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Probabilistic reachability and control synthesis for stochastic switched systems using the tamed Euler method

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Abstract

In this paper, we explain how, under the one-sided Lipschitz (OSL) hypothesis, one can find a mean square error bound for a variant of the Euler-Maruyama approximation method for stochastic switched systems. Subsequently, we explain how this bound can be used to control a stochastic switched system in order to make it reach a target zone with guaranteed minimum probability. The method is illustrated on several examples from the literature.

Keywords: Stochastic systems, numerical simulation, control system synthesis, switched control systems, nonlinear control systems.

1. Introduction

Symbolic methods for the verification and control synthesis of hybrid systems (and, particularly, "switched systems") have received significant attention in the past few years. However, control systems involving stochastic differential equations remain difficult to handle with symbolic methods, and few methods have been developed for these systems. One distinguishes two main classes of symbolic methods for hybrid systems: indirect methods and direct methods [2].

Indirect methods proceed by constructing a finite abstraction of the original system by discretization of the dense state space $\mathbb{R}^d$ (where d is the dimension of

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the state space). Among the indirect methods, one of the most successful proceeds by *approximate bisimulation* [10]. This method originally designed for deterministic switched systems has been recently extended for *stochastic switched systems* [30, 31, 32]. This approach relies on the hypothesis of *incremental stability* of the stochastic switched system (or existence of a common/multiple Lyapunov function). Another method is presented in [7]. The associated tool [25] allows to generate formal abstractions for discrete-time Markov processes defined over uncountable (continuous) state spaces.

Indirect methods have also been developed for the verification of safety. Safety verification has been studied using the concept of barrier certificates [23]. The original inquiry has been later modified to cope with stochastic systems [29], and to the design of a controller that makes the closed-loop system safe [28, 24]. Also related is the subject of stability and safety verification is the reach avoidance problem. It has been studied using dynamic programming. This approach has been developed both for deterministic hybrid system in [18], and for discrete-time stochastic hybrid systems in [26].

A direct method proceeds by working directly at the level of the dense state of the system $\mathbb{R}^d$; it computes "trajectory tubes", which are over-approximations of the set of all the controlled trajectories starting at a given subregion of $\mathbb{R}^d$. In previous work, we have followed such a direct approach first in [9], then in [15], using the Euler approximation scheme for calculating over-approximations of tubes of trajectories. We show here how to extend this direct method to *stochastic* switched systems. The method extends the deterministic approach by replacing the classical Euler approximation scheme with a variant of the stochastic Euler-Maruyama scheme [13]. The correctness of these Euler-based methods does not rely on the hypothesis of incremental stability as in [30, 32], but on the hypothesis of 'one-sided Lipschitz (OSL)' condition with constant $\lambda \in \mathbb{R}^d$ (also called 'monotonicity'/'dissipativity', see [27]). It can be seen that if a stochastic switched system satisfies an OSL condition with $\lambda < 0$, then the function $V(x, x') = \|x - x'\|^2$ is a *common incremental Lyapunov function* in the sense of [31], from which it follows that the switched system is incrementally stable, and can be analysed by approximate bisimulation. However, Euler-based methods also apply when the system is not incrementally stable, in which case the constant λ is necessarily positive. We thus consider a class of systems different from that of [31].

The plan of the paper is as follows: In Section 3, we give an explicit upper bound on the mean square error of the tamed Euler method for SDEs under OSL condition (Theorem 2). We apply the result in order to ensure reachability properties of stochastic switched systems with guaranteed minimum probability (Section 4). We conclude in Section 5.

This paper is an extended version of a paper (A. Le Coënt, L. Fribourg and J. Vacher. Control Synthesis for Stochastic Switched Systems using the Tamed Euler Method. In ADHS'18, IFAC-PapersOnLine 51(16), pages 259-264. Elsevier Science Publishers, 2018). The main additional material of the extended version is:

- Section 3.3, which gives analytical upper bounds on expressions occurring in Theorem 2,
- Section 3.4, which gives a lower bound on the probability of reaching a given set R (Proposition 3), and
- Section 4.4, which proposes an optimized method of control synthesis.

2. Preliminaries

2.1. Notations

The symbols $\mathbb{N}$, $\mathbb{N}_{\geqslant 0}$, $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geqslant 0}$ denote the set of natural, nonnegative integer, real, positive, and nonnegative real numbers.

The symbol $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. The symbol $\langle\cdot,\cdot\rangle$ denotes the scalar product of two vectors of $\mathbb{R}^d$. Given a point $x \in \mathbb{R}^d$ and a positive real $r > 0$, the ball $B(x, r)$ of centre x and radius r is the set $\{y \in \mathbb{R}^d \mid \|x - y\| \leqslant r\}$.

Throughout this paper, the set U denotes a finite set of switched modes. Given a sequence of k modes $a = (a_1 \cdots a_k) \in U^k$, and a sequence of p modes $b = (b_1 \cdots b_p) \in \mathbb{N}^p$, we denote the concatenation $a * b$ of a and b the sequence of length $k + p$: $a * b = (a_1 \cdots a_k \cdot b_1 \cdots b_p)$.

A sequence of non-negative integers is written in bold when it is used as a multi-index, *e.g.* $\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{N}^p_{\geqslant 0}$, the sum of the components of $\mathbf{a}$ is denoted by $|\mathbf{a}| = a_1 + \cdots + a_p$, and $\binom{b}{\mathbf{a}} = \frac{b!}{a_1! a_2! \ldots a_p!}$ is the multinomial coefficient.

2.2. Systems considered and assumptions

Let $\tau \in \mathbb{R}_{>0}$ be a fixed real number, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with normal filtration $(\mathcal{F}_t)_{t\in[0,\tau]}$, let $d, m \in \mathbb{N}$ let $W = (W^{(1)}, \ldots, W^{(m)}) : [0, \tau] \times \Omega \to \mathbb{R}^m$ be an m-dimensional standard $(W_t)_{t\in[0,\tau]}$-Brownian motion and let $x_0 : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|x_0\|^p] < \infty$ for all $p \in [1, \infty)$. Moreover, let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable function whose derivative grows at most polynomially. Formally, let us suppose the existence of constants $D \in \mathbb{R}_{\geqslant 0}$ and $q \in \mathbb{N}$ such that, for all $x, y \in \mathbb{R}^d$

$$\|f(x) - f(y)\|^2 \leqslant D\|x - y\|^2(1 + \|x\|^q + \|y\|^q) \tag{H1}$$

Let $g = (g_{i,j})_{i\in\{1,\ldots,d\}, j\in\{1,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be a globally Lipschitz continuous function: there exists $L_g \in \mathbb{R}_{\geqslant 0}$ such that, for all $x, y \in \mathbb{R}^d$

$$\|g(x) - g(y)\| \leqslant L_g\|x - y\| \tag{H2}$$

Finally, let us suppose that f is globally one-sided Lipschitz with constant $\lambda \in \mathbb{R}$:

$$\exists\lambda \in \mathbb{R}\ \forall x, y \in \mathbb{R}^d :\ \langle f(y) - f(x), y - x\rangle \leqslant \lambda\, \|y - x\|^2 \tag{H3}$$

Then consider the Stochastic Differential Equations (SDE):

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}W_t, \qquad X_0 = x_0 \tag{1}$$

for $t \in [0, \tau]$. The drift coefficient f is the infinitesimal mean of the process X and the diffusion coefficient g is the infinitesimal standard deviation of the process X. Under the above assumptions, the SDE (1) is known to have a unique strong solution. More formally, there exists an adapted stochastic process $(X_{t,x_0}) : [0, \tau] \times \Omega \to \mathbb{R}^d$ with continuous sample paths fulfilling

$$X_{t,x_0} = x_0 + \int_0^t f(X_s)\mathrm{d}s + \int_0^t g(X_s)\mathrm{d}W_s$$

for all $t \in [0, \tau]$ $\mathbb{P}$-a.s. (see, e.g., ([21])).

We denote by X_{t,x_0} the solution of Equation (1) at time t from initial condition $X_{0,x_0} = x_0$ $\mathbb{P}$-a.s., in which x_0 is a random variable that is measurable in $\mathcal{F}_0$.

Remark 1. *Constants λ, L_g and D can be computed using (constrained) optimization algorithms (see ([15])).*

2.3. Overview of the approach

We aim at synthesizing reachability controllers for the class of systems described above. Our approach consists in adapting reachability analysis based control systhesis methods to stochastic systems. In [15], we successfully used the Euler method for guaranteeing controlled reachability of deterministic systems using the Euler method in association to a new error bound relying on the OSL property of the vector field.

Consider equation (1) with $g = 0$. Classically, one knows that, if the function f is Lipschitz continuous with Lipschitz constant L, the solution of the ODE starting at a given initial value exists and is unique. Besides, one has:

$$\|X_{t,x_0} - X_{t,x_1}\| \leqslant e^{Lt}\|x_0 - x_1\|, \tag{2}$$

with two initial values x_i $(i = 0, 1)$. This gives a rough *growth bound*, i.e. a function bounding the distance of neighboring trajectories as t evolves. In [6], it is proven that, if f is continuous and OSL with OSL constant λ, then the solution of the ODE starting at a given initial value exists and is unique, and, for all $x_0, x_1 \in \mathbb{R}^n$:

$$\|X_{t,x_0} - X_{t,x_1}\| \leqslant e^{\lambda t}\|x_0 - x_1\|. \tag{3}$$

This gives a more accurate growth bound because a Lipschitz function f is always OSL, and the associated OSL constant λ is always less than or equal to its Lipschitz counterpart L. Using the OSL constant λ, it is also possible to bound the error $\|X_{t,x_0} - \tilde{X}_{t,x_1}\|$ in function of $\|x_0 - x_1\|$, where $\tilde{X}_{t,x_1}$ denotes the *Euler approximate* trajectory of the solution X_{t,x_1}. We can then conclude that any trajectory starting from x_0 close enough from a given initial condition x_1 $(\|x_0 - x_1\| \leqslant \delta_0)$ will remain within some bound (denoted by δ_{t,δ_0}) of the Euler trajectory $\tilde{X}_{t,x_1}$. Our objective is to use a similar approach for stochastic systems, while making as few additional hypotheses on the dynamics as possible.

Therefore, we first explain how we can use a variant of the *Euler-Maruyama scheme* called *tamed Euler scheme* in order to guarantee reachability of the expectation of the state $\mathbb{E}[X_{t,x_0}]$ of the stochastic system (1) in a given set. It consists in computing an approximate Euler-based trajectory $\tilde{X}_{t,x_1}$, we then provide an error bound for $\mathbb{E}[\|X_{t,x_0} - \tilde{X}_{t,x_1}\|^2]$ (Theorem 2) which allows to write a result for the expectation of the non approximated trajectory $\mathbb{E}[X_{t,x_0}]$ (Proposition 2). This result can be extended further as a probability result (the probability with which the state reaches the set, Proposition 3).

This reachability analysis method is then used in association with tiling based control synthesis algorithms to ensure that the system is stabilised in a given region of interest with a given probability (Section 4). The first algorithm we use is based on an adaptive state-space decomposition algorithm [15, 14], and uses an exhaustive enumeration of all possible control sequences (up to a given length) in order to find stabilising sequences (see Section 4.3). We then propose a faster approach (in Section 4.4) which relies on state-space discretization and dynamic programming techniques in order to minimize the average distance of the state with some given objective after a given number of steps. The second algorithm is merely an enhancement of the control sequence search of the first one, using a static tiling. Note that with both algorithms, there is no need to compute a finite-state abstraction of the system, which means that computing a controller is done with the application of a single algorithm, unlike [31] which requires the computation of a finite-state abstraction before computing controllers.

Example 1. *Throughout the paper, we illustrate our approach on a 2-mode switched system borrowed from [30]. This case-study is a stochastic version of a well-known system originally introduced as an illustrative example in [16]. The dynamics is given by the system of equations*

$\mathrm{d}x_1 = (-0.25x_1 + px_2 + (-1)^p 0.25)\mathrm{d}t + wx_1\mathrm{d}W_t^1,$

$\mathrm{d}x_2 = ((p-3)x_1 - 0.25x_2 + (-1)^p(3-p))\mathrm{d}t + wx_2\mathrm{d}W_t^2,$

where $p = 1, 2$ is the mode, and we consider different values of noise $w \in \mathbb{R}_{>0}$.

3. Bounding the Error of the tamed Euler method

3.1. Tamed Euler approximation

The standard way to extend the classical Euler method for ordinary differential equations to the SDE (1) is the Euler-Maruyama scheme ([19]). More precisely, given $z : \Omega \to \mathbb{R}^d$ an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|z\|^p] < \infty$ for all $p \in [1, \infty)$, the explicit Euler-Maruyama (EM) method for the SDE (1) is given by the mappings $Y^N_{n,z} : \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, which satisfy $Y^N_{0,z} = z$ and

$$Y^N_{n+1,z} = Y^N_{n,z} + \frac{\tau}{N} \cdot f(Y^N_{n,z}) + g(Y^N_{n,z})(W_{(n+1)\tau/N} - W_{n\tau/N})$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. See ([19]). Unfortunately, the convergence results for the EM scheme does not hold when the drift function

f of the SDE (1) is polynomial (not an affine function). In order to consider non affine drift functions, we adopt a refined scheme, which has been proposed recently to overcome this difficulty ([13]). Let $\underline{X}^N_{n,z} : \Omega \to \mathbb{R}^d$,

$$\begin{aligned} \underline{X}^N_{n+1,z} = \underline{X}^N_{n,z} &+ \frac{\frac{\tau}{N} \cdot f(\underline{X}^N_{n,z})}{1 + \frac{\tau}{N} \cdot \|f(\underline{X}^N_{n,z})\|} \\ &+ g(\underline{X}^N_{n,z})(W_{\frac{(n+1)\tau}{N}} - W_{\frac{n\tau}{N}}) \end{aligned} \tag{4}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. We refer to the numerical method (4) as a *tamed Euler scheme* ([13]). In this method the drift term $\frac{\tau}{n} \cdot f(\underline{X}^N_{n,z})$ is "tamed" by the factor $1/(1 + \frac{\tau}{N} \cdot \|f(\underline{X}^N_{n,z})\|)$ for $n \in \{0, 1, \ldots, N-1\}$ and $N \in \mathbb{N}$ in (4). In order to establish results for the time continuous system, we use the following time continuous interpolation of the time discrete numerical approximations (4), also introduced in ([13]). Let $\tilde{X}^N_z : [0, \tau] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of stochastic processes given by

$$\tilde{X}^N_{t,z} = \tilde{X}^N_{n,z} + \frac{(t - n\tau/N) \cdot f(\tilde{X}^N_{n,z})}{1 + \tau/N \cdot \|f(\tilde{X}^N_{n,z})\|} + g(\tilde{X}^N_{n,z})(W_t - W_{\frac{n\tau}{N}}) \tag{5}$$

for all $t \in [\frac{n\tau}{N}, \frac{(n+1)\tau}{N}]$, $n \in \{0, 1 \ldots, N-1\}$ and all $N \in \mathbb{N}$. Note that $\tilde{X}^N_{t,z} : [0, \tau] \times \Omega \to \mathbb{R}^d$ is an adapted stochastic process with continuous sample paths for every $N \in \mathbb{N}$.

We finally introduce a piecewise constant interpolation $\underline{X}^N_{t,z}$ of the tamed Euler scheme as follows. It is mainly used in some constants when establishing the error bound and for proving Theorem 2.

$$\underline{X}^N_{t,z} := \underline{X}^N_{n,z} \quad \text{for } t \in [\frac{n\tau}{N}, \frac{(n+1)\tau}{N}). \tag{6}$$

Note that $\tilde{X}^N_{t,z} = \underline{X}^N_{t,z} = \underline{X}^N_{n,z}$ at time $t = \frac{n\tau}{N}$ for $n \in \{0, 1, \ldots, N\}$.

The following theorem is proven in ([13]):

Theorem 1. *Let us suppose* (H1)-(H2)-(H3). *Let $z : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|z\|^p] < \infty$ for all $p \in [1, \infty)$. Then, for all $p \in [1, \infty)$, the tamed Euler scheme* (4) *satisfies:*

$$\sup_{N \in \mathbb{N}} \sup_{n \in \{0,1,\ldots,N\}} \mathbb{E}[\|\underline{X}^N_{n,z}\|^p] < \infty$$

This theorem allows to ensure the strong convergence of the tamed Euler method. Any number N of subsampling steps can thus be used. This number is now left implicit for the sake of simplicity. From Theorem 1, it follows (cf. Lemma 4.3, ([11])):

Lemma 1. *Let us suppose* (H1)-(H2)-(H3). *Let $z : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|z\|^p] < \infty$ for all $p \in [1, \infty)$. Let us consider the time continuous interpolation of the tamed Euler scheme as defined in* (5)*, and*

the piecewise constant interpolation as defined in (6)*. Then, for any even integer $r \geqslant 2$, there exist two constants $E_{r,z}$ and $F_{r,z}$ such that*

$$\sup_{0\leqslant t\leqslant\tau} \mathbb{E}\|\underline{X}_{t,z} - \tilde{X}_{t,z}\|^r \leqslant (\Delta_t)^{\frac{r}{2}}(E_{r,z}(\Delta_t)^{\frac{r}{2}} + F_{r,z}d).$$

with $\Delta_t = \tau/N$ and:

$$E_{r,z} = 2^r(\|f(0)\|^r + D2^{\frac{r+1}{2}} (1 + \mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{qr})^{\frac{1}{2}}(\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2r})^{\frac{1}{2}}),$$

$$F_{r,z} = 2^r(\|g(0)\|^{2r} + L_g^r\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{\frac{r}{2}}).$$

Remark 2. *Constants $E_{r,z}$ and $F_{r,z}$ are computed using the constants λ and L_g (see Remark 1), and the expected values of $\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^p$ for the required values of p at each time $t = 0,\ \Delta t,\ 2\Delta t,\ \ldots,\ N\Delta t$. These expected values can either be computed numerically by using a Monte Carlo method, or by evaluating analytically the expectations $\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^p$ for all required values of p. Proposition 2 explains how such expectations can be computed for the piecewise linear interpolation $\tilde{X}_{t,z}$, the computation for $\underline{X}^p_{t,z}$ is the same but much simpler since there is no time dependent term in* (6)*. In this paper, we use the latter approach.*

3.2. Mean square error bounding

The following Theorem holds for SDE (1). This corresponds to a stochastic version of Theorem 1 of ([15]), showing that a similar result holds on average, using the tamed Euler method of ([13]). It is an adaptation of Theorem 4.4 in ([11]).

Theorem 2. *Given the SDE system* (1) *satisfying* (H1)-(H2)-(H3)*. Let $\delta_0 \in \mathbb{R}_{\geqslant 0}$. Suppose that z is a random variable on $\mathbb{R}^d$ such that*

$$\mathbb{E}[\|x_0 - z\|^2] \leqslant \delta_0^2.$$

Then, we have, for all $\tau \geqslant 0$:

$$\mathbb{E}[\sup_{0\leqslant t\leqslant\tau}\|X_{t,x_0} - \tilde{X}_{t,z}\|^2] \leqslant \delta^2_{\tau,\delta_0},$$

with $\delta^2_{\tau,\delta_0} := \beta(\tau)e^{\gamma\tau}$, where:

$$\gamma = 2(\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4), \text{ and}$$

$$\begin{aligned}\beta(\tau) &= 2\delta_0^2 + 2\tau\Delta_t L_g^2(1 + 128L_g^2)(F_{2,z}d + E_{2,z}\Delta_t)\\ &+ 4\tau\sqrt{\Delta_t}D(F_{4,z}d + E_{4,z}\Delta_t^2)^{\frac{1}{2}}\\ &(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}.\end{aligned} \tag{7}$$

with $\Delta_t = \tau/N$.

Proof. The proof closely follows the proof of Theorem 4.4 in ([11]). Let $e_t = X_{t,x_0} - \tilde{X}_{t,z}$. We have, for all $0 \leqslant t \leqslant \tau$:

$$\mathrm{d}e_t = (f(X_{t,x_0}) - f(z))\mathrm{d}t + (g(X_{t,x_0}) - g(z))\mathrm{d}W_t. \tag{8}$$

Then, by using Equation (8) and the integral version of Itô formula applied to function $x \mapsto \|x\|^2$ we obtain

$$\begin{aligned} \|e_t\|^2 = \|e_0\|^2 &+ \int_0^t 2\langle e_s,\, f(X_{s,x_0}) - f(\underline{X}_{s,z})\rangle \mathrm{d}s \\ &+ \int_0^t \|g(X_{s,x_0}) - g(\underline{X}_{s,z})\|^2 \mathrm{d}s + M(t), \end{aligned} \tag{9}$$

where $e_0 = x_0 - z$, and

$$M(t) = \int_0^t 2\langle e_s,\, g(X_{s,x_0}) - g(\underline{X}_{s,z})\rangle \mathrm{d}W_s.$$

So we have using (H2):

$$\begin{aligned} \|e_t\|^2 \leqslant \|e_0\|^2 &+ \int_0^t 2\langle e_s,\, f(X_{s,x_0}) - f(\tilde{X}_{s,z})\rangle \mathrm{d}s \\ &+ L_g^2 \int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2 \mathrm{d}s \\ &+ \int_0^t 2\langle e_s, f(\tilde{X}_{s,z}) - f(\underline{X}_{s,z})\rangle \mathrm{d}s + M(t). \end{aligned} \tag{10}$$

So we have using (H3) and Young's inequality:

$$\begin{aligned} \|e_t\|^2 \leqslant \|e_0\|^2 &+ \int_0^t (2\lambda\|e_s\|^2 + L_g^2\|e_s\|^2)\mathrm{d}s \\ &+ L_g^2 \int_0^t \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2 \mathrm{d}s \\ &+ \int_0^t (\frac{1}{\sqrt{\Delta_t}}\|f(\tilde{X}_{s,z}) - f(\underline{X}_{s,z})\|^2 + \sqrt{\Delta_t}\|e_s\|^2)\mathrm{d}s + M(t). \end{aligned} \tag{11}$$

So we have using (H1), for all $0 \leqslant t \leqslant \tau$:

$$\begin{aligned} \|e_t\|^2 \leqslant \|e_0\|^2 &+ (\sqrt{\Delta_t} + 2\lambda + L_g^2) \int_0^t \|e_s\|^2 \mathrm{d}s \\ &+ L_g^2 \int_0^t \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2 \mathrm{d}s \\ &+ \frac{D}{\sqrt{\Delta_t}} \int_0^t (1 + \|\underline{X}_{s,z}\|^q + \|\tilde{X}_{s,z}\|^q)\|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2 \mathrm{d}s \\ &+ M(t). \end{aligned} \tag{12}$$

It follows using Lemma 1 for $r = 2$, and Cauchy-Schwarz inequality:

$$\begin{aligned}
&\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] \leqslant \mathbb{E}\|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2)\int_0^t \mathbb{E}\|e_s\|^2 \mathrm{d}s \\
&+ L_g^2\tau\Delta_t(E_{2,z}\Delta_t + F_{2,z}d) \\
&+ \frac{D}{\sqrt{\Delta_t}} \\
&\int_0^t (\mathbb{E}(1 + \|\underline{X}_{s,z}\|^q + \|\tilde{X}_{s,z}\|^q)^2)^{\frac{1}{2}}(\mathbb{E}\|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^4)^{\frac{1}{2}}\mathrm{d}s \\
&+ m(t),
\end{aligned} \tag{13}$$

where

$$m(t) = \mathbb{E}[\sup_{0\leqslant s\leqslant t}\|M(s)\|].$$

Hence, using using Lemma 1 for $r = 4$, and inequality $(a + b)^p \leqslant 2^p(a^p + b^p)$:

$$\begin{aligned}
&\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] \leqslant \mathbb{E}\|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2))\int_0^t \mathbb{E}\|e_s\|^2 \mathrm{d}s \\
&+ L_g^2\tau\Delta_t(E_{2,z}\Delta_t + F_{2,z}d) \\
&+ 2D\tau\sqrt{\Delta_t}(E_{4,z}\Delta_t^2 + F_{4,z}d)^{\frac{1}{2}} \\
&(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}} + m(t).
\end{aligned} \tag{14}$$

On the other hand, from the Burkholder-Davis-Gundy inequality, we get:

$$m(t) \leqslant 16\mathbb{E}[\int_0^t \|e_s\|^2\|g(X_{s,x_0}) - g(\underline{X}_{s,z})\|^2\mathrm{d}s]^{\frac{1}{2}}$$

Hence, using (H2):

$$m(t) \leqslant 16L_g^2\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2\int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2\mathrm{d}s]^{\frac{1}{2}}$$

Then, using Young's inequality (for any $\alpha > 0$):

$$m(t) \leqslant 8L_g^2(\alpha\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] + \frac{1}{\alpha}\mathbb{E}[\int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2\mathrm{d}s]).$$

Hence, by using Lemma 1 for $r = 2$:

$$\begin{aligned}
m(t) &\leqslant 8\alpha L_g^2\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] \\
&+ \frac{8L_g^2}{\alpha}\int_0^t \mathbb{E}[\sup_{0\leqslant r\leqslant s}\|e_r\|^2]\mathrm{d}s \\
&+ \frac{8L_g^2}{\alpha}\tau\Delta_t(E_{2,z}\Delta_t + F_{2,z}d).
\end{aligned} \tag{15}$$

Hence, letting $\alpha = \frac{1}{16L_g^2}$, we have by replacing in (14):

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] \leqslant \\
&\delta_0^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4)\int_0^t \mathbb{E}[\sup_{0\leqslant r\leqslant s}\|e_r\|^2]\mathrm{d}s \\
&+ \tau(L_g^2 + 128L_g^4)\Delta_t(E_{2,z}\Delta_t + F_{2,z}d) \\
&+ \tau 2D\sqrt{\Delta_t}(E_{4,z}\Delta_t^2 + F_{4,z}d)^{\frac{1}{2}} \\
&(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}.
\end{aligned} \tag{16}$$

It results from Gronwall's inequality:

$$\mathbb{E}[\sup_{0\leqslant t\leqslant\tau}\|e_t\|^2] = \beta(\tau)e^{\gamma\tau},$$

with
$\gamma = 2(\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4)$, and

$$\begin{aligned}
&\beta(\tau) = 2\delta_0^2 + 2\tau(\Delta_t L_g^2(1 + 128L_g^2)(F_{2,z}d + E_{2,z}\Delta_t) \\
&+ 4\tau\sqrt{\Delta_t}D(F_{4,z}d + E_{4,z}\Delta_t^2)^{\frac{1}{2}} \\
&(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}.
\end{aligned} \tag{17}$$

□

It follows from Theorem 2 and Jensen's inequality:

Proposition 1. *Consider two points x_0 and z in $\mathbb{R}^d$, and a positive real number δ_0. Suppose that $x_0 \in B(z, \delta_0)$. Then $\mathbb{E}X_{t,x_0} \in B(\tilde{X}_{t,z}, \delta_{t,\delta_0})$ for all $t \in [0, \tau]$, where δ_{t,δ_0} is defined in (7).*

3.3. Numerical pre-computation of expectations

Theorem 2 requires the knowledge of $\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q}$. In this section, we give a pre-computable upper bound of this quantity.

Proposition 2. *Let $\tilde{X}_{t,z}$ be the piecewise linear interpolation of the solution to equation (1) as defined in equation (5), written in the form $\tilde{X}_{t,z} = \alpha(z) + \beta(z)t + \gamma(z)W_t$ with $\alpha(z) = \tilde{X}_{n,z}^N - \frac{(n\tau/N)\cdot f(\tilde{X}_{n,z}^N)}{1+\tau/N\cdot\|f(\tilde{X}_{n,z}^N)\|}$, $\beta(z) = \frac{f(\tilde{X}_{n,z}^N)}{1+\tau/N\cdot\|f(\tilde{X}_{n,z}^N)\|}$, $\gamma(z) = g(\tilde{X}_{n,z}^N)$. Then*

$$\mathbb{E}[\|\tilde{X}_{t,z}\|^{2q}] = \sum_{|\mathbf{k}|=q}\binom{q}{\mathbf{k}}C_{k_1,k_2,k_3}(z)\mu_{k_4,k_5,k_6}(z,1)t^{k_2+2k_3+k_4+\frac{k_4+k_5}{2}+k_6}, \tag{18}$$

where $\mathbf{k} = (k_1, \ldots, k_6)$ *and*

$$C_{k_1,k_2,k_3}(z) = \Big(\alpha(z)^T\alpha(z)\Big)^{k_1}\Big(2\alpha(z)^T\beta(z)\Big)^{k_2}\Big(\beta(z)^T\beta(z)\Big)^{k_3},$$

and

$$\mu_{\boldsymbol{\kappa}}(z,t) = \begin{cases} \lambda_{\boldsymbol{\kappa}}^{(1)} D_{\frac{k_5-k_4}{2},k_4,k_6}(b(z)b(z)^T, P_{a,b}(z), Q(z))t^{\frac{k_4+k_5}{2}+k_6} \\ \quad \textit{if} \quad k_4 < k_5 \quad \textit{and} \quad k_4 + k_5 \equiv 0 \ (\text{mod } 2) \\ \lambda_{\boldsymbol{\kappa}}^{(2)} D_{\frac{k_4-k_5}{2},k_5,k_6}(a(z)a(z)^T, P_{a,b}(z), Q(z))t^{\frac{k_4+k_5}{2}+k_6} \\ \quad \textit{if} \quad k_4 > k_5 \quad \textit{and} \quad k_4 + k_5 \equiv 0 \ (\text{mod } 2) \\ \lambda_{\boldsymbol{\kappa}}^{(3)} D_{k_5,k_6}(P_{a,b}(z), Q(z))t^{k_5+k_6} \\ \quad \textit{if} \quad k_4 = k_5 \quad \textit{and} \quad k_5 \equiv 0 \ (\text{mod } 2) \\ 0 \quad \textit{if} \quad k_4 + k_5 \equiv 1 \ (\text{mod } 2) \end{cases}$$

where $a(z) = 2\beta(z)^T\gamma(z)$, $b(z) = 2\alpha(z)^T\gamma(z)$, $Q(z) = \gamma(z)^T\gamma(z)$ *and* $D_{n_1,\ldots,n_i}(A_1, \ldots, A_i)$ *being the normalized Davis polynomials [5, 12] and*

$$\begin{aligned} P_{a,b}(z) &= \frac{a(z)b(z)^T + b(z)a(z)^T}{2}, \\ \lambda_{\boldsymbol{\kappa}}^{(1)} &= 2^{\frac{k_4+k_5}{2}+k_6-1}k_4k_6(k_5-k_4), \\ \lambda_{\boldsymbol{\kappa}}^{(2)} &= 2^{\frac{k_4+k_5}{2}+k_6-1}k_5k_6(k_4-k_5), \\ \lambda_{\boldsymbol{\kappa}}^{(3)} &= 2^{k_5+k_6}k_5k_6. \end{aligned}$$

.

Proof. First, note that we can write $\tilde{X}_{t,z}$ as the sum of a mean value, a linear drift and a standard Wiener process

$$\tilde{X}_{t,z} = \alpha(z) + \beta(z)t + \gamma(z)W_t,$$

for $z \in \mathbb{R}^d$. Then, the squared norm of $\tilde{X}_{t,z}$ is

$$\begin{aligned} \tilde{X}_{t,z}^T\tilde{X}_{t,z} &= (\alpha(z) + \beta(z)t + \gamma(z)W_t)^T(\alpha(z) + \beta(z)t + \gamma(z)W_t) \\ &= \alpha(z)^T\alpha(z) + 2(\alpha(z)^T\beta(z) + \beta(z)^T\gamma(z)W_t)t \\ &\quad + \beta(z)^T\beta(z)t^2 + 2\alpha(z)^T\gamma(z)W_t + W_t^T\gamma(z)^T\gamma(z)W_t. \end{aligned}$$

The Newton multinomial formula allows to express the $2q^{\text{th}}$ power of the $\tilde{X}_{t,z}$ norm

$$\begin{aligned} \|\tilde{X}_{t,z}\|^{2q} = \sum_{|\mathbf{k}|=q} \binom{q}{\mathbf{k}} &\Big(\alpha(z)^T\alpha(z)\Big)^{k_1}\Big(2\alpha(z)^T\beta(z)t\Big)^{k_2}\Big(\beta(z)^T\beta(z)t^2\Big)^{k_3} \\ &\Big(2\beta(z)^T\gamma(z)W_t t\Big)^{k_4}\Big(2\alpha(z)^T\gamma(z)W_t\Big)^{k_5}\Big(W_t^T\gamma(z)^T\gamma(z)W_t\Big)^{k_6} \end{aligned}$$

where $\mathbf{k} = (k_1, \ldots, k_6)$. The previous equation can be written as follows

$$\|\tilde{X}_{t,z}\|^{2q} = \sum_{|\mathbf{k}|=q} \binom{q}{\mathbf{k}} C_{k_1,k_2,k_3}(z) t^{k_2+2k_3+k_4} (a(z)^T W_t)^{k_4} (b(z)^T W_t)^{k_5} (W_t^T Q(z) W_t)^{k_6}$$

where $a(z) = 2\beta(z)^T\gamma(z)$, $b(z) = 2\alpha(z)^T\gamma(z)$, $Q(z) = \gamma(z)^T\gamma(z)$ and

$$C_{k_1,k_2,k_3}(z) = \Big(\alpha(z)^T\alpha(z)\Big)^{k_1} \Big(2\alpha(z)^T\beta(z)\Big)^{k_2} \Big(\beta(z)^T\beta(z)\Big)^{k_3}.$$

We denote

$$\mu_{\boldsymbol{\kappa}}(z,t) \stackrel{\text{def.}}{=} \mathbb{E}\left[(a(z)^T W_t)^{k_4} (b(z)^T W_t)^{k_5} (W_t^T Q(z) W_t)^{k_6}\right]$$

where $\boldsymbol{\kappa} = (k_4, k_5, k_6)$. The expectation $\mu_{\boldsymbol{\kappa}}(z,t)$ can always be written as the high-order moment of a product of three quadratic forms [17], therefore we have

$$\mu_{\boldsymbol{\kappa}}(z,t) = \begin{cases} \lambda_{\boldsymbol{\kappa}}^{(1)} D_{\frac{k_5-k_4}{2},k_4,k_6}(b(z)b(z)^T, P_{a,b}(z), Q(z)) t^{\frac{k_4+k_5}{2}+k_6} \\ \quad \text{if} \quad k_4 < k_5 \quad \text{and} \quad k_4 + k_5 \equiv 0 \pmod 2 \\ \lambda_{\boldsymbol{\kappa}}^{(2)} D_{\frac{k_4-k_5}{2},k_5,k_6}(a(z)a(z)^T, P_{a,b}(z), Q(z)) t^{\frac{k_4+k_5}{2}+k_6} \\ \quad \text{if} \quad k_4 > k_5 \quad \text{and} \quad k_4 + k_5 \equiv 0 \pmod 2 \\ \lambda_{\boldsymbol{\kappa}}^{(3)} D_{k_5,k_6}(P_{a,b}(z), Q(z)) t^{k_5+k_6} \\ \quad \text{if} \quad k_4 = k_5 \quad \text{and} \quad k_5 \equiv 0 \pmod 2 \\ 0 \quad \text{if} \quad k_4 + k_5 \equiv 1 \pmod 2 \end{cases}$$

where $D_{n_1,\ldots,n_i}(A_1, \ldots, A_i)$ denotes the normalized Davis polynomials [5, 12] and

$$\begin{aligned} P_{a,b}(z) &= \frac{a(z)b(z)^T + b(z)a(z)^T}{2}, \\ \lambda_{\boldsymbol{\kappa}}^{(1)} &= 2^{\frac{k_4+k_5}{2}+k_6-1} k_4 k_6 (k_5 - k_4), \\ \lambda_{\boldsymbol{\kappa}}^{(2)} &= 2^{\frac{k_4+k_5}{2}+k_6-1} k_5 k_6 (k_4 - k_5), \\ \lambda_{\boldsymbol{\kappa}}^{(3)} &= 2^{k_5+k_6} k_5 k_6. \end{aligned}$$

Finally, the result holds. □

The normalized Davis polynomials $D_{n_1,\ldots,n_i}(A_1, \ldots, A_i)$ can be computed efficiently using recursion formulas, see [12].

From Proposition 2, we obtain an upper bound of $\mathbb{E}\sup_{0\leqslant t\leqslant\tau} \|\tilde{X}_{t,z}\|^{2q}$ using the following lemma.

Lemma 2. *For every martingale $Y_t \in L^p$ (where $p > 1$), we have:*

$$\mathbb{E}[\sup_{0\leqslant t\leqslant\tau} \|Y_t\|^p] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}[\|Y_\tau\|^p].$$

Proof. See [1]. □

We thus have:

$$\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q} \leqslant \left(\frac{2q}{2q-1}\right)^{2q}\sum_{|\mathbf{k}|=q}\binom{q}{\mathbf{k}}C_{k_1,k_2,k_3}(z)\mu_{k_4,k_5,k_6}(z,1)t^{k_2+2k_3+k_4+\frac{k_4+k_5}{2}+k_6}$$

where $C_{k_1,k_2,k_3}(z)$ and $\mu_{k_4,k_5,k_6}(z,1)$ are given in Proposition 2. The computation of this deterministic upper-bound is much faster than the direct Monte-Carlo approximation of the expectation. Furthermore, we get rid of the approximations inherent to the use of Monte-Carlo simulations.

3.4. Probabilistic reachability

Let us suppose for simplicity that R is a ball of the form $B(O,\rho)$ where O is the origin and $\rho > 0$. Suppose that $x_0 \in B(z,\delta_0)$. We have:

$$Pr(X_{\tau,x_0} \notin R) = Pr(\|X_{\tau,x_0}\| > \rho).$$

Furthermore:

Proposition 3. $Pr(X_{\tau,x_0} \in R) \;\geqslant\; 1 - \dfrac{1}{\rho^2}\;\left(\delta_{\tau,\delta_0} + \sqrt{\mathbb{E}[\|\tilde{X}_{\tau,z}\|^2]}\;\right)^2.$

Proof. Using Chebyshev's inequality and triangular inequality, it follows

$$Pr(X_{\tau,x_0} \notin R) \leqslant \frac{1}{\rho^2}\mathbb{E}[\|X_{\tau,x_0}\|^2] \leqslant \frac{1}{\rho^2}\left(\sqrt{\mathbb{E}[\|\tilde{X}_{\tau,z} - X_{\tau,x_0}\|^2]} + \sqrt{\mathbb{E}[\|\tilde{X}_{\tau,z}\|^2]}\right)^2.$$

We thus get, using the inequality of the previous section (namely, $\mathbb{E}[\|\tilde{X}_{\tau,z} - X_{\tau,x_0}\|^2 \leqslant \delta^2_{\tau,\delta_0}$):

$$Pr(X_{\tau,x_0} \notin R) \;\leqslant\; \frac{1}{\rho^2}\;\;(\delta_{\tau,\delta_0})^2 \;\;+\;\; \sqrt{\mathbb{E}[\|\tilde{X}_{\tau,z}\|^2]}.$$

Alternatively, we have:

$$Pr(X_{\tau,x_0} \in R) \;\geqslant\; 1 - \frac{1}{\rho^2}\;\left(\delta_{\tau,\delta_0} + \sqrt{\mathbb{E}[\|\tilde{X}_{\tau,z}\|^2]}\;\right)^2.$$

□

NB: If we have $R = B(O',\rho)$ where O' is *not* the origin, we have:

$$Pr(X_{\tau,x_0} \in R) \;\geqslant\; 1 - \frac{1}{\rho^2}\;\left(\delta_{\tau,\delta_0} + \sqrt{\mathbb{E}[\|\tilde{X}_{\tau,z} - O'\|^2]}\;\right)^2.$$

3.5. Implementation

This method has been implemented in the interpreted language Octave, and the experiments performed on a 2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory. The implementation is an adaptation of the program described in ([15]) for controlling deterministic switched systems, but makes use of the tamed Euler scheme for SDEs (with the error function δ_{t,δ_0} given in Theorem 2) instead of the classical Euler scheme.

Example 2. *Consider the system of Example 1, for mode $u = 1$ and $w = 0.05$:*

$\mathrm{d}x_1 = (-0.25x_1 + x_2 + 0.25)\mathrm{d}t + 0.05x_1\mathrm{d}W_t^1$

$\mathrm{d}x_2 = (-2x_1 - 0.25x_2 - 2)\mathrm{d}t + 0.05x_2\mathrm{d}W_t^2$

The program gives (for $\tau = 1$, $\Delta_t = \tau/10^4$): $q = 0$, $D = 1.36$, $L_g = 0.05$, $\lambda = 0.25$; and for $z = (-4, -3.8)$: $E_{2,z} = 893.3$, $E_{4,z} = 2.14 \cdot 10^5$, $F_{2,z} = 0.002$, $F_{4,z} = 4.9 \cdot 10^{-6}$.

Consider now the system corresponding to Example 1 for mode $u = 2$ and $w = 0.05$:

$\mathrm{d}x_1 = (-0.25x_1 + 2x_2 - 0.25)\mathrm{d}t + 0.05x_1\mathrm{d}W_t^1$

$\mathrm{d}x_2 = (-x_1 - 0.25x_2 + 1)\mathrm{d}t + 0.05x_2\mathrm{d}W_t^2$

The program gives (for $\tau = 1$, $\Delta_t = \tau/10^4$): $q = 0$, $D = 1.36$, $L_g = 0.05$, $\lambda = 0.25$, and, for $z = (0, 3)$: $E_{2,z} = 543.2$, $E_{4,z} = 7.94 \cdot 10^4$, $F_{2,z} = 0.0442$, $F_{4,z} = 0.00178$.

Both computations take less than 10s of CPU time. Simulations of the two systems are given in Figure 1 for mode $u = 1$ and starting point $z = (-4, 3.8)$, and mode $u = 2$ and starting point $z = (0, 3)$. The initial ball $B(z, \delta_0)$ is depicted in black, the final ball $B(\mathbb{E}[\tilde{X}_{\tau,z}], \delta_{\tau,\delta_0})$ in red, and 200 random sampling trajectories in blue for $t \in [0, \tau]$.

Example 3. *We now consider a nonlinear model of a pendulum on a cart borrowed from [4, 31]. The dynamics is given by:*

$\mathrm{d}x_1 = x_2\mathrm{d}t + 0.03x_1\mathrm{d}W_t^1$

$\mathrm{d}x_2 = (-\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}u)\mathrm{d}t + 0.03x_2\mathrm{d}W_t^2$

where the state (x_1, x_2) represents the angular position and velocity of the point mass on the pendulum, u is the control input (torque applied to the cart). The parameters are $g = 9.8$, $l = 0.5$, $m = 0.6$, $k = 2$. The program gives for control input $u = 0.5$, $\tau = 0.1$, $\Delta_t = \tau/10^2$: $q = 1$, $D = 2.56$, $L_g = 0.03$, $\lambda = 1.446$; and for $z = (0, 0)$: $E_{2,z} = 47.5$, $E_{4,z} = 1977.2 \cdot 10^5$, $F_{2,z} = 9.8 \cdot 10^{-4}$, $F_{4,z} = 9.73 \cdot 10^{-7}$.

The computation takes less than 10s of CPU time. Simulations of the system are given in Figure 2 for control input $u = 0.5$ and starting point $z = (0, 0)$. The initial ball $B(z, \delta_0)$ is depicted in black, the final ball $B(\mathbb{E}[\tilde{X}_{\tau,z}], \delta_{\tau,\delta_0})$ in red, and 200 random sampling trajectories in blue for $t \in [0, \tau]$.

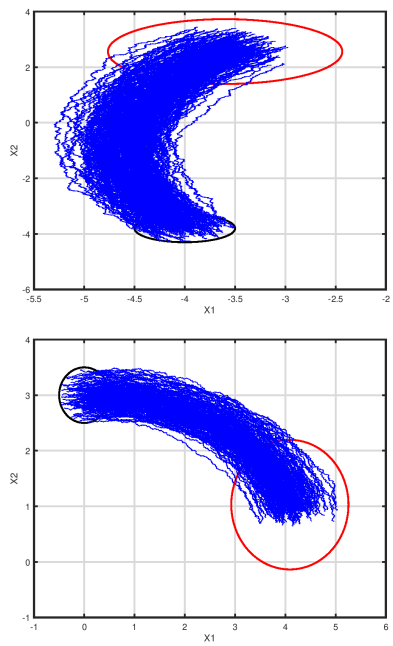

Figure 1: Simulations of Example 2 with mode $u = 1$ and initial ball $B((-4, 3.8), 0.5)$, and mode $u = 2$ and initial ball $B((0, 3), 0.5)$; $\tau = 1$.

4. Sampled stochastic switched systems

4.1. Stochastic switched system as a finite collection of SDEs

We now consider a finite number of SDEs. Each SDE is referred to as a *mode* j, and the set of modes is referred to as $U = \{1, \ldots, M\}$. We will denote by X^j_{t,x_0} the solution at time t of the system:

$$\begin{aligned} \mathrm{d}x(t) &= f_j(x(t)) + g_j(x(t))\mathrm{d}W^j_t, \\ x(0) &= x_0. \end{aligned} \tag{19}$$

where x_0 is a random variable that is measurable in $\mathcal{F}_0$. Hypotheses (H1)-(H2)-(H3), as defined in Section 3, are naturally extended to every mode j of U. Accordingly, constants L_g, λ, F associated to SDE (1) in Section 3, now become L_{g_j}, λ_j, F_j respectively, for each $j \in U$.

Likewise, for each $j \in U$, the nonnegative real $(\delta_{t,\delta_0})^2$ becomes $(\delta^j_{t,\delta_0})^2$ for each mode j; the approximate continuous-time solution of (19) starting from z, is denoted by $\tilde{X}^j_{t,z}$, and the approximate staircase solution by $\underline{X}^j_{t,z}$.

4.2. Control patterns

The control laws that we now consider are "piecewise constant of duration τ" in the sense that, every τ seconds, they select a given mode (see ([30])). We call

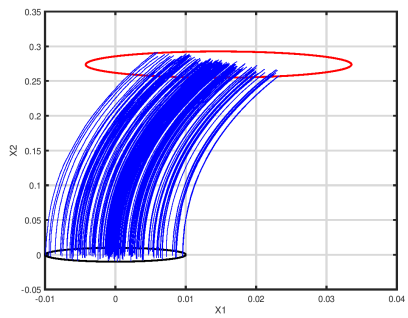

Figure 2: Simulations of Example 3 with control input $u = 0.5$, initial ball $B((0,0), 0.01)$, $\tau = 0.1$.

"(control) pattern of length k" a sequence of k modes (i.e., an element of U^k). Each pattern π of the form $j_1 j_2 \cdots j_k$ corresponds to the selection of mode j_1 for time $t \in [0, \tau)$, then mode j_2 for $t \in [\tau, 2\tau)$, and so on, until $t = k\tau$. We assume that the solution of the system is continuous at sampling instants $t = \tau, 2\tau, \ldots$ (which means that there is no "reset" of the system at sampling instants).

Given a stochastic switched system, a pattern π of length k and an initial random variable z, one constructs the "approximate solution controlled by π" by composing together the approximations obtained by successive application of the modes of the pattern π. Formally, the "continuous" approximate solution $\tilde{X}^{\pi}_{t,z}$ is defined at time $t \in [0, k\tau]$ as follows:

- $\tilde{X}^{\pi}_{t,z} = \tilde{X}^{j}_{t,z}$ if $\pi = j \in U$, $k = 1$ and $t \in [0, \tau]$, and
- $\tilde{X}^{\pi}_{(k-1)\tau+t',z} = \tilde{X}^{j}_{t,z'}$ with $z' = \tilde{X}^{\pi'}_{(k-1)\tau,z}$ if $k \geqslant 2$, $t' \in [0, \tau]$, $\pi = \pi' * j$ for some $j \in U$ and $\pi' \in U^{k-1}$.

The "staircase" approximate solution $\underline{X}^{\pi}_{t,z}$ is defined analogously. Likewise, given an initial error radius $\delta_0 > 0$ and a pattern π of length $k \geqslant 1$, one defines the error radius $\delta^{\pi}_{t,\delta_0}$ using (7) as follows:

- $\delta^{\pi}_{t,\delta_0} = \delta^{j}_{t,\delta_0}$ if $\pi = j \in U$, $k = 1$ and $t \in [0, \tau]$, and
- $\delta^{\pi}_{(k-1)\tau+t',\delta_0} = \delta^{j}_{t',\delta'}$ with $\delta' = \delta^{\pi'}_{(k-1)\tau,\delta_0}$, if $k \geqslant 2$, $t' \in [0, \tau]$, $\pi = \pi' * j$ for some $j \in U$ and $\pi' \in U^{k-1}$.

4.3. Controlled probabilistic reachability

Given a ball $R = B(O, \rho) \subset \mathbb{R}^d$ with $\rho > 0$, we define the problem of "controlled probabilistic reachability inside R" as follows:

Given a starting point $z \in \mathbb{R}^d$ and a positive real δ_0, find a pattern π of length k such that, at $t = k\tau$, the image of the points located initially in $B(z, \delta_0)$ be, on average, located as close as possible to O, and, in particular:

$$B(\mathbb{E}[\tilde{X}^{\pi}_{t,z}], \delta^{\pi}_{t,\delta_0}) \subseteq R \quad \text{for} \quad t = k\tau.$$

The program mentioned in Section 3.5, has been extended in order to find, by *exhaustive enumeration*[1], such a pattern π. This guarantees that, for all $x_0 \in B(z, \delta_0)$:

$$Pr(X^{\pi}_{k\tau,x_0} \in R) \geqslant 1 - \frac{1}{\rho^2}(\sqrt{\mathbb{E}[\|\tilde{X}^{\pi}_{k\tau,z}\|^2]} + \delta^{\pi}_{k\tau,\delta_0})^2,$$

We now give two examples of application of this program.

Example 4. *Consider the system of Example 1 with $w = 0.01$:*
$\mathrm{d}x_1 = (-0.25x_1 + ux_2 + (-1)^u 0.25)\mathrm{d}t + 0.01x_1\mathrm{d}W^1_t$
$\mathrm{d}x_2 = ((u-3)x_1 - 0.25x_2 + (-1)^u(3-u))\mathrm{d}t + 0.01x_2\mathrm{d}W^2_t$
where $u = 1, 2$.
For $\tau = 0.5$, $\Delta_t = 10^{-4}$, one finds (for all modes $u = 1, 2$):
$q = 0$, $D = 1.36$, $L_g = 0.01$, $\lambda = 0.25$; for $z = (-4, -3.8)$: $E_{2,z} = 893.31$, $E_{4,z} = 2.14 \cdot 10^5$, $F_{2,z} = 0.002$, $F_{4,z} = 4.9 \cdot 10^{-6}$; and for $z = (0, 3)$: $E_{2,z} = 543.22$, $E_{4,z} = 7.94 \cdot 10^4$, $F_{2,z} = 0.0442$, $F_{4,z} = 0.00178$.
Our program shows the probabilistic reachability inside $R = B((0,0), \rho)$, for all point $x_0 \in B((5,4), \delta_0)$ and all point $y_0 \in B((-5,-4), \delta_0)$, with $\delta_0 = 0.1$ and $\rho = 7$. We have, for $\pi_1 = (2 \cdot 2 \cdot 2 \cdot 2)$

$$Pr(X^{\pi_1}_{4\tau,x_0} \in R) \geqslant 1 - \frac{1}{\rho^2}(\sqrt{\mathbb{E}[\|\tilde{X}^{\pi_1}_{4\tau,(5,4)}\|^2]} + \delta^{\pi_1}_{4\tau,\delta_0})^2,$$

$$Pr(X^{\pi_1}_{4\tau,x_0} \in R) \geqslant 0.689,$$

and for $\pi_2 = (1 \cdot 1 \cdot 1 \cdot 1 \cdot 1)$:

$$Pr(X^{\pi_2}_{5\tau,y_0} \in R) \geqslant 1 - \frac{1}{\rho^2}(\sqrt{\mathbb{E}[\|\tilde{X}^{\pi_2}_{5\tau,(-5,-4)}\|^2]} + \delta^{\pi_2}_{5\tau,\delta_0})^2.$$

$$Pr(X^{\pi_2}_{5\tau,y_0} \in R) \geqslant 0.713$$

Figure 3 depicts in black the initial balls (at $t = 0$) centered at $(5,4)$ and $(-5,-4)$; and for each initial ball, the pattern that sends the ball inside R (at time $t = k\tau$); the intermediate balls (at $t = \tau, 2\tau, \ldots, (k-1)\tau$) are depicted in red, and 200 sampling trajectories drawn in black.

Example 5. *(the slit problem)*
The problem is adapted from ([20]). The controlled dynamics is:

$$\mathrm{d}X = u\mathrm{d}t + \mathrm{d}W, \qquad X_0 = 1$$

with mode $u \in \{-6, -5, -4, -3, -2, 1, 0, 1, 2, 3, 4, 5, 6\}$. We have (at $t = 0.5$) a slit at $x \in [-1, -3]$. The objective is thus to control the system so that $x(t) \in R = [-1, -3]$ (i.e., $x(t) \in R = B(O', \rho)$ with $O' = -2$ and $\rho = 1$) at $t = 0.5$.

[1] In Section 4.4 a strategy based on the search of an *optimal* pattern is suggested rather than an exhaustive search.

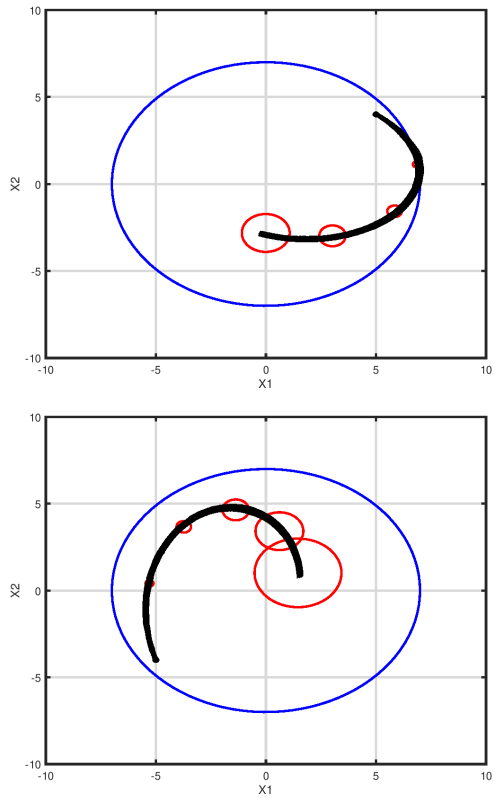

Figure 3: Simulations of Example 4 from the initial balls $B((5,4),0.1)$ and $B((-5,-4),0.1)$ using patterns $(2\cdot 2\cdot 2\cdot 2)$ and $(1\cdot 1\cdot 1\cdot 1\cdot 1)$ resp., with $\tau = 0.5$.

One has, for all modes: $q = 0$, $D = 0$, $L_g = 0$, $\lambda = 0$. *For* $\delta_0 = 0.2$, *an initial point* $z = 1$ *and a sampling time* $\tau = 0.5$ *with subsampling* $\Delta_t = 2.5\times 10^{-3}$, *one has for mode* $u = -6$: $E_{2,z} = 144$, $E_{4,z} = 20736$, $F_{2,z} = 4$, $F_{4,z} = 16$; *and for mode* $u = 0$: $E_{2,z} = 0, E_{4,z} = 0, F_{2,z} = 4, F_{4,z} = 16$.

Suppose that all the trajectories start at $x_0 \in B(z,\delta_0)$, *with* $z = 1$ *and* $\delta_0 = 0.2$. *When there is no control* ($u = 0$), *at time* $t = 0.5$, *the expected value of* X_{t,x_0} *is in* $B(z_1,\delta_{t,\delta_0})$ *with* $z_1 = 1$ *and* $\delta_{t,\delta_0} = 0.29$. *From Markov's inequality, it follows that the trajectories pass by* $R = B(-2,1)$ *at* $t = 0.5$ *with* low probability*: see Figure 4. On the other hand, with control* $u = -6$, *at time* $t = \tau = 0.5$, *the expected value of* X_{t,x_0} *is now in* $B(z_1',\delta_{\tau,\delta_0}')$ *with* $z_1' = -2$ *and* $\delta_{\tau,\delta_0}' = 0.29$. *This explains why the trajectories pass by* $R = B(-2,1)$ *at* $t = 0.5$ *with* high probability. *We have, for* $x_0 \in B(z,\delta_0)$ *with* $z = 1$, $\delta_0 = 0.2$, $t = \tau = 0.5$, $\rho = 1$, $O' = -2$:

$$Pr(X^u_{\tau,x_0} \in R = B(O',\rho)) \geqslant 1 - \frac{1}{\rho^2}(\sqrt{\mathbb{E}[\|\tilde{X}^u_{\tau,z} - O'\|^2]} + \delta^u_{\tau,\delta_0})^2. \tag{20}$$

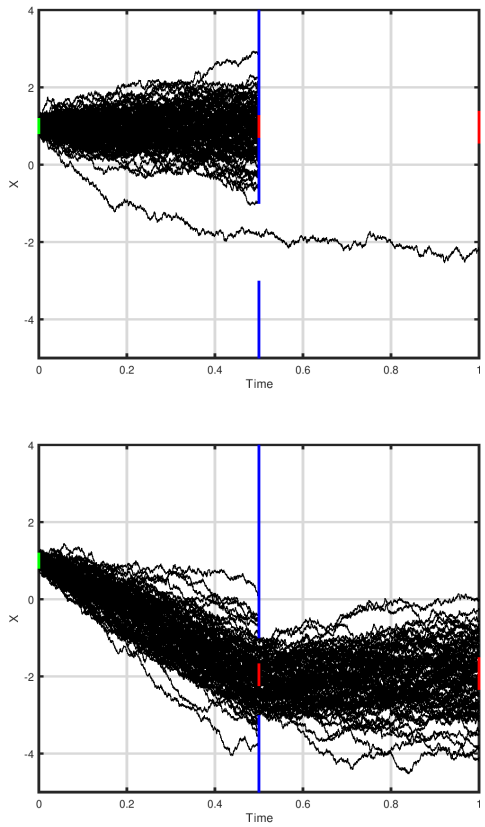

Figure 4: Simulations of Example 5 without control ($\pi = (0 \cdot 0)$) and with control pattern $(-6 \cdot 0)$ from the initial ball $B(1, 0.2)$. The initial region is in green. The red regions are $B(\mathbb{E}[\tilde{X}^{\pi}_{t,z}], \delta^{\pi}_{t,\delta_0})$. with $z = 1$, $\delta_0 = 0.2$ for the two patterns $\pi = (0 \cdot 0)$ (top) and $\pi = (-6 \cdot 0)$ (bottom).

For mode $u = -6$, this gives: $\delta^{u}_{\tau,\delta_0} = 0.29$ and $\mathbb{E}[\|\tilde{X}^{u}_{\tau,z} - O'\|^2] = 0.048$. It follows: $Pr(X^{u}_{\tau,x_0} \in R) \geqslant 0.871$.

4.4. Optimal control by state space discretization

In this Subsection, we are interested in solving an optimal control problem in order to minimize the average distance of the state reached after a given number of steps to the origin. In order to solve such optimal control problems, it is classical to *spatially discretize* the set S into a finite number of cells of equal size. Consider a partition of a compact subset S of R^d in cells. For the sake of simplicity, we suppose that each cell C is a hypercube of R^d of side length $2\varepsilon/\sqrt{d}$, and we consider the centre of the cell, denoted $\sharp x$, as the "representative" of the cell. Note that any point z located in the cell is such that $\|z - \sharp x\| \leqslant \varepsilon$. In [3] (cf. [8]), it is explained how a discretized procedure,

based on classical techniques of dynamic programming, constructs a mapping $\mathbf{u}$ which associates to each representative $\sharp x$ a mode $\mathbf{u}(\sharp x) \in U$, corresponding to the *optimal policy* at position $\sharp x$. More precisely, for any representative point $\sharp x_{0,\varepsilon}$, the discretized procedure $PROC(\sharp x_{0,\varepsilon})$ constructs a sequence of modes $u_{1,\varepsilon}, \ldots, u_{k,\varepsilon}$ and a sequence of representatives $\sharp x_{1,\varepsilon}, \ldots, \sharp x_{k,\varepsilon}$ such that, for all $1 \leqslant i \leqslant k$:

- $u_{i,\varepsilon} = \mathbf{u}(\sharp x_{i-1,\varepsilon})$.
- $\sharp x_{i,\varepsilon}$ is the representative of the cell containing $\mathbb{E}[\tilde{X}^{u_{i,\varepsilon}}_{\tau,\sharp x_{i-1,\varepsilon}}]$.

For the sake of notation simplicity, we will now drop the index ε in symbols $u_{i,\varepsilon}$ and $\sharp x_{i,\varepsilon}$. Let us define the function $next^{u_i}$ by: $next^{u_i}(\sharp x_{i-1}) = \sharp x_i$, where $\sharp x_i$ is the representative of the cell containing $\mathbb{E}[\tilde{X}^{u_i}_{\tau,\sharp x_{i-1}}]$. We define the function $next^{\pi}$ as follows:

- if $\pi = u$ for some $u \in U$ then $next^{\pi}(\sharp x) = next^{u}(\sharp x)$,
- if π is of the form $u \cdot \pi'$ then $next^{\pi}(\sharp x) = next^{\pi'}(next^{u}(\sharp x))$.

Given a point $z_0 \in \mathbb{R}^d$ and a positive integer k, we focus on the problem of finding the optimal pattern $\pi^*(z_0) \in U^k$, which, at time $T = k\tau$, maps z_0 to a distribution located at a *minimal distance* of the origin, i.e., such that:

$$\pi^*(z_0) = argmin_{\pi \in U^k} \mathbb{E}[\|X^{\pi}_{T,z_0}\|^2].$$

Let $\sharp x_0$ be the representative of z_0. The sequences $\{u_i\}_{1 \leqslant i \leqslant k}$ and $\{\sharp x_i\}_{1 \leqslant i \leqslant k}$ given by $PROC(\sharp x_0)$, are *optimal* in the following sense: (see, e.g., [3, 8]):

- the pattern $u_1 \cdots u_k$ coincides with $\pi^*(z_0)$ when ε is sufficiently small, and
- the discrete random variable $\sharp x_k$ converges in distribution to the random variable $X^{u_1 \cdots u_k}_{T,z_0}$ when ε tends to 0.

This justifies the employment of the pattern $u_1 \cdots u_k$ generated by $PROC(\sharp x_0)$ as a (sub)optimal control pattern π for the original stochastic diffusion system [2].

Let us now, in our context, give a precise upper bound to $\mathbb{E}[\|X^{u_1 \cdots u_k}_{T,z_0} - \sharp x_k\|^2]$, where $\sharp x_k$ is the representative point defined by $next^{u_1 \cdots u_k}(\sharp x_0)$. We first define $\lambda_i \geqslant 0$, for $1 \leqslant i \leqslant k$:

$$\lambda_i := \sqrt{\mathbb{E}[\|\tilde{X}^{u_i}_{\tau,\sharp x_{i-1}} - \sharp x_i\|^2]}.$$

Let us also define $\Delta(u_1 \cdots u_k)$ recursively by:

- $\Delta(u_1 \cdots u_i) = \delta^{u_i}_{\tau,\lambda_{i-1}+\Delta(u_1 \cdots u_{i-1})}$ for $i = 2, \ldots, k$, and

[2] see, e.g., [3]

- $\Delta(u_1) = \delta^{u_1}_{\tau,\varepsilon}$.

Without loss of understanding, we abbreviate $\Delta(u_1 \cdots u_i)$ by Δ_i. We have

Lemma 3. $\sqrt{\mathbb{E}[\|X^{u_1\cdots u_k}_{k\tau,z_0} - \tilde{X}^{u_k}_{\tau,\sharp x_{k-1}}\|^2]} \leqslant \Delta_k$.

Proof. The proof is by induction on k. The case $k = 1$ is trivial. Suppose $k \geqslant 2$. We have:

$\sqrt{\mathbb{E}[\|X^{u_1\cdots u_k}_{k\tau,z_0} - \tilde{X}^{u_k}_{\tau,\sharp x_{k-1}}\|^2]} \leqslant \delta^{u_k}_{\tau,\sqrt{\mathbb{E}[\|X^{u_1\cdots u_{k-1}}_{(k-1)\tau,z_0} - \sharp x_{k-1}\|^2]}}$

On the other hand, using the triangular inequality and the induction hypothesis, we have:

$\sqrt{\mathbb{E}[\|X^{u_1\cdots u_{k-1}}_{(k-1)\tau,z_0} - \sharp x_{k-1}\|^2]}$
$\leqslant \sqrt{\mathbb{E}[\|X^{u_1\cdots u_{k-1}}_{(k-1)\tau,z_0} - \tilde{X}^{u_{k-1}}_{\tau,\sharp x_{k-2}}\|^2]} + \sqrt{\mathbb{E}[\|\tilde{X}^{u_{k-1}}_{\tau,\sharp x_{k-2}} - \sharp x_{k-1}\|^2]}$
$\leqslant \Delta_{k-1} + \lambda_{k-1}$.

Hence: $\sqrt{\mathbb{E}[\|X^{u_1\cdots u_k}_{k\tau,z_0} - \tilde{X}^{u_k}_{\tau,\sharp x_{k-1}}\|^2]} \leqslant \delta^{u_k}_{\tau,\Delta_{k-1}+\lambda_{k-1}} = \Delta_k$. This achieves the proof of the induction step. □

We have

Theorem 3. *Consider a point $z_0 \in \mathbb{R}^d$ of representative $\sharp x_0$, and let $\sharp x_k = next^{u_1\cdots u_k}(\sharp x_0)$ for sequence $u_1 \cdots u_k \in U^k$ (generated by $PROC(\sharp x_0)$). We have:*

$$\sqrt{\mathbb{E}[\|X^{u_1\cdots u_k}_{T,z_0} - \sharp x_k\|^2]} \leqslant \Delta_k + \lambda_k.$$

Proof. We have

1. $\mathbb{E}[\|X^{u_1\cdots u_k}_{k\tau,z_0} - \tilde{X}^{u_k}_{\tau,\sharp x_{k-1}}\|^2] \leqslant \Delta_k^2$ by Lemma 3, and
2. $\mathbb{E}[\|\tilde{X}^{u_k}_{\tau,\sharp x_{k-1}} - \sharp x_k\|^2] \leqslant \lambda_k^2$ by definition.

It follows, using triangular inequality: $\sqrt{\mathbb{E}[\|X^{u_1\cdots u_k}_{T,z_0} - \sharp x_k\|^2]} \leqslant \Delta_k + \lambda_k$. □

Note that the term Δ_k of the inequality right-hand side takes into account the error introduced by the Euler scheme (with respect to the real solution of the SDE); in contrast, in the literature (e.g. [8, 22]), the Euler error is often disregarded, the analysis focusing just on the convergence of $\sharp x_k$ to the Euler approximation $\tilde{X}^{u_1\cdots u_k}_{T,z_0}$.

Example 6. *We apply the above procedure $PROC(\sharp x_0)$ on Example 1 with $w = 0.01$. Consider the interest set $R = B((0,0),\rho)$ with $\rho = 7$, discretized with an accuracy $\varepsilon = 0.57$. We compute (sub)optimal patterns for the entire set, using different lengths of patterns, and simulate the induced controller for 200 initial conditions randomly selected in R. Simulations are given in Figure 5. The procedure took* 11.8 *seconds of computation for patterns of length* 1, 47.4 *seconds of computation for patterns of length* 3, 152.0 *seconds of computation for patterns of length* 5.

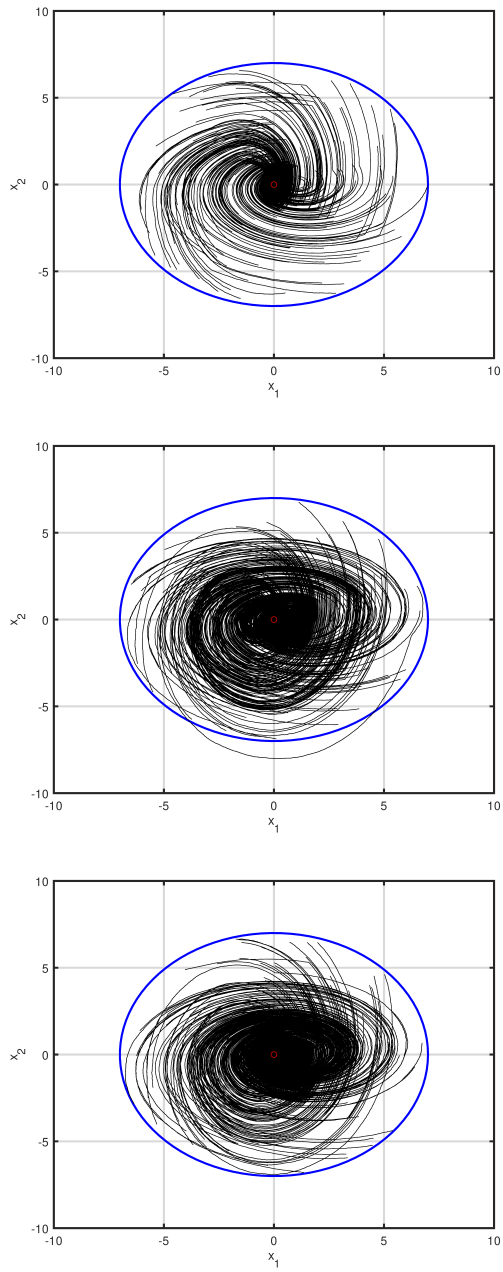

Figure 5: Simulations of Example 6 with the controller induced by $PROC(\sharp x_0)$, for patterns of length 1 (top), length 3 (center), and length 5 (bottom). The blue circle is the set $R = B((0,0),7)$, the red marker is the target state (the origin), the black lines are the controlled trajectories.

We now compare the results yielded by $PROC(\sharp x_0)$ with those of Example 4. The sub-optimal procedure returns patterns ensuring the probabilistic reachability inside $R = B((0,0),\rho)$, for all point $x_0 \in B((5,4),\delta_0)$ and all point $y_0 \in B((-5,-4),\delta_0)$, with $\delta_0 = 0.1$ and $\rho = 7$. For $x_0 \in B((5,4),\delta_0)$, the procedure returns pattern $\pi_1 = (1 \cdot 2 \cdot 2 \cdot 2)$ ensuring a probability

$$Pr(X^{\pi_1}_{4\tau,x_0} \in R) \geqslant 0.801,$$

and for $x_0 \in B((-5,-4),\delta_0)$, the procedure returns pattern $\pi_2 = (2 \cdot 2 \cdot 2 \cdot 1 \cdot 1)$, with probability

$$Pr(X^{\pi_2}_{5\tau,y_0} \in R) \geqslant 0.802.$$

The probabilities are thus higher than in Example 4. Simulations are shown in Figure 6, depicting in black the initial balls (at $t = 0$) centered at $(5,4)$ and $(-5,-4)$; and for each initial ball, the pattern that sends the ball inside R (at time $t = k\tau$); the intermediate balls (at $t = \tau, 2\tau, \ldots, (k-1)\tau$) are depicted in red, and 200 sampling trajectories drawn in black.

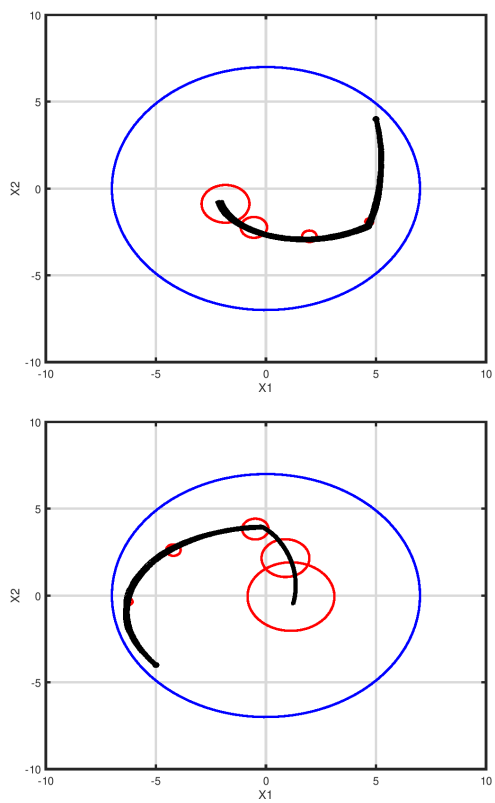

Figure 6: Simulations of Example 6 from the initial balls $B((5,4),0.1)$ and $B((-5,-4),0.1)$ using patterns $(1 \cdot 2 \cdot 2 \cdot 2)$ and $(2 \cdot 2 \cdot 2 \cdot 1 \cdot 1)$ resp., with $\tau = 0.5$.

Example 7. *We consider nonlinear Example 3. Let us recall the dynamics:*
$\mathrm{d}x_1 = x_2\mathrm{d}t + 0.03x_1\mathrm{d}W_t^1$
$\mathrm{d}x_2 = (-\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}u)\mathrm{d}t + 0.03x_2\mathrm{d}W_t^2$
We consider the set of control inputs $u \in \{-1.5, -1, -0.5, 0, 0.5, 1, 1.5\}$. *In order to reproduce the results of [31], we slightly modify the procedure so that the interest set is* $R = B((0,0),\rho)$ *with* $\rho = 1.0$, *but the optimization objective is to minimize the average distance with the target state* $x^{obj} = (0.5, 0)$. *The state space is discretized with an accuracy* $\varepsilon = 0.02$. *We compute (sub)optimal patterns for the entire set* R, *using patterns of length 1, and simulate the induced controller for 30 initial conditions randomly selected in* R. *Simulations are given in Figure 7. The procedure took* 127 *seconds of computation for patterns of length* 1.

5. Final remarks and future work

We have established a mean-square error bound on an Euler-like method to simulate stochastic differential equations that satisfy a one-sided Lipschitz (OSL) condition. An advantage of our method is that the OSL assumption is weaker than the condition under which classical methods converge. From the mean square error bound, we have inferred a lower bound on the probability to reach a given set R at a given time. This allows us to determine a control pattern (in term of sequences of control modes) which makes the stochastic switched system reach a target with guaranteed minimum probability. Besides, instead of searching over all possible sequences of modes (which is a process exponential in the length of sequences), we have proposed an optimized method inferring a 'best' pattern, which may entail a significant gain in time and space. In the future, we plan to experiment our method with significantly larger examples than those given here.

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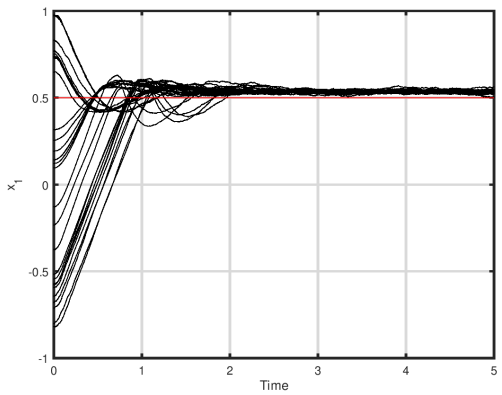

Figure 7: Simulations of Example 7 projected on the first dimension, with the controller induced by $PROC(\sharp x_0)$ for patterns of length 1. The red line is the target state $x^{obj} = (0.5, 0)$, the black lines are the controlled trajectories.