilde{\mathcal{F}}$. Thus it induces a suspension foliation $\mathcal{F}$ on the closed manifold $M = \Gamma\backslash\widetilde{M}$ [8, Section 3.1]. $\mathcal{F}$ is a transversely projective foliation, whose developing map is $D$ and with $\operatorname{Hol}\mathcal{F} = \Gamma$ (Section 1.8). It has two compact leaves, which are diffeomorphic to $L$, and all other leaves are diffeomorphic to $\widetilde{L}$.

Example 2.17. In Example 2.4 (iii), the model (1) foliation $\mathcal{G}$ is transversely projective. It is transversely affine if and only if $\operatorname{sign}(f(x))$ has the same limit as $x \to 1$ and as $x \to -1$, which is another description of the transversely affine Reeb component of Example 2.13 for $n = 2$ and $\lambda = e^{1/\mu}$.

In Example 2.5 (iii), using the above model (1) foliations to make tangential gluing, all of them with the same $\mu$, the result is a transversely projective foliation if it is transversely oriented, which means that the number of transversely affine models is even. It is transversely affine if and only if all models are transversely affine.

Example 2.18. In Example 2.6, if $\mathcal{F}$ and $\mathcal{G}$ are also transversely projective, and induce the same projective structure on $S^1$ via $c$ and $d$, then $\mathcal{F}'$ clearly becomes transversely projective. (See [19, Appendix to Section 2] for the classification of projective circles.)

## 3. Transversely simple foliated flows

Let $\mathcal{F}$ be a smooth foliation of codimension one on a manifold $M$. For the sake of simplicity, assume that $\mathcal{F}$ is transversely oriented. Let $Z \in \mathfrak{X}_{\mathrm{com}}(M,\mathcal{F})$ with foliated flow

$\phi$. Let $M^0$ be the union of leaves preserved by $\phi$. The $\phi$-invariant set $M^0$ is closed in $M$ because it is the zero set of $\overline{Z} \in \overline{\mathfrak{X}}(M, \mathcal{F}) \subset C^\infty(M; N\mathcal{F})$. Moreover $\phi$ is transverse to the leaves on the open set $M^1 := M \smallsetminus M^0$. So there is a canonical isomorphism $N\phi \cong T\mathcal{F}$ on $M^1$, and $\mathcal{F}$ is TC at every point of $M^1$ (Section 1.7); in particular, the leaves in $M^1$ have no holonomy. With the notation of Sections 1.4–1.6, let $\bar{\phi}$ be the $\mathcal{H}$-equivariant local flow on $\Sigma$ generated by $\overline{Z} \in \mathfrak{X}(\Sigma, \mathcal{H})$. Via the homeomorphism $M/\mathcal{F} \to \Sigma/\mathcal{H}$ defined by the maps $x'_k$, the leaves preserved by $\phi$ correspond to the $\mathcal{H}$-orbits preserved by $\bar{\phi}$, whose union is $\mathrm{Fix}(\bar{\phi})$ because they are totally disconnected.

Definition 3.1. The leaves preserved by $\phi$ that correspond to simple fixed points of $\bar{\phi}$ are called *transversely simple*. If all leaves preserved by $\phi$ are transversely simple, then $\phi$ (or $Z$) is called *transversely simple*.

Since $\dim \Sigma = 1$, for all simple $\bar{p} \in \mathrm{Fix}(\bar{\phi})$, there is some $\varkappa = \varkappa_{\bar{p}} \in \mathbb{R}^\times$ such that $\bar{\phi}^t_* \equiv e^{\varkappa t}$ on $T_{\bar{p}}\Sigma \equiv \mathbb{R}$. By the $\mathcal{H}$-equivariance of $\bar{\phi}$, we easily get $\varkappa_{\bar{p}} = \varkappa_{\bar{q}}$ for all $\bar{q} \in \mathcal{H}(\bar{p}) \subset \mathrm{Fix}(\bar{\phi})$. Thus we can use the notation $\varkappa_L = \varkappa_{\bar{p}}$ if $\mathcal{H}(\bar{p})$ corresponds to the simple preserved leaf $L$.

Lemma 3.2. *Let $\psi$ be a local flow on $\mathbb{R}$ with infinitesimal generator $X \in \mathfrak{X}(\mathbb{R})$. If $0$ is a simple fixed point of $\psi$ with $\varkappa_0 = \varkappa$, then there is a coordinate $x$ around $0$ in $\mathbb{R}$ so that $x(0) = 0$ and $X = \varkappa x \partial_x$, and therefore $\psi^t(x) = e^{\varkappa t} x$.*

Proof. Let $u$ denote the standard coordinate of $\mathbb{R}$. The condition on $0$ means that $X = f(u)\partial_u$ for some $f \in C^\infty(\mathbb{R})$ with $f(0) = 0$ and $f'(0) = \varkappa$. Then $f(u) = uh(u)$ for some $h \in C^\infty(\mathbb{R})$ with $h(0) = \varkappa$. Hence there is some $g \in C^\infty(\mathbb{R})$ such that $\varkappa - h(u) = ug(u)$. We look for some smooth function $x = x(u)$ around $0$ so that $x(0) = 0$, $x'(0) \neq 0$ and $\varkappa x \partial_x = X$. Thus $x(u) = ue(u)$ for some smooth function $e = e(u)$ defined around $0$ with $e(0) \neq 0$. Since $\partial_u = x'(u)\partial_x$, we need $\varkappa u e(u) = uh(u)(e(u) + ue'(u))$ around $0$; i.e., $e'(u)/e(u) = (\varkappa - h(u))/uh(u) = g(u)/h(u)$. Any $e(u) = C \exp(\int_0^u g(v)/h(v)\, dv)$ with $C \neq 0$ will do the job. □

Remark 3.3. (i) Since $\bar{\phi}$ and $\overline{Z} \in \mathfrak{X}(\Sigma, \mathcal{H}) \equiv \overline{\mathfrak{X}}(M, \mathcal{F})$ determine each other, the condition on the preserved leaves of $\phi$ to be transversely simple depends only on $\overline{Z} \in \overline{\mathfrak{X}}(M, \mathcal{F})$.

(ii) By Lemma 3.2, around any point $p$ in a transversely simple leaf $L \subset M^0$, there are foliated coordinates $(x, y)$ with $x(p) = 0$ and $\overline{Z} = \varkappa_L x \partial_x$.

(iii) If $\phi$ is transversely simple, then every closed orbit is contained in either $M^0$ or $M^1$, and all fixed points belong to $M^0$.

From now on, suppose that $\phi$ is transversely simple and $M$ is compact, unless otherwise stated.

Proposition 3.4. *$M^0$ is a finite union of compact leaves.*

Proof. Since $\mathrm{Fix}(\bar{\phi})$ has no accumulation points in $\Sigma$ (Section 1.1), every leaf $L$ in $M^0$ has a neighborhood $V$ such that $V \cap M^0 = L$. Thus the result follows using that $M$ is compact, and $M^0$ is closed in $M$. □

By Proposition 3.4 and since the leaves in $M^1$ have no holonomy, $\mathcal{F}$ is almost without holonomy (Section 2.3), and only a finite number of leaves may have holonomy. According to Remark 2.2 (vi), we can consider Hector's description with this choice of $M^0$ and $M^1$, even though there may be leaves without holonomy in $M^0$. Consider also the rest of the notation of Section 2.3. If the leaves in $M^1$ are not compact, then $M^1$ is just the transitive point set of $\mathcal{F}$.

Given any leaf $L \subset M^0$ and $p \in L$, let $(x, y) : U \to \Sigma \times B''$ be a foliated chart around $p$ like in Remark 3.3 (ii), where $\Sigma$ is some open interval containing 0. Let $\mathbf{h} : \pi_1 L \to \mathrm{Hol}\, L$, $\sigma \mapsto \mathbf{h}_\sigma$, be the holonomy homomorphism of $L$ at $p$. Via the projection $x : U \to \Sigma$, we can regard $\mathrm{Hol}\, L$ as a subgroup of the group of germs at 0 of local transformations of $\Sigma$ such that 0 is a fixed point in their domains.

Proposition 3.5. $\mathrm{Hol}\, L$ *consists of germs at* 0 *of homotheties on* $\mathbb{R}$.

Proof. All elements of $\mathrm{Hol}\, L$ can be represented by elements of the group $\mathrm{Diffeo}^+(\mathbb{R}, 0)$ of orientation-preserving diffeomorphisms of $\mathbb{R}$ that fix 0. According to Remark 3.3 (ii), for the above foliated coordinates $(x, y)$ around $p$, we have $\overline{Z} = \varkappa x \partial_x$ for $\varkappa = \varkappa_L$. Then, by Lemma 2.9 (ii) and Remark 2.10 (ii), any element of $\mathrm{Hol}\, L$ is the germ at 0 of a homothety. □

According to Proposition 3.5, $\mathbf{h} = \mathbf{h}_L$ is induced by the homomorphism $\hat{h} = \hat{h}_L : \pi_1 L \to \mathrm{Diffeo}^+(\mathbb{R}, 0)$ whose image consists of homotheties. We get an induced monomorphism $h = h_L : \Gamma := \pi_1 L / \ker \hat{h} \to \mathrm{Diffeo}^+(\mathbb{R}, 0)$, $\gamma \mapsto h_\gamma$, with $h_\gamma(x) = a_\gamma x$ for some monomorphism $\Gamma \to \mathbb{R}^+ \equiv (\mathbb{R}^+, \times)$, $\gamma \mapsto a_\gamma = a_{L,\gamma}$. The holonomy cover $\pi = \pi_L : \widetilde{L} \to L$ is determined by $\pi_1 \widetilde{L} \equiv \ker \hat{h} = \ker \mathbf{h}$. On some neighborhood of $L$, $\mathcal{F}$ can be described with the suspension defined by $\pi$ and $h$, recalled in Section 4.

Every $\mathcal{F}_l^1$ becomes a complete $\mathbb{R}$-Lie foliation with the structure induced by $Z_l \in \mathfrak{X}_{\mathrm{com}}(M_l^1, \mathcal{F}_l^1)$, with the original differentiable structure (see Remark 2.2 (iv)). We use the following notation for its Fedida's description (Sections 1.7 and 2.2): $\pi_l : \widetilde{M}_l^1 \to M_l^1$, $h_l : \Gamma_l := \mathrm{Aut}(\pi_l) \to \mathbb{R}$, $D_l : \widetilde{M}_l^1 \to \mathbb{R}$ and $\widetilde{\mathcal{F}}_l^1 = \pi_l^* \mathcal{F}_l^1$. The abelian and torsion free group $\Gamma_l$ has finite rank because $\pi_1 M_l^1 \equiv \pi_1 \mathring{M}_l \cong \pi_1 M_l$ and $M_l$ is compact. The action of any $\gamma \in \Gamma_l$ on $\widetilde{M}_l^1$ is denoted by $\tilde{p} \mapsto \gamma \cdot \tilde{p}$ or by $T_\gamma$. Let $\widetilde{Z}_l$ and $\tilde{\phi}_l$ be the lifts of $Z_l$ and $\phi_l$ to $\widetilde{M}_l^1$. Recall that $\widetilde{Z}_l$ is $D_l$-projectable, and we can assume that $D_{l*} \widetilde{Z}_l = \partial_x$ (Section 2.2).

By Remark 2.2 and Proposition 2.3, we have the following cases for $\mathcal{F}$:

(a) $\mathcal{F}$ is given by a fiber bundle $M \to S^1$ with connected fibers.
(b) $\mathcal{F}$ is an $\mathbb{R}$-Lie foliation with dense leaves.

(c) $M^0 \neq \emptyset$, $\operatorname{Hol} L \cong \mathbb{Z}$ for all leaves $L \subset M^0$, and the foliations $\mathcal{F}_l^1$ are given by fiber bundles $M_l^1 \to S^1$ with connected fibers.
(d) $M^0 \neq \emptyset$, $\operatorname{Hol} L$ is a finitely generated abelian group of rank $> 1$ for all leaves $L \subset M^0$, and all foliations $\mathcal{F}_l^1$ are minimal $\mathbb{R}$-Lie foliations.

The case (a) can be considered as a model (1) with empty boundary, avoiding the use of models (0), or it can be cut into models (0) by adding a finite number of leaves without holonomy to $M^0$ (Remark 2.2 (vi)).

Remark 3.6. The results and observations of this section hold without requiring $M$ to be compact, assuming only that $M^0$ is compact.

Example 3.7. By Proposition 3.5, the Reeb foliation $\mathcal{F}$ on $S^3$ does not admit any transversely simple foliated flow because it has a leaf with holonomy but no infinitesimal holonomy. Actually, this proves that its Reeb components of [8, Example 3.3.11] cannot show up as models in Hector's description of any foliation on a closed manifold with a simple foliated flow. Similarly, this realization is impossible for Example 2.4 (i),(ii).

## 4. Case of a suspension foliation

4.1. **Basic definitions.** For a connected closed manifold $L$, let $\hat{h} : \pi_1 L \to \operatorname{Diffeo}^+(\mathbb{R}, 0)$ be a homomorphism whose image consists of homotheties (like in Section 3). It induces a monomorphism $h : \Gamma := \pi_1 L / \ker \hat{h} \to \operatorname{Diffeo}^+(\mathbb{R}, 0)$, $\gamma \mapsto h_\gamma$. We have $h_\gamma(x) = a_\gamma x$ for some monomorphism $\Gamma \to \mathbb{R}^+$, $\gamma \mapsto a_\gamma$; in particular, $\Gamma$ is abelian, torsion free and finitely generated. Let $\pi = \pi_L : (\widetilde{L}, \tilde{p}) \to (L, p)$ be the pointed regular covering map with $\pi_1 \widetilde{L} = \pi_1(\widetilde{L}, \tilde{p}) \equiv \ker \hat{h}$, and therefore $\operatorname{Aut}(\pi) \equiv \Gamma$. We may use the notation $[\tilde{y}] = \pi(\tilde{y})$ for $\tilde{y} \in \widetilde{L}$. The canonical left action of every $\gamma \in \Gamma$ on $\widetilde{L}$ is denoted by $T_\gamma$ or $\tilde{y} \mapsto \gamma \cdot \tilde{y}$. For the diagonal left action of $\Gamma$ on $\widetilde{M} = \mathbb{R} \times \widetilde{L}$, $\gamma \cdot (x, \tilde{y}) = (a_\gamma x, \gamma \cdot \tilde{y})$, let $M = \Gamma \backslash \widetilde{M}$. The canonical projection $\pi_M : \widetilde{M} \to M$ is a $\Gamma$-cover with deck transformations $h_\gamma \times T_\gamma$ ($\gamma \in \Gamma$). Write $[x, \tilde{y}] = \pi_M(x, \tilde{y})$ for $(x, \tilde{y}) \in \widetilde{M}$. Let $\widetilde{\varpi} : \widetilde{M} \to \widetilde{L}$ denote the second factor projection, and let $\widetilde{\mathcal{F}}$ be the foliation on $\widetilde{M}$ with leaves $\{x\} \times \widetilde{L}$ ($x \in \mathbb{R}$). Since $\widetilde{\varpi}$ is $\Gamma$-equivariant, it induces a fiber bundle map $\varpi : M \to L$, defined by $\varpi([x, \tilde{y}]) = [\tilde{y}]$. On the other hand, since $\widetilde{\mathcal{F}}$ is $\Gamma$-invariant, it induces a foliation $\mathcal{F}$ on $M$ so that $\pi^* \mathcal{F} = \widetilde{\mathcal{F}}$, which is transverse to the fibers of $\varpi$. $(M, \mathcal{F})$ is called the *suspension* defined by $\hat{h}$ (or $h$) and $\pi$ [8, Section 3.1]. Note that the typical fiber of $\varpi$ is $\mathbb{R}$ because the corresponding fibers of $\widetilde{\varpi}$ and $\varpi$ can be identified via $\pi_M$. Since 0 is fixed by the $\Gamma$-action on $\mathbb{R}$, the leaf $\{0\} \times \widetilde{L} \equiv \widetilde{L}$ of $\widetilde{\mathcal{F}}$ is $\Gamma$-invariant, and $\pi_M(\{0\} \times \widetilde{L}) \equiv L$ is a compact leaf of $\mathcal{F}$. The other leaves of $\widetilde{\mathcal{F}}$ are diffeomorphic via $\pi_M$ to the corresponding leaves of $\mathcal{F}$ because the elements of $\Gamma \smallsetminus \{e\}$ have no fixed points in $\mathbb{R}^\times$. Given $\tilde{y} \in \widetilde{L}$ and $y = [\tilde{y}] \in L$, the fiber $\varpi^{-1}(y) \equiv \widetilde{\varpi}^{-1}(\tilde{y}) = \mathbb{R} \times \{\tilde{y}\} \equiv \mathbb{R}$ is a global transversal of $\mathcal{F}$ through $[0, \tilde{y}] \equiv y$. Note that the holonomy homomorphism $\mathbf{h} : \pi_1 L \to \operatorname{Hol} L$ is induced by $h$, and therefore

$\widetilde{L}^{\mathrm{hol}} \equiv \widetilde{L}$. The standard orientation of $\mathbb{R}$ induces a transverse orientation of $\widetilde{\mathcal{F}}$, which is $\Gamma$-invariant, giving rise to a transverse orientation of $\mathcal{F}$.

$\mathcal{F}$ is transversely affine foliation on an open manifold. Its description of Section 1.8 is given by $\pi_M : \widetilde{M} \to M$, the first factor projection $D : \widetilde{M} \to \mathbb{R}$ and $h : \Gamma \to \mathrm{Aff}^+(\mathbb{R})$. In this case, $D$ induces an identity $\widetilde{M}/\widetilde{\mathcal{F}} \equiv \mathbb{R}$, and therefore the inclusions of (1.2) and (1.3) are equalities (cf. Proposition 2.12 for the case where $M$ is closed).

4.2. **Transversely simple vector fields on a suspension foliation.** Given any $\varkappa \in \mathbb{R}^\times$, consider the transversely simple foliated flow $\tilde{\xi}$ on $(\widetilde{M}, \widetilde{\mathcal{F}})$ given by $\tilde{\xi}^t(x, \tilde{y}) = (e^{\varkappa t}x, \tilde{y})$, whose infinitesimal generator is $\widetilde{Y} = (\varkappa x\partial_x, 0) \in \mathfrak{X}_{\mathrm{com}}(\widetilde{M}, \widetilde{\mathcal{F}})$. With the notation of Section 3 for $\tilde{\xi}$, we have $\mathrm{Fix}(\tilde{\xi}^t) = \widetilde{M}^0 = \{0\} \times \widetilde{L} \equiv \widetilde{L}$, and the orbits on $\widetilde{M}^1$ are the fibers of the restriction $\widetilde{\varpi} : \widetilde{M}^1 \to \widetilde{L}$. Since $\tilde{\xi}^t$ is $\Gamma$-equivariant and $\widetilde{Y}$ is $\Gamma$-invariant, they can be projected to $M$ obtaining a transversely simple foliated flow $\xi^t$ with infinitesimal generator $Y \in \mathfrak{X}(M, \mathcal{F})$, satisfying $\mathrm{Fix}(\xi^t) = M^0 = \pi_M(\widetilde{M}^0) \equiv L$, and the orbits on $M^1$ are the fibers of the restriction $\varpi : M^1 \to \widetilde{L}$. Moreover $\overline{Y} \equiv \varkappa x\partial_x$ on $\mathbb{R}$ via (1.2), whose flow $\bar{\xi}$ is given by $\bar{\xi}^t(x) = e^{\varkappa t}x$.

$\mathcal{F}^1_\pm \equiv \mathring{\mathcal{F}}_\pm$ on $M^1_\pm \equiv \mathring{M}_\pm$ is a transversely complete $\mathbb{R}$-Lie foliation with the structure defined by $Y_\pm \in \mathfrak{X}_{\mathrm{com}}(M^1_\pm, \mathcal{F}^1_\pm)$ (see Remark 3.6). In its Fedida's description (Section 1.7), $\widetilde{M}^1_\pm$ is the holonomy covering of $M^1_\pm$, whose group of deck transformations is also $\Gamma$. The developing map $D_\pm : \widetilde{M}^1_\pm \to \mathbb{R}$ and holonomy homomorphism $h_\pm : \Gamma \to \mathbb{R}$ can be chosen to be given by $D_\pm(x, y) = \varkappa^{-1} \ln |x| =: t$ and $h_\pm(\gamma) = \varkappa^{-1} \ln a_\gamma$, and therefore $\mathrm{Hol}\, \mathcal{F}_\pm = \{\, \varkappa^{-1} \ln a_\gamma \mid \gamma \in \Gamma \,\}$. In this way, $(D_\pm)_* \widetilde{Y}_\pm = \partial_t$, like in Section 2.2.

Let $\phi$ be any transversely simple foliated flow on $M$, with infinitesimal generator $Z \in \mathfrak{X}_{\mathrm{com}}(M, \mathcal{F})$, such that $M^0 = L$. According to Remark 3.3 (ii), we can assume $\bar{\phi} = \bar{\xi}$ and $\overline{Z} = \overline{Y}$. Then the lifts to $\widetilde{M}$, $\tilde{\phi}$ of $\phi$ and $\widetilde{Z}$ of $Z$, are of the form

$$\tilde{\phi}^t(x, \tilde{y}) = (e^{\varkappa t}x, \tilde{\phi}^t_x(\tilde{y})) \ , \quad \widetilde{Z} = (\varkappa x\partial_x, \widetilde{Z}_x) \ , \tag{4.1}$$

for smooth families, $\{\, \tilde{\phi}^t_x \mid x, t \in \mathbb{R} \,\} \subset \mathrm{Diffeo}(\widetilde{L})$ and $\{\, \widetilde{Z}_x \mid x \in \mathbb{R} \,\} \subset \mathfrak{X}(\widetilde{L})$. In particular, $\widetilde{Z}_0$ is the restriction of $\widetilde{Z}$ to $\widetilde{L} \equiv \{0\} \times \widetilde{L}$, and its flow is $\phi_0 = \{\tilde{\phi}^t_0\}$. Thus $\widetilde{Z}_0$ is $\Gamma$-invariant and $\tilde{\phi}_0$ is $\Gamma$-equivariant, inducing the restrictions of $Z$ and $\phi$ to $L$, denoted by $Z_0$ and $\phi_0$.

Proposition 4.1. *The flow $\phi_0$ is simple if and only if the fixed points and closed orbits of $\phi$ in $M^0$ are simple.*

Proof. Let $\tilde{y} \equiv (0, \tilde{y}) \in \widetilde{L} \equiv \widetilde{M}^0$ and $y = [\tilde{y}] \equiv [0, \tilde{y}] \in L \equiv M^0$. Suppose that $y \in \mathrm{Fix}(\phi_0) \equiv \mathrm{Fix}(\phi) \cap M^0$, and therefore $\tilde{y} \in \mathrm{Fix}(\tilde{\phi}_0) \equiv \mathrm{Fix}(\tilde{\phi}) \cap \widetilde{M}^0$. By (4.1),

$$T_{[0,\tilde{y}]}M \equiv T_{(0,\tilde{y})}\widetilde{M} \equiv \mathbb{R} \oplus T_{\tilde{y}}\widetilde{L} \equiv \mathbb{R} \oplus T_y L \ ,$$
$$\phi^t_{*[0,\tilde{y}]} \equiv \tilde{\phi}^t_{*(0,\tilde{y})} \equiv e^{\varkappa t} \oplus \tilde{\phi}^t_{0*\tilde{y}} \equiv e^{\varkappa t} \oplus \phi^t_{0*y} \ .$$

So $p$ is simple for $\phi$ if and only if $y$ is simple for $\phi_0$.

Now suppose that $y$ is in some closed orbit $c$ of $\phi_0$, which can be also considered as a closed orbit of $\phi$ in $M^0$. Then there is some $\gamma \in \Gamma$ such that $\tilde{\phi}_0^{\ell(c)}(\tilde{y}) = \gamma \cdot \tilde{y}$. As before,

$$N_{[0,\tilde{y}]}\phi \equiv N_{(0,\tilde{y})}\tilde{\phi} \equiv \mathbb{R} \oplus N_{\tilde{y}}\tilde{\phi}_0 \equiv \mathbb{R} \oplus N_y\phi_0 \ ,$$
$$N_{[0,\tilde{y}]}\phi \equiv N_{(0,\gamma\cdot\tilde{y})}\tilde{\phi} \equiv \mathbb{R} \oplus N_{\gamma\cdot\tilde{y}}\phi_0 \equiv \mathbb{R} \oplus N_y\phi_0 \ ,$$
$$\phi^{\ell(c)}_{*[0,\tilde{y}]} \equiv \tilde{\phi}^{\ell(c)}_{*(0,\tilde{y})} \equiv e^{\varkappa\ell(c)} \oplus \tilde{\phi}^{\ell(c)}_{0*\tilde{y}} \equiv e^{\varkappa\ell(c)} \oplus \phi^{\ell(c)}_{0*y} \ .$$

So $c$ is simple for $\phi$ if and only if it is simple for $\phi_0$. □

Proposition 4.2. *For every simple $A \in \mathfrak{X}(L)$ without closed orbits, there is some simple $B \in \mathfrak{X}_{\mathrm{com}}(M, \mathcal{F})$ without closed orbits such that $\overline{B} = \overline{Y}$ and $B_0 \equiv A$.*

Proof. Let $\widetilde{A} \in \mathfrak{X}_{\mathrm{com}}(\widetilde{L})$ be the lift of $A$, whose flow is denoted by $\tilde{\zeta}$, and let $\widetilde{B} = (\varkappa x\partial_x, \widetilde{A}) \in \mathfrak{X}(\widetilde{M}, \widetilde{\mathcal{F}})$. Clearly, $\widetilde{B}_0 \equiv \widetilde{A}$ and $\overline{\widetilde{B}} = \overline{\widetilde{Y}}$. Moreover $\widetilde{B}$ is complete because its flow $\tilde{\eta}$ is given by $\tilde{\eta}^t(x, \tilde{y}) = (e^{\varkappa t}x, \tilde{\zeta}^t(\tilde{y}))$. Since $\widetilde{B}$ is $\Gamma$-invariant, it induces some $B \in \mathfrak{X}_{\mathrm{com}}(M, \mathcal{F})$ with flow $\eta$.

Claim 4.3. *The flow $\eta$ has neither fixed points nor closed orbits in $M^1$.*

By absurdity, suppose that $\eta^t([x, \tilde{y}]) = [x, \tilde{y}]$ for some $[x, \tilde{y}] \in M^1$ and $t > 0$. Then there is some $\gamma \in \Gamma$ such that $\tilde{\eta}^t(x, \tilde{y}) = \gamma \cdot (x, \tilde{y})$. Since $x \neq 0$, this means that $e^{\varkappa t} = a_\gamma$ and $\tilde{\zeta}^t(\tilde{y}) = \gamma \cdot \tilde{y}$. Thus $\zeta^t(y) = y$ for $y = [\tilde{y}]$. Hence $y \in \mathrm{Fix}(\zeta)$ because $\zeta$ has no closed orbits, and therefore $\tilde{y} \in \mathrm{Fix}(\tilde{\zeta})$. It follows that $\gamma \cdot \tilde{y} = \tilde{y}$, yielding $\gamma = e$. So $e^{\varkappa t} = 1$, obtaining $\varkappa t = 0$, a contradiction.

By Proposition 4.1, Claim 4.3 and since $\eta_0 \equiv \zeta$, it follows that $\eta$ is simple without closed orbits. □

4.3. **Differential forms defining a suspension foliation.** For $k = \operatorname{rank}\Gamma$, fix generators $\gamma_1, \ldots, \gamma_k$ of $\Gamma$. Let $c_i$ be a piecewise smooth loop in $L$ based at $p$ such that $[c_i] \in \pi_1(L, p)$ defines $\gamma_i$, and let $a_i = a_{\gamma_i}$. By the universal coefficients and Hurewicz theorems, there are closed 1-forms $\beta_1, \ldots, \beta_k$ on $L$ so that $\delta_{ij} = \langle [\beta_i], [c_j] \rangle = \int_0^1 c_j^*\beta_i$ and $\langle [\beta_i], \ker \hat{h} \rangle = 0$. Thus every $\pi^*\beta_i$ is exact on $\widetilde{L}$. Let $\theta = -\ln(a_1)\,\beta_1 - \cdots - \ln(a_k)\,\beta_k$. Then $\tilde{\theta} = \pi^*\theta = dF$ for some $F \in C^\infty(\widetilde{L})$. With some abuse of notation, let $\theta \equiv \varpi^*\theta$, $\tilde{\theta} \equiv \widetilde{\varpi}^*\tilde{\theta}$ and $F \equiv \widetilde{\varpi}^*F$. It is easy to check that $T_\gamma^*F = F - \ln a_\gamma$ on $\widetilde{L}$ for all $\gamma \in \Gamma$. Thus $\tilde{\rho} = e^F x$ and $\tilde{\omega} = |\varkappa|^{-1}e^F\,dx$ are $\Gamma$-invariant on $\widetilde{M}$. Furthermore $\tilde{\rho}$ is a defining function of $\widetilde{L}$ on $\widetilde{M}$, $\tilde{\omega}$ defines $\widetilde{\mathcal{F}}$, $d\tilde{\omega} = \tilde{\theta} \wedge \tilde{\omega}$ and $d\tilde{\rho} = \tilde{\rho}\tilde{\theta} + |\varkappa|\,\tilde{\omega}$. We get an induced defining function $\rho$ of $L$ on $M$, and an induced form $\omega$ defining of $\mathcal{F}$, so that $d\omega = \theta \wedge \omega$ and $d\rho = \rho\theta + |\varkappa|\,\omega$. We also get $M \equiv \mathbb{R}_\rho \times L_\varpi$, giving rise to smaller tubular neighborhoods $T_\epsilon \equiv (-\epsilon, \epsilon)_\rho \times L_\varpi$ $(\epsilon > 0)$.

4.4. **Change of the differentiable structure.** Given $0 < \alpha \neq 1$, let $f_\alpha : \mathbb{R} \to \mathbb{R}$ be the homeomorphism defined by $f_\alpha(x) = \mathrm{sign}(x)|x|^\alpha = x\,|x|^{\alpha-1}$. The restrictions $f_\alpha : \mathbb{R}^\pm \to \mathbb{R}^\pm$ are diffeomorphisms, but $f_\alpha$ is not diffeomorphism around 0. Clearly, $f_\alpha(a_\gamma x) =$

$a^\alpha_\gamma f_\alpha(x)$, and it is easy to check that $f_{\alpha*}(x\partial_x) = \alpha u\partial_u$ on $\mathbb{R}^\pm$, using the coordinate $u = f_\alpha(x)$. Like in Section 4.1, let $h_\alpha : \Gamma \to \mathrm{Diffeo}^+(\mathbb{R}, 0)$ be the monomorphism defined by $h_{\alpha,\gamma}(u) = a^\alpha_\gamma u$, and let $(M_\alpha, \mathcal{F}_\alpha)$ be the suspension defined with $h_\alpha$ and $\pi$. The foliated homeomorphism $\widetilde{\Upsilon}_\alpha = f_\alpha \times \mathrm{id}$ of $(\widetilde{M}, \widetilde{\mathcal{F}})$ is equivariant with respect to the $\Gamma$-actions defined by $h$ and $h_\alpha$, and therefore it induces a foliated homeomorphism $\Upsilon_\alpha : (M, \mathcal{F}) \to (M_\alpha, \mathcal{F}_\alpha)$. The restriction $\Upsilon_\alpha : (M^1, \mathcal{F}^1) \to (M^1_\alpha, \mathcal{F}^1_\alpha)$ is a diffeomorphism.

A transversely simple foliated flow $\xi_\alpha$ on $(M_\alpha, \mathcal{F}_\alpha)$, with infinitesimal generator $Y_\alpha$, can be defined like $\xi$ and $Y$ in Section 4.2, using $\varkappa\alpha$ instead of $\varkappa$, and we get $\Upsilon_{\alpha*}Y = Y_\alpha$ on $M^1_\alpha$. With more generality, for any transversely simple foliated flow $\phi$ on $(M, \mathcal{F})$, with infinitesimal generator $Z \in \mathfrak{X}_{\mathrm{com}}(M, \mathcal{F})$, such that $\bar{\phi} = \bar{\xi}$ and $\overline{Z} = \overline{Y}$, there is a transversely simple foliated flow $\phi_\alpha$ on $(M_\alpha, \mathcal{F}_\alpha)$, with infinitesimal generator $Z_\alpha$, such that $\bar{\phi}_\alpha = \bar{\xi}_\alpha$, $\overline{Z_\alpha} = \overline{Y_\alpha}$, and $\Upsilon_{\alpha*}Z = Z_\alpha$ on $M^1_\alpha$. Precisely, using (4.1), their lifts $\tilde{\phi}_\alpha$ and $\widetilde{Z}_\alpha$ to $\widetilde{M}$ are given by

$$\tilde{\phi}^t_\alpha(u, \tilde{y}) = (e^{\varkappa\alpha t}u, \tilde{\phi}^t_u(\tilde{y})) \ , \quad \widetilde{Z}_\alpha = (\varkappa\alpha u\partial_u, \widetilde{Z}_u) \ .$$

In other words, we get a new differentiable structure on $(M, \mathcal{F})$ via $\Upsilon_\alpha$, which agrees with the original one on $M^1$. This will be called a *transverse power change* of the differentiable structure (around the leaf $L$). With this point of view, $\phi$ is a smooth transversely simple foliated flow with both differentiable structures, replacing $\varkappa$ with $\varkappa\alpha$. In this way, we can change $|\varkappa|$ arbitrarily, but keeping $\mathrm{sign}(\varkappa)$ invariant.

With the new differentiable structure, $C^\infty(M)$ is generated by $\rho_\alpha := \rho\,|\rho|^{\alpha-1}$ and $C^\infty(L) \equiv \varpi^* C^\infty(L)$. Moreover $\rho_\alpha$ is a defining function of $L$, $\omega_\alpha := \rho^{\alpha-1}\omega$ and $\theta_\alpha$ have smooth extensions to $M$, $\omega_\alpha$ defines $\mathcal{F}$, $d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha$ and $d\rho_\alpha = \rho_\alpha\theta_\alpha + |\alpha\varkappa|\,\omega_\alpha$.

## 5. Global structure

Consider the notation of Section 3, where $M$ is compact, $\mathcal{F}$ is transversely oriented, and $\phi$ is transversely simple.

5.1. **Tubular neighborhoods of the components of $M^0$.** In the following, $L$ runs in $\pi_0 M^0$ (the set of leaves in $M^0$), and we have corresponding objects $\hat{h}_L$, $h_L$, $\Gamma_L$, $\pi_L : \widetilde{L} \to L$, $a_{L,\gamma}$ and $\varkappa_L$, defined by $\mathcal{F}$ and $\phi$. Consider the constructions of Sections 4.1–4.3, using this data, adding a prime and the subindex "$L$" to their notation: the suspension $(M'_L, \mathcal{F}'_L)$ defined with $h_L$, with projection $\varpi'_L : M'_L \to L$, the transversely simple foliated flow $\xi'_L$ with infinitesimal generator $Y'_L$, the differential forms $\omega'_L$ and $\theta'_L$, the defining function $\rho'_L$, and the tubular neighborhoods $T'_{\epsilon,L}$.

By the Reeb's local stability, there are foliated diffeomorphisms between the restrictions of $\mathcal{F}$ and $\mathcal{F}'$ to tubular neighborhoods, $T_{L,0}$ of $L$ in $M$ and $T'_{L,0} := T'_{L,\epsilon_0}$ ($\epsilon_0 > 0$) of $L$ in $M'_L$, so that the projection $\varpi_L$ of $T_{L,0}$ corresponds to the projection $\varpi'_L$ of $T'_{L,0}$. We will simply write $\mathcal{F} \equiv \mathcal{F}'_L$ and $\varpi_L \equiv \varpi'_L$ on $T_{L,0} \equiv T'_{L,0}$. We can assume that the sets $\overline{T_{L,0}}$ are

disjoint in $M$, and $\bar{\phi} \equiv \bar{\xi}'_L$ and $\overline{Z} \equiv \overline{Y'_L}$ on $T_{L,0} \equiv T'_{L,0}$ (Remark 3.3 (ii)). Fix also smaller tubular neighborhoods, $T_L \equiv T'_L := T'_{L,\epsilon}$ $(0 < \epsilon < \epsilon_0)$.

Let $M' = \bigsqcup_L M'_L$, where we consider the combinations of all of the above objects, removing $L$ from the notation: $\mathcal{F}'$, $\varpi'$, $\xi'$, $Y'$, $\omega'$, $\theta'$ and $\rho'$. Similarly, let $T' = \bigsqcup_L T'_L$, $T'_0 = \bigsqcup_L T'_{L,0}$, $T = \bigcup_L T_L$ and $T_0 = \bigcup_L T_{L,0}$.

Proposition 5.1. *(i) There is some $Z' \in \mathfrak{X}_{\mathrm{com}}(M', \mathcal{F}')$ such that $\overline{Z'} = \overline{Y'}$, $Z' \equiv Z$ on $T \equiv T'$, and $Z' = Y'$ on $M' \smallsetminus T'_0$.*

*(ii) For any $A \in \mathfrak{X}(M', \mathcal{F}')$ with $\overline{A} = \overline{Y'}$, there is some $B \in \mathfrak{X}(M, \mathcal{F})$ with $\overline{B} = \overline{Z}$, $B \equiv A$ on $T \equiv T'$, and $B = Z$ on $M \smallsetminus T_0$.*

*(iii) There are $\omega, \theta \in C^\infty(M; \Lambda^1)$ such that $\omega$ defines $\mathcal{F}$, $\omega \equiv \omega'$ on $T \equiv T'$ and $d\omega = \theta \wedge \omega$ on $M$.*

Proof. Let $\lambda \in C^\infty(M)$ such that $0 \leq \lambda \leq 1$, $\lambda = 1$ on $T$, and $\operatorname{supp} \lambda \subset T_0$, and let $\lambda' \in C^\infty_{\mathrm{c}}(M')$ such that $\operatorname{supp} \lambda \subset T'_0$ and $\lambda' \equiv \lambda$ on $T'_0 \equiv T_0$.

To prove (i), let $Z'_0 \equiv Z$ on $T'_0 \equiv T_0$, and take $Z' \equiv Y' + \lambda'(Z'_0 - Y')$.

To prove (ii), let $B_0 \equiv A$ on $T_0 \equiv T'_0$, and take $B = Z + \lambda(B_0 - Z)$.

To prove (iii), take $\omega_0 \equiv \omega'$ and $\theta_0 \equiv \theta'$ on $T_0$. Take $\omega_1 \in C^\infty(M; \Lambda^1)$ defining $\mathcal{F}$. Then $\omega = \lambda\omega_0 + (1-\lambda)\omega_1$ also defines $\mathcal{F}$. Thus $d\omega = \theta_1 \wedge \omega$ for some $\theta_1 \in C^\infty(M; \Lambda^1)$. We get $(\theta_0 - \theta_1) \wedge \omega = 0$ on $T$, and therefore (iii) is satisfied $\theta = \theta_1 + \lambda(\theta_0 - \theta_1)$. □

We can also consider a transverse power change of the differential structure on every $M'_L$ around $L$ (Section 4.4). The corresponding new differentiable structure on every $T_L \equiv T'_L$ can be combined with the differentiable structure of $M^1$ to produce a new differentiable structure on $M$, also called a *transverse power change* of the differentiable structure (around $M^0$), and keeping $Z \in \mathfrak{X}(M, \mathcal{F})$ after this change. In this way, the absolute values $|\varkappa_L|$ can be changed arbitrarily, but keeping every $\operatorname{sign}(\varkappa_L)$ invariant.

Consider the forms $\omega$ and $\theta$ of Proposition 5.1 (iii), and let $\rho \equiv \rho'$ on $T \equiv T'$. With the new differentiable structure, $C^\infty(T)$ is generated by $\rho_\alpha := \rho\,|\rho|^{\alpha-1}$ and $C^\infty(M^0) \equiv \varpi^* C^\infty(M^0)$. Moreover $\omega_\alpha := \rho^{\alpha-1}\omega$ and $\theta_\alpha := \alpha\theta$ have smooth extensions to $T$, $\omega_\alpha$ defines $\mathcal{F}|_T$, $d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha$ on $T$, and $d\rho_\alpha = \rho_\alpha\theta_\alpha + |\alpha\varkappa_L|\,\omega_\alpha$ on $T_L$. Like in Proposition 5.1 (iii), the restrictions of $\omega_\alpha$ and $\theta_\alpha$ to some smaller tubular neighborhood of $L$ can be extended to $M$, keeping the relation $d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha$.

5.2. **Transverse structure.** Let $\mathcal{P}$ be the pseudogroup on $S^1_\infty = \mathbb{R} \cup \{\infty\}$ generated by the projective rotation $x \mapsto -1/x$, the hyperbolic projective transformations $x \mapsto \lambda x$ $(\lambda > 0)$, and the diffeomorphisms $x \mapsto x^\alpha$ of $\mathbb{R}^+$ $(\alpha > 0)$. $\mathcal{F}$ is called a *$\mathcal{P}$-foliation* if $\{U_k, x_k\}$ can be chosen such that every $\Sigma_k$ is realized as an open subset of $S^1_\infty$ and the maps $h_{kl}$ belong to $\mathcal{P}$.

Proposition 5.2. *$\mathcal{F}$ is a $\mathcal{P}$-foliation.*

Proof. Since $\mathcal{F} \equiv \mathcal{F}'$ on every $T_{L,0} \equiv T'_{L,0}$ (Section 5.1), the restriction of $\mathcal{F}$ to any $T_{L,0}$ has a regular foliated atlas $\{U_a, (x_a, y_a)\}$ such that the corresponding elementary holonomy transformations are restrictions of homotheties. For $\varkappa = \varkappa_L \in \mathbb{R}^\times$, we have $\overline{Z} = \varkappa x_a \, \partial_{x_a}$ on $x_a(U_a)$ (Section 5.1), whose local flow $\bar{\phi}_a$ is given by $\bar{\phi}_a(x,t) = e^{\varkappa t} x$.

Now the restrictions $\mathcal{F}_l^1$ are $\mathbb{R}$-Lie foliations according to Sections 3. Then any $\mathcal{F}_l^1$ has a regular foliated atlas $\{V_i, (w_i, v_i)\}$ whose elementary holonomy transformations are given by translations, $w_i = h_{ij}(w_j) = w_j + c_{ij}$, between open intervals of $\mathbb{R}$. Taking the new transverse coordinates $u_i = e^{w_i}$, we get another regular foliated atlas $\{V_i, (u_i, v_i)\}$ of $\mathcal{F}_l^1$, whose elementary holonomy transformations are given by homotheties, $u_i = e^{c_{ij}} u_j$, between open intervals of $\mathbb{R}^+$. Thus $\{V_i, (u_i, v_i)\}$ defines a transversely affine structure of $\mathcal{F}_l^1$. With the notation of Section 3, we can indeed assume that $\pi_l : \widetilde{V}_i \to V_i$ is a diffeomorphism for some open $\widetilde{V}_i \subset \widetilde{M}_l^1$, and $u_i \pi_l = D_l$ on $\widetilde{V}_i$. Hence $\overline{Z} = \partial_{w_i}$ on $w_i(V_i)$, and therefore $\overline{Z} = u_i \partial_{u_i}$ on $u_i(V_i)$, whose local flow $\bar{\phi}_i$ is given by $\bar{\phi}_i^t(u) = e^t u$.

For any nonempty intersection $U_a \cap V_i$, via the corresponding elementary holonomy transformation $h_{ai} = x_a u_i^{-1}$, the vector field $\varkappa x_a \, \partial_{x_a}$ corresponds to $u_i \partial_{u_i}$, and therefore $\bar{\phi}_a$ corresponds to $\bar{\phi}_i$. Take any $p \in U_a \cap V_i$, and let $\bar{p}_a = x_a(p) \in \mathbb{R}^\times$ and $\bar{p}_i = u_i(p) \in \mathbb{R}^+$. Then, for $|t|$ small enough,

$$h_{ai}(e^t \bar{p}_i) = h_{ai} \bar{\phi}_i^t(\bar{p}_i) = \bar{\phi}_a^t(\bar{p}_a) = e^{\varkappa t} \bar{p}_a = \bar{p}_a \bar{p}_i^{-\varkappa} (e^t \bar{p}_i)^\varkappa \ ,$$

yielding $h_{ai}(u) = \bar{p}_a \bar{p}_i^{-\varkappa} u^\varkappa$ for u close enough to $\bar{p}_i$. Since $h_{ai}$ preserves the orientation, $\bar{p}_a$ and $\varkappa$ must have the same sign. Then $h_{ai}$ can be expressed as a composition of generators of $\mathcal{P}$:

$$u \mapsto \tilde{u} := u^\varkappa \mapsto \bar{p}_a \bar{p}_i^{-\varkappa} \tilde{u} \qquad \text{if } \bar{p}_a, \varkappa > 0 \ , \tag{5.1}$$

$$u \mapsto \tilde{u} := u^{-\varkappa} \mapsto \hat{u} := |\bar{p}_a|^{-1} \bar{p}_i^\varkappa \tilde{u} \mapsto -1/\hat{u} \qquad \text{if } \bar{p}_a, \varkappa < 0 \ . \tag{5.2}$$

Thus a union of foliated atlases of these types, for all $L \in \pi_0 M^0$ and foliations $\mathcal{F}_l$, is a foliated atlas of $\mathcal{F}$ defining a structure of $\mathcal{P}$-foliation. □

Proposition 5.3. *After performing some transverse power change of the differentiable structure around $M^0$, $\mathcal{F}$ becomes transversely projective.*

Proof. Using a transverse power change of the differentiable structure around $M^0$, we can assume that $\varkappa_L = \pm 1$ for all $L \in \pi_0 M^0$. Then, in the proof of Proposition 5.2, the elementary holonomy transformations (5.1) and (5.2) are also restrictions of elements of $\mathrm{PSL}(2, \mathbb{R})$. □

## 6. Existence and description of simple foliated flows

Now let $\mathcal{F}$ be any smooth transversely oriented foliation of codimension one on a closed manifold $M$.

6.1. **Existence of simple foliated flows.**

PROPOSITION 6.1. *If $(M, \mathcal{F})$ admits some transversely simple foliated flow $\phi$, then it also admits some simple foliated flow $\psi$ with $\bar{\phi} = \bar{\psi}$.*

PROOF. Let $Z \in \mathfrak{X}(M, \mathcal{F})$ be the infinitesimal generator of $\phi$, and consider the notation of Section 5.1. Take some simple flow $\zeta$ on $M^0$ without closed orbits (Example 1.2), and let $A$ denote its infinitesimal generator. By Proposition 4.2, there is some simple $B \in \mathfrak{X}_{\mathrm{com}}(M', \mathcal{F}')$, without closed orbits, such that $B|_{M^0} = A$ and $\overline{B} = \overline{Z'}$. Then, by Proposition 5.1 (ii), there is some $C \in \mathfrak{X}(M, \mathcal{F})$ with $\overline{C} = \overline{Z}$, $C \equiv B$ on $T \equiv T'$, and $C = Z$ on $M \smallsetminus T_0$.

By Peixoto's extension to open manifolds of a theorem of Kupka and Smale (Section 1.1), there is some generic $D \in \mathfrak{X}(M^1)$ as close as desired to $C|_{M^1}$ in the strong $C^\infty$ topology; in particular, $D$ is simple. If $D$ close enough to $C|_{M^1}$ in the strong $C^\infty$ topology, then $D$ has an extension $E \in \mathfrak{X}(M)$ with $E|_{M^0} = A$, and $\overline{C} = f\overline{E}$ in $C^\infty(M; N\mathcal{F})$ for some $0 < f \in C^\infty(M)$ with $f = 1$ on $M^0$. Thus $fE \in \mathfrak{X}(M, \mathcal{F})$ and $\overline{fE} = \overline{Z}$, and therefore the foliated flow $\psi$ of $fE$ satisfies $\bar{\psi} = \bar{\phi}$. So $\psi$ is transversely simple and has the same preserved leaves as $\phi$ (the leaves in $M^0$); in particular, $\psi$ has no fixed points in $M^1$. Since $fE = E = C \equiv B = A$ on $M^0$, we get that $\psi$ agrees with $\zeta$ on $M^0$, and therefore its fixed points are simple by Proposition 4.1. Moreover $fE|_{M^1} = fD$ is simple by Remark 1.1. □

DEFINITION 6.2. It is said that $\phi$ (or $Z$) is *weakly simple* if its preserved leaves are transversely simple and its closed orbits are simple.

By Proposition 4.1, simple foliated flows are weakly simple.

PROPOSITION 6.3. *If $(M, \mathcal{F})$ has some transversely simple foliated flow $\phi$, then it also has some weakly simple foliated flow $\zeta$ such that $\bar{\phi} = \bar{\zeta}$, $\zeta^t = \mathrm{id}$ on $M^0$ for all $t$, and $\zeta$ has no closed orbit in some neighborhood of $M^0$.*

PROOF. Apply Proposition 5.1 (ii) with some transversely simple $Z \in \mathfrak{X}(M, \mathcal{F})$ and $A = Y'$. □

6.2. **Description of foliations with simple foliated flows.** Now, without requiring the existence of any special foliated flow a priori, assume that $\mathcal{F}$ satisfies the following properties:

(A) $\mathcal{F}$ is almost without holonomy with finitely many leaves with holonomy.
(B) The holonomy groups of the compact leaves can be described as groups of germs at 0 of homotheties on $\mathbb{R}$.

By (A), we can use the notation of Section 2.3. In the following, we refer to the possibilities (a)–(d) of Section 3 for transversely simple flows.

Example 6.4. Suppose that $\mathcal{F}$ is given by a fiber bundle $M \to S^1$ with connected fibers. For any even number of points, $x_1, \dots, x_{2m} \in S^1$ ($m \geq 0$), in cyclic order, and numbers $\varkappa_1, \dots, \varkappa_{2m} \in \mathbb{R}^\times$, with alternate sign, there is some simple flow $\bar{\phi}$ on $S^1$ such that $\mathrm{Fix}(\bar{\phi}) = \{x_1, \dots, x_{2m}\}$ and $\bar{\phi}^t_* = e^{\kappa_j t}$ on $T_{x_j}S^1 \equiv \mathbb{R}$. By Proposition 6.1, there is a simple foliated flow $\phi$ on $(M, \mathcal{F})$ whose preserved leaves are fibers $L_1, \dots, L_{2m}$ over $x_1, \dots, x_{2m}$. If $m > 0$, then $\phi$ has no closed orbits in $M^1$. If $m = 0$, then $\phi$ has no preserved leaves, and therefore no fixed points. This is of type (a).

Example 6.5. If $\mathcal{F}$ is an $\mathbb{R}$-Lie foliation with dense leaves, $\overline{\mathfrak{X}}(M, \mathcal{F})$ is of dimension 1 and generated by a non-vanishing transverse vector field. Hence there are simple foliated flows by Proposition 6.1, all of them without preserved leaves. This is of type (b),

Example 6.6. Suppose that $\mathcal{F}$ is a transversely affine foliation that is not an $\mathbb{R}$-Lie foliation. Then, according to Section 2.4, to get (A), $\mathcal{F}$ is elementary, and we can assume that $\operatorname{im} D = \mathbb{R}$ and $\operatorname{Hol} \mathcal{F}$ is a non-trivial group of homotheties. Then, by Lemma 2.9 (i) and Proposition 2.12, $\overline{\mathfrak{X}}(M, \mathcal{F})$ is generated by a transverse vector field $\overline{Z}$ such that the foliated flow $\phi$ of $Z$ is transversely simple. By Proposition 6.1, there is a simple foliated flow $\phi'$ with $\bar{\phi}' = \bar{\phi}$. It also follows from Lemma 2.9 (i) and Proposition 2.12 that there is some $\varkappa \in \mathbb{R}^\times$ such that $\{\, \varkappa_L \mid L \in \pi_0 M^0 \,\} = \{\varkappa\}$.

Example 6.7. Assume that $\mathcal{F}$ is a transversely projective foliation that is not transversely affine. Then, according to Section 2.5, to get (A) and (B), we can assume that $\operatorname{im} D = S^1_\infty$ and $\operatorname{Hol} \mathcal{F}$ consists of the identity and hyperbolic elements with common fixed point set $\{0, \infty\}$ and possible elliptic elements that keep $\{0, \infty\}$ invariant. By Lemma 2.15 and the projective version of Proposition 2.12, to get $\overline{\mathfrak{X}}(M, \mathcal{F}) \neq 0$, there must be no elliptic element in $\operatorname{Hol} \mathcal{F}$. Moreover, in this case, $\overline{\mathfrak{X}}(M, \mathcal{F})$ is generated by a transverse vector field $\overline{Z}$ such that the foliated flow $\phi$ of $Z$ is transversely simple. By Proposition 6.1, there is some simple foliated flow $\phi'$ with $\bar{\phi}' = \bar{\phi}$. By Lemma 2.15 and the projective version of Proposition 2.12, there is some $\varkappa \in \mathbb{R}^+$ such that $\{\, \varkappa_L \mid L \in \pi_0 M^0 \,\} = \{\pm\varkappa\}$.

Example 6.8. In Examples 6.6 and 6.7, we can consider any transverse power change of the differentiable structure around $M^0$ (Sections 4.4 and 5.1). With the new differentiable structure, the foliation has the same simple foliated flows, but the absolute values $|\varkappa_L|$ can be arbitrary, keeping the same signs $\operatorname{sign}(\varkappa_L)$. Thus $\{\, \operatorname{sign}(\varkappa_L) \mid L \in \pi_0 M^0 \,\}$ is $\{1\}$ (respectively, $\{\pm 1\}$) if and only if the foliation is obtained by a change of the differential structure from Example 6.6 (respectively, Example 6.7).

Examples 6.6–6.8 can be of type (c) or (d).

Theorem 6.9. *For any smooth transversely oriented foliation of codimension one on a closed manifold, the following conditions are equivalent:*

(*i*) *It satisfies* (*A*) *and* (*B*).

(*ii*) *It is described by one of Examples* 6.4–6.8.
(*iii*) *It admits a transversely simple foliated flow.*
(*iv*) *It admits a weakly simple foliated flow (trivial on its preserved leaves).*
(*v*) *It admits a simple foliated flow.*

Proof. We already know that (iii) yields (i) (Section 3). By Proposition 5.3, Examples 6.4–6.8 cover all cases (a)–(d), and therefore (i) yields (ii). Proposition 6.1 states that (iii) yields (v), which was used in Examples 6.4–6.8, showing that (ii) yields (v). Proposition 6.3 states that (iii) yields (iv). The remaining implications are obvious. □

According to Theorem 6.9, the foliations of Examples 2.1, 2.13, 2.14 and 2.16–2.18 admit simple foliated flows.

## References


1. J. A. Álvarez López and Y. A. Kordyukov, *Long time behavior of leafwise heat flow for Riemannian foliations*, Compos. Math. **125** (2001), no. 2, 129–153, `http://dx.doi.org/10.1023/A:1002492700960`. MR 1815391
2. J. A. Álvarez López and Y. A. Kordyukov, *Distributional Betti numbers of transitive foliations of codimension one*, Foliations: geometry and dynamics. Proceedings of the Euroworkshop, Warsaw, Poland, May 29–June 9, 2000 (Singapore), World Sci. Publ., 2002, pp. 159–183. MR 1882768
3. J. A. Álvarez López and Y. A. Kordyukov, *Lefschetz distribution of Lie foliations*, $C^*$-algebras and elliptic theory II (Basel), Trends Math., Birkhäuser, 2008, `http://dx.doi.org/10.1007/978-3-7643-8604-7_1`, pp. 1–40. MR 2408134
4. J. A. Álvarez López, Y. A. Kordyukov, and E. Leichtnam, *Analysis on Riemannian foliations of bounded geometry*, Münster J. Math. 13 (2020), 221–265, `https://www.uni-muenster.de/FB10/mjm/vol_13/mjm_vol_13_07.pdf`.
5. J. A. Álvarez López, Y. A. Kordyukov, and E. Leichtnam, *A trace formula for foliated flows*, in preparation, 2019.
6. R. A. Blumenthal, *Transversely homogeneous foliations*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 4, 143–158, `http://www.numdam.org/item?id=AIF_1979__29_4_143_0`. MR 558593
7. C. Camacho and A. Lins Neto, *Geometric theory of foliations*, Birkhäuser, Boston-Basel-Stuttgart, 1985, Translated from the Portuguese by Sue E. Goodman, `http://dx.doi.org/10.1007/978-1-4612-5292-4`. MR 824240
8. A. Candel and L. Conlon, *Foliations. I*, Graduate Studies in Mathematics, vol. 23, American Mathematical Society, Providence, RI, 2000. MR 1732868
9. A. Candel and L. Conlon, *Foliations. II*, Graduate Studies in Mathematics, vol. 60, American Mathematical Society, Providence, RI, 2003. MR 1994394
10. C. Deninger, *Some analogies between number theory and dynamical systems on foliated spaces*, Doc. Math. **Extra Vol. I** (1998), 163–186, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). MR 1648030
11. C. Deninger, *Number theory and dynamical systems on foliated spaces*, Jahresber. Deutsch. Math.-Verein. **103** (2001), no. 3, 79–100. MR 1873325
12. C. Deninger, *On the nature of the "explicit formulas" in analytic number theory—a simple example*, Number theoretic methods (Iizuka, 2001), Dev. Math., vol. 8, Kluwer Acad. Publ., Dordrecht, 2002, `https://doi.org/10.1007/978-1-4757-3675-5_7`, pp. 97–118. MR 1974137

13. C. Deninger, *Arithmetic geometry and analysis on foliated spaces*, arXiv:math/0505354, 2005.
14. C. Deninger, *Analogies between analysis on foliated spaces and arithmetic geometry*, Groups and analysis, London Math. Soc. Lecture Note Ser., vol. 354, Cambridge Univ. Press, Cambridge, 2008, `https://doi.org/10.1017/CBO9780511721410.010`, pp. 174–190. MR 2528467
15. D. B. A. Epstein, K. C. Millett, and D. Tischler, *Leaves without holonomy*, J. London Math. Soc. (2) **16** (1977), no. 3, 548–552, `http://dx.doi.org/10.1112/jlms/s2-16.3.548`. MR 0464259
16. E. Fedida, *Sur les feuilletages de Lie*, C. R. Acad. Sci. Paris Sér. A **272** (1971), 999–1001. MR 0285025
17. E. Fedida, *Feuilletages de Lie, feuilletages du plan*, Lecture Notes in Math., vol. 352, Springer, 1973, pp. 183–195.
18. C. Godbillon, *Feuilletages: études géométriques*, Progress in Math., vol. 98, Birkhäuser Verlag, Boston-Basel-Stuttgart, 1991. MR 1120547
19. W. M. Goldman, *Discontinuous groups and the Euler class*, Ph.D. thesis, University of California, Berkeley, 1980, `http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:8029408`.
20. V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J. **44** (1977), no. 3, 485–517, `http://projecteuclid.org/euclid.dmj/1077312384`. MR 0448452
21. A. Haefliger, *Variétés feuilletées*, Ann. Scuola Norm. Sup. Pisa (3) **16** (1962), 367–397. MR 0189060
22. G. Hector, *Sur les feuilletages presque sans holonomie*, C. R. Acad. Sci. Paris Sér. A **274** (1972), 1703–1706. MR 0303550
23. G. Hector, *Feuilletages en cylindres*, Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Math., vol. 597, Springer, Berlin, 1977, pp. 252–270. MR 0451260
24. G. Hector, *Croissance des feuilletages presque sans holonomie*, Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976) (Berlin), Lecture Notes in Math., vol. 652, Springer, 1978, pp. 141–182. MR 505659
25. G. Hector and U. Hirsch, *Introduction to the geometry of foliations. Part A: Foliations on compact surfaces, fundamentals for arbitrary codimension, and holonomy*, Aspects of Mathematics, vol. E1, Friedr. Vieweg & Sohn, Braunschweig, 1981. MR 639738
26. G. Hector and U. Hirsch, *Introduction to the geometry of foliations. Part B*, Aspects of Mathematics, vol. E3, Friedr. Vieweg & Sohn, Braunschweig, 1983.
27. T. Inaba, *Resilient leaves in transversely affine foliations*, Tohoku Math. J. (2) **41** (1989), no. 4, 625–631, `http://dx.doi.org/10.2748/tmj/1178227733`. MR 1025328
28. T. Inaba and S. Matsumoto, *Resilient leaves in transversely projective foliations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990), no. 1, 89–101. MR 1049020
29. J. Kim, M. Morishita, T. Noda, and Y. Terashima, *On* 3*-dimensional foliated dynamical systems and Hilbert type reciprocity law*, arXiv:1906.02424, 2019.
30. N. H. Kuiper, *Locally projective spaces of dimension one*, Michigan Math. J. **2** (1953), 95–97, `http://projecteuclid.org/euclid.mmj/1028989908`. MR 0064443
31. I. Kupka, *Contribution à la théorie des champs génériques*, Contributions to Differential Equations **2** (1963), 457–484. MR 0165536
32. I. Kupka, *Addendum et corrections au mémoire: "Contributions à la théorie des champs génériques"*, Contributions to Differential Equations **3** (1964), 411–420. MR 0168878
33. P. Molino, *Riemannian foliations*, Progress in Mathematics, vol. 73, Birkhäuser Boston, Inc., Boston, MA, 1988, Translated from the French by G. Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu, `http://dx.doi.org/10.1007/978-1-4684-8670-4`. MR 932463

34. M. M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology **1** (1962), 101–120, https://doi.org/10.1016/0040-9383(65)90018-2. MR 0142859
35. M. M. Peixoto, *On an approximation theorem of Kupka and Smale*, J. Differential Equations **3** (1967), 214–227, https://doi.org/10.1016/0022-0396(67)90026-5. MR 0209602
36. M. Schwarz, *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel, 1993, https://doi.org/10.1007/978-3-0348-8577-5. MR 1239174
37. B. Seke, *Sur les structures transversalement affines des feuilletages de codimension un*, Ann. Inst. Fourier (Grenoble) **30** (1980), no. 1, 1–29, http://www.numdam.org/item?id=AIF_1980__30_1_1_0. MR 576071
38. S. Smale, *On gradient dynamical systems*, Ann. of Math. (2) **74** (1961), 199–206, https://doi.org/10.2307/1970311. MR 0133139
39. S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 97–116. MR 0165537
40. I. Tamura, *Topology of foliations: an introduction*, Translations of Mathematical Monographs, vol. 97, American Mathematical Society, Providence, RI, 1992, Translated from the 1976 Japanese edition and with an afterword by Kiki Hudson. With a foreword by Takashi Tsuboi. MR 1151624
41. P. Walczak, *Dynamics of foliations, groups and pseudogroups*, Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), vol. 64, Birkhäuser Verlag, Basel, 2004, https://doi.org/10.1007/978-3-0348-7887-6. MR 2056374

Department/Institute of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain, ☎ +34 881813149, ORCID: 0000-0001-6056-2847
*E-mail address*: jesus.alvarez@usc.es

Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Science, 112 Chernyshevsky str., 450008 Ufa, Russia, ORCID: 0000-0003-2957-2873
*E-mail address*: yurikor@matem.anrb.ru

Institut de Mathématiques de Jussieu-PRG, CNRS, Batiment Sophie Germain (bureau 740), Case 7012, Paris Cedex 13, France
*E-mail address*: eric.leichtnam@imj-prg.fr