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Rankine–Hugoniot conditions obtained by using the space–time Hamilton action

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Abstract In the *quadri-dimensional space-time*, the variation of Hamilton’s action is a powerful tool to study the process equations for conservative fluid media. In this framework, Hamilton’s principle allows to obtain equation of motions, equation of energy but also Rankine–Hugoniot conditions. The variational method may be a versatile key to obtain the shock-wave conditions for complex media when the equations of processes are not expressed by linear or quasi-linear differential equations.

Keywords Hamilton’s action in space–time · Hamilton’s principle · Moving surfaces of discontinuity · Rankine–Hugoniot conditions

Mathematics Subject Classification (2000) 70H25 · 76L05 · 76M30

1 Introduction

In the literature, an ideal shock wave is mainly associated with hyperbolic systems of conservation laws and represented by a moving surface dividing space in two parts where a continuous solution exists, with a jump across this surface [1]. The main problem of shocks for hyperbolic systems was proposed by Riemann but only in the case of one spatial dimension and shock solutions are weak solutions that must satisfy the conditions of Rankine–Hugoniot [2]. Although shock waves are an important research topic for many years, the

Aix Marseille Univ, CNRS, IUSTI, UMR 7343, Marseille, France.
E-mail: henri.gouin@univ-amu.fr; henri.gouin@yahoo.fr
ORCID iD: 0000-0003-4088-1386

importance of the Riemann problem is not so clear in multidimensional situations. In case of complex medium, the Rankine–Hugoniot conditions are often the subject of discussions [3].

The paper is destined to didactically present the tool associated with variations of Hamilton’s action in the physical quadri–dimensional space–time (*4-D space–time*) to obtain the Rankine–Hugoniot conditions of shock waves for conservative (i.e non dissipative) media.

The variation of Hamilton’s action is related to the theory of distributions where a decomposition theorem is associated with a linear functional of *virtual displacements* [4]. The virtual displacements, which are well-known in variational methods, are considered as *test functions* whose supports are compact manifolds [5]. The variation of Hamilton’s action can be written in a unique canonical zero order form with respect both to the test functions and their transverse (normal) derivatives to sub–manifolds corresponding to successive boundaries and edges [6].

The equations of motion and energy, and boundary conditions of continuous media are deduced from Hamilton’s principle : the motion is such that the action is zero for any virtual displacement [7]. When the Lagrangian depends on the strain tensor, Hamilton’s action depends on the gradient of the virtual displacement and leads to existence of the Cauchy stress–tensor. When the Lagrangian also depends on over–strain tensor, then the Hamilton action depends on second gradient of the virtual displacement and we obtain second–gradient media model like for van der Waals fluids [8,9].

To understand the proposed tool, we simply present the case of conservative fluids (as elastic media) and we prove that Hamilton’s principle is able to determine shock conditions when the variations of Hamilton’s action are associated with virtual displacements in the *4-D* space–time. The method is able to determine the Rankine–Hugoniot conditions as it was already performed for mixtures of fluids [10].

Generally applied in relativity, the Hamilton method – as commented in conclusion – will be used for more complex media, when the equations of motions are not linear or quasi-linear, in a forthcoming article [11] ; they are cases when the Rankine–Hugoniot conditions cannot be obtained by classical methods associated with problems of hyperbolic equations of motions as in [12].

The paper is presented as follows :

In Section 2, Hamilton’s action and Hamilton’s principle are developed in the *4-D* space–time.

In section 3, fluids (and more generally elastic media) are considered, and two forms of virtual displacements associated with variations of Hamilton’s action in *4-D* space–time and *4-D* reference–space are analyzed and compared. The form of virtual displacements in space–time allows to obtain the equations of motion and energy; the form in reference–space allows to obtain the equation of motion in thermodynamic form and the specific entropy conservation along trajectories.

Section 4 introduces the moving surfaces of discontinuity.

In Section 5, the virtual displacements in the 4- D space-time yield the complete set of Rankine–Hugoniot conditions; it is not the same for virtual displacements in the 4- D reference-space.

In conclusion, the virtual displacements in the 4- D space-time are highlighted.

2 The Hamilton principle

2.1 The Hamilton action

In the 4- D space-time \mathcal{W} , we consider a continuous medium of position variables $\mathbf{z} = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \equiv \{z^i\}$, ($i = 0, 1, 2, 3$), where t is the time and $\mathbf{x} \equiv \{x^i\}$, ($i = 1, 2, 3$) denote the Euler variables; we also write :

$$\mathbf{z}^* = (z^0, z^1, z^2, z^3) \quad \text{where} \quad z^0 = t \quad \text{and} \quad z^1 = x^1, \quad z^2 = x^2, \quad z^3 = x^3$$

We also consider the most general 4- D reference-space \mathcal{W}_0 of position variables $\mathbf{Z} = \begin{pmatrix} \lambda \\ \mathbf{X} \end{pmatrix} \equiv \{Z^i\}$, ($i = 0, 1, 2, 3$), where λ is a real scalar parameter and $\mathbf{X} \equiv \{X^i\}$, ($i = 1, 2, 3$) denote the Lagrange variables; we also write :

$$\mathbf{Z}^* = (Z^1, Z^2, Z^3, Z^4) \quad \text{where} \quad Z^0 = \lambda \quad \text{and} \quad Z^1 = X^1, \quad Z^2 = X^2, \quad Z^3 = X^3$$

A continuous-medium motion is represented by the mapping :

$$\mathbf{z} = \Phi(\mathbf{Z}) \tag{1}$$

The Hamilton action a of an elastic medium is expressed as [7,13] :

$$a = \int_{\mathcal{W}} L dw \quad \text{with} \quad L = \mathcal{L} \left(\mathbf{z}, \mathbf{Z}, \frac{\partial \mathbf{z}}{\partial \mathbf{Z}} \right)$$

where $dw = dv \times dt$ is the volume-time measure in the 4- D space-time \mathcal{W} , L is the Lagrangian, and

$$\mathcal{B} = \frac{\partial \mathbf{z}}{\partial \mathbf{Z}} \equiv \frac{\partial \Phi(\mathbf{Z})}{\partial \mathbf{Z}} \quad \text{where} \quad \frac{\partial \mathbf{z}}{\partial \mathbf{Z}} \equiv \left\{ \frac{\partial z^i}{\partial Z^j} \right\} \quad \text{with} \quad i, j \in \{0, 1, 2, 3\} \tag{2}$$

denotes the tangent linear application of motion Φ in the 4- D space-time ⁽¹⁾.

¹ We notice that L has the dimension of an energy per unit volume and consequently, $a = \int_{\mathcal{W}} L dw$ has the dimension of an action.

2.2 Variation of the Hamilton action

To vary a motion of the medium, we consider a family Ψ associated with a real parameter ε belonging to the vicinity of 0 [4, 7, 14] :

$$z = \Psi(\mathbf{Z}, \varepsilon) \quad \text{such that} \quad \Psi(\mathbf{Z}, 0) = \Phi(\mathbf{Z}) \quad (3)$$

Then,

$$a = f(\varepsilon) \text{ and the variation } \delta a \text{ is defined by } \delta a = f'(0),$$

where for a variation, the differential is denoted δ in place of d . Two possibilities can be considered to obtain the variation of action a . From

$$\delta z = \frac{\partial \Psi}{\partial \mathbf{Z}} \delta \mathbf{Z} + \frac{\partial \Psi}{\partial \varepsilon} \delta \varepsilon \quad \text{with} \quad \delta \varepsilon = 1, \quad \text{we deduce :} \quad (4)$$

- A first variation :

$$\delta z = \tilde{\zeta} \quad \text{when} \quad \delta \mathbf{Z} = \mathbf{0}$$

- A second variation :

$$\delta \mathbf{Z} = \hat{\zeta} \quad \text{when} \quad \delta z = \mathbf{0}$$

where symbols *tilde* and *hat* respectively denote the first and second variations associated with (3).

Remark 1 : The two variations respectively denoted by $\tilde{\delta}$ and $\hat{\delta}$ are not independent. In fact (3) and (4) imply :

$$\frac{\partial \Psi(\mathbf{Z}, 0)}{\partial \varepsilon} + \frac{\partial \Psi(\mathbf{Z}, 0)}{\partial \mathbf{Z}} \hat{\zeta} = \mathbf{0}$$

which can be written :

$$\tilde{\zeta} + \mathcal{B} \hat{\zeta} = \mathbf{0} \quad (5)$$

We use the notations ⁽²⁾ :

Operators div and Div , $\nabla = \text{grad} = \left(\frac{\partial}{\partial \mathbf{x}} \right)^*$ and $\text{Grad} = \left(\frac{\partial}{\partial \mathbf{z}} \right)^*$, are the divergence and gradient in the 3- D and 4- D physical spaces, respectively.

The divergence of a second order tensor \mathcal{A} is a covector (i.e. a form) defined as :

$$\text{Div}(\mathcal{A} \mathbf{h}_0) = \text{Div}(\mathcal{A}) \mathbf{h}_0$$

² For vectors \mathbf{a} and \mathbf{b} , $\mathbf{a}^* \mathbf{b}$, where superscript $*$ denotes the transposition, is the scalar product (line vector \mathbf{a}^* is multiplied by column vector \mathbf{b}); for the sake of simplicity, we also denote $\mathbf{a}^* \mathbf{a} = \mathbf{a}^2$. Tensor $\mathbf{a} \mathbf{b}^*$ (or $\mathbf{a} \otimes \mathbf{b}$) is the product of column vector \mathbf{a} by line vector \mathbf{b}^* . Tensor $\mathbf{1}$ is the identity. In the physical 3- D space, we denote the zero matrix by \mathbf{O} and the zero vector by $\mathbf{0}$, respectively.

where \mathbf{h}_0 is a constant vector field in the 4- D space-time. In particular, for any linear transformation \mathcal{A} and any vector field \mathbf{h} , we get :

$$\text{Div}(\mathcal{A}\mathbf{h}) = (\text{Div } \mathcal{A})\mathbf{h} + \text{Tr} \left(\mathcal{A} \frac{\partial \mathbf{h}}{\partial \mathbf{z}} \right)$$

where Tr denotes the trace operator. If $f(\mathcal{A})$ is any scalar function of \mathcal{A} , we denote :

$$\nabla_{\mathcal{A}} f = \left(\frac{\partial f}{\partial \mathcal{A}} \right)^* \quad \text{with} \quad \left(\frac{\partial f}{\partial \mathcal{A}} \right)_j^i = \left(\frac{\partial f}{\partial \mathcal{A}_i^j} \right)$$

where i is the line index and j the column index of \mathcal{A} and we denote :

$$df(\mathcal{A}) = \nabla_{\mathcal{A}} f : d\mathcal{A} \equiv \left(\frac{\partial f}{\partial \mathcal{A}} \right)_j^i d\mathcal{A}_i^j$$

where repeated indices correspond to the summation.

The notations are similar for 4- D reference space \mathcal{W}_0 ; in this case, we simply add the subscript 0.

Equation (1) allows to write the variations of Hamilton's action a as well in variable \mathbf{Z} than in variable \mathbf{z} :

$\delta \mathbf{z} = \tilde{\zeta}, \quad \delta \mathbf{Z} = \mathbf{0} \quad \Longrightarrow \quad \tilde{\delta} \mathcal{B} = \frac{\partial \tilde{\zeta}}{\partial \mathbf{z}} \mathcal{B}$	$\delta \mathbf{Z} = \hat{\zeta}, \quad \delta \mathbf{z} = \mathbf{0} \quad \Longrightarrow \quad \hat{\delta} \mathcal{B} = -\mathcal{B} \frac{\partial \hat{\zeta}}{\partial \mathbf{Z}}$
$a = \int_{\mathcal{W}_0} L \det \mathcal{B} dw_0$	$a = \int_{\mathcal{W}} L dw$
$\delta a = \int_{\mathcal{W}_0} \left\{ \frac{\partial L}{\partial \mathbf{z}} \det \mathcal{B} \tilde{\zeta} + \text{Tr} \left(\frac{\partial(L \det \mathcal{B})}{\partial \mathcal{B}} \tilde{\delta} \mathcal{B} \right) \right\} dw_0$	$\delta a = \int_{\mathcal{W}} \left\{ \frac{\partial L}{\partial \mathbf{Z}} \hat{\zeta} + \text{Tr} \left(\frac{\partial L}{\partial \mathcal{B}} \hat{\delta} \mathcal{B} \right) \right\} dw$
$\Longrightarrow \quad \delta a = \int_{\mathcal{W}} \left\{ \mathcal{F}^* \tilde{\zeta} + \text{Tr} \left(T \frac{\partial \tilde{\zeta}}{\partial \mathbf{z}} \right) \right\} dw =$	$\Longrightarrow \quad \delta a = \int_{\mathcal{W}_0} \left\{ \mathcal{F}_0^* \hat{\zeta} + \text{Tr} \left(T_0 \frac{\partial \hat{\zeta}}{\partial \mathbf{Z}} \right) \right\} dw_0 =$
$\int_{\mathcal{W}} \left\{ \mathcal{F}^* - \text{Div } T \right\} \tilde{\zeta} dw + \int_{\partial \mathcal{W}} \mathbf{N}^* T \tilde{\zeta} d\sigma$	$\int_{\mathcal{W}_0} \left\{ \mathcal{F}_0^* - \text{Div}_0 T_0 \right\} \hat{\zeta} dw_0 + \int_{\partial \mathcal{W}_0} \mathbf{N}_0^* T_0 \hat{\zeta} d\sigma_0$
$\text{with} \quad \mathcal{F}^* = \frac{\partial L}{\partial \mathbf{z}} \quad \text{and} \quad T = L \mathbf{1} + \mathcal{B} \frac{\partial L}{\partial \mathcal{B}}$	$\text{with} \quad \mathcal{F}_0^* = \det \mathcal{B} \frac{\partial L}{\partial \mathbf{Z}} \quad \text{and} \quad T_0 = -\det \mathcal{B} \frac{\partial L}{\partial \mathcal{B}} \mathcal{B}$

In relativity, T is called energy-impulsion tensor (see [15]) and \mathcal{F} is the extended force; $\text{Div } T$ is a form of \mathcal{W} ; \det denotes the determinant. The boundary of \mathcal{W} is denoted $\partial \mathcal{W}$ with measure $d\sigma$; \mathbf{N}^* denotes the linear form such

that $\mathbf{N}^* \tilde{\zeta} d\sigma = \det \left(\tilde{\zeta}, d_1 \mathbf{z}, d_2 \mathbf{z}, d_3 \mathbf{z} \right)$, where $d_1 \mathbf{z}, d_2 \mathbf{z}, d_3 \mathbf{z}$ are the differentials associated with coordinate lines of $\partial \mathcal{W}$ in the 4- D space-time. Similar notations are used for \mathcal{W}_0 and its boundary $\partial \mathcal{W}_0$ with the additional subscript 0, but T_0 and \mathcal{F}_0 don't have any names.

Thanks to the Stokes formula, integrals $\int_{\partial \mathcal{W}} \mathbf{N}^* T \tilde{\zeta} d\sigma$ and $\int_{\partial \mathcal{W}_0} \mathbf{N}_0^* T_0 \hat{\zeta} d\sigma_0$ correspond to the integration of $\text{Div} (T \tilde{\zeta})$ and $\text{Div}_0 (T_0 \hat{\zeta})$ on boundaries $\partial \mathcal{W}$ and $\partial \mathcal{W}_0$, respectively.

3 Motions of fluid media

From (2), we note that \mathcal{B} can be written :

$$\mathcal{B} = \begin{pmatrix} \mu & \mathbf{w}^* \\ \mathbf{r} & \mathbf{B} \end{pmatrix} \text{ corresponding to the relations } \begin{cases} dt = \mu d\lambda + \mathbf{w}^* d\mathbf{X} \\ d\mathbf{x} = \mathbf{r} d\lambda + \mathbf{B} d\mathbf{X} \end{cases} \quad (6)$$

where μ is a scalar, \mathbf{w}^* a form, \mathbf{r} a vector and \mathbf{B} is a 3- D linear application. Eliminating $d\lambda$ between the two relations of system (6), we obtain :

$$\mathbf{v} \equiv \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} \equiv \frac{\mathbf{r}}{\mu} \quad \text{and} \quad \mathbf{F} \equiv \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial \mathbf{X}} = \mathbf{B} - \frac{\mathbf{r}}{\mu} \mathbf{w}^* = \mathbf{B} - \mathbf{v} \mathbf{w}^* \quad (7)$$

where \mathbf{v} is the fluid velocity and \mathbf{F} denotes the tangent linear deformation of the medium.

In the particular case when we choose $\lambda = t$, we obtain :

$$\mathcal{A} \equiv \frac{\partial \mathbf{z}}{\partial \mathbf{Z}_0} = \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{v} & \mathbf{F} \end{pmatrix} \quad \text{where} \quad \mathbf{Z}_0 = \begin{pmatrix} t \\ \mathbf{X} \end{pmatrix}$$

and we denote \mathcal{W}'_0 the associated 4- D reference space of coordinates (t, \mathbf{X}) . Second order tensors \mathcal{A} and \mathcal{B} are connected by the relation:

$$\mathcal{B} = \mathcal{A} \mathbf{A} \quad \text{where} \quad \mathbf{A} \equiv \frac{\partial \mathbf{Z}_0}{\partial \mathbf{Z}} = \begin{pmatrix} \mu & \mathbf{w}^* \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

with $dt(\lambda, \mathbf{X}) = \mu d\lambda + \mathbf{w}^* d\mathbf{X}$ and $d\mathbf{X}(\lambda, \mathbf{X}) = \mathbf{0} d\lambda + \mathbf{1} d\mathbf{X}$, which implies:

$$\mu = \frac{\partial t(\lambda, \mathbf{X})}{\partial \lambda} \quad \text{and} \quad \mathbf{w}^* = \frac{\partial t(\lambda, \mathbf{X})}{\partial \mathbf{X}}$$

3.1 The Lagrangian of a fluid and its consequences

If we denote $\mathcal{V} = \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix}$ the 4- D velocity field in space-time associated with the material derivative of $\mathbf{z} = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$, conservative fluids are elastic media with

Lagrangian. The Lagrangian is a function of \mathbf{z} , \mathbf{Z} and \mathcal{B} such that [16,17] (3):

$$L = \frac{1}{2} \rho \mathcal{V}^2 - \rho \alpha(\rho, s) - \rho \Omega(\mathbf{z})$$

where :

- The volume kinetic energy is $\frac{1}{2} \rho \mathbf{v}^2$, ρ being the mass density.

$$\text{In fact, } \frac{1}{2} \mathcal{V}^2 \equiv \frac{1}{2} \frac{\mathbf{I}^* \mathcal{B}^* \mathcal{B} \mathbf{I}}{\mu^2} \quad \text{where } \mathbf{I} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \quad \text{and } \mu = \mathbf{I}^* \mathcal{B} \mathbf{I} \quad (8)$$

- The specific internal energy $\alpha(\rho, s)$ is function of the mass density ρ and the specific entropy s . Due to the mass conservation, the image of the density in the reference space \mathcal{W}_0 is conserved. For conservative motions, the entropy s is attached to the reference space. Then :

$$\frac{\rho \det \mathcal{B}}{\mu} = f(\mathbf{X}) \quad \text{and} \quad s = g(\mathbf{Z})$$

where f and g are two real scalar functions defined in Lagrange variables.

- The specific potential energy Ω due to body forces is defined in \mathcal{W} :

$$\Omega = \Omega(\mathbf{z})$$

We obtain :

$$\frac{\partial L}{\partial \rho} = \frac{1}{2} \mathcal{V}^2 - h - \Omega \equiv m, \quad \frac{\partial L}{\partial s} = -\rho \theta \quad \text{and} \quad \frac{\partial L}{\partial \left(\frac{1}{2} \mathcal{V}^2 \right)} = \rho$$

where $h = \alpha + \rho \frac{\partial \alpha(\rho, s)}{\partial \rho}$ is the specific enthalpy, θ the Kelvin temperature and $p = \rho^2 \frac{\partial \alpha}{\partial \rho}$ the pressure of the fluid.

Let us calculate $\frac{\partial L}{\partial \mathcal{B}}$ or firstly $\frac{\partial \rho}{\partial \mathcal{B}}$ and $\frac{\partial \left(\frac{1}{2} \mathcal{V}^2 \right)}{\partial \mathcal{B}}$.

From $d(\det \mathcal{B}) = \det \mathcal{B} \text{Tr}(\mathcal{B}^{-1} d\mathcal{B})$, $d\mu = \mathbf{I}^* d\mathcal{B} \mathbf{I} = \text{Tr}(\mathbf{I} \mathbf{I}^* d\mathcal{B})$ and (8), we obtain the three relations :

$$\frac{\partial \rho}{\partial \mathcal{B}} = f(\mathbf{X}) \mu \frac{\partial}{\partial \mathcal{B}} \left(\frac{1}{\det \mathcal{B}} \right) + f(\mathbf{X}) \frac{1}{\det \mathcal{B}} \frac{\partial \mu}{\partial \mathcal{B}} = -\frac{f(\mathbf{X}) \mu}{\det \mathcal{B}} \mathcal{B}^{-1} + f(\mathbf{X}) \frac{\mathbf{I} \mathbf{I}^*}{\det \mathcal{B}}$$

³ Let us note that $\frac{1}{2} \rho \mathcal{V}^2 = \frac{1}{2} \rho (1 + \mathbf{v}^2)$. Because potential energy Ω is only defined to within an arbitrary additive constant, the term $\frac{1}{2} \rho \times 1$ where 1 has the physical dimension of a velocity square can be added to $\rho \Omega(\mathbf{z})$, without changing the Hamilton action. Consequently, the Lagrangian can also be written $L = \frac{1}{2} \rho \mathbf{v}^2 - \rho \alpha(\rho, s) - \rho \Omega(\mathbf{z})$.

$$\det \mathcal{B} \frac{\partial \rho}{\partial \mathcal{B}} \mathcal{B} = -f(\mathbf{X}) \mu \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & \mathbf{1} \end{pmatrix} + f(\mathbf{X}) \begin{pmatrix} \mu & \mathbf{w}^* \\ \mathbf{0} & \mathbf{O} \end{pmatrix} = f(\mathbf{X}) \begin{pmatrix} 0 & \mathbf{w}^* \\ \mathbf{0} & -\mu \mathbf{1} \end{pmatrix}$$

$$\mathcal{B} \frac{\partial \rho}{\partial \mathcal{B}} = -\rho (\mathbf{1} - \nu \mathbf{I}^*), \quad \frac{\partial \left(\frac{1}{2} \nu^2 \right)}{\partial \mathcal{B}} = \frac{\mathbf{I} \mathbf{I}^*}{\mu} \left(\frac{\mathcal{B}^*}{\mu} - \frac{1}{2} \nu^2 \mathbf{1} \right)$$

Moreover, from (7), we obtain the three relations :

$$\mathbf{I} \mathbf{I}^* = \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & \mathbf{O} \end{pmatrix} \implies \det \mathcal{B} \frac{\partial \left(\frac{1}{2} \nu^2 \right)}{\partial \mathcal{B}} \mathcal{B} = \frac{\det \mathcal{B}}{\mu} \begin{pmatrix} 0 & \mathbf{v}^* \mathbf{F} \\ \mathbf{0} & \mathbf{O} \end{pmatrix}$$

$$\mathcal{B} \frac{\partial \left(\frac{1}{2} \nu^2 \right)}{\partial \mathcal{B}} = \begin{pmatrix} -\mathbf{v}^2 & \mathbf{v}^* \\ -(\mathbf{v}^2) \mathbf{v} & \mathbf{v} \mathbf{v}^* \end{pmatrix} = \nu \mathcal{V}^* (\mathbf{1} - \nu \mathbf{I}^*)$$

$$\frac{\partial L}{\partial \mathbf{Z}} = \left(\frac{1}{2} \nu^2 - \alpha - \rho \frac{\partial \alpha}{\partial \rho} - \Omega \right) \frac{\partial \rho}{\partial \mathbf{Z}} - \rho \frac{\partial \alpha}{\partial s} \frac{\partial s}{\partial \mathbf{Z}}$$

Additively, we obtain :

$$\frac{\partial L}{\partial \mathcal{B}} = \rho \frac{\partial \left(\frac{1}{2} \nu^2 \right)}{\partial \mathcal{B}} + \left(\frac{1}{2} \nu^2 - \alpha - \rho \frac{\partial \alpha}{\partial \rho} - \Omega \right) \frac{\partial \rho}{\partial \mathcal{B}} \implies$$

$$T_0 = -\det \mathcal{B} \left(\rho \frac{\partial \left(\frac{1}{2} \nu^2 \right)}{\partial \mathcal{B}} + m \frac{\partial \rho}{\partial \mathcal{B}} \right) \mathcal{B}$$

3.2 The Hamilton principle

Principle [7,14]: *For all virtual displacements which are null on the boundary $\partial \mathcal{W}$, (respectively on the boundary $\partial \mathcal{W}_0$), the variations of Hamilton's action are zero.*

From Hamilton's principle and calculations in Section 3.1, we obtain :

$\mathcal{F}^* = -\rho \frac{\partial \Omega}{\partial \mathbf{z}}, \quad T = \begin{pmatrix} -e & \rho \mathbf{v}^* \\ -(e+p)\mathbf{v} & \rho \mathbf{v} \mathbf{v}^* + p \mathbf{1} \end{pmatrix}$ $e = \rho \left(\frac{1}{2} \mathbf{v}^2 + \alpha + \Omega \right), \quad p = \rho^2 \frac{\partial \alpha(\rho, s)}{\partial \rho}$	$\mathcal{F}_0^* = -\mu f(\mathbf{X}) \theta \frac{\partial s}{\partial \mathbf{Z}} + \mu m \left(0, \frac{\partial f}{\partial \mathbf{X}} \right)$ $T_0 = f(\mathbf{X}) \begin{pmatrix} 0 & -\mathbf{v}^* \mathbf{F} - m \mathbf{w}^* \\ \mathbf{0} & m \mu \mathbf{1} \end{pmatrix}$
<p>We obtain: $\mathcal{F}^* - \text{Div } T = \mathbf{0}^* \iff$</p> $\frac{\partial e}{\partial t} + \text{div}((e+p)\mathbf{v}) - \rho \frac{\partial \Omega}{\partial t} = 0,$ $\frac{\partial \rho \mathbf{v}^*}{\partial t} + \text{div}(\rho \mathbf{v} \mathbf{v}^* + p \mathbf{1}) + \rho \frac{\partial \Omega}{\partial \mathbf{x}} = \mathbf{0}^*$	<p>We obtain: $\mathcal{F}_0^* - \text{Div}_0 T_0 = \mathbf{0}^* \iff$</p> $\mu f(\mathbf{X}) \theta \frac{\partial s}{\partial \lambda} = 0,$ $\frac{\partial}{\partial \lambda} (\mathbf{v}^* \mathbf{F} + m \mathbf{w}^*) = \mu \theta \frac{\partial s}{\partial \mathbf{X}} + \frac{\partial(\mu m)}{\partial \mathbf{X}}$

In the first column, e is the total volume energy of the fluid ; we firstly obtain the classical *equation of energy* and secondly the *equation of motions*.

In the second column, although it is not necessary, we can choose the parameter $\lambda = t$ and we obtain $\mu = 1$, $\mathbf{w} = \mathbf{0}$. Consequently in Lagrange variables, we get :

$$\frac{\partial s(t, \mathbf{X})}{\partial t} = 0 \iff \dot{s} = 0 \quad \text{and} \quad \frac{\partial \mathbf{v}^* \mathbf{F}}{\partial t} = \theta \frac{\partial s}{\partial \mathbf{X}} + \frac{\partial m}{\partial \mathbf{X}} \quad (9)$$

The first equation (9)¹ is the conservation of the specific entropy along fluid trajectories. Due to

$$\frac{\partial \mathbf{v}^* \mathbf{F}}{\partial t} = \mathbf{a}^* \mathbf{F} + \frac{1}{2} \frac{\partial \mathbf{v}^2}{\partial \mathbf{x}} \mathbf{F}$$

where \mathbf{a} is the acceleration vector, the second equation (9)² writes in variables (t, \mathbf{X}) :

$$\left(\mathbf{a}^* + \frac{\partial (h + \Omega)}{\partial \mathbf{x}} - \theta \frac{\partial s}{\partial \mathbf{x}} \right) \mathbf{F} = \mathbf{0}^*$$

and finally :

$$\mathbf{a} + \text{grad}(h + \Omega) - \theta \text{grad } s = \mathbf{0}$$

This equation is a thermodynamic form of the equation of motion [4].

4 Moving surface

4.1 Generality [18]

In (1), one can choose the representation $t = \ell(\lambda, \mathbf{X})$, and consequently its derivatives

$$\mu = \frac{\partial \ell(\lambda, \mathbf{X})}{\partial \lambda} \quad \text{and} \quad \mathbf{w}^* = \frac{\partial \ell(\lambda, \mathbf{X})}{\partial \mathbf{X}}$$

such that the image by application Φ^{-1} of a moving surface Σ (which is a 3- D manifold of 4- D space-time \mathcal{W}) is the hyperplan Σ_0 of equation $\lambda = 0$. The surface Σ_0 can be considered as a 3- D material-manifold of \mathcal{W}'_0 with equation $t = \ell(0, \mathbf{X})$. Along Σ_0 we obtain :

$$dt = \mathbf{w}^* d\mathbf{X}$$

and we can write :

$$\mathbf{N}_0^* d\mathbf{Z}_0 = 0 \quad \text{with} \quad \mathbf{N}_0^* = \pi(-1, \mathbf{w}^*) \quad (10)$$

where π is a coefficient of proportionality. Along Σ , we also have:

$$\mathbf{N}^* dz = 0 \quad \text{with} \quad \mathbf{N}^* = (-D_n, \mathbf{n}^*) \quad (11)$$

where \mathbf{n} is the unit normal vector to S_t section of Σ at time t and D_n is the velocity of S_t . From (11), we deduce $\mathbf{N}^* \mathcal{A} d\mathbf{Z}_0 = 0$. Then, from (10)

$$\mathbf{N}^* \mathcal{A} = \kappa \mathbf{N}_0^* \quad \text{with} \quad \mathbf{N}^* \mathcal{A} = (u, \mathbf{n}^* \mathbf{F}) \quad (12)$$

where $u = \mathbf{n}^* \mathbf{v} - D_n$ is the medium velocity with respect to the moving surface Σ and κ is a Lagrange multiplier. By similarity with \mathbf{N} , we can write :

$$\mathbf{N}_0^* = (-D_{n_0}, \mathbf{n}_0^*) \quad \text{and consequently} \quad \mathbf{w}^* = \frac{\mathbf{n}_0^*}{D_{n_0}}$$

where D_{n_0} is the velocity of S_0 image of S_t in the 3- D reference-space of variables \mathbf{X} and $u_0 = -D_{n_0}$ denotes the material velocity with respect to S_0 . From (12), we deduce :

$$u \left(1, \frac{\mathbf{n}^* \mathbf{F}}{u} \right) = k (1, \mathbf{n}_0^*) \quad \text{with} \quad \mathbf{n}_0^* = \frac{\mathbf{n}_0^*}{u_0}$$

where k is a coefficient of proportionality ($k = u$). Consequently,

$$\frac{\mathbf{n}^* \mathbf{F}}{u} = \frac{\mathbf{n}_0^*}{u_0} = \mathbf{n}_0^* = -\mathbf{w}^* \quad (13)$$

Expression (13) yields u and \mathbf{n} as function of \mathbf{w}^* and \mathbf{F} . From the knowledge of \mathbf{v} we deduce $\mathbf{n}^* \mathbf{v}$ and consequently $D_n = \mathbf{n}^* \mathbf{v} - u$.

4.2 Surfaces of discontinuity

We assume that motion Φ is a continuous function but with discontinuous derivatives on the moving surface Σ represented in \mathcal{W}_0 by its image Σ_0 of equation $\lambda = 0$. This hypothesis implies :

$$\mathbf{N}_0^* d\mathbf{Z} = 0 \quad \implies \quad \left[\frac{\partial z}{\partial \mathbf{Z}} \right] d\mathbf{Z} = \mathbf{0}$$

where brackets $[\]$ indicate the discontinuity jump across Σ . Consequently on the surface of discontinuity Σ_0 of equation $\lambda = 0$ in \mathcal{W}_0 ,

$$\{d\lambda \equiv \mathbf{I}^* d\mathbf{Z} = 0 \implies [\mathcal{B}] d\mathbf{Z} = \mathbf{0}\} \implies \{\forall d\mathbf{X}, [\mathbf{w}^*] d\mathbf{X} = 0 \text{ and } [\mathcal{B}] d\mathbf{X} = \mathbf{0}\}$$

Then, $[\mathbf{w}^*] = \mathbf{0}^*$ and $[\mathcal{B}] = \mathbf{O}$. Consequently :

$$[\mathcal{B}] = \begin{pmatrix} [\mu] & \mathbf{0}^* \\ [\mu \mathbf{v}] & \mathbf{O} \end{pmatrix}$$

From (6) and (7),

$$[\mathbf{w}^*] = \mathbf{0}^* \quad \text{and} \quad [\mathbf{v}] \mathbf{w}^* + [\mathbf{F}] = \mathbf{O} \quad (14)$$

From (13),

$$\frac{\mathbf{n}^* \mathbf{F}_1}{u_1} = \frac{\mathbf{n}^* \mathbf{F}_2}{u_2} = \frac{\mathbf{n}_0^*}{u_0}, \quad \left[\frac{\mathbf{n}^* \mathbf{F}}{u} \right] = \mathbf{0}^* \quad \text{and} \quad [\mathbf{F}] = [\mathbf{v}] \frac{\mathbf{n}_0^*}{u_0} \quad (15)$$

where subscripts 1 and 2 indicate the upstream and downstream values of the discontinuity surface Σ . It is possible to calculate the discontinuities of each tensor, image in \mathcal{W} of a tensor defined on \mathcal{W}_0 .

We can interpret (15): applying $d\mathbf{X}$ to (15)¹, we obtain :

$$\frac{\mathbf{n}^* d\mathbf{x}_1}{u_1} = \frac{\mathbf{n}^* d\mathbf{x}_2}{u_2}$$

The projections on \mathbf{n} of vectors $d\mathbf{x}_1 = \mathbf{F}_1 d\mathbf{X}$ and $d\mathbf{x}_2 = \mathbf{F}_2 d\mathbf{X}$ are proportional to u_1 and u_2 , respectively.

Equation(14)² yields $[\mathbf{F}] = -[\mathbf{v}] \mathbf{w}^*$; then $d_2 \mathbf{x} - d_1 \mathbf{x} = -[\mathbf{v}] \mathbf{w}^* d\mathbf{X}$ and the direction of vector $d_2 \mathbf{x} - d_1 \mathbf{x}$ corresponds to the direction of $[\mathbf{v}]$.

Let us notice that $[\mathbf{v}]$ is not necessary normal to the surface of discontinuity.

5 Shock waves

When $u \neq 0$, surfaces of discontinuity are shock waves [18,19].

5.1 Mass conservation

The reference mass density being given in \mathcal{W}'_0 and the mass conservation corresponding to $\rho \det \mathbf{F} = f(\mathbf{X})$, we get across a shock wave in \mathcal{W}'_0 :

$$[f(\mathbf{X})] = 0$$

By using (4), one deduces :

$$[\rho] = f(\mathbf{X}) \left[\frac{1}{\det \mathbf{F}} \right] = -f(\mathbf{X}) \frac{[\det \mathbf{F}]}{\det \mathbf{F}_1 \det \mathbf{F}_2} = -f(\mathbf{X}) \frac{[u]}{u_2 \det \mathbf{F}_1}$$

Then, $[\rho] = -\frac{[u]}{u_2} \rho_1$, which implies the geometrical property (which is not related with Hamilton's principle) :

$$[\rho u] = 0 \quad (16)$$

5.2 Hamilton's principle

When $\mathcal{F}^* - \text{Div } T = \mathbf{0}^*$ and $\mathcal{F}_0^* - \text{Div}_0 T_0 = \mathbf{0}^*$ corresponding to conservative motion equations (see Section 3.2), the variation of Hamilton's action writes :

$$\delta a = \int_{\Sigma} \mathbf{N}^* [T] \tilde{\zeta} d\sigma = \int_{\Sigma_0} \mathbf{N}_0^* [T_0] \hat{\zeta} d\sigma_0$$

5.2.1 First variation

From Hamilton's principle, we obtain :

$$\begin{aligned} \mathbf{N}^* [T] &= \mathbf{0}^* \\ [D_n p + (e + p)u] &= 0 \quad \text{and} \quad [\rho \mathbf{v} + p \mathbf{n}] = \mathbf{0} \end{aligned} \quad (17)$$

Equation (17)² corresponds to the *conservation of impulsion* across the shock wave. We can write :

$$[\rho \mathbf{v} + p \mathbf{n}] = \mathbf{0} \quad \iff \quad \begin{cases} [p + \rho u^2] = 0 \\ [\mathbf{v}_{tg}] = \mathbf{0} \end{cases}$$

where \mathbf{v}_{tg} is the tangential velocity component at the shock wave. Equivalently, we obtain :

$$[\mathbf{v}] = [u] \mathbf{n}$$

From (14)² we additively obtain :

$$[F] = -[u] \mathbf{n} \mathbf{w}^* = \frac{[u]}{u_0} \mathbf{n} \mathbf{n}_0^* \quad (18)$$

⁴ From Relation (15), we get :

$$[F] = [v] \frac{\mathbf{n}_0^*}{u_0} \implies \mathbf{F}_2 \mathbf{F}_1^{-1} = \mathbf{1} + [v] \frac{\mathbf{n}_0^*}{u_0} \mathbf{F}_1^{-1} \implies \mathbf{F}_2 \mathbf{F}_1^{-1} = \mathbf{1} + [v] \frac{\mathbf{n}^*}{u_1}.$$

From $\det(\mathbf{1} + \mathbf{K} \mathbf{L}^*) = 1 + \mathbf{L}^* \mathbf{K}$, where \mathbf{L} and \mathbf{K} are two 3-D vectors, we get :

$$\frac{\det \mathbf{F}_2}{\det \mathbf{F}_1} = 1 + \frac{[u]}{u_1} = \frac{u_2}{u_1} \implies \frac{\det \mathbf{F}_i}{u_i} = \frac{[\det \mathbf{F}]}{[u]}, \quad i \in \{1, 2\}.$$

From (16) and $[p + \rho u^2] = 0$, (17)¹ is equivalent to $\left[\frac{1}{2}v^2 + h - D_n u\right] = 0$ and $[\mathbf{v}_{tg}] = \mathbf{0}$ implies $[\mathbf{v}^2] = [u^2 + 2D_n u]$. Consequently,

$$\left[\frac{1}{2}u^2 + h\right] = 0$$

which corresponds to the conservation of energy. We can resume the conditions associated with the first variation corresponding to $\tilde{\zeta}$:

$$[\rho u] = 0, \quad [p + \rho u^2] = 0, \quad [\mathbf{v}] = [u] \mathbf{n}, \quad \left[\frac{1}{2}u^2 + h\right] = 0 \quad (19)$$

Equations (19) represent the Rankine-Hugoniot conditions for fluid shock-waves in the space-time \mathcal{W} , where we can add geometrical condition (18).

5.2.2 Second variation

We assume the shock wave Σ_0 is represented by equation $\lambda = 0$ in \mathcal{W}_0 . The mapping Φ being continuous along the shock wave, we obtain again (14).

The principle of Hamilton implies :

$$N_0^* [T_0] \equiv \mathbf{I}^* [T_0] = \mathbf{0}^*$$

From the value of T_0 obtained in Section 3.2, we obtain :

$$[\mathbf{v}^* F + m \mathbf{w}^*] = \mathbf{0}^* \quad (20)$$

which is the only shock condition associated with the virtual displacement $\hat{\zeta}$. Equation (20) corresponds to three scalar equations through the shock wave Σ_0 . From (14)², (15)² and (20), we respectively deduce :

$$\mathbf{w}^* d\mathbf{X} = 0 \quad \implies \quad [F] d\mathbf{X} = \mathbf{0} \quad \text{and} \quad [\mathbf{v}^* F] d\mathbf{X} = 0$$

or

$$\mathbf{w}^* d\mathbf{X} = 0 \quad \implies \quad d_1 \mathbf{x} = d_2 \mathbf{x} \quad \text{and} \quad \mathbf{v}_1^* d_1 \mathbf{x} = \mathbf{v}_2^* d_2 \mathbf{x}$$

From (13),

$$\mathbf{w}^* d\mathbf{X} = 0 \quad \iff \quad \mathbf{n}^* d\mathbf{x} = 0$$

Consequently,

$$\mathbf{n}^* d\mathbf{x} = 0 \quad \implies \quad [\mathbf{v}^*] d\mathbf{x} = 0$$

and there exists a Lagrange multiplier α such that $[\mathbf{v}] = \alpha \mathbf{n}$; consequently the discontinuity of \mathbf{v} is normal to the shock wave.

Due to $\alpha = [\mathbf{n}^* \mathbf{v}] = [u]$, we deduce :

$$[\mathbf{v}] = [u] \mathbf{n} \quad \text{and} \quad [\mathbf{v}_{tg}] = \mathbf{0} \quad (21)$$

and we obtain again (18).

From $[\Omega] = 0$, we get $[m \mathbf{w}^*] = \left[\frac{1}{2}v^2 - h\right] \mathbf{w}^*$. Moreover, $[\mathbf{v}^* F] = \mathbf{v}_1^* [F] +$

$$[\mathbf{v}^*] F_2 = \mathbf{v}_1^* [F] + [u] \mathbf{n}^* F_2.$$

From $\mathbf{n}^* \mathbf{v}_1 = u_1 + D_n$ and because (15)² and (21), $[F] = [\mathbf{v}] \mathbf{n}_0'^* = [u] \mathbf{n} \mathbf{n}_0'^*$ and from (15)¹, we obtain $\mathbf{n}^* F_2 = u_2 \mathbf{n}_0'^*$. We deduce :

$$[\mathbf{v}^* F] = [u] \mathbf{v}_1^* \mathbf{n} \mathbf{n}_0'^* + [u] u_2 \mathbf{n}_0'^* = [u] (u_1 + D_n + u_2) \mathbf{n}_0'^* = - [u^2 + D_n u] \mathbf{w}^*$$

Consequently, $[\mathbf{v}^* F + m \mathbf{w}^*] = - \left[u^2 + D_n u + h - \frac{1}{2} \mathbf{v}^2 \right] \mathbf{w}^* = \mathbf{0}^*$ and finally,

$$\left[\frac{1}{2} u^2 + h \right] \mathbf{w}^* = \mathbf{0}^* \iff \left[\frac{1}{2} u^2 + h \right] = 0$$

We can resume the conditions associated with the second variation :

$$[\rho u] = 0, \quad [\mathbf{v}] = [u] \mathbf{n}, \quad \left[\frac{1}{2} u^2 + h \right] = 0 \quad (22)$$

Equations (22) represent the Rankine–Hugoniot conditions for fluid shock wave in \mathcal{W}_0 .

6 Conclusion

Only the first variation of Hamilton's action yields the well-known Rankine–Hugoniot conditions for fluids. The second variation yields only a part of the Rankine–Hugoniot conditions by missing the condition $[p + \rho u^2] = 0$.

This difference of results can be explained : in the 4 - D space-time, along surface Σ , the test function (or virtual displacement) $\tilde{\zeta}$ is a continuous function. But \mathcal{B} is discontinuous through the shock wave and due to (5) the virtual displacement $\tilde{\zeta}$ is discontinuous; consequently we cannot apply the fundamental lemma of variation calculus which must be used with continuous test functions. Then, only the first variation associated with $\tilde{\zeta}$ in the 4 - D space-time is able to get the whole Rankine–Hugoniot conditions on shock waves.

It is noticeable that the variation $\tilde{\delta}$ gives the conservation of energy, while the variation $\hat{\delta}$ does not give the conservation of entropy. This fact was the subject of strong discussions in the shock waves' studies and Hamilton's principle directly yields the result without any ambiguity. The problem of Lax's condition for entropy is a different problem relevant of the second law of thermodynamics [12, 20, 21].

To conclude, the Hamilton action in the 4 - D space-time is a powerful tool to obtain shock conditions in multi-dimensional spaces. It unambiguously allows us to obtain the equations already known and to consider the study of more complex cases not yet considered in the literature [11]. The forthcoming article [11] also uses Hamilton's principle and obtain complementary relations to Rankine–Hugoniot conditions for second-gradient or bubbly fluids. In this paper devoted to classical fluids, we can nevertheless observe the importance of the virtual displacements in the 4 - D space-time to obtain the correct Rankine–Hugoniot conditions; this is an important difference with the

virtual displacements in the 4 - D reference-space associated with Lagrange variables. Another important observation is that the virtual displacements in the 4 - D space-time naturally yield the conservation of energy through a shock wave. However, the Hamilton action cannot be directly used for hyperbolic dissipative systems; in such cases, the principle of virtual power in space-time may be a useful extension of the Hamilton principle as it was considered in [9].

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