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LARGE TIME ASYMPTOTIC BEHAVIORS OF TWO TYPES OF FAST DIFFUSION EQUATIONS

CHUQI CAO, XINGYU LI

ABSTRACT. Consider two types of nonlinear fast diffusion equations in \mathbb{R}^N . For the external drift case, it is a natural extension of the simple case that the external is harmonic. In this paper we can prove the large time asymptotic behavior to the stationary states by using entropy methods. For the more complicated mead-field type case with the convolution term, we prove that for some special cases, it also exists large time asymptotic behaviour.

Keywords: nonlinear diffusion; mean field equations; free energy; large time asymptotics; Hardy-Poincaré inequality.

AMS subject classifications: 35K55; 35B40; 35P15.

1. INTRODUCTION

We study two different types of non-linear fast diffusion equations in \mathbb{R}^N , and

$$V_\lambda(x) := \frac{1}{\lambda}|x|^\lambda (\lambda > 0), \quad \frac{N}{N+\lambda} < q < 1 :$$

1. Fast diffusion equation with external drift

$$n_t = \Delta(n^q) + \nabla(n\nabla V_\lambda), \quad n(0, \cdot) = n_0 > 0. \quad (1)$$

Obviously, this equation is mass conserved. Set $\int_{\mathbb{R}^N} n(x) dx = m$. Notice that there is a stationary solution N_h of (1) as the form

$$N_h = \left(\frac{1-q}{q} (h + V_\lambda) \right)^{\frac{1}{q-1}}.$$

here $h(m) > 0$ is uniquely decided by the equation

$$\int_{\mathbb{R}^N} N_h dx = m. \quad (2)$$

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From now on, we always suppose that n satisfies the assumption if no specially mentioned.
(H1) There exist constants $0 < h_1 < h_2$, such that

$$N_{h_2}(x) \leq n_0(x) \leq N_{h_1}(x), \quad \forall x \in \mathbb{R}^d$$

According to the maximum principle, $N_{h_2} \leq n(t, \cdot) \leq N_{h_1}$ for any $t > 0$, see page 8-10 from [3] for more details.

2. Mean-filed type fast diffusion equation

We consider the equation

$$\rho_t = \Delta(\rho^q) + \nabla(\rho \nabla V_\lambda * \rho), \quad \rho(0, \cdot) = \rho_0 > 0. \quad (3)$$

Notice that the stationary solution ρ_∞ satisfies

$$\frac{q}{1-q} \rho_\infty^{q-1} = V_\lambda * \rho_\infty + C$$

For some constant $C > 0$. In this paper, we focus on the case that conserved mass

$$\int_{\mathbb{R}^N} \rho(t, x) dx = \int_{\mathbb{R}^N} \rho_\infty dx = 1$$

for the equation (3), Notice that for $i = 1, \dots, N$,

$$\frac{d}{dt} \int_{\mathbb{R}^N} x_i \rho dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho(x) |x - y|^{\lambda-2} (x_i - y_i) \rho(y) dy dx = 0$$

under some certain assumptions, the stationary solution is radial symmetric after transition. so we can suppose that

$$\int_{\mathbb{R}^N} x_i \rho dx = 0$$

and the solution converges to $\rho_\infty(|x|)$ without losing generality. We also assume that

(H2) There exist two stationary solutions (possibly with different mass) ρ_1, ρ_2 such that

$$\rho_2(x) \leq \rho_0(x) \leq \rho_1(x), \quad \forall x \in \mathbb{R}^d$$

When $\lambda = 2$, we know from [13] that after modulo translations, ρ_∞ has the form

$$\frac{q}{1-q} \rho_\infty^{q-1} = \frac{1}{2} |x|^2 + C$$

here C satisfies

$$C^{\frac{1}{1-q} - \frac{N}{2}} = (2\pi)^{\frac{N}{2}} \left(\frac{1-q}{q} \right)^{\frac{1}{q-1}} \frac{\Gamma\left(\frac{1}{1-q} - \frac{N}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)}$$

with direct calculation,

$$\lim_{q \rightarrow 1} \left(C(q) - \frac{q}{1-q} \right) = \frac{N}{2} \log(2\pi)$$

And we will show the large time asymptotic to the stationary solution ρ_∞ . For $\lambda \neq 2$, the form of ρ_∞ is much more complicated. In this paper, we will try to analyse the exact result about ρ_∞ for the special case $\lambda = 4$. We will also show the similar result about the asymptotic behavior if q is close enough to 1. The same talk would work for at least $\lambda = 2N$, N integer and may have the possibility to extend to all $\lambda \geq 2$.

1.1. **Main tools and results.** For equation (1), we consider the *free energy*

$$\mathcal{F}[n] := \frac{1}{q-1} \int_{\mathbb{R}^N} n^q dx + \int_{\mathbb{R}^N} V n dx \quad (4)$$

and the *relative entropy* with respect to N_{h^*} defined as

$$\mathcal{J}[n] := -\frac{d}{dt} \mathcal{F}[n] = \int_{\mathbb{R}^N} n \left| \nabla \left(\frac{q}{q-1} n^{q-1} + V \right) \right|^2 dx = \frac{q^2}{(1-q)^2} \int_{\mathbb{R}^N} n |\nabla(n^{q-1} - N_{h^*}^{q-1})|^2 dx \quad (5)$$

we will prove in Section 2 that N_{h^*} is the unique minimizer of \mathcal{F} , which is $\mathcal{F}[n] - \mathcal{F}[N_{h^*}] \geq 0$. If n solves (1), we obtain that

$$\frac{d}{dt} \mathcal{F}[n] = -\mathcal{J}[n]$$

For equation (3), the relative free energy is

$$\mathbb{F}[\rho] = -\frac{1}{1-q} \int_{\mathbb{R}^N} \rho^q dx + \frac{1}{2\lambda} I_\lambda[\rho] \quad (6)$$

where

$$I_\lambda[\rho] = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^\lambda \rho(x) \rho(y) dx dy$$

and

$$\mathbb{J}[\rho] := -\frac{d}{dt} \mathbb{F}[\rho] = \int_{\mathbb{R}^N} \rho \left| \frac{q}{1-q} \nabla \rho^{q-1} - \nabla W_\lambda * \rho \right|^2 dx \quad (7)$$

Our goal is to show that

Theorem 1.1. *Suppose that the solution n of the equation (1) with initial data n_0 satisfying (H1), $q \in (\frac{N}{N+\lambda}, 1)$ and $\mathcal{F}[n_0] < \infty$. Then for $\lambda \geq 2$, there exist constants $C, \mu > 0$, such that for any $t > 0$*

$$\int_{\mathbb{R}^N} N_{h^*}^{q-2} (n(t, \cdot) - N_{h^*})^2 dx \leq C e^{-\mu t}.$$

For $\lambda \in (0, 2)$, if we assume in addition $q > \frac{N+2}{N+2+\lambda}$ then there exist constants $C, \mu > 0$, such that for any $t > 0$

$$\int_{\mathbb{R}^N} N_{h^*}^{q-2} (n(t, \cdot) - N_{h^*})^2 dx \leq C(1+t)^{-\mu}.$$

For equation (3) we have similar results when $\lambda = 2$.

Theorem 1.2. *For $\lambda = 2$, Suppose that the solution ρ of the equation (3) with initial data ρ_0 that satisfies (H1) and $\mathbb{F}[\rho_0] < \infty$. Then for all q , there exist constants $C_0, \gamma > 0$, such that for any $t > 0$,*

$$\int_{\mathbb{R}^N} \rho_\infty^{q-2} (\rho(t, \cdot) - \rho_\infty)^2 dx \leq C_0 e^{-2\gamma t}.$$

For general λ , we have the similar result when q near 1.

Theorem 1.3. *Suppose that $\lambda = 2N$ for some integer $N \geq 2$. Suppose that the solution ρ of the equation (3) with initial data ρ_0 that satisfies (H1) and $\mathbb{F}[\rho_0] < \infty$. Then for q close enough to 1, there exist constants $C_0, \gamma > 0$, such that for any $t > 0$,*

$$\int_{\mathbb{R}^N} \rho_\infty^{q-2} (\rho(t, \cdot) - \rho_\infty)^2 dx \leq C_0 e^{-2\gamma t}.$$

Remark 1.4. We remark here that for equation (1) if we assume further that (H2) There exists a constant $h_* \in [h_1, h_2]$, such that

$$p(x) := n_0(x) - N_{h_*}(x) \in L^1(\mathbb{R}^d).$$

Then we can extend the Theorem 1.1 to the case $q \in (0, 1)$ with the same proof and some preliminaries. The preliminaries are the same as [3] Section 2.

Remark 1.5. For equation (3), $\frac{N}{N+\lambda}$ is the critical point for the reverse Hardy-Littlewood-Sobolev inequality [13] holds.

Theorem 1.6. (Reverse Hardy-Littlewood-Sobolev inequality) Let $N \geq 1, \lambda > 0, q \in (0, 1)$, define

$$\alpha := \frac{2N - q(2N + \lambda)}{N(1 - q)}$$

then the inequality

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^\lambda \rho(x) \rho(y) dx dy \geq C_{N, \lambda, q} \left(\int_{\mathbb{R}^N} \rho(x) dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q dx \right)^{\frac{2-\alpha}{q}}$$

holds for any nonnegative function $\rho \in L^1 \cap L^q(\mathbb{R}^N)$ and for some positive constant $C_{N, \lambda, q}$ if and only if $q > \frac{N}{N+\lambda}$, or equivalently $\alpha < 1$.

This theorem implies that the free energy $\mathbb{F}[\rho]$ for equation (3) is bounded from below.

1.2. Background. The nonlinear fast diffusion equations have caught many attentions, see [24] for a more precise introduction. And studying the asymptotic rates of convergence to the stationary states is an important theme. For the equation (1), the harmonic potential case that $\lambda = 2$ has been studied by [13]. The Hardy-Poincaré inequality plays a key role. See [2] for more details. The mean-field equation (3) is more complicated, because mean field potential $W_\lambda = V_\lambda * \rho$ depends on the regular part ρ , and for more general λ , there is no explicit form of ρ_∞ for the estimate. This equation behaves different with different choice of λ and q . Recall the functional energy

$$\mathbb{F}[\rho] = -\frac{1}{1-q} \int_{\mathbb{R}^N} \rho^q dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} V_\lambda(x-y) \rho(x) \rho(y) dx dy$$

If we take the mass-preserving dilations

$$\rho_\lambda(x) = \beta^N \rho(\beta x)$$

so

$$\mathbb{F}[\rho] = -\frac{\beta^{N(q-1)}}{1-q} \int_{\mathbb{R}^N} \rho^q dx + \beta^{-\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W_\lambda(x-y) \rho(x) \rho(y) dx dy$$

and one observes different types of behavior depending on the relation between the parameters N, q and λ . The energy functional is homogeneous if attraction and repulsion are in balance, so that the two terms of the energy scale with the same power, that is, $q = q_*$ with

$$q_* = 1 - \frac{\lambda}{N}$$

This motivates the definition of three different regimes: the diffusion-dominated regime $q > q_*$, the fair-competition regime $q = q_*$, and the attraction-dominated regime $0 < q < q_*$. We refer to [17] for a complete summary of existing works. The fair competition case has been studied in several papers [7, 8].

In our paper we consider the case $q > \frac{N}{N+\lambda} > 1 - \frac{\lambda}{N} = q_*$, which correspond to the diffusion-dominated regime. In the diffusion dominated regime, several results has been for the case $-N < \lambda < 0$ and $q > 1$, the logarithmic case $\lambda = 0, q > 1$ in two dimensions [10], and the Newtonian case $\lambda = 2 - N$ in [4, 19]. For our case $\lambda > 0, q > \frac{N}{N+\lambda}$, the existence and uniqueness of ρ_∞ has been given in [3], we remark here that the asymptotic behavior of the case $\lambda > 0, q > \frac{N}{N+\lambda}$ is to our knowledge new.

As we introduced above, the free energy and the related Fisher information are the key tools, but we need the propositions about the bound and the minimizers of the free energy. For $q < 1$, the reverse Hardy-Littlewood-Sobolev inequalities in [13] provide the sufficient conditions. The reader can check [13] for more information.

1.3. Sketch of the proof. In this paper, the proof for the paper is to use linearization around equilibrium plus nonlinear stability, roughly speaking, for a equation

$$\partial_t f = Lf$$

with equilibrium f_∞ , we linearized it around equilibrium $g = f - f_\infty$

$$\partial_t g = L_1 g + L_2 g$$

with L_1 is a linear operator. Then we first prove the convergence for the linearized equation

$$\partial_t g = L_1 g$$

then we use the nonlinear stability to prove that the convergence results still holds for the nonlinear equation when the initial data f_0 is close to equilibrium

$$\|f_0 - f_\infty\|_X \leq \epsilon$$

for some space X and some $\epsilon > 0$ small, then we use a (weak) global convergence to prove that, for any $f_0 \in X$, we can find a time $t_0 > 0$ such that

$$\|f(t_0) - f_\infty\|_X \leq \epsilon$$

the global convergence can be very weak, in this paper we use

$$\lim_{t \rightarrow \infty} \|f(t) - f_\infty\|_X = 0$$

which can be proved by the entropy method. Such method can improve convergence rate and get better rate of convergence at large time. It's largely used in the asymptotic behavior of many nonlinear equations, see [9, 15] for its use in Boltzmann and Landau equation for example.

1.4. Plan of the paper. Sections 2-4 are devoted to the fast diffusion with external drift. In Section 2, we give the results about the free energy and the comparison principle. In Section 3, we prove the result about the convergence without rate and the convergence with rate in Section 4. Sections ??-7.2 are about the mean-field equation with convolution term. Section ?? is about the linearized equation with the free energy and Fisher information. The case that $\lambda = 2$ is simple, we prove the convergence without rate in Section 5 and the convergence with rate in Section 6, And in Section 7 and 7.2, we deal with more general λ .

1.5. Notations. We denote

$$h_k(x) := \frac{x^{k-1} - 1}{k - 1}$$

and S_N denotes the area of unit N-dimension sphere

2. FAST DIFFUSION EQUATION WITH EXTERNAL DRIFT: SOME PREPARATIONS

2.1. The free energy and its minimizer. We first prove the basic propositions about the free energy $\mathcal{F}[n]$ defined in (4), and the existence of the minimizers of $\mathcal{F}[n]$ under the condition $\int_{\mathbb{R}^N} n(x) dx = m$.

Proposition 2.1. *The free energy $\mathcal{F}[n]$ satisfies*

(1) *For $h \geq 0$, the free energy $\mathcal{F}[N_h]$ is increasing by h , which means that it is decreasing by the mass m .*

(2) *For any $n > 0$, $\int_{\mathbb{R}^N} n(x) dx = m$, we have $\mathcal{F}[n] \geq \mathcal{F}[N_h]$, and equality fits if and only if $n = N_h$. Here h is decided by the equation (2). In particular, $\mathcal{F}[n]$ is bounded from below.*

Proof. (1) Remind that

$$N_h = \left(\frac{1-q}{q} \right)^{\frac{1}{q-1}} \cdot (h+V)^{\frac{1}{q-1}}$$

so

$$\begin{aligned} \mathcal{F}[N_h] &= \frac{1}{q-1} \int_{\mathbb{R}^N} N_h^q dx + \int_{\mathbb{R}^N} V N_h dx \\ &= \left(\frac{1-q}{q} \right)^{\frac{1}{q-1}} \cdot \left(-\frac{1}{q} \int_{\mathbb{R}^N} (h+V)^{\frac{q}{q-1}} dx + \int_{\mathbb{R}^N} V (h+V)^{\frac{1}{q-1}} dx \right) \end{aligned}$$

which means that

$$\begin{aligned} \frac{d}{dh} \mathcal{F}[N_h] &= \left(\frac{1-q}{q} \right)^{\frac{1}{q-1}} \cdot \left(-\frac{1}{q-1} \int_{\mathbb{R}^N} (h+V)^{\frac{1}{q-1}} dx + \frac{1}{q-1} \int_{\mathbb{R}^N} V (h+V)^{\frac{2-q}{q-1}} dx \right) \\ &= \left(\frac{1-q}{q} \right)^{\frac{1}{q-1}} \cdot \frac{1}{1-q} \int_{\mathbb{R}^N} h (h+V)^{\frac{2-q}{q-1}} dx \geq 0. \end{aligned}$$

(2) Notice that

$$\mathcal{F}[n] - \mathcal{F}[N_h] = \frac{1}{1-q} \int_{\mathbb{R}^N} q N_h^{q-1} (n - N_h) - (n^q - N_h^q) dx$$

and from the inequality

$$q b^{q-1} (a-b) - (a^q - b^q) \geq 0$$

for any $a, b \geq 0$, we get the result. □

2.2. Comparison principle. Before proving the main theorem, we still need that

Lemma 2.2. *For any two non-negative solutions u_1 and u_2 of equation (1), defined on a time interval $[0, T]$ with initial data in $L^1(\mathbb{R}^d)$, and any two times t_1 and t_2 such that $0 \leq t_1 \leq t_2 \leq T$, we have*

$$\int_{\mathbb{R}^d} |u_1(t_2) - u_2(t_2)| dx \leq \int_{\mathbb{R}^d} |u_1(t_1) - u_2(t_1)| dx$$

and even stronger

$$\int_{\mathbb{R}^d} |u_1(t_2) - u_2(t_2)|_+ dx \leq \int_{\mathbb{R}^d} |u_1(t_1) - u_2(t_1)|_+ dx$$

The lemma implies

Lemma 2.3. (*Comparison principle*) For any two non-negative solutions u_1 and u_2 of equation (1) on $[0, T)$, $T > 0$, with initial data satisfying $u_{0,1} \leq u_{0,2}$ almost everywhere, $u_{0,2} \in L^1_{loc}(\mathbb{R}^d)$, then we have $u_1(t) \leq u_2(t)$ for almost every $t \in [0, T)$.

The proof can be found in [3].

3. CONVERGENCE WITHOUT RATE

In this section we mainly prove convergence without rate, which will allow us to assume that $|h_0 - h_*|$ and $|h_1 - h_*|$ is arbitrarily small. Define relative entropy by

$$\mathcal{F}[n|N_{h_*}] = \int_{\mathbb{R}^N} \phi(n) - \phi(N_{h_*}) - \phi'(n)(n - N_{h_*}) dx, \quad \phi(x) := \frac{x^q}{q-1}$$

Define

$$w := \frac{n}{N_{h_*}}$$

then we have

$$\begin{aligned} w_t &= \frac{1}{N_{h_*}} n_t = \frac{1}{N_{h_*}} \nabla(qn^{q-1} \nabla n + n \nabla V_\lambda) \\ &= \frac{1}{N_{h_*}} \nabla(q(N_{h_*} w)^{q-1} \nabla(N_{h_*} w) + N_{h_*} w \nabla V_\lambda) \\ &= \frac{1}{N_{h_*}} \nabla(N_{h_*} w (qw^{q-2} N_{h_*}^{q-1} \nabla w + qw^{q-1} N_{h_*}^{q-2} \nabla N_{h_*} + \frac{q}{1-q} \nabla(N_{h_*}^{q-1}))) \quad (8) \\ &= \frac{1}{N_{h_*}} \nabla(N_{h_*} w (\frac{q}{q-1} N_{h_*}^{q-1} \nabla(w^{q-1}) + w^{q-1} \nabla(N_{h_*}^{q-1}) + \frac{q}{1-q} \nabla(N_{h_*}^{q-1}))) \\ &= \frac{1}{N_{h_*}} \nabla \left[N_{h_*} w \nabla \left(\frac{q}{q-1} N_{h_*}^{q-1} (w^{q-1} - 1) \right) \right] \end{aligned}$$

recall that in the third line we use that

$$\nabla V_\lambda = \frac{q}{1-q} \nabla(N_{h_*}^{q-1})$$

by homogeneity of ϕ the relative entropy we rewrite that

$$\mathcal{F}[n|N_{h_*}] = \int_{\mathbb{R}^N} \phi(w) - \phi(1) - \phi'(1)(w-1) dx, \quad w = \frac{n}{N_{h_*}}$$

so we define the relative entropy by

$$\mathcal{F}[w] = \frac{1}{1-q} \int_{\mathbb{R}^N} \left[(w-1) - \frac{1}{q}(w^q - 1) \right] N_{h_*}^q dx$$

and the relative Fisher information

$$\mathcal{I}[w] = \frac{q}{(1-q)^2} \int_{\mathbb{R}^N} \left| \nabla \left[(w^{q-1} - 1) N_{h_*}^{q-1} \right] \right|^2 w N_{h_*} dx$$

it's easily seen that

$$\mathcal{F}[w] = \frac{1}{q} \mathcal{F}[n|N_{h_*}], \quad \mathcal{I}[w] = \frac{1}{q} \mathcal{I}[n|N_{h_*}]$$

and

$$\frac{d}{dt} \mathcal{F}[w(t)] = -\mathcal{I}[w(t)]$$

we omit the regularity here, see Proposition 2 in [3] for more details. Define

$$W_0 := \inf_{x \in \mathbb{R}^N} \frac{N_{h_2}(x)}{N_{h_*}(x)} = \left(\frac{h_*}{h_2} \right)^{\frac{1}{1-q}} < 1, \quad W_1 := \inf_{x \in \mathbb{R}^N} \frac{N_{h_1}(x)}{N_{h_*}(x)} = \left(\frac{h_*}{h_1} \right)^{\frac{1}{1-q}} > 1,$$

With such notations, we can rewrite the assumptions as follows:

(H1') n_0 is a non-negative function in $L^1_{loc}(\mathbb{R}^d)$ and there exist positive constants $h_1 < h_2$ such that

$$0 < W_0 \leq \frac{N_{h_2}(x)}{N_{h_*}(x)} \leq w(x) \leq \frac{N_{h_1}(x)}{N_{h_*}(x)} \leq W_1 \leq \infty, \quad \forall x \in \mathbb{R}^d$$

Lemma 3.1. (Uniform C^k regularity) Let $q \in (0, 1)$ and $w \in L^\infty_{loc}((0, T) \times \mathbb{R}^d)$ be a solution of the nonlinear equation. Then for any $k \in \mathbb{N}$ and $t_0 \in (0, T)$,

$$\sup_{t \geq t_0} \|w(t)\|_{C^k(\mathbb{R}^d)} < +\infty.$$

Proof. See [3] Theorem 4. □

Lemma 3.2. If $q \in (0, 1)$, w satisfies (H1') above, then we have

$$\frac{1}{2} W_1^{q-2} \int_{\mathbb{R}^N} |w-1|^2 N_{h_*}^q dx \leq \mathcal{F}[w] \leq \frac{1}{2} W_0^{q-2} \int_{\mathbb{R}^N} |w-1|^2 N_{h_*}^q dx$$

Proof. For some $a > 0$ to be fixed later, we define

$$\phi_a(w) := \frac{1}{1-q} \left[(w-1) - \frac{1}{q}(w^m - 1) \right] - a(w-1)^2$$

we compute

$$\phi_a(w) = \frac{1}{q-1} [1 - w^{m-1}] - 2a(w-1), \quad \phi_a''(w) = w^{m-2} - 2a.$$

Note here $\phi_a(1) = \phi_a'(1) = 0$, recall that $0 < W_0 < 1 < W_1$ and $w \in [W_0, W_1]$. So let $a = W_1^{q-2}/2$, then

$$\phi_a''(w) > 0, \quad w \in (W_0, W_1),$$

which implies

$$\phi_a(w) \geq 0, \quad w \in (W_0, W_1),$$

so the lower bound is proved after multiplying N_{h_*} and integrating over \mathbb{R}^d . Similarly taking $a = W_0^{q-2}/2$ we can prove the upper bound. □

Corollary 3.3. If w_0 satisfies (H1'), then the free energy $\mathcal{F}[w(t)]$ is finite for all $t \geq 0$.

Proof. By the Lemma 3.2, we have

$$\frac{2}{W_0^{q-2}} \mathcal{F}[w] \leq \frac{2}{W_0^{q-2}} \mathcal{F}[w_0] \leq \int_{\mathbb{R}^N} |f|^2 |N_{h_*}^{q-2}| dx \leq \int_{\mathbb{R}^N} |f| |N_{h_1} - N_{h_0}| |N_{h_*}^{q-2}| dx$$

and it is easily seen that

$$|N_{h_1} - N_{h_0}| \leq C(1 + |x|)^{-\frac{\lambda(2-q)}{1-q}}$$

for some constant $C > 0$. So the proof is concluded since f is integrable and $|N_{h_1} - N_{h_0}| |N_{h_*}^{q-2}|$ is bounded. □

Lemma 3.4. *Let $q \in (0, 1)$. If w is a solution and w_0 satisfying (H1') and (H2'), then*

$$\lim_{t \rightarrow \infty} w(t, x) = 1, \quad \forall x \in \mathbb{R}^d.$$

Proof. Let $w_\tau(t, x) = w(t + \tau, x)$. Since the functions are uniformly C^1 continuous, we have there exists a sequence $\tau_n \rightarrow \infty$ such that w_{τ_n} converges to a function w_∞ uniformly in every compact set. By interior regularity of the solutions, the derivatives also converge everywhere. Since $w(x) \geq W_0$ we have $w_\infty \geq W_0 > 0$.

Since $\mathcal{F}[w]$ is finite, we compute

$$\mathcal{F}[w(\tau_n)] - \mathcal{F}[w(\tau_n + 1)] = \int_0^1 \mathcal{F}[w(t + \tau_n)] dt$$

by Fatou's Lemma we have

$$\int_0^1 \mathcal{F}[w_\infty] dt \leq \liminf_{n \rightarrow \infty} \int_0^1 \mathcal{F}[w(t + \tau_n)] dt = \lim_{n \rightarrow \infty} \mathcal{F}[w(\tau_n)] - \mathcal{F}[w(\tau_n + 1)] = 0$$

which is

$$\int_0^1 \int_{\mathbb{R}^N} \left| \nabla \left[(w_\infty^{p-1}(t, x) - 1) N_{h_*}^{q-1}(x) \right] \right|^2 w_\infty(t, x) N_{h_*}(x) dx dt = 0$$

which implies that $w_\infty = C$ for some constant $C > 0$. By the conservation of the mass, we deduce $w_\infty = 1$. Since the limit is unique, the whole $w(t)$ converges to 1 as $t \rightarrow \infty$. \square

Corollary 3.5. *Let $q \in (0, 1)$. If w is a solution of (8) and w_0 satisfying (H1) and (H1'), then*

$$\lim_{t \rightarrow \infty} \|w(t) - 1\|_{L^\infty} = 0.$$

Proof. First we compute

$$|w(t, x) - 1| = |n(t, x) - N_{h_*}| |N_{h_*}^{-1}| \leq |N_{h_1} - N_{h_0}| |N_{h_*}^{-1}| \leq C(1 + |x|)^{-\frac{\lambda}{1-q}} \in L^p,$$

for some constant $C > 0$ and some p large. By dominated convergence theorem, we have

$$\lim_{t \rightarrow \infty} \|w(t) - 1\|_{L^p} = 0.$$

by the inequality in [21] P. 126

$$\|f\|_{L^\infty} \leq \|f\|_{C^1}^q \|f\|_{L^p}^{1-q}$$

for $q = \frac{p}{p+N}$, we deduce the result. \square

4. CONVERGENCE WITH RATE

In this section we prove the convergence with rate around steady state, together with the convergence without rate in the former section we are able to give the convergence rate for the equation (1). The fact that $w(t)$ converges uniformly to 1 as $t \rightarrow \infty$ allows us to improve the lower and upper bounds W_0 and W_1 for the function $w(t)$, at the price of waiting some time. For any $\epsilon > 0$ there exists a time $t_0 = t_0(\epsilon)$ such that

$$1 - \epsilon \leq w(t, x) \leq 1 + \epsilon, \quad \forall (t, x) \in (t_0, \infty) \times \mathbb{R}^d.$$

Define

$$m = \begin{cases} \frac{1}{1 + |x|^{\lambda-2}}, & \lambda > 2 \\ 1 + |x|^{2-\lambda}, & \lambda < 2 \\ 1, & \lambda = 2 \end{cases} \quad (9)$$

and

$$m_1 := \Delta(N_{h_*}^{q-1})m + \nabla m \cdot \nabla(N_{h_*}^{q-1}) \quad (10)$$

We have

$$m_1 \leq C,$$

for some constant $C > 0$ for all the three cases.

Lemma 4.1. *For any differentiable function $\alpha(x)$ and the functions $m(x), m_1(x)$ defined above, we have*

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla(\alpha(w)N_{h_*}^{q-1})|^2 N_{h_*} m \, dx \\ &= \int_{\mathbb{R}^N} |\alpha'(w)|^2 |\nabla w|^2 N_{h_*}^{2q-1} m \, dx + \frac{1}{1-q} \int_{\mathbb{R}^N} \alpha^2(w) |\nabla(N_{h_*}^{q-1})|^2 N_{h_*} m \, dx - \int_{\mathbb{R}^N} \alpha^2(w) N_{h_*}^q m_1 \, dx. \end{aligned}$$

Proof. We have

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla(\alpha(w)N_{h_*}^{q-1})|^2 N_{h_*} m \, dx \\ &= \int_{\mathbb{R}^N} |N_{h_*}^{q-1} \nabla \alpha(w) + \alpha(w) \nabla(N_{h_*}^{q-1})|^2 N_{h_*} m \, dx \\ &= \int_{\mathbb{R}^N} |\alpha'(w)|^2 |\nabla w|^2 N_{h_*}^{2q-1} m \, dx + \int_{\mathbb{R}^N} \alpha^2(w) |\nabla(N_{h_*}^{q-1})|^2 N_{h_*} m \, dx + \int_{\mathbb{R}^N} \nabla(\alpha^2(w)) N_{h_*}^q m \nabla(N_{h_*}^{q-1}) \, dx \\ &= \int_{\mathbb{R}^N} |\alpha'(w)|^2 |\nabla w|^2 N_{h_*}^{2q-1} m \, dx + \int_{\mathbb{R}^N} \alpha^2(w) |\nabla(N_{h_*}^{q-1})|^2 N_{h_*} m \, dx - \int_{\mathbb{R}^N} \alpha^2(w) \nabla(N_{h_*}^q) \nabla(N_{h_*}^{q-1}) m \, dx \\ &\quad - \int_{\mathbb{R}^N} \alpha^2(w) N_{h_*}^q \Delta(N_{h_*}^{q-1}) m \, dx - \int_{\mathbb{R}^N} \alpha^2(w) N_{h_*}^q \nabla m \cdot \nabla(N_{h_*}^{q-1}) \, dx \\ &= \int_{\mathbb{R}^N} |\alpha'(w)|^2 |\nabla w|^2 N_{h_*}^{2q-1} m \, dx + \frac{1}{1-q} \int_{\mathbb{R}^N} \alpha^2(w) |\nabla(N_{h_*}^{q-1})|^2 N_{h_*} m \, dx - \int_{\mathbb{R}^N} \alpha^2(w) N_{h_*}^q m_1 \, dx \end{aligned}$$

where in the third line we use that

$$\nabla(N_{h_*}^q) = \frac{q}{q-1} N_{h_*} \nabla(N_{h_*}^{q-1}).$$

□

Next, we define two functionals

$$\Phi_1[f] := \frac{1}{2} \int_{\mathbb{R}^N} |f|^2 N_{h_*}^{2-q} \, dx \quad (11)$$

and

$$\Phi_2[f] := q \int_{\mathbb{R}^N} |\nabla f|^2 N_{h_*} m \, dx \quad (12)$$

Lemma 4.2. (Hardy Poincaré inequality) Define $q_* = \frac{d-2-\lambda}{d-2}$. For any $q \in (0, 1)$, $q \neq q_*$ and any suitable function f satisfies

$$\int_{\mathbb{R}^N} f N_{h_*}^{2-q} dx = 0$$

we have

$$\Phi_1[f] \leq C_{q,d} \Phi_2[f]$$

for some constant $C_{q,d} > 0$.

Since $\frac{1}{C_\lambda}(1 + |x|^2)^{\lambda/2} \leq 1 + |x|^\lambda \leq C_\lambda(1 + |x|^2)^{\lambda/2}$, for some $C_\lambda > 0$ so this is just the classical Hardy-Poincaré inequality, see for example [3, 2] for the full proof.

Lemma 4.3. Let w be the solution to the equation and the function $\eta := (w-1)N_{h_*}^{q-1}$. There exist constants $\beta_1, \beta_2 > 0$, such that

$$\Phi_2[\eta] \leq \beta_1 \mathcal{I}_1[w] + \beta_2 \Phi_1[\eta].$$

with

$$\mathcal{I}_1[w] := q \int_{\mathbb{R}^N} |\nabla(h_q(w)N_{h_*}^{q-1})|^2 N_{h_*} m w dx$$

here m is defined in (9) and β_1, β_2 can be arbitrarily small if t large.

Proof. For convenience, define $w = 1 + f$ and $h_k(w) := \frac{w^{k-1}-1}{k-1}$. Let $\alpha_0 = W_0^{2(2-q)}$, $\alpha_1 = W_1^{2(2-q)}$. Since $|h_2/h_m|$ is non-decreasing, we have

$$\alpha_0 \leq \left| \frac{h_2(W_0)}{h_q(W_0)} \right|^2 \leq \left| \frac{h_2(w)}{h_q(w)} \right|^2 \leq \left| \frac{h_2(W_1)}{h_q(W_1)} \right|^2 \leq \alpha_1$$

and

$$\alpha_0 \leq \left| \frac{h'_2(w)}{h'_q(w)} \right|^2 \leq \alpha_1$$

Note that $\alpha_0 = \alpha_0(W_0) < 1 < \alpha_1 = \alpha_1(W_1)$ and both converges to 1 as $W_0, W_1 \rightarrow 1$. By Lemma 4.1, take $\alpha(w) = h_2(w)$ we have

$$\begin{aligned} \Phi_1[\eta] &= q \int_{\mathbb{R}^N} |\nabla(h_2(w)N_{h_*}^{q-1})|^2 N_{h_*} m dx \\ &= \int_{\mathbb{R}^N} |h'_2(w)|^2 |\nabla w|^2 N_{h_*}^{2q-1} m dx + \frac{1}{1-q} \int_{\mathbb{R}^N} h_2^2(w) |\nabla(N_{h_*}^{q-1})|^2 N_{h_*} m dx - \int_{\mathbb{R}^N} h_2^2(w) N_{h_*}^q m_1 dx \\ &\leq q\alpha_1 \int_{\mathbb{R}^N} |h'_q(w)|^2 |\nabla w|^2 N_{h_*}^{2q-1} m dx + \frac{\alpha_1 q}{1-q} \int_{\mathbb{R}^N} h_q^2(w) |\nabla(N_{h_*}^{q-1})|^2 N_{h_*} m dx - q \int_{\mathbb{R}^N} h_q^2(w) N_{h_*}^q m_1 dx \end{aligned}$$

and take $\alpha(w) = h_q(w)$ in Lemma 4.1, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |h'_q(w)|^2 |\nabla w|^2 N_{h_*}^{2q-1} m dx \\ &= \int_{\mathbb{R}^N} |\nabla(h_q(w)N_{h_*}^{q-1})|^2 N_{h_*} m dx - \frac{1}{1-q} \int_{\mathbb{R}^N} h_q^2(w) |\nabla(N_{h_*}^{q-1})|^2 N_{h_*} m dx + \int_{\mathbb{R}^N} h_q^2(w) N_{h_*}^q m_1 dx \end{aligned}$$

so

$$\begin{aligned} \Phi_1[\eta] &\leq q\alpha_1 \int_{\mathbb{R}^N} |\nabla(h_q(w)N_{h_*}^{q-1})|^2 N_{h_*} m dx + q \int_{\mathbb{R}^N} (\alpha_1 |h_q(w)|^2 - |h_2(w)|^2) m_1 N_{h_*}^q dx \\ &\leq q\alpha_1 \int_{\mathbb{R}^N} |\nabla(h_q(w)N_{h_*}^{q-1})|^2 N_{h_*} m dx + q \int_{\mathbb{R}^N} \left(\frac{\alpha_1}{\alpha_0} - 1 \right) |h_2(w)|^2 m_1 N_{h_*}^q dx \end{aligned}$$

next, since $0 < W_0 \leq w$,

$$q \int_{\mathbb{R}^N} |\nabla(h_q(w)N_{h_*}^{q-1})|^2 N_{h_*} m \, dx \leq \frac{1}{W_0} \mathcal{I}_1[w]$$

recall that

$$\eta = (w-1)N_{h_*}^{q-1}$$

so we have

$$q \int_{\mathbb{R}^N} \left(\frac{\alpha_1}{\alpha_0} - 1 \right) |h_2(w)|^2 N_{h_*}^q m_1 \, dx = q \left(\frac{\alpha_1}{\alpha_0} - 1 \right) \int_{\mathbb{R}^N} |\eta|^2 N_{h_*}^{2-q} m_1 \, dx \leq Cq \left(\frac{\alpha_1}{\alpha_0} - 1 \right) \mathcal{F}[\eta]$$

for some constant $C > 0$, since $\frac{\alpha_1}{\alpha_2} - 1 \rightarrow 0$ as $t \rightarrow \infty$, we finally we obtain the result. \square

Corollary 4.4. *With the notations above, we have*

$$\mathcal{F}[w] \leq \gamma \mathcal{I}_1[w]$$

for some $\gamma > 0$.

Proof. By

$$\Phi_2[g] \leq \beta_1 \mathcal{I}_1[w] + \beta_2 \Phi_1[g].$$

and

$$\Phi_1[g] \leq C_{q,d} \Phi_2[g]$$

we have

$$\Phi_1[g] \leq \frac{\beta_1 C_{q,d}}{1 - \beta_2 C_{q,d}} \mathcal{I}_1[w]$$

since we can pick β_2 small. So the theorem ends since

$$\Phi_1[g] \geq W_0^{2-q} \mathcal{F}[w]$$

by Lemma 3.2. \square

Corollary 4.5. *For $\lambda \geq 2$, we have*

$$\mathcal{F}[w] \leq C e^{-\beta t}$$

for some $C, \beta > 0$. For $\lambda \in (0, 2)$, if $\frac{d+2}{d+2+\lambda} < q < 1$, we have

$$\mathcal{F}[w] \leq C(1+t)^{-\beta}$$

for some $C, \beta > 0$.

Proof. For $\lambda \geq 2$, $m \leq 1$, so

$$\mathcal{F}[w] \leq C \mathcal{I}_1[w] \leq C \mathcal{I}[w]$$

so the conclusion follows. For $\lambda \in (0, 2)$, by Hölder inequality

$$\mathcal{F}[w] \leq C \mathcal{I}_1[w] \leq C \mathcal{I}[w]^\alpha \mathcal{I}_2[w]^{1-\alpha}$$

for some $\alpha > 0$ with

$$\mathcal{I}_2[w] = q \int_{\mathbb{R}^N} |\nabla(h_q(w)N_{h_*}^{q-1})|^2 N_{h_*} (1+|x|^a) \, dx w$$

for some $a > 2 - \lambda$. Recall that

$$\sup_{t \geq t_0} \|w(t)\|_{C^k(\mathbb{R}^d)} < +\infty$$

so

$$\mathcal{F}_2[w] = q \int_{\mathbb{R}^N} |\nabla(h_q(w)N_{h_*}^{q-1})|^2 N_{h_*} (1 + |x|^\alpha) w \, dx < +\infty$$

is the same as

$$\int_{\mathbb{R}^N} |N_{h_*}^{q-1}|^2 N_{h_*} (1 + |x|^\alpha) \, dx < +\infty$$

which is equivalent to $q > \frac{d+2}{d+2+\lambda}$.

□

Theorem 1.1 can be directly deduced by Corollary 4.5 and Lemma 3.2.

5. PRELIMINARIES FOR THE EQUATION (3)

We first make the change of variable

$$\rho = \rho_\infty \nu = \rho_\infty (1 + g) = \rho_\infty + j \quad (13)$$

The existence of minimizers of the free energy is proved in [13]. To continue our proof, we still need the following theorems, the proofs are similar as the fast diffusion equation with external drift (1) and thus omitted.

Lemma 5.1. (*Comparison principle*) For any two non-negative solutions ρ_1 and ρ_2 of the equation (3) on $[0, T)$, $T > 0$, with initial data satisfying $\rho_{0,1} \leq \rho_{0,2}$ almost everywhere, $\rho_{0,2} \in L^1_{loc}(\mathbb{R}^d)$, then we have $\rho_1(t) \leq \rho_2(t)$ for almost every $t \in [0, T)$.

Lemma 5.2. (*Uniform C^k regularity*) Let $q \in (0, 1)$ and $\nu \in L^\infty_{loc}((0, T) \times \mathbb{R}^d)$ be a solution of equation (3). Then for any $k \in \mathbb{N}$ and any $t_0 \in (0, T)$,

$$\sup_{t \geq t_0} \|\nu(t)\|_{C^k(\mathbb{R}^d)} < +\infty.$$

We can also similarly define

$$\mathcal{W}_0 = \inf_{x \in \mathbb{R}^N} \frac{\rho_2(x)}{\rho_*(x)}, \quad \mathcal{W}_1 = \inf_{x \in \mathbb{R}^N} \frac{\rho_1(x)}{\rho_*(x)}$$

Lemma 5.3. If $q \in (0, 1)$, $0 < \mathcal{W}_0 \leq \nu \leq \mathcal{W}_1 < +\infty$, then we have

$$\frac{1}{2} \mathcal{W}_1^{q-2} \int_{\mathbb{R}^N} |\nu - 1|^2 \rho_\infty^q \, dx \leq -\frac{1}{1-q} \int_{\mathbb{R}^N} \rho_\infty^q [\nu^q - 1 - q(\nu - 1)] \, dx \leq \frac{1}{2} \mathcal{W}_0^{q-2} \int_{\mathbb{R}^N} |\nu - 1|^2 \rho_\infty^q \, dx$$

Lemma 5.4. Let $q \in (0, 1)$. If ρ is a solution of (3) satisfying (H2), then for $\nu = \frac{\rho}{\rho_\infty}$,

$$\lim_{t \rightarrow \infty} \|\nu(t) - 1\|_{L^\infty} = 0.$$

Lemma 5.4 can be proved the same way as the fast diffusion equation, so we focus now on the convergence with rate. The fact that $\nu(t)$ converges uniformly to 1 as $t \rightarrow \infty$ allows us to improve the lower and upper bounds \mathcal{W}_0 and \mathcal{W}_1 for the function $\nu(t)$, at the price of waiting some time. For any $\epsilon > 0$ there exists a time $t_0 = t_0(\epsilon)$ such that

$$1 - \epsilon \leq \nu(t, x) \leq 1 + \epsilon, \quad \forall (t, x) \in (t_0, \infty) \times \mathbb{R}^d.$$

by the conservation of mass we have

$$\int_{\mathbb{R}^N} g \rho_\infty(x) \, dx = 0$$

Define three functionals $\Psi_1(g), \Psi_2(g), \Psi_3(g)$ by

$$\Psi_1(g) := \int_{\mathbb{R}^N} \rho_\infty^q g^2 dx$$

$$\Psi_2(g) := q^2 \int_{\mathbb{R}^N} \rho_\infty |\nabla(\rho_\infty^{q-1} g)|^2 dx$$

and

$$\Psi_3(g) := \sum_{i=1}^N \left(\int_{\mathbb{R}^N} x_i g \rho_\infty(x) dx \right)^2$$

note that $\Psi_1(g), \Psi_2(g)$ are actually the similar forms as (11), (12) from Section 4.

The associated free energy and Fisher information defined in (6), (7) become

$$\mathbb{F}[v] := -\frac{1}{1-q} \int_{\mathbb{R}^N} \rho_\infty^q [v^q - 1 - q(v-1)] dx + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} V_\lambda(x-y) \rho_\infty(x) g(x) \rho_\infty(y) g(y) dx dy \quad (14)$$

and

$$\mathbb{I}[v] := \int_{\mathbb{R}^N} \rho_\infty v \left| \frac{q}{1-q} \nabla(\rho_\infty^{q-1} (v^{q-1} - 1)) - \nabla W_\lambda * (\rho_\infty g) \right|^2 dx \quad (15)$$

We study the quadratic forms associated with the expansion of the \mathbb{F} and \mathbb{I} around ρ_∞ . For a smooth perturbation g of ρ_∞ such that $\int_{\mathbb{R}^N} g \rho_\infty(x) dx = 0$, define

$$\begin{aligned} \mathbb{Q}_1[g] &:= \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} (\mathbb{F}[\rho_\infty(1 + \varepsilon g)] - \mathbb{F}[\rho_\infty]) \\ &= q \int_{\mathbb{R}^N} \rho_\infty^q g^2 dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} V_\lambda(x-y) \rho_\infty(x) g(x) \rho_\infty(y) g(y) dx dy \end{aligned} \quad (16)$$

and

$$\mathbb{Q}_2[g] := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathbb{I}[\rho_\infty(1 + \varepsilon g)] = \int_{\mathbb{R}^N} \rho_\infty \left| q \nabla(\rho_\infty^{q-1} g) - \nabla V_\lambda * (\rho_\infty g) \right|^2 dx \quad (17)$$

6. ASYMPTOTIC BEHAVIOR FOR $\lambda = 2$

6.1. The coercivity result.

Lemma 6.1. *When $\lambda = 2$, for $\mathbb{Q}_1[g]$ and $\mathbb{Q}_2[g]$ we have*

$$\mathbb{Q}_1[g] = \Psi_1(g) - \Psi_3(g)$$

and

$$\mathbb{Q}_2[g] = \Psi_2(g) + 3\Psi_3(g).$$

Proof. Recall that

$$\int_{\mathbb{R}^N} g \rho_\infty dx = 0,$$

so, it's easily seen that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} V_\lambda(x-y) \rho_\infty(x) g(x) \rho_\infty(y) g(y) dx dy = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^2 g \rho_\infty(x) g \rho_\infty(y) dx dy = -\Psi_3(g)$$

next, for the $\mathbb{Q}_2[g]$ term

$$\begin{aligned}\mathbb{Q}_2[g] &:= \int_{\mathbb{R}^N} \rho_\infty \left| q \nabla(\rho_\infty^{q-1} g) - \nabla V_\lambda * (g \rho_\infty) \right|^2 dx \\ &= q^2 \int_{\mathbb{R}^N} \rho_\infty |\nabla(\rho_\infty^{q-1} g)|^2 dx - 2q \int_{\mathbb{R}^N} \rho_\infty \nabla(\rho_\infty^{q-1} g) \cdot \nabla V_\lambda * (g \rho_\infty) dx + \int_{\mathbb{R}^N} \rho_\infty |\nabla V_\lambda * (g \rho_\infty)|^2 dx\end{aligned}$$

for the second term we have

$$\begin{aligned}&- 2q \int_{\mathbb{R}^N} \rho_\infty \nabla(\rho_\infty^{q-1} g) \cdot \nabla V_\lambda * (g \rho_\infty) dx \\ &= 2q \int_{\mathbb{R}^N} \rho_\infty^{q-1} g \nabla \rho_\infty \cdot \nabla V_\lambda * (g \rho_\infty) dx + 2q \int_{\mathbb{R}^N} \rho_\infty^q g \Delta V_\lambda * (g \rho_\infty) dx\end{aligned}$$

recall that

$$\rho_\infty^{q-1} = \left(\frac{1-q}{q} (C + W_\lambda) \right)$$

we have

$$2q \int_{\mathbb{R}^N} \rho_\infty^{q-1} h g \nabla \rho_\infty \cdot \nabla V_\lambda * (g \rho_\infty) dx = 2 \int_{\mathbb{R}^N} \frac{q}{q} g \nabla(\rho_\infty^{q-1}) \cdot \nabla V_\lambda * (g \rho_\infty) dx = -2 \int_{\mathbb{R}^N} g \rho_\infty \nabla V_\lambda \cdot \nabla V_\lambda * (g \rho_\infty) dx$$

and we compute

$$\begin{aligned}- \int_{\mathbb{R}^N} g \rho_\infty \nabla V_\lambda \cdot \nabla V_\lambda * (g \rho_\infty) dx &= - \sum_{i=1}^N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g \rho_\infty(x) g \rho_\infty(y) x_i (x_i - y_i) dy dx \\ &= \sum_{i=1}^N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g \rho_\infty(x) g \rho_\infty(y) x_i y_i dy dx = \sum_{i=1}^N \left(\int_{\mathbb{R}^N} x_i g \rho_\infty(x) dx \right)^2\end{aligned}$$

since the mass of $g \rho_\infty$ is 0, we have

$$2q \int_{\mathbb{R}^N} \rho_\infty^q g \Delta V_\lambda * (g \rho_\infty) dx = 0$$

for the third term we have

$$\int_{\mathbb{R}^N} \rho_\infty |\nabla V_\lambda * (g \rho_\infty)|^2 dx = \int_{\mathbb{R}^N} \rho_\infty \sum_{i=1}^N \left(\int_{\mathbb{R}^N} (x_i - y_i) g \rho_\infty(y) dy \right)^2 dx = \sum_{i=1}^N \left(\int_{\mathbb{R}^N} x_i g \rho_\infty(x) dx \right)^2$$

we conclude by gathering all the terms. \square

Suppose also that $\mathcal{W}_0 \leq v \leq \mathcal{W}_1$ and $\mathcal{W}_0, \mathcal{W}_1$ is close enough to 1, then we have

Lemma 6.2. *For some $\epsilon > 0$ small, we have*

$$2 \int_{\mathbb{R}^N} \rho_\infty \left| \nabla \left(\rho_\infty^{q-1} \frac{(v^{q-1} - 1)}{q-1} - \nabla(\rho_\infty^{q-1} (v-1)) \right) \right|^2 dx \leq C\epsilon \Psi_1(g) + \epsilon \Psi_2(g)$$

for some constant $C > 0$.

Proof. We use the similar result as Lemma 4.1. For any function β we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \rho_\infty \left| \nabla \left(\rho_\infty^{q-1} \beta(v) \right) \right|^2 dx \\ &= \int_{\mathbb{R}^N} |\beta'(v)|^2 |\nabla v|^2 \rho_\infty^{2q-1} dx + \frac{1}{1-q} \int_{\mathbb{R}^N} |\beta(v)|^2 |\nabla(\rho_\infty^{q-1})|^2 \rho_\infty dx - \int_{\mathbb{R}^N} |\beta(v)|^2 \rho_\infty^q \Delta(\rho_\infty^{q-1}) dx\end{aligned}$$

take $\beta = h_q(v) - h_2(v)$ and $\beta = h_2(v)$ separately, and recall that $h_k(v) := \frac{v^{k-1}-1}{k-1}$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} \rho_\infty \left| \nabla \left(\rho_\infty^{q-1} h_q(w) - h_2(w) \right) \right|^2 dx \\
& \leq \int_{\mathbb{R}^N} |h'_q(v) - h'_2(v)|^2 |\nabla v|^2 \rho_\infty^{2q-1} dx + \frac{1}{1-q} \int_{\mathbb{R}^N} |h_q(v) - h_2(v)|^2 |\nabla(\rho_\infty^{q-1})|^2 \rho_\infty dx \\
& \leq \epsilon \int_{\mathbb{R}^N} |h'_2(v)|^2 |\nabla v|^2 \rho_\infty^{2q-1} dx + \epsilon \frac{1}{1-q} \int_{\mathbb{R}^N} |h_2(v)|^2 |\nabla(\rho_\infty^{q-1})|^2 \rho_\infty dx \\
& \leq \epsilon \int_{\mathbb{R}^N} \rho_\infty \left| \nabla \left(\rho_\infty^{q-1} h_2(v) \right) \right|^2 dx + \epsilon \int_{\mathbb{R}^N} |h_2(v)|^2 \rho_\infty^q \Delta(\rho_\infty^{q-1}) dx \leq C\epsilon \Psi_1(g) + \epsilon \Psi_2(g)
\end{aligned}$$

The lemma is thus proved. \square

6.2. Linearization operator and large time asymptotic behaviour.

$$\begin{aligned}
v_t &= \frac{1}{\rho_\infty} \rho_t = \frac{1}{\rho_\infty} \nabla (q \rho^{q-1} \nabla \rho + \rho \nabla V_\lambda * \rho) \\
&= \frac{1}{\rho_\infty} \nabla (q (\rho_\infty v)^{q-1} \nabla (\rho_\infty v) + \rho_\infty v \nabla W_\lambda * (\rho_\infty v)) \\
&= \frac{1}{\rho_\infty} \nabla \left[\rho_\infty v \left(\frac{q}{q-1} \rho_\infty^{q-1} \nabla (v^{q-1}) + \frac{q}{q-1} v^{q-1} \nabla (\rho_\infty^{q-1}) + \nabla W_\lambda * (\rho_\infty v) \right) \right] \\
&= \frac{1}{\rho_\infty} \nabla \left[\rho_\infty v \nabla \left(\frac{q}{q-1} \rho_\infty^{q-1} v^{q-1} + W_\lambda * (\rho_\infty v) \right) \right]
\end{aligned}$$

Corollary 6.3. *With the notations above, we have*

$$\mathbb{F}[v] \leq \gamma \mathbb{I}[v]$$

for some $\gamma > 0$.

Proof. We prove by talking on the relationship between \mathbb{Q}_1 and \mathbb{F} and the relationship between \mathbb{Q}_2 and \mathbb{I} , by Lemma 3.2 above we have

$$\frac{q}{2} \mathcal{W}_1^{q-2} \int_{\mathbb{R}^N} |v-1|^2 \rho_\infty^q dx \leq -\frac{1}{1-q} \int_{\mathbb{R}^N} \rho_\infty^q [v^q - 1 - q(v-1)] dx \leq \frac{q}{2} \mathcal{W}_0^{q-2} \int_{\mathbb{R}^N} |v-1|^2 \rho_\infty^q dx$$

which implies

$$\mathbb{F}[v] \leq 2\Psi_1(g) - \frac{1}{2}\Psi_3(g).$$

Since

$$|a+b|^2 \leq 2|a|^2 + 2|b|^2$$

we have

$$\mathbb{Q}_2[g] \leq 2\mathbb{I}[v] + 2q^2 \int_{\mathbb{R}^N} \rho_\infty \left| \nabla \left(\rho_\infty^{q-1} \frac{(v^{q-1}-1)}{q-1} - \nabla(\rho_\infty^{q-1}(v-1)) \right) \right|^2 dx$$

By Hardy-Poncaré inequality we have

$$\Psi_1(g) \leq \mathcal{C}_{q,d} \Psi_2(g)$$

for some constant $\mathcal{C}_{q,d}$, then by Lemma 6.2 we have

$$\mathbb{I}[v] \geq \frac{1}{2} \mathbb{Q}_2[g] - \epsilon \Psi_2(g) - C\epsilon \Psi_1(g) \geq \frac{1}{4} \Psi_2[g] \geq \frac{\mathcal{C}_{q,d}}{4} \Psi_1[g] \geq \frac{\mathcal{C}_{q,d}}{8} \mathbb{F}[v]$$

If we take $\epsilon > 0$ small. \square

7. LARGE TIME ASYMPTOTIC BEHAVIOUR FOR GENERAL λ

In this section, we consider general λ .

7.1. A key lemma.

Lemma 7.1. *If there exists a constant \mathcal{C} (independent of q), such that for all $q \in (0, 1)$*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{2\lambda} \rho_\infty^{2-q}(x) \rho_\infty^{2-q}(y) dx dy \leq \mathcal{C}, \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{2\lambda-2} \rho_\infty(x) \rho_\infty^{2-q}(y) dx dy \leq \mathcal{C} \quad (18)$$

and if the constant in the Hardy-Poincaré inequality

$$\int_{\mathbb{R}^N} \rho_\infty |\nabla(\rho_\infty^{q-1} g)|^2 dx \geq \mathcal{C}_{q,d} \int_{\mathbb{R}^N} \rho_\infty^q g^2 dx$$

satisfies $\mathcal{C}_{q,d} \rightarrow \infty$ when $q \rightarrow 1$, then there exists $q(\lambda) \in (0, 1)$ near 1 such that for all $1 > q > q(\lambda)$,

$$\mathbb{F}[v] \leq \gamma \|v\|$$

for some $\gamma > 0$.

Proof. We still use the quadratic forms defined in (16),(17). By Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^N} |x-y|^\lambda g \rho_\infty(y) dx \leq \left(\int_{\mathbb{R}^N} |x-y|^{2\lambda} \rho_\infty^{2-q}(y) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \rho_\infty^q g^2 dy \right)^{\frac{1}{2}}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^\lambda g \rho_\infty(x) g \rho_\infty(y) dx dy \right| &\leq \left(\int_{\mathbb{R}^N} \rho_\infty^q g^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \rho_\infty^{2-q}(x) \left| \int_{\mathbb{R}^N} |x-y|^\lambda g \rho_\infty(y) dy \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{2\lambda} \rho_\infty^{2-q}(x) \rho_\infty^{2-q}(y) dx dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \rho_\infty^q g^2 dy \right) \end{aligned}$$

so we have

$$\mathbb{Q}_1[g] \leq C \int_{\mathbb{R}^N} \rho_\infty^q g^2 dx$$

for some constant $C > 0$ as $q \rightarrow 1$.

Still by Cauchy-Schwarz inequality,

$$|\nabla V_\lambda * (g \rho_\infty)|(x) = \left| \int_{\mathbb{R}^N} |x-y|^{\lambda-2} (x-y) (g \rho_\infty)(y) dx \right| \leq \left(\int_{\mathbb{R}^N} |x-y|^{2\lambda-2} \rho_\infty^{2-q}(y) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \rho_\infty^q g^2 dy \right)^{\frac{1}{2}}$$

so we have

$$\int_{\mathbb{R}^N} \rho_\infty |\nabla V_\lambda * (g \rho_\infty)|^2 dx \leq \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{2\lambda-2} \rho_\infty(x) \rho_\infty^{2-q}(y) dx dy \right) \left(\int_{\mathbb{R}^N} \rho_\infty^q g^2 dy \right)$$

so

$$\int_{\mathbb{R}^N} \rho_\infty |\nabla V_\lambda * (g \rho_\infty)|^2 dx \leq C \int_{\mathbb{R}^N} \rho_\infty^q g^2 dx$$

and since

$$|a+b|^2 \leq 2|a|^2 + 2|b|^2$$

so we have

$$\begin{aligned} \mathbb{Q}_2[g] &= \int_{\mathbb{R}^N} \rho_\infty \left| q \nabla(\rho_\infty^{q-1} g) - \nabla V_\lambda * (g \rho_\infty) \right|^2 dx \geq \frac{1}{2} q^2 \int_{\mathbb{R}^N} \rho_\infty |\nabla(\rho_\infty^{q-1} g)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \rho_\infty |\nabla V_\lambda * (g \rho_\infty)|^2 dx \\ &\geq \frac{1}{2} q^2 \int_{\mathbb{R}^N} \rho_\infty |\nabla(\rho_\infty^{q-1} g)|^2 dx - \frac{C}{2} \int_{\mathbb{R}^N} \rho_\infty^q g^2 dx \end{aligned}$$

by our assumption,

$$\int_{\mathbb{R}^N} \rho_\infty |\nabla(\rho_\infty^{q-1} g)|^2 dx \geq \mathcal{C}_{q,d} \int_{\mathbb{R}^N} \rho_\infty^q g^2 dx, \quad \mathcal{C}_{q,d} \rightarrow +\infty$$

as $q \rightarrow 1$, so we have

$$\mathbb{I}[v] \geq \frac{1}{2} \mathbb{Q}_2[g] - \epsilon \Psi_2[g] \geq \frac{1}{4} \Psi_2[g] - \mathcal{C} \Psi_1[g] \geq \frac{\mathcal{C}_{q,d}}{8} \Psi_1[g] \geq \frac{\mathcal{C}_{q,d}}{8C} \mathbb{Q}_1[g] \geq \frac{\mathcal{C}_{q,d}}{16C} \mathbb{F}[v]$$

the lemma is thus proved. \square

7.2. A special case $\lambda = 4$. In this subsection we provide enough materials on the case $\lambda = 4$, which meets the requirements of Lemma 7.1 so that we can conclude the asymptotic behavior. We remark here that the same computation would be true for $\lambda = 2\mathbb{N}$ and maybe possible to extend to all $\lambda \geq 2$). Remind

$$\frac{q}{1-q} \rho_\infty^{q-1} = \frac{1}{4} |x|^4 + \frac{3a}{2} |x|^2 + C \quad (19)$$

here a satisfies $\int_{\mathbb{R}^N} (|x|^2 - a) \rho_\infty dx = 0$, which is

$$\int_0^\infty (x^{N+1} - ax^{N-1}) \left(\frac{1}{4} x^4 + \frac{3a}{2} x^2 + C \right)^{\frac{1}{q-1}} dx = 0$$

define

$$p_n := \int_0^\infty x^n \left(\frac{1}{4} x^4 + \frac{3a}{2} x^2 + C \right)^{\frac{1}{q-1}} dx$$

for convenience. We first prove the lemma below.

Lemma 7.2. (i) $a(q)$ is uniformly bounded. (ii) $\lim_{q \rightarrow 1} C(q)(1-q) = 1$.

Proof. Notice that

$$\rho_\infty < \left(\frac{1-q}{q} \right)^{\frac{1}{q-1}} \left(\frac{|x|^4}{4} + C \right)^{\frac{1}{q-1}} \quad (20)$$

from $\int_{\mathbb{R}^N} \rho_\infty dx = 1$, we have

$$\left(\frac{1-q}{q} \right)^{\frac{1}{q-1}} (4C)^{\frac{N}{4}} C^{\frac{1}{q-1}} \int_{\mathbb{R}^N} (1+|x|^4)^{\frac{1}{q-1}} dx > 1$$

from direct calculation,

$$\int_{\mathbb{R}^N} (1+|x|^\lambda)^{\frac{1}{q-1}} dx = \frac{S_N}{4} B\left(\frac{N}{4}, \frac{1}{1-q} - \frac{N}{4}\right)$$

so

$$C^{\frac{1}{1-q} - \frac{N}{4}} < S_N 4^{\frac{N}{4}-1} \Gamma\left(\frac{N}{4}\right) \left(\frac{1-q}{q}\right)^{\frac{1}{q-1}} \frac{\Gamma\left(\frac{1}{1-q} - \frac{N}{4}\right)}{\Gamma\left(\frac{1}{1-q}\right)} \sim \frac{S_N}{e} 4^{\frac{N}{4}-1} \Gamma\left(\frac{N}{4}\right) \left(\frac{1}{1-q}\right)^{\frac{1}{1-q} - \frac{N}{4}}$$

here we use the facts that as $q \rightarrow 1$,

$$\lim_{q \rightarrow 1} q^{\frac{1}{1-q}} = \frac{1}{e}, \quad \frac{\Gamma\left(\frac{1}{1-q} - \frac{N}{4}\right)}{\Gamma\left(\frac{1}{1-q}\right)} \sim (1-q)^{\frac{N}{4}}$$

so we have

$$\lim_{q \rightarrow 1} C(1-q) \leq 1 \quad (21)$$

on the other hand, from integrating by parts,

$$p_{N+3} + 3ap_{N+1} = \frac{N(1-q)}{q} \left(\frac{1}{4}p_{N+3} + \frac{3a}{2}p_{N+1} + Cp_{N-1} \right) \quad (22)$$

notice that

$$p_{N+1} = ap_{N-1}, \quad p_{N+3}p_{N-1} \geq p_{N+1}^2$$

so

$$\left(1 - \frac{1-q}{4q}N\right)p_{N+3} = 3a \left(\frac{N(1-q)}{2q} - 1 \right) p_{N+1} + \frac{CN(1-q)}{q} p_{N-1} \geq \left(1 - \frac{1-q}{4q}N\right)a^2 p_{N-1} \quad (23)$$

so as $q \rightarrow 1$.

$$C \geq \frac{16q - 7N(1-q)}{4N(1-q)} a^2 \sim \frac{4}{N(1-q)} a^2 \quad (24)$$

so part (i) is deduced from (21), (24). Next, notice that for any x ,

$$1 + \frac{\frac{3a|x|^2}{2}}{\frac{|x|^4}{4} + C} \leq 1 + \frac{3a}{2\sqrt{C}}$$

so from (24), as $q \rightarrow 1$

$$\begin{aligned} \left(\frac{1}{C(1-q)}\right)^{\frac{1}{1-q} - \frac{N}{4}} \frac{1}{e} 4^{\frac{N}{4}-1} \Gamma\left(\frac{N}{4}\right) &\sim \left(\frac{1-q}{q}\right)^{\frac{1}{q-1}} \int_{\mathbb{R}^N} \left(\frac{|x|^4}{4} + C\right)^{\frac{1}{q-1}} dx \\ &\leq \left(1 + \frac{3a}{2\sqrt{C}}\right)^{\frac{1}{1-q}} \left(\frac{1-q}{q}\right)^{\frac{1}{q-1}} \int_{\mathbb{R}^N} \left(\frac{|x|^4}{4} + \frac{3a}{2}|x|^2 + C\right)^{\frac{1}{q-1}} dx \\ &= \left(1 + \frac{3a}{2\sqrt{C}}\right)^{\frac{1}{1-q}} \int_{\mathbb{R}^N} \rho_\infty dx = \left(1 + \frac{3a}{2\sqrt{C}}\right)^{\frac{1}{1-q}} \lesssim \left(1 + \frac{3\sqrt{N}}{4}\sqrt{1-q}\right)^{\frac{1}{1-q}} \end{aligned}$$

and we deduce that

$$\lim_{q \rightarrow 1} C(1-q) \geq 1.$$

□

Proposition 7.3. *The inequalities (18) are true for $\lambda = 4$.*

Proof. Notice that

$$|x - y|^p \leq 2^{p-1}(|x|^p + |y|^p) \quad \text{for } p \geq 1$$

so we need to compute

$$\int_{\mathbb{R}^N} |x|^8 \rho_\infty^{2-q} dx \int_{\mathbb{R}^N} \rho_\infty^{2-q} dx, \quad \int_{\mathbb{R}^N} |x|^6 \rho_\infty dx \int_{\mathbb{R}^N} \rho_\infty^{2-q} dx + \int_{\mathbb{R}^N} |x|^6 \rho_\infty^{2-q} dx$$

after interpolation, we only need to estimate the integrals

$$\int_{\mathbb{R}^N} |x|^6 \rho_\infty dx, \quad \int_{\mathbb{R}^N} \rho_\infty^{2-q} dx, \quad \int_{\mathbb{R}^N} |x|^8 \rho_\infty^{2-q} dx,$$

first, from (19), for any x ,

$$\frac{q}{1-q} \rho_\infty^{q-1} \geq C$$

so as $q \rightarrow 1$

$$\rho_\infty^{1-q} \leq \frac{q}{C(1-q)} \rightarrow 1$$

which is

$$\int_{\mathbb{R}^N} \rho_\infty^{2-q} dx \lesssim \int_{\mathbb{R}^N} \rho_\infty dx = 1, \quad \int_{\mathbb{R}^N} |x|^8 \rho_\infty^{2-q} dx \lesssim \int_{\mathbb{R}^N} |x|^8 \rho_\infty dx$$

so after interpolation, we calculate $\int_{\mathbb{R}^N} |x|^8 \rho_\infty dx$.

$$p_{N+7} + 3ap_{N+5} = \frac{(N+4)(1-q)}{q} \left(\frac{1}{4} p_{N+7} + \frac{3a}{2} p_{N+5} + C p_{N+3} \right) \quad (25)$$

so

$$\left(1 - \frac{1-q}{4q} (N+4) \right) p_{N+7} = 3a \left(\frac{(N+4)(1-q)}{2q} - 1 \right) p_{N+5} + \frac{C(N+4)(1-q)}{q} p_{N+3}$$

take q close enough to 1, we have

$$\begin{aligned} \frac{1}{2} p_{N+7} &< \left(1 - \frac{1-q}{4q} (N+4) \right) p_{N+7} = 3a \left(\frac{(N+4)(1-q)}{2q} - 1 \right) p_{N+5} + \frac{C(N+4)(1-q)}{q} p_{N+3} \\ &< \frac{C(N+4)(1-q)}{q} p_{N+3} \sim (N+4) p_{N+3} \end{aligned}$$

similarly from (23),

$$\begin{aligned} \frac{1}{2} p_{N+3} &< \left(1 - \frac{1-q}{4q} N \right) p_{N+3} = 3a \left(\frac{N(1-q)}{2q} - 1 \right) p_{N+1} + \frac{CN(1-q)}{q} p_{N-1} \\ &< \frac{CN(1-q)}{q} p_{N-1} \sim N p_{N-1} \end{aligned}$$

so finally

$$p_{N+7} \lesssim 4N(N+4) p_{N-1}$$

which is

$$\int_{\mathbb{R}^N} |x|^8 \rho_* dx \lesssim 4N(N+4)$$

as $q \rightarrow 1$. □

Lemma 7.4. For ρ_∞ in this section, we have

$$\int_{\mathbb{R}^N} \rho_\infty |\nabla h|^2 dx \geq \mathcal{C}_{q,d} \int_{\mathbb{R}^N} \rho_\infty^{2-q} h^2 dx$$

for all h such that

$$\int_{\mathbb{R}^N} \rho_\infty^{2-q} h dx = 0$$

with

$$\mathcal{C}_{q,d} \sim \frac{1}{(1-q)^{\frac{1}{2}}} \quad \text{as } q \rightarrow 1.$$

Proof. The proof is the same as the Hardy-Poincaré inequality, so we omit some details and focusing on the asymptotic behavior, the full proof can be seen in [3] Appendix A. We only prove this for radical functions, for the proof of general functions, we use the results in radical functions plus the Poincaré inequality on the unit sphere

$$\int_{\mathbb{S}^{d-1}} |u - \hat{u}|^2 d\theta \leq \frac{1}{d-1} \int_{\mathbb{S}^{d-1}} |\nabla_{\theta} u|^2 d\theta$$

with

$$\hat{u} = \int_{\mathbb{S}^{d-1}} u d\theta$$

The details can also be found in [3]. Recall that

$$\frac{q}{1-q} \rho_{\infty}^{q-1} = \frac{1}{4}|x|^4 + \frac{3a}{2}|x|^2 + C$$

so we only need to prove that

$$\mathcal{C}_{q,d} \frac{q}{1-q} \int_0^{\infty} |h(r) - \bar{h}| r^{d-1} \left(\frac{1}{4} r^4 + \frac{3a}{2} r^2 + C \right)^{-\frac{2-q}{1-q}} dr \leq \int_0^{\infty} |h'(r)|^2 r^{d-1} \left(\frac{1}{4} r^4 + \frac{3a}{2} r^2 + C \right)^{-\frac{1}{1-q}} dr$$

if we make the change of variable $r^4 = Cs^4$, the former inequality turns to

$$\begin{aligned} & \mathcal{C}_{q,d} \frac{q}{(1-q)C^{1/2}} \int_0^{\infty} |h(s) - \bar{h}| s^{d-1} \left(\frac{1}{4} s^4 + \frac{3a}{2C^{1/2}} s^2 + 1 \right)^{-\frac{2-q}{1-q}} ds \\ & \leq \int_0^{\infty} |h'(s)|^2 s^{d-1} \left(\frac{1}{4} s^4 + \frac{3a}{2C^{1/2}} s^2 + 1 \right)^{-\frac{1}{1-q}} ds \end{aligned}$$

since for any $x \in \mathbb{R}^N$,

$$\frac{1}{16}(1+x^2) \leq \frac{1}{4}x^4 + \frac{3a}{2C^{1/2}}x^2 + 1$$

we only need to prove that

$$\begin{aligned} & \mathcal{C}_{q,d} \frac{q}{(1-q)C^{1/2}} \int_0^{\infty} |h(s) - \bar{h}| s^{d-1} \left(\frac{1}{4} s^4 + \frac{3a}{2C^{1/2}} s^2 + 1 \right)^{-\frac{1}{1-q}} (1+s^2)^{-1} ds \\ & \leq \int_0^{\infty} |h'(s)|^2 s^{d-1} \left(\frac{1}{4} s^4 + \frac{3a}{2C^{1/2}} s^2 + 1 \right)^{-\frac{1}{1-q}} ds \end{aligned}$$

By [1] Theorem 2, we have that

$$\mathcal{C}_{q,d} \frac{q}{(1-q)C^{1/2}} \geq \frac{1}{4K}$$

with

$$K = \max_{r>0} \int_0^r s^{1-d} \left(\frac{1}{4} s^4 + \frac{3a}{2C^{1/2}} s^2 + 1 \right)^{\frac{1}{1-q}} (1+s^2)^{-1} dx \cdot \int_r^{\infty} s^{d-1} \left(\frac{1}{4} s^4 + \frac{3a}{2C^{1/2}} s^2 + 1 \right)^{-\frac{1}{1-q}} ds$$

it's easily seen that

$$K \sim \max_{r>0} \int_0^r s^{1-d+\frac{4}{1-q}-2} ds \cdot \int_r^{\infty} s^{d-1-\frac{4}{1-q}} ds$$

so we have

$$K \leq C_2(1-q)^2$$

which ens the proof of our theorem by recalling

$$C \sim \frac{1}{1-q}.$$

□

REFERENCES

- [1] Barthe, Franck, Roberto, Cyril. *Modified Logarithmic Sobolev Inequalities on R* . Potential Analysis. 29. 167-193, 2006
- [2] Blanchet, Adrien and Bonforte, Matteo and Dolbeault, Jean and Grillo, Gabriele and Vázquez, Juan Luis, *Hardy–Poincaré inequalities and applications to nonlinear diffusions*, Comptes Rendus Mathématique (2007), no. 7, 431–436.
- [3] Blanchet, Adrien and Bonforte, Matteo and Dolbeault, Jean and Grillo, Gabriele and Vázquez, Juan Luis, *Asymptotics of the fast diffusion equation via entropy estimates*, Archive for Rational Mechanics and Analysis. **191** (2009), no. 2, 347–385.
- [4] Blanchet A., Carrillo J.A., and Laurencot P. *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions*. Calc. Var. Partial Differential Equations, 35(2):133-168, 2009.
- [5] Bonforte, Matteo and Vázquez, Juan Luis, *Fine asymptotics near extinction and elliptic Harnack inequalities for the fast diffusion equation*, Preprint, April, 2006.
- [6] Bonforte, Matteo and Vázquez, Juan Luis, *Global positivity estimates and Harnack inequalities for the fast diffusion equation*, Journal of Functional Analysis(2006), no. 2, 399–428.
- [7] Calvez, Vincent and Carrillo, Jose Antonio and Hoffmann, Franca, *Equilibria of homogeneous functionals in the fair-competition regime*, Nonlinear Analysis (2017), vol 159, 85-128.
- [8] Calvez, Vincent and Carrillo, Jose Antonio and Hoffmann, Franca, *Uniqueness of stationary states for singular Keller-Segel type models*, arXiv preprint arXiv:1905.07788, 2019.
- [9] Carrapatoso, K., Mischler, S. *Landau Equation for Very Soft and Coulomb Potentials Near Maxwellians*. Ann. PDE 3, 1 (2017)
- [10] Carrillo J. A, Castorina D., and Volzone B. *Ground states for diffusion dominated free energies with logarithmic interaction*. SIAM J. Math. Anal., 47(1):1-25, 2015.
- [11] Carrillo, J. A and Delgado, MG, *Free energies and the reversed HLS inequality*, arXiv preprint arXiv:1803.06232, 2018.
- [12] Catrina, Florin and Wang, Zhi-Qiang, *On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences (2001), no. 2, 229-258.
- [13] Dolbeault, Jean and Frank, Rupert and Hoffmann, Franca, *Reverse Hardy-littlewood-sobolev inequalities*, arXiv preprint arXiv:1803.06151, 2018.
- [14] Dou, Jingbo and Zhu, Meijun, *Reversed Hardy-Littlewood-Sobolev inequality*, Int. Math. Res. Not.(2015), no. 19, 9696-9726.
- [15] Hérau, F, Tonon, D., Tristani, I. *Regularization Estimates and Cauchy Theory for Inhomogeneous Boltzmann Equation for Hard Potentials Without Cut-Off*. Commun. Math. Phys. 377, 697-771 (2020)
- [16] Herrero, Miguel A and Pierre, Michel, *The Cauchy problem for $u_t = \Delta(u^m)$ when $0 < m < 1$* , Transactions of the American Mathematical Society(1985),no. 1,145–158.
- [17] Hoffmann, Franca, *Keller-Segel-Type Models and Kinetic Equations for Interacting Particles: Long-Time Asymptotic Analysis*, University of Cambridge, 2017.
- [18] Li, Xingyu, *Flocking: Phase transition and asymptotic behaviour*, arXiv preprint arXiv:1906.07517, 2019.
- [19] Kim I. and Yao Y.. *The Patlak-Keller-Segel model and its variations: properties of solutions via maximum principle*. SIAM Journal on Mathematical Analysis, 44(2):568-602, 2012.
- [20] Lopes, Orlando, *Uniqueness and radial symmetry of minimizers for a nonlocal variational problem*, Commun. Pure Appl. Anal. 18 (5), 2265-2282.
- [21] Nirenberg, L. *On elliptic partial differential equations*. Ann. Scuola Norm. Sup. Pisa 13(3), 115-162, 1959.
- [22] Ngô, Quốc Anh and Van Hoang Nguyen, *Sharp reversed Hardy–Littlewood–Sobolev inequality on \mathbb{R}^n* , Israel Journal of Mathematics (2017), no 1, 189–223.
- [23] Persson, Arne, *Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator*, Mathematica Scandinavica (1960),no. 8,143-153.
- [24] Vázquez, Juan Luis, *Smoothing and decay estimates for nonlinear diffusion equations: equations of porous medium type*, Oxford University Press, 2006.