



HAL
open science

Lambert's proof of the irrationality of Pi: Context and translation

Denis Roegel

► **To cite this version:**

Denis Roegel. Lambert's proof of the irrationality of Pi: Context and translation. [Research Report] LORIA. 2020. hal-02984214

HAL Id: hal-02984214

<https://hal.science/hal-02984214>

Submitted on 30 Oct 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Lambert's proof of the irrationality of π :

Context and translation

**this is a preliminary draft
please check for the final version**

Denis Roegel
LORIA, Nancy¹

30 October 2020

¹Denis Roegel, LORIA, BP 239, 54506 Vandœuvre-lès-Nancy cedex,
roegel@loria.fr

In this document, I give the first complete English translation of Johann Heinrich Lambert's memoir on the irrationality of π published in 1768 [92], as well as some contextual elements, such as Legendre's proof [98] and more recent proofs such as Niven's [109]. Only a small part of Lambert's memoir has been translated before, namely in Struik's source book [143]. My translation is not based on that of Struik and it is supplemented with notes and indications of gaps or uncertain matters.

The purpose of this document is not to give a complete treatment of such a vast subject, but rather to provide interested people with a better access to the sources. Several accounts of Lambert's proof actually do not refer to Lambert's original memoir, but to more recent expositions, such as those of Legendre or Lebesgue, which may be misleading. Some authors suggest that Lambert gave his results without proofs, which is not true.

Lambert's memoir still contains some gaps and I have tried to identify them in a number of notes. Some of these gaps do not seem to have been reported by others. They may be either genuine gaps in Lambert's proof, or merely things that should be clarified. I hope that these points can be improved in a future version of this document.

Contents

1	Introduction	7
1.1	Irrational numbers	7
1.2	Lambert's work on π	8
1.2.1	Lambert's article on the transformation of fractions . .	9
1.2.2	Lambert's article on circle squarers (1766)	10
1.2.3	Lambert's main article (1767)	11
1.3	Lambert's work on e	15
1.4	After Lambert: a quick review	15
1.4.1	Legendre's analysis (1794)	15
1.4.2	More concise proofs by Gauss and Hermite	16
1.4.3	Various summaries of earlier proofs	16
1.4.4	Pringsheim's renewal (1899) and the rediscovery of Gauss (1932)	18
1.4.5	More concise proofs by Niven and others	18
1.4.6	Recent summaries and comments	18
2	Lambert's memoir (1767)	21
3	Legendre's proof (1794)	75
4	Modern proofs	83
4.1	Gauss (1850s?)	83
4.2	Glaisher (1872)	84
4.3	Hermite (1873)	85
4.4	Hermite (1882)	86
4.5	Hobson (1918)	87
4.6	Popken (1940)	88
4.7	Cartwright (1945)	90
4.8	Niven (1947)	91
4.9	Popken (1948)	93
4.10	Breusch (1954)	94

4.11	Laczkovich (1997)	96
4.12	Other proofs	97
4.13	Irrationality of powers of π and other results	98
4.14	Transcendence	98
	Bibliography	99

Chapter 1

Introduction

In this introduction¹, I give a few additional elements for the understanding of Lambert's proof of the irrationality of π . My purpose is however not to give a complete coverage of this vast and rich subject.

1.1 Irrational numbers

Lambert's work deals with the classification of numbers. The simplest numbers are the natural numbers 0, 1, 2, etc. Next come the rational numbers which can be expressed as ratios of two natural numbers, for instance $5/2$, $7/3$, $1/8$, etc. Real numbers correspond to the continuum and this set contains the rational numbers, which themselves contain the natural numbers:

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R}$$

An irrational number is a real number which cannot be expressed as a ratio of two integers. The set of irrational numbers can be denoted by $\mathbb{R} \setminus \mathbb{Q}$. The simplest irrational number is probably $\sqrt{2}$ and it is easy to prove that it is not rational. For assume that $\sqrt{2} = a/b$, where a and b are two positive integers. We can also assume that the fraction is irreducible, that is, that it cannot further be simplified, and that a has the smallest possible value (in case the irreducible fractions were not unique). In that case we find that $a^2 = 2b^2$ which implies that $a = 2a'$ and therefore that $\sqrt{2} = b/a'$. This is a new ratio with $b < a$ and therefore it contradicts our assumptions. Consequently, $\sqrt{2}$ cannot be expressed as a ratio of two integers. Likewise, it is easy to prove that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$, etc., are not rational numbers.

¹It is a pleasure to thank Gérald Tenenbaum for commenting upon a previous version of this document.

But there are many more irrational numbers and proving their irrationality is not so simple. For some numbers such as π , the irrationality has been suspected. For instance, Fantet de Lagny [48, p. 141], who computed π to many places, also conjectured it being irrational.

Furthermore, irrational numbers can be divided into algebraic numbers and transcendental numbers. The set of algebraic numbers is made of those real numbers which are solutions of polynomial equations with integers coefficients. Lambert suspected that π is a transcendental number but this result was only proven in 1882 by Lindemann.

For more context on irrational numbers and their construction, the reader should turn in particular to Brandenberger [23], Niven [110] and Havil [65].

1.2 Lambert's work on π

Johann Heinrich Lambert (1728-1777) was born in Mulhouse (France), then in the Swiss confederacy.² He was the son of a tailor of French ancestry and he was only schooled until the age of 12. Later he became the preceptor of the Count de Salis' children, and this gave him access to a vast library and also the ability to travel through Germany and get in touch with many scientists and mathematicians, in particular Leonhard Euler (1707-1783).

Lambert was a polymath and made contributions to mathematics, optics, astronomy, map projections and philosophy. He died in 1777 in Berlin.

In mathematics, Lambert is the author of the first³ explicit proof of the irrationality of π . Lambert's work relies on continued fractions and his proof of the irrationality of π leads to questions of convergence which are not treated fully rigorously, although fixes have been provided by several authors. It is in part because of these difficulties, as noted by Laczkovich [91], but also because Lambert's proof is very long, that usual proofs of the irrationality of π now avoid continued fractions. It is therefore all the more important to be able to examine carefully Lambert's work, to compare it with that of Legendre, and to put them in the context of various more recent studies.

Here, I go through Lambert's writings related to the irrationality of π , and in the next chapters the translations of Lambert's main memoir, of Legendre's

²For biographical sketches of Lambert, see the notices of Formey [50], Huber [73], Wolf [151] and Scriba [135]. See also the proceedings of the international conference held in 1977 for the second century of Lambert's death [112].

³However, according to Youschkevitch [152, p. 216] and Aycock [5], Euler wrote a letter in 1739 in which he claimed that "it would be easy to prove" that the ratio of the circumference to the diameter is not rational. The fact is that Euler did not leave any such proof.

proof and the main modern proofs will be given.

1.2.1 Lambert's article on the transformation of fractions

In a long article on the transformation of fractions [94], probably written in 1765,⁴ Lambert gave some interesting results related to π . I shall only give one example and not summarize the entire article.

Starting with the arctan series

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

Lambert obtained the continued fraction [94, p. 82]

$$\arctan z = \frac{1}{1 : z + \frac{1}{3 : z + \frac{1}{5 : (4z) + \frac{1}{28 : (9z) + \frac{1}{81 : (64z) + \frac{1}{704 : (225z) + \&c.}}}}}}$$

This fraction, as observed by Bauer [10], can be rewritten

$$\arctan z = \frac{z}{1 + \frac{z^2}{3 + \frac{z^2}{5 : 4 + \frac{z^2}{28 : 9 + \frac{z^2}{81 : 64 + \frac{z^2}{704 : 225 + \&c.}}}}}}$$

⁴In the introduction of the volume containing this article, Lambert writes that all the articles were written in 1765 or later, and the article described in the next section was written in 1766.

and

$$\arctan z = \frac{z}{1 + \frac{z^2}{3 + \frac{(2z)^2}{5 + \frac{(3z)^2}{7 + \frac{(4z)^2}{9 + \frac{(5z)^2}{11 + \&c.}}}}}}$$

and eventually

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{7 + \frac{4^2}{9 + \frac{5^2}{11 + \&c.}}}}}}$$

although Lambert did not give this fraction in his article. In their articles [11, 15, 12, 13, 14], Bauer and Haenel explore a number of ideas on the theme of arctan-based approximations of π .

1.2.2 Lambert's article on circle squarers (1766)

In the 1760s, Lambert worked on the quadrature of the circle, the irrationality of π and hyperbolic geometry. The subject of the quadrature of the circle is a very old one⁵ and Lambert himself became interested in it already in the 1750s.⁶

Lambert wrote two related articles in 1766 (published in 1770)⁷ and 1767 (published in 1768) on the quadrature. The first is examined here. In this

⁵For some leads on the vast literature on the quadrature of the circle, see in particular Montucla [106], De Morgan [35], Hobson [71], Jacob [78], and Crippa [32]. The quadrature of the circle is sometimes confused with the rationality of π , even by mathematicians who should know better. For instance, even Glaisher wrote in 1871 that the “arithmetical quadrature of the circle, that is to say, the expression of the ratio of the circumference to the diameter in the form of a vulgar fraction with both numerator and denominator finite quantities, was shown to be impossible by Lambert” [56, p. 12].

⁶See Jacob [78, p. 402-409].

⁷The date 1766 is given explicitly by Lambert in the foreword of the second volume of the *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, as observed by Rudio [131, p. 165]. Lambert must already have had his proof of irrationality, except that the details were only made public in 1767. However, because the proof was published in

article, Lambert's focus was on "circle squarers" (those who claim to have solved the quadrature of the circle) and it was a somewhat popular article, not venturing into any actual proof [95].⁸

In his article, Lambert first writes [95, p. 155] that he is not aware of any result on the rationality of π up to now. He cleverly does not state his own result in advance. He writes however that if π is rational, it must be a fraction of large numerators and denominators. He then gives a series of 27 approaching ratios, from $\frac{3}{1}$ to $\frac{1019514486099146}{324521540032945}$ [95, p. 156-157] and he refers to his article on fractions [94] for their obtention.⁹

Later, Lambert does state that $e^{p/q}$ is irrational for integers $p > 0$ and $q > 0$ [95, p. 161].

Regarding π , Lambert writes [95, p. 162] that he found the expression

$$\tan v = \frac{1}{1 : v - \frac{1}{3 : v - \frac{1}{5 : v - \frac{1}{7 : v - \frac{1}{9 : v - \&c.}}}}}$$

And he deduces immediately that since this continued fraction doesn't end when $1/v$ is an integer, and in particular when $v = 1$, that $\tan v$ is then irrational.

In fact, Lambert does state that $\tan \frac{m}{n}$ is irrational for integer m and n and therefore that π is irrational.

Lambert's article concludes with a kind of riddle, which is in fact some mundane observation later clarified by Unger [146, p. 326-327] (see also [127]).

1.2.3 Lambert's main article (1767)

The article read in 1767 and published in 1768 is the one containing the details of Lambert's proof of the irrationality of π . This article is fully trans-

a volume covering communications from 1761, several authors, for instance Brezinski [25, p. 110], or Havil [65, p. 104], mistakenly have dated Lambert's proof to 1761.

⁸For a summary of this article, see Jacob [78, p. 50-61] and Viola [148, p. 243].

⁹Lambert's ratios are correct, except for the last two. This error has been notified by Schulz in 1803 [134, p. 159] and later by Hirsch [70, p. 90] and Egen [42, p. 444]. Lambert's last ratio is not as accurate as he claimed, but the cause of his incorrect statement is that Lambert used an incorrect continued development of π (see <https://oeis.org/A001203> for a more extensive development). In fact, in his last two steps, Lambert assumed that $\pi = [3; 7, 15, 1, 292, \dots, 1, 37, 3, \dots]$ (see [95, p. 158]) instead of the correct $\pi = [3; 7, 15, 1, 292, \dots, 1, 15, 3, \dots]$. Surprisingly, these errors have *not* been corrected by Rudin [131, p. 147-148] who edited Lambert's article.

lated in the next chapter.

It is interesting to note that Lambert never uses the symbol π to denote the ratio of the circumference to the diameter, although the first use of π in its modern meaning goes back to William Jones (1675-1749) in 1706 [85, 130]. Lambert however uses the symbol π in §. 40, but with a different meaning.

Lambert builds upon earlier work by Leonhard Euler (1707-1783), in particular his *De fractionibus continuis dissertatio* published in 1744 [44] and his *Introductio in analysin infinitorum* published in 1748 [45].

Lambert starts (§. 4) with the series for the sines and cosines and obtains a continued fraction¹⁰ for the tangent (§. 7):

$$\tan v = \frac{1}{1 : v - 1} \cfrac{}{3 : v - 1} \cfrac{}{5 : v - 1} \cfrac{}{7 : v - 1} \cfrac{}{9 : v - \&c.}$$

Lambert's expression can be rewritten as

$$\tan v = \frac{v}{1 - v^2} \cfrac{}{3 - v^2} \cfrac{}{5 - v^2} \cfrac{}{7 - v^2} \cfrac{}{9 - \&c.}$$

although Lambert doesn't use it.

Lambert's expression is proven by induction and he obtains series expansions for R' , R'' , R''' , etc., such that

$$\tan v = \frac{\sin v}{\cos v} = \frac{A}{B} = \frac{1}{Q' + \frac{R'}{A}} = \frac{1}{Q' + \frac{1}{\frac{R''}{Q'' + \frac{R'''}{R''}}}} = \frac{1}{Q' + \frac{1}{\frac{1}{Q'' + \frac{1}{\frac{R'''}{Q'' + \frac{R'''}{R''}}}}}} = \dots$$

with $Q' = 1/v$, $Q'' = -3/v$, $Q''' = 5/v$, etc. This leads us up to §. 14 in Lambert's memoir but it assumed that $v < 1$ and that $1/v$ was an integer (v is an aliquot part of 1).

¹⁰In most cases, when I cite continued fractions from Lambert or Legendre's work, I try to use their notation.

Now, starting with

$$\tan v = \frac{1}{1 : v - 1 \over 3 : v - 1 \over 5 : v - 1 \over 7 : v - 1 \over 9 : v - \&c.}$$

and setting $w = 1/v$ (and still keeping implicitly $v < 1$), Lambert considers the convergents obtained by restricting the continued fraction to a finite number of terms. These convergents are fractions of two polynomials in w and Lambert obtains general expressions for the numerators and the denominators, which he also proves by induction (§.17-28). He observes that the quotients of the numerators to the denominators seem to tend to the quotients of the sines to the cosines (§. 29-30).

Next, Lambert seeks to find how these convergents are approaching the value of the tangent (§. 32). Lambert obtains the general expression of the differences of these convergents, and then he is able to write the tangent as a sum of terms (§. 34). Then, assuming $w = \frac{\varphi}{\omega}$, Lambert also obtains the analog general expressions with φ and ω (§. 37). Finally, assuming that $\tan \frac{\varphi}{\omega}$ is equal to some fraction $\frac{M}{P}$ he defines $D > 0$ to be the greatest common divisor of M and P . The values M and P need not be integers, they might be for instance 2.793 and 6.23, in which case D would be equal to 0.007, as $2.793 = 399 \times 0.007$ and $6.23 = 890 \times 0.007$.

Lambert then defines a sequence of remainders R' , R'' , R''' , etc., such that D divides each of them (§. 41-45). But these remainders are decreasing towards 0 (§. 49)¹¹, without ever being equal to 0. We therefore have a decreasing sequence of integers which cannot be equal to 0, which is a contradiction. This is a so-called “proof by infinite descent,” or “Fermat’s method of descent,” although it predates Fermat. In Lambert’s case, this contradiction shows that there is no such $D > 0$ and therefore that $\tan \frac{\varphi}{\omega}$ is not rational when $\frac{\varphi}{\omega} \neq 0$.

And then Lambert immediately concludes (§. 51) that π is not rational.

In sections §. 52-71, Lambert introduces the notion of *prime tangent* and obtains some of their properties. In section §. 72, Lambert gives a continued fraction whose convergents approach $\tan v$ by excess and by default, whereas the convergents of the initial continued fraction (in §. 7) only approach $\tan v$ by default.

¹¹This was not clearly proved by Lambert, as observed by Gauss [123, 124], and more recently by Baltus [7].

Sections §. 73-88 develop the theory of the hyperbolic functions and in particular prove that $e^{p/q}$ is irrational if $\frac{p}{q} > 0$ is rational. Eventually in §. 89-91, Lambert suggests the existence of numbers which are not solutions of algebraic equations, hence transcendental numbers.

There are however some gaps in Lambert's proof. Lambert's proof is difficult to read, there are some missing steps and some unproven assumptions, and the text is furthermore crippled by a number of typographical errors. Many of the errors have been corrected by Speiser in 1948 [96], but some errors remain and some parts of the original memoir have been altered although they could have been saved.¹² Even Struik's partial translation [143] contains some new typos, and makes it very difficult to have a complete overview of Lambert's article.

Lambert then expands on the work of Vincenzo Riccati (1707-1775) on hyperbolic trigonometry¹³ and obtains an expression analogous to that of $\tan v$ for the hyperbolic tangent (§. 73).

Like above, Lambert's expression

$$\tan v = \frac{1}{1 : v + 1} \frac{1}{3 : v + 1} \frac{1}{5 : v + 1} \frac{1}{7 : v + 1} \frac{1}{9 : v + \&c.}$$

can be rewritten as

$$\tan v = \frac{v}{1 + v^2} \frac{1}{3 + v^2} \frac{1}{5 + v^2} \frac{1}{7 + v^2} \frac{1}{9 + \&c.}$$

which Lambert also doesn't use.

As mentioned earlier, Lambert's memoir contains some inaccuracies or gaps, or at least things that are in need of clarifications. In my translation, I did not provide fixes other than those for typos and simple clarifications. There is still a need to improve things, in particular in sections 49, 61, 65, 67, 68, 72, 73, and 88. The interested reader should check the notes in these sections where the gaps or obscure passages are marked.

¹²Inkeri [75] claims that Speiser, the editor of Lambert's Opera, filled a gap in Lambert's proof but Speiser has in fact only corrected some typos, not any fundamental gap.

¹³See Barnett [9] for the details of the connection with Riccati.

1.3 Lambert's work on e

In his main memoir, Lambert also proves the irrationality of e . This had been proven earlier by Euler [105, 31, 132, 140] in *De fractionibus continuis dissertatio* [44], written in 1737 and published in 1744, and which is the first comprehensive account of the properties of continued fractions.

In his work, Euler used for instance the continued fraction [45, p. 319]:

$$\frac{e-1}{2} = \frac{1}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \frac{1}{18 + \dots}}}}}$$

and this led Lambert [95, p. 162] to research a continued fraction for $\frac{e-1}{e+1}$, or rather of $\frac{e^x-1}{e^x+1}$ which he gave in §. 74.

This enabled him to prove that $e^{p/q}$ is irrational, and also conversely that the hyperbolic logarithm of a rational number is irrational.

1.4 After Lambert: a quick review

After the publication of Lambert's proof of the irrationality of π , a number of authors have published comments or summaries of the proof. Some of them provided new and simpler proofs which will be examined in chapter 4.

1.4.1 Legendre's analysis (1794)

Adrien-Marie Legendre (1752-1833)'s *Éléments de géométrie*, published in 1794 [98, p. 296-304], contains an appendix where π is proven to be irrational, using a convergence lemma for continued fractions. Legendre doesn't describe Lambert's proof in detail and his proof is much simpler. In chapter 3, I give a complete translation of Legendre's proof.

But, even though Legendre never lays any such claim, several later authors, for instance Youschkevitch [152, p. 216], have incorrectly claimed that Legendre fixed some gaps in Lambert's proof, as noted by Pringsheim [122]. Rudio [131], who published a German translation of Legendre's note, also claimed that Legendre gave the lemma which was missing in Lambert's proof [131, p. 56].

It is true that Lambert's demonstration has some gaps, but Legendre has only made two contributions to Lambert's work. First, he simplified

Lambert's proof, making it much easier to follow. And second, he extended it by proving the irrationality of π^2 . Legendre was also convinced of the transcendence of π , but did not prove it.

Various authors have summarized or adapted the proof of Legendre. For instance, Chrystal [29] has proven the irrationality of π by first deriving the continued fraction

$$\tan x = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \dots}}}}$$

using a technique from Lambert [94], then basically used Legendre's lemma to show that the continued fraction converges towards an irrational value if π were rational, hence a contradiction.

Other uses of Legendre's proof are found in Hobson [72, p. 374-375], Serfati [136, 137], Delahaye [37], etc.

In particular Lebesgue [97, p. 108] has observed that Legendre could also have considered $\tan \frac{\pi}{2} = \infty$ to prove that π^2 is not rational.

Notice should also be made of a manuscript in Gauss's Nachlass [124] which contains some comments on Legendre's lemma.

1.4.2 More concise proofs by Gauss and Hermite

In the 1850s, Gauss worked on a more concise proof of the irrationality of π , but did not publish it. It was only published in 1900 as part of his Nachlass. The first concise proof to be published was therefore that of Hermite in 1873. More details on these proofs will be found in chapter 4.

1.4.3 Various summaries of earlier proofs

Comments and summaries of Lambert's results appeared as early as the beginning of the 19th century. For instance, in 1803, Schulz [134, p. 161-178] gave a summary of Lambert's proof. In 1824, Eytelwein [47, p. 364-367] gave the computation of Lambert's convergents as an exercise. In 1856, Eugène Prouhet (1817-1867) [125] provided extensions to Lambert's results on primary tangents, but did not add anything to the question of the irrationality of π . In 1911, Vahlen [147, p. 319-325] gave a summary of Lambert's proof, of Legendre's proof, as well as of those of Gauss and of Hermite.

In his lectures on geometric constructions [97, p. 103-109], Henri Lebesgue

(1875-1941) recast Lambert's proof. He defined A_1, A_2, A_3 , etc., such that

$$\begin{aligned} \tan x &= \frac{x}{1 + A_1} \\ A_1 &= -\frac{x^2}{3 + A_2} \\ A_2 &= -\frac{x^2}{5 + A_3} \\ &\dots\dots \\ A_k &= -\frac{x^2}{2k + 1 + A_{k+1}} \\ &\dots\dots \end{aligned}$$

with for A_k the general expression

$$A_k = \frac{\sum_{n=0}^{\infty} (-1)^{n+1} x^{2n+2} \frac{(2n+2)(2n+4)\cdots(2n+2k)}{(2n+2k+1)!}}{\sum_{n=0}^{\infty} (-1)^n x^{2n} \frac{(2n+2)(2n+4)\cdots(2n+2k-2)}{(2n+2k-1)!}}$$

Given that Lambert defines (§. 6) $B = AQ' + R'$, $A = R'Q'' + R''$, $R' = R''Q''' + R'''$, etc., it is easy to see that $\frac{B}{A} = Q' + \frac{A_1}{v}$, $\frac{A}{R'} = Q'' + \frac{A_2}{v}$, etc., and that in fact Lebesgue's A_k is related to Lambert's R^k by

$$A_k = v \frac{R^k}{R^{k-1}}$$

where the series for R^k are given in §. 8, with the exception of the signs, the alternating initial signs having vanished because all the quotients Q^i are taken positive.

Lebesgue then gave the general expression of the convergents, but as a fraction of polynomials in v , whereas Lambert gave the series for the numerators and denominators as functions of w (§. 24):

$$\begin{aligned} P_n &= 1 \cdot 3 \cdot 5 \cdots (2n-1) \times \sum_{k=1}^n (-1)^{k-1} \frac{v^{2k-1}}{(2k-1)!} \frac{(2n-2k)(2n-2k-2)\cdots(2n-4k+4)}{(2n-1)(2n-3)\cdots(2n-2k+3)} \\ Q_n &= 1 \cdot 3 \cdot 5 \cdots (2n-1) \times \sum_{k=0}^n (-1)^k \frac{v^{2k}}{(2k)!} \frac{(2n-2k)(2n-2k-2)\cdots(2n-4k+2)}{(2n-1)(2n-3)\cdots(2n-2k+1)} \end{aligned}$$

the sums being extended until the terms equal 0.

Lebesgue observes then that the expressions in the sums converge uniformly towards $\sin v$ and $\cos v$, and that P_n/Q_n therefore converges uniformly towards $\tan v$.

Lebesgue seems also to be the only one to comment upon Lambert's notion of "primary tangent" [97, p. 107].

Lebesgue's derivation was used by several authors [149, 114, 46] as a way of providing a more concise summary of Lambert's proof.

Eymard and Lafon [46] also follow Lebesgue, but they incorrectly attribute Legendre's lemma to Lambert. Lambert did not deduce the irrationality of a continued fraction from Legendre's first lemma, but instead used a much more complex reasoning.

1.4.4 Pringsheim's renewal (1899) and the rediscovery of Gauss (1932)

In 1899, Alfred Pringsheim (1850-1941), spurred by what he considered to be unfair uncounts of Lambert's merits, gave a summary of Lambert's memoir and claimed [122, p. 326] that Lambert had completely proved the irrationality of π and that Legendre's work instead lacked at proofs of existence and convergence of the infinite continued fractions. I refer the reader to Pringsheim's article on the convergence of continued fractions [121].

Pringsheim returned to Lambert in the 1930s when commenting Gauss's Nachlass published in 1900 [123, 124]. For some reason, Gauss's observations were pretty much ignored until they were rediscovered at the beginning of the 21st century.

1.4.5 More concise proofs by Niven and others

Several new proofs of the irrationality of π were published in the 1940s and 1950s. One of the more popular ones was the proof published by Niven in 1947 [109]. It was in turn borrowed or adapted by others [137, 1].

1.4.6 Recent summaries and comments

As part of the 1977 international conference celebrating the 200th anniversary of Lambert's death, Viola [148, p. 239-242] gave another summary of Lambert's main memoir.

In 2000, Wallisser [150] gave a summary of the proofs of Lambert and Legendre, but with some inaccuracies (for instance on the definitions of P and M in Lambert's §. 38), and omitting Pringsheim's second article on Gauss's work [124].

In 2003, Baltus [7] sought to settle the matter of the completeness of Lambert's proof. Apparently in ignorance of Gauss's earlier findings [123, 124], Baltus rediscovered¹⁴ the gap in Lambert's proof in §. 49, where Lambert does not clearly prove that the sequence R' , R'' , R''' , etc., tends to 0. Baltus provided a fix for the "gap."

¹⁴Not having seen Baltus's 2004 article [8], there is the possibility that he there has a reference to Gauss.

Chapter 2

Lambert's memoir (1767)

Memoir
on
some remarkable properties
of the circular and logarithmic
transcendental quantities¹
by Mr. Lambert²

§. 1.

Proving that the diameter of the circle is not to its circumference like an integer to an integer, this is something which geometers will hardly take as a surprise. The numbers of *Ludolph*,³ the ratios found by *Archimedes*,⁴

¹“Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques,” *Histoire de l'Académie Royale des Sciences et Belles-Lettres, Année 1761*, tome XVII, Berlin, 1768, pp. 265-322 [92]. Translated from the French by Denis Roegel, 2010-2012. With the exception of the first note, all the notes are from the translator. This translation tries to be as faithful as possible, but intends to remain intelligible. In order to clarify certain passages, a number of notes, as well as several appropriate figures have been added.

²Read in 1767. (original footnote)

³Ludolph van Ceulen (1540–1610) published a 20-decimal value of π in 1596 (see *Vanden circkel daer in gheleert werdt te vinden de naeste proportie des circkels-diameter teghen synen omloop etc.*, 1596, Folio 14 [26], and 1615, Folio 24 [28]). In his *Arithmetische en geometrische fundamenten* (1615), p. 163 [27], published by his widow, he reached thirty-two decimal places. Finally, Willebrord Snell, in his *Cyclometricus* (1621), p. 55 [139], published Van Ceulen's final triumph, π to 35 places. See also [58, 57, 62] on the early history of the computation of π .

⁴Archimedes (c. 287 BC—c. 212 BC) found the approximation $\frac{\pi}{4} \approx \frac{11}{14}$, therefore $\pi \approx$

by *Metius*⁵ etc., as well as a great number of infinite series are known, and they are all related to the quadrature of the circle.⁶ And if the sum of these series is a rational amount, it is rather natural to conclude that it will either be an integer, or a very simple fraction. Indeed, if a very complex fraction were needed, why would it be rather this fraction instead of any other? For instance, it is the case that the sum of the series

$$\frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \frac{2}{7 \cdot 9} + \&c.$$

is equal to one,⁷ which of all the rational quantities is the simplest one. But, by omitting alternatively the 2nd, 4th, 6th, 8th &c. terms, the sum of the remaining ones⁸

$$\frac{2}{1 \cdot 3} + \frac{2}{5 \cdot 7} + \frac{2}{9 \cdot 11} + \frac{2}{13 \cdot 15} + \&c.$$

gives the surface of the circle, when the diameter is = 1. It seems therefore that, if this sum were rational, it should also be expressed by a most simple fraction, such as would be $\frac{3}{4}$ or $\frac{4}{5}$ &c. Indeed, the diameter being = 1, the radius = $\frac{1}{2}$, the square of the radius = $\frac{1}{4}$, it is obvious that these expressions being so simple would not be an obstacle to it. And since we are dealing with the whole circle, which represents a kind of unit, and not with some sector, which by its nature would require very large fractions, it is consequently again obvious that there is no reason to expect a very complex fraction. But since, after the fraction $\frac{11}{14}$ found by *Archimedes*, which gives only an approximation, we go to that of *Metius*, $\frac{355}{452}$, which is also not exact, and in which the numbers are considerably larger, one must be led to conclude, that the sum of this series, far from being equal to a simple fraction, is an irrational quantity.

§. 2. However vague this reasoning be, there are nevertheless cases where nothing more is required. But this is not the case of the quadrature of

²²₇. See *Archimedes' Measurement of a circle* in Heath, *The works of Archimedes*, 1897 [3] and Knorr [89].

⁵Metius (1571–1635) found the approximation $\frac{\pi}{4} \approx \frac{355}{452}$, therefore $\pi \approx \frac{355}{113}$. See his *Arithmeticae et Geometriae practica*, 1611 [103], (attested by Prouhet [126]).

⁶The quadrature of the circle is the process of squaring the circle, that is, constructing $\sqrt{\pi}$ using a compass and a straightedge, which is impossible because π is transcendental. This was proven by Lindemann in 1882 [101, 69, 51]. See also the work of Sylvester [145, 144] who proved the transcendence as an extension of Lambert's proof.

⁷This is easily seen given that the partial sums are $(\frac{1}{1} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{7}) + \dots + (\frac{1}{2n+1} - \frac{1}{2n+3}) = 1 - \frac{1}{2n+3}$.

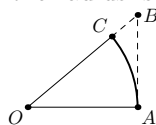
⁸This sum is equal to $\frac{\pi}{4}$.

the circle.⁹ Most of those who are trying to find it, do so with an ardour which sometimes drives them to cast doubt on the most fundamental and well established truths of geometry. Would one believe that they would be satisfied by what I just wrote? Much more is needed. And if the purpose is to prove that the diameter is not to the circumference like an integer to an integer, this proof must be so solid, that it does not give in to a geometric proof. And with all that, I come back saying that geometers will not be surprised by it. They must have long been accustomed not to expect anything else. But here is what will deserve more attention, and which will make up a good part of this memoir. We want to show that, *whenever a circle arc is commensurable to the radius, the tangent of the arc is not commensurable with it*;¹⁰ and reciprocally, *every commensurable tangent is not that of a commensurable arc*. Now we have a reason to be more surprised. This proposition would seem to admit an infinity of exceptions, but it admits none. Moreover it shows how much the transcendental circular quantities are transcendental, and removed beyond all commensurability. Since the proof which I will give requires the entire geometric rigour, and that in addition it will weave a number of other theorems which have to be proven with as much rigour, these reasons will excuse me when I will not hasten to reach the end, or when I will stop along the way to whatever remarkable thing shall present itself.

§. 3. Let thus be given *any arc commensurable to the radius*: and we have to find *if this arc will at the same time be commensurable to its tangent or not*. Consider now a fraction such that its numerator be equal to the given arc, and its denominator be equal to the tangent of this arc. It is clear that no matter how this arc and its tangent are expressed, this fraction must be equal to another fraction, whose numerator and denominator are integers, whenever the given arc happens to be commensurable to its tangent. It is also clear that this second fraction can be deduced from the first one, by the same method as the one which is used in arithmetic to reduce a fraction to its smallest denominator. This method is known since *Euclid*, who made it the 2nd proposition of his 7th book¹¹, and I will not stop proving it again.

⁹If π were rational, $\sqrt{\pi}$ would be constructible and the quadrature of the circle would be possible.

¹⁰The two lengths are those of the arc AC and of the part AB of the tangent delimited by the line from the center of the circle to the end of the arc. This tangent is only equal to the modern tangent function when the radius is equal to 1.



¹¹See T. L. Heath, *The thirteen books of Euclid's elements*, Cambridge, 1908, volume 2,

But it should be noted that, whereas *Euclid* only applies it to integer and rationals, I will have to use it in a different way, when it has to be applied to quantities, of which it is still ignored whether they are rational or not. Here is therefore the procedure which fits the present case.

§. 4. Let the radius = 1, and v = any given arc.¹² And we will have the two well known infinite series¹³

$$\begin{aligned}\sin v &= v - \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}v^7 + \&c. \\ \cos v &= 1 - \frac{1}{2}v^2 + \frac{1}{2 \cdot 3 \cdot 4}v^4 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}v^6 + \&c.\end{aligned}$$

Since I will give in the sequel two series for the hyperbola which will only differ from these two series in that all signs are positive,¹⁴ I will defer until that point the proof of the progression of these series, and I will in fact only prove it so that nothing required by geometric rigour be omitted. It is sufficient to have warned the readers beforehand.

§. 5. Since

$$\tan v = \frac{\sin v}{\cos v},$$

we will have, substituting these two series, the fraction

$$\tan v = \frac{v - \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 - \&c.}{1 - \frac{1}{2}v^2 + \frac{1}{2 \cdot 3 \cdot 4}v^4 - \&c.}$$

For more brevity, I will set

$$\tan v = \frac{A}{B},$$

so that

$$\begin{aligned}A &= \sin v, \\ B &= \cos v.\end{aligned}$$

Here is now the procedure prescribed by *Euclid*.

§. 6. B is divided by A ; let the quotient = Q' , the remainder = R' . A is divided by R' ; let the quotient = Q'' , the remainder = R'' . R' is divided by

p. 298 [43].

¹² v is therefore the angle of the arc expressed in radians.

¹³These series were first obtained by Newton in 1669, but only published in 1711 (*Analysis per quantitatum series, fluxiones, ac differentias cum enumeratione linearum tertii ordinis*, London) [108]. See Katz, "The calculus of the trigonometric functions", *Historia mathematica* 14 (1987), 311-324 [88], and Ferraro, *The rise and development of the theory of series up to the early 1820s*, Springer, 2008 [49].

¹⁴See §. 73.

R'' ; let the quotient = Q''' , the remainder = R''' . R'' is divided by R''' ; let the quotient = Q^{IV} , the remainder = R^{IV} . &c. so that by continuing these divisions, one successively finds¹⁵

$$\begin{array}{llllll} \text{the quotients} & Q', Q'', Q''' & \dots & Q^n, Q^{n+1}, Q^{n+2} & \dots & \&c. \\ \text{the remainders} & R', R'', R''' & \dots & R^n, R^{n+1}, R^{n+2} & \dots & \&c. \end{array}$$

and it goes without notice that the exponents $n, n + 1, n + 2$ &c. only serve to indicate the position of the quotient or remainder. This being set, here is what must be proven.

§. 7. First, *not only can the division be continued forever, but the quotients follow a very simple law in that*¹⁶

$$\begin{aligned} Q' &= +1 : v, \\ Q'' &= -3 : v, \\ Q''' &= +5 : v, \\ Q^{IV} &= -7 : v, \&c. \end{aligned}$$

and in general

$$Q^n = \pm(2n - 1) : v,$$

where the + sign stands for odd n exponents,¹⁷ the - sign for even n exponents, and that therefore we will have for the tangent expressed by the arc the very simple continued fraction¹⁸

$$\tan v = \frac{1}{1 : v - \frac{1}{3 : v - \frac{1}{5 : v - \frac{1}{7 : v - \frac{1}{9 : v - \dots}}}}}}$$

§. 8. Second, *that the remainders R', R'', R''' &c will be expressed by the*

¹⁵ $B = AQ' + R', A = R'Q'' + R'', R' = R''Q''' + R'''$, etc.

¹⁶Lambert divides the power series, keeping the smallest degree for the quotient. The proofs are given in §. 9.

¹⁷The original article had mistakenly swapped “even” and “odd.”

¹⁸This is obtained in §. 14.

following series, whose progression laws are also very simple:¹⁹

$$\begin{aligned} R' &= -\frac{2}{2 \cdot 3}v^2 + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5}v^4 - \frac{6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}v^6 + \&c. \\ R'' &= -\frac{2 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5}v^3 + \frac{4 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}v^5 - \frac{6 \cdot 8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}v^7 + \&c. \\ R''' &= +\frac{2 \cdot 4 \cdot 6}{2 \cdot \dots \cdot 7}v^4 - \frac{4 \cdot 6 \cdot 8}{2 \cdot \dots \cdot 9}v^6 + \frac{6 \cdot 8 \cdot 10}{2 \cdot \dots \cdot 11}v^8 - \&c. \\ R^{iv} &= +\frac{2 \cdot 4 \cdot 6 \cdot 8}{2 \cdot \dots \cdot 9}v^5 - \frac{4 \cdot 6 \cdot 8 \cdot 10}{2 \cdot \dots \cdot 11}v^7 + \frac{6 \cdot 8 \cdot 10 \cdot 12}{2 \cdot \dots \cdot 13}v^9 - \&c. \end{aligned}$$

that is, the signs of the first terms change according to the quaternary sequence $- - + +$, and in general we will have²⁰

$$\begin{aligned} \pm R^n &= -\frac{2^n(1 \cdot 2 \cdot \dots \cdot n)}{1 \cdot 2 \cdot \dots \cdot (2n+1)}v^{n+1} + \frac{2^n(2 \cdot \dots \cdot (n+1))}{1 \cdot 2 \cdot \dots \cdot (2n+3)}v^{n+3} - \&c. \\ \pm R^{n+1} &= -\frac{2^{n+1}(1 \cdot 2 \cdot \dots \cdot (n+1))}{1 \cdot 2 \cdot \dots \cdot (2n+3)}v^{n+2} + \frac{2^{n+1}(2 \cdot \dots \cdot (n+2))}{1 \cdot 2 \cdot \dots \cdot (2n+5)}v^{n+4} - \&c. \\ \mp R^{n+2} &= +\frac{2^{n+2}(1 \cdot 2 \cdot \dots \cdot (n+2))}{1 \cdot 2 \cdot \dots \cdot (2n+5)}v^{n+3} - \frac{2^{n+2}(2 \cdot \dots \cdot (n+3))}{1 \cdot 2 \cdot \dots \cdot (2n+7)}v^{n+5} + \&c. \end{aligned}$$

§. 9. In order to give to the proof of these theorems all the needed brevity, let us consider that each remainder R^{n+2} is obtained by dividing by the remainder R^{n+1} , which precedes it immediately, the antepenultimate one

¹⁹See §. 13.

²⁰The original expressions contained several errors. I have slightly altered the second terms to make the expressions more understandable by comparison with those of §. 11. As observed by Speiser [96, p. 116], more general expressions are

$$\begin{aligned} R^{4n+1} &= -\frac{2^{4n+1}(1 \cdot 2 \cdot \dots \cdot (4n+1))}{1 \cdot 2 \cdot \dots \cdot (8n+3)}v^{4n+2} + \frac{2^{4n+1}(2 \cdot \dots \cdot (4n+2))}{1 \cdot 2 \cdot \dots \cdot (8n+5)}v^{4n+4} - \&c. \\ R^{4n+2} &= -\frac{2^{4n+2}(1 \cdot 2 \cdot \dots \cdot (4n+2))}{1 \cdot 2 \cdot \dots \cdot (8n+5)}v^{4n+3} + \frac{2^{4n+2}(2 \cdot \dots \cdot (4n+3))}{1 \cdot 2 \cdot \dots \cdot (8n+7)}v^{4n+5} - \&c. \\ R^{4n+3} &= +\frac{2^{4n+3}(1 \cdot 2 \cdot \dots \cdot (4n+3))}{1 \cdot 2 \cdot \dots \cdot (8n+7)}v^{4n+4} - \frac{2^{4n+3}(2 \cdot \dots \cdot (4n+4))}{1 \cdot 2 \cdot \dots \cdot (8n+9)}v^{4n+6} + \&c. \\ R^{4n+4} &= +\frac{2^{4n+4}(1 \cdot 2 \cdot \dots \cdot (4n+4))}{1 \cdot 2 \cdot \dots \cdot (8n+9)}v^{4n+5} - \frac{2^{4n+4}(2 \cdot \dots \cdot (4n+5))}{1 \cdot 2 \cdot \dots \cdot (8n+11)}v^{4n+7} + \&c. \end{aligned}$$

This can be abridged as

$$R^n = (-1)^{n(n+1)/2} 2^n \sum_{m=0}^{\infty} (-1)^m \frac{(n+m)!}{m!(2n+2m+1)!} v^{n+2m+1}$$

which was given by Popken in 1948 [119], apart from a slight change of notation.

R^n .²¹ This consideration entails that the proof can be split in two parts. In the first, we must show that *if two remainders R^n , R^{n+1} , which are in immediate sequence, have the form I assigned to them, the remainder R^{n+2} which follows immediately shall have the same form.* Once this is proven, it only remains to show in the second part of the proof, *that the form of the first two remainders is the one they must have.* For, in that manner, it is obvious that the form of all the following ones will be established like by itself.²²

§. 10. Let us start to divide the first term of the remainder R^n by the first term of the remainder R^{n+1} , in order to obtain the quotient

$$\begin{aligned} Q^{n+2} &= \frac{2^n(1 \cdot 2 \cdot 3 \cdots n)}{1 \cdot 2 \cdot 3 \cdots (2n+1)} v^{n+1} : \frac{2^{n+1}(1 \cdot 2 \cdot 3 \cdots (n+1))}{1 \cdot 2 \cdot 3 \cdots (2n+3)} v^{n+2} \\ &= 1 : \frac{2(n+1)v}{(2n+2) \cdot (2n+3)} = (2n+3) : v. \end{aligned}$$

And it is clear²³ that the remainder R^{n+1} being multiplied by this quotient

$$Q^{n+2} = (2n+3) : v,$$

and the product being subtracted from the remainder R^n , there must remain the remainder R^{n+2} .

§. 11. But in order to avoid having to do this operation separately for each term and consequently in order to restrict ourselves to a mere induction, let us take the general term of each of the series which express the remainders R^n , R^{n+1} , R^{n+2} , so that by taking the m -th term²⁴ of the remainders R^n , R^{n+1} , we take the $(m-1)$ -th term of the remainder R^{n+2} . This being set,

²¹In other words, $R^n = R^{n+1}Q^{n+2} + R^{n+2}$.

²²The above three expressions are not totally general, in that R^{n+3} is not given, and the signs complicate the induction. A more general proof is given in the next paragraph, but it is not totally complete.

²³This is in fact not totally clear, and is only clarified in the next paragraph. In general, as mentioned above (§. 7.), the sign of Q^{n+2} is alternating: $B(+)$ $\xrightarrow{1/v}$ $A(+)$ $\xrightarrow{-3/v}$ $R'(-)$ $\xrightarrow{5/v}$ $R''(-)$ $\xrightarrow{-7/v}$ $R'''(+)$ $\xrightarrow{9/v}$ $R^{IV}(+)$ $\xrightarrow{-11/v}$ $R^V(-)$ $\xrightarrow{13/v}$ $R^{VI}(-)$ $\xrightarrow{-15/v}$ $R^{VII}(+)$ $\xrightarrow{17/v}$ $R^{VIII}(+)$ etc., where the signs of the first terms are given between parentheses and the values of Q' , Q'' , Q''' , Q^{IV} , etc., are given over the arrows.

²⁴This is because of $R^n = R^{n+1}Q^{n+2} + R^{n+2}$, because $Q^{n+2} = (2n+3)/v$ and because of the expressions of R^n , R^{n+1} and R^{n+2} seen above.

these terms will be²⁵

$$\begin{aligned}\pm r^n &= -\frac{2^n(m \cdot (m+1) \cdot (m+2) \cdots (n+m-1))v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+2m-1)} \\ \pm r^{n+1} &= -\frac{2^{n+1}(m \cdot (m+1) \cdot (m+2) \cdots (n+m))v^{n+2m}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+2m+1)} \\ \pm r^{n+2} &= -\frac{2^{n+2}((m-1) \cdot m \cdot (m+1) \cdots (n+m))v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+2m+1)}\end{aligned}$$

But, since we must have

$$r^n - r^{n+1} \cdot (2n+3) : v = r^{n+2},$$

and that we have indeed²⁶

$$\begin{aligned}r^n - r^{n+1}(2n+3) : v &= -\frac{2^n \cdot (m \cdots (n+m-1))v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdots (2n+2m-1)} \\ &\quad + \frac{2^{n+1} \cdot (m \cdots (n+m))v^{n+2m}}{1 \cdot 2 \cdot 3 \cdots (2n+2m+1)} \cdot \frac{2n+3}{v} \\ &= \frac{2^n \cdot (m \cdots (n+m-1))}{1 \cdot 2 \cdots (2n+2m-1)} v^{n+2m-1} \cdot \left(-1 + \frac{2 \cdot (n+m) \cdot (2n+3)}{(2n+2m) \cdot (2n+2m+1)} \right) \\ &= -\frac{2^n \cdot (m \cdots (n+m-1))}{1 \cdot 2 \cdots (2n+2m-1)} v^{n+2m-1} \cdot \frac{(2m-2) \cdot (2n+2m)}{(2n+2m) \cdot (2n+2m+1)} \\ &= -\frac{2^{n+2} \cdot ((m-1) \cdot m \cdot (m+1) \cdots (n+m))v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdots (2n+2m+1)},\end{aligned}$$

and therefore

$$= \pm r^{n+2}.$$

It appears that, given the form that I gave to the remainders R^n , R^{n+1} , the remainder R^{n+2} will have the same form.²⁷ It therefore only remains to check the form of the first two remainders R' , R'' , in order to establish what this first part of our proof had admitted as true as an hypothesis. And this will be the second part of the proof.

²⁵The first terms of R^n and R^{n+1} correspond to $m = 1$, but the first term of R^{n+2} corresponds to $m = 2$. Lambert therefore associates R_{m+1}^n , R_{m+1}^{n+1} and R_m^{n+2} . The expressions are given here so that the terms can be associated with the same value of m . The original expressions contained errors.

²⁶The original expressions contained errors.

²⁷However, the remainders do not always appear in this order of signs, and the sign of Q^{n+2} also changes. The above has made the implicit assumption that Q^{n+2} is always positive. There are only two configurations. Either the signs of r^n , r^{n+1} and r^{n+2} are

§. 12. For that purpose, let us recall that the first remainder R' is the one remaining when dividing²⁸

$$\cos v = 1 - \frac{1}{2}v^2 + \frac{1}{2 \cdot 3 \cdot 4}v^4 \dots \frac{1}{1 \dots m}v^m \dots \&c.$$

by

$$\sin v = v - \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 \dots \frac{1}{1 \dots (m+1)}v^{m+1} \dots \&c.$$

But the quotient²⁹ resulting from the division of the first term, being $= 1 : v$, we see that we will have

$$R' = \cos v - \frac{1}{v} \cdot \sin v.$$

Multiplying thus the general term of the divider,

$$\pm \frac{1}{1 \cdot 2 \dots (m+1)}v^{m+1},$$

by $1 : v$, and subtracting the product

$$\pm \frac{1}{1 \cdot 2 \dots (m+1)} \cdot v^m,$$

from the general term of the dividend

$$\pm \frac{1}{1 \cdot 2 \dots m} \cdot v^m,$$

we will have the general term of the first remainder R'

$$r' = \pm \frac{m \cdot v^m}{1 \dots (m+1)}.$$

identical, or they are alternating. The second configuration is that of

$$\begin{aligned} \pm r^n &= + \frac{2^n(m \cdot (m+1) \cdot (m+2) \dots (n+m-1))v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \dots (2n+2m-1)} \\ \pm r^{n+1} &= - \frac{2^{n+1}(m \cdot (m+1) \cdot (m+2) \dots (n+m))v^{n+2m}}{1 \cdot 2 \cdot 3 \cdot 4 \dots (2n+2m+1)} \\ \pm r^{n+2} &= + \frac{2^{n+2}((m-1) \cdot m \cdot (m+1) \dots (n+m))v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \dots (2n+2m+1)} \end{aligned}$$

We then have $Q^{n+2} = -(2n+3) : v$ and it is easy to verify that we have $r^n - r^{n+1}Q^{n+2} = r^n + r^{n+1} \cdot (2n+3) : v = r^{n+2}$, the computation being almost identical to that of §. 11.

²⁸See §. 5.

²⁹Hence $Q' = \frac{1}{v}$.

$(m + 1)$ being always an odd number, m will be an even number, and the first remainder will be

$$R' = -\frac{2}{2 \cdot 3}v^2 + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5}v^4 - \frac{6}{2 \cdot \dots \cdot 7}v^6 + \&c.$$

as we did assume it.³⁰

§. 13. The second remainder R'' results from the division of

$$\sin v = v - \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 - \&c. \dots \pm \frac{1}{1 \cdot 2 \cdot \dots \cdot (m-1)}v^{m-1}$$

by the first remainder that we just obtained³¹

$$R' = -\frac{2}{2 \cdot 3}v^2 + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5}v^4 - \frac{6}{2 \cdot \dots \cdot 7}v^6 + \dots \mp \frac{mv^m}{1 \cdot \dots \cdot (m+1)}$$

However the quotient resulting from the division from the first term, being $= -3 : v$, we see that it will be³²

$$R'' = \sin v - \left(-\frac{3}{v}\right) \cdot R'.$$

Multiplying thus the general term of the divisor

$$\mp \frac{mv^m}{1 \cdot \dots \cdot (m+1)},$$

by $-3 : v$, and subtracting the product

$$\pm \frac{3mv^{m-1}}{1 \cdot \dots \cdot (m+1)},$$

from the general term of the dividend

$$\pm \frac{1}{1 \cdot \dots \cdot (m-1)}v^{m-1},$$

the general term from the second remainder will be

$$\begin{aligned} r'' &= \pm \frac{v^{m-1}}{1 \cdot \dots \cdot (m-1)} \mp \frac{3mv^{m-1}}{1 \cdot \dots \cdot (m+1)} \\ &= \pm \frac{(m-2) \cdot m \cdot v^{m-1}}{1 \cdot \dots \cdot (m+1)}. \end{aligned}$$

³⁰See §. 8.

³¹Here \mp means that the first, third, fifth, etc., terms are negative. The original expression contained an error.

³²The original expression has a sign error. This error was apparently not corrected by Speiser [96, p. 119].

Substituting thus the even numbers for m , we will have the second remainder³³

$$R'' = -\frac{2 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5}v^3 + \frac{4 \cdot 6}{2 \cdot \dots \cdot 7}v^5 - \frac{6 \cdot 8}{2 \cdot \dots \cdot 9}v^7 + \&c.$$

again as we have assumed it.³⁴ Consequently, the form of the first two remainders being proven, it follows, from the first part of our proof, that the form of all the following remainders is also proven.³⁵

§. 14. It is now no longer necessary to prove separately the progression law of the quotients Q' , Q'' , Q''' &c.³⁶ Indeed, the law of the remainders being proven, it is also proven that any quotient will be (§. 10)³⁷

$$\pm Q^{n+2} = (2n + 3) : v,$$

which, by virtue of the theory of continued fractions,³⁸ gives

$$\tan v = \frac{1}{1 : v - 1} \cfrac{3 : v - 1}{5 : v - 1} \cfrac{7 : v - 1}{9 : v - 1} \cfrac{11 : v - 1}{\&c.}$$

³³ \pm meaning that the first term is positive, but there being no term in v , we start with a negative sign.

³⁴See §. 8.

³⁵This is not totally true, because the first part of the proof is not complete.

³⁶A more rigorous proof would have included the general form of the quotients, with their sign, in the induction. Right now, Lambert has only provided an incomplete induction. Once this proof is complete, the law of the quotients follows from the law of the remainders, but the latter needs the law of the quotients. . .

³⁷Here Q^{n+2} is positive for $n = 1, 3, 5$, etc.

³⁸We have $\frac{A}{B} = \tan v$, $\frac{B}{A} = Q' + \frac{R'}{A}$, $\frac{A}{R'} = Q'' + \frac{R''}{R'}$, $\frac{R'}{R''} = Q''' + \frac{R'''}{R''}$, etc., from which it follows that $\tan v = \frac{1}{Q' + \frac{1}{Q'' + \frac{1}{Q''' + \dots}}}$, which corresponds to the given expression, when the

signs are factored out :

$$\tan v = \frac{1}{1 : v - 1} \cfrac{3 : v - 1}{5 : v - 1} \cfrac{7 : v - 1}{9 : v - 1} \cfrac{11 : v - 1}{\&c.} = \frac{1}{1 : v + 1} \cfrac{-3 : v + 1}{5 : v + 1} \cfrac{-7 : v + 1}{9 : v + 1} \cfrac{-11 : v + 1}{\&c.}$$

This does of course assume that the continued fraction converges.

from which we see at the same time, that *whenever the arc v is equal to an aliquot part of the radius, all these quotients will be integers increasing in an arithmetic progression.*³⁹

And this is what must be observed, because in *Euclid's* theorem mentioned above (§. 3.) all the quotients are assumed to be integers. Hence up to now the method prescribed by *Euclid* will be applicable to all these cases where the arc v is an aliquot part of the radius. But, again in these cases, there is another circumstance that must be observed.

§. 15. The problem proposed by *Euclid* is *to find the greatest common divisor of two integer numbers, which are not mutually prime.* This problem can be solved whenever one of the remainders R', R'', R''' &c. $\dots R^n$ becomes $= 0$, the previous remainder R^{n-1} being different from unity, which case, according to the 1st Proposition⁴⁰ of the same book only occurs when the two given numbers are mutually prime, assuming of course that all quotients Q', Q'', Q''' &c. are integers. But we have just seen that the latter assumption is true in the present case, whenever $\frac{1}{v}$ is an integer. But, regarding the remainders R', R'', R''' &c., none is becoming $= 0$.⁴¹ On the contrary, considering the law of progression of the remainders that we have found, we see that they not only decrease without interruption, but that they even decrease more than any geometrical progression.⁴² Although this continues for ever, we will nevertheless be able to apply *Euclid's* proposition. Indeed, by virtue of this proposition, *the greatest common divisor of A, B , is at the same time the greatest common divisor of all the remainders R', R'', R''' &c.*⁴³ But since these remainders decrease in such a way that they become smaller than any assignable amount, it follows that *the greatest common divisor of A, B , is smaller than any assignable amount*; this means that there is none, and that therefore A, B , being incommensurable quantities,

$$\tan v = \frac{A}{B}$$

*will be an irrational quantity whenever the arc v will be an aliquot part of the radius.*⁴⁴

³⁹An aliquot part of an integer is a proper divisor of that integer. In other words, the radius being 1 and the arc being v (in radians), if v is an aliquot part of the radius, $1 : v$ is an integer, and so are $3 : v, 5 : v$, etc.

⁴⁰See Heath 1908, volume 2, p. 296 [43].

⁴¹It is easy to see that the series are alternating series and that the terms are decreasing. Their sum can therefore not be equal to 0.

⁴²If $v < 1$, $|R^n| \leq \frac{2^n}{(n+1)\dots(2n+1)} \leq \frac{2^n}{n!}$.

⁴³The manuscript had R, R', R'' .

⁴⁴For instance, $\tan(1), \tan(1/2), \tan(1/3)$, etc., are irrational, the angles being expressed in radians.

§. 16. This is therefore the extent to which we can make use of *Euclid's* proposition. We must now extend it to all cases where the arc v is commensurable with the radius. To this effect, and in order to prove a few other theorems, I will consider again the continued fraction

$$\tan v = \frac{1}{1 : v - 1} \frac{1}{\frac{3 : v - 1}{5 : v - 1} \frac{1}{7 : v - 1} \&c.}$$

and by letting $1 : v = w$, I will transform it into

$$\tan v = \frac{1}{w - 1} \frac{1}{\frac{3w - 1}{5w - 1} \frac{1}{7w - 1} \&c.}$$

§. 17. *But, by keeping as many quotients w , $3w$, $5w$ &c. as desired, one will have merely to simplify them, in order to have fractions expressing the tangent of v the more precisely that a greater number of the quotients were retained.* So, it is for instance by retaining 1, 2, 3, 4 &c. quotients that we obtain the fractions

$$\frac{1}{w}, \frac{3w}{3w^2 - 1}, \frac{15w^2 - 1}{15w^3 - 6w}, \frac{105w^3 - 10w}{105w^4 - 45w^2 + 1},$$

§. 18. *But, in order to do all these simplifications in order, & at the same time in order to prove the progression law followed by these fractions, we will first set*

$$\tan v = \frac{1}{w - a} = \frac{1}{w - 1} \frac{1}{\frac{3w - a'}{5w - a''}} = \frac{1}{w - 1} \frac{1}{\frac{3w - 1}{5w - a''}} = \&c.$$

expressing by a , a' , a'' , $a''' \dots a^n$, a^{n+1} , a^{n+2} , \dots &c. the quantities resulting from the quotients that we wish to be omitted, so that in order to omit them, it will suffice to set a , a' , a'' , \dots a^n &c. = 0.

§. 19. Now, I say, *that by setting $a^n = 0$, the fraction resulting from the*

simplification of the quotients retained, will have the form⁴⁵

$$\tan v = \frac{A - ma^{n-1}}{B - pa^{n-1}},$$

in which m, p, A, B do not contain a^{n-1} . Let us first assume this form to be the true one, and we will easily prove that by retaining an additional quotient, the fraction resulting from the reduction will have the same form. Indeed since we have⁴⁶

$$a^{n-1} = \frac{1}{(2n+1)w - a^n},$$

it will suffice to substitute this value in the proposed form, and it will be transformed into

$$\tan v = \frac{A(2n+1)w - m - A \cdot a^n}{B(2n+1)w - p - B \cdot a^n},$$

Since this is the same form, it will be sufficient to show that it is true for the member a' , because it will then be true for all following members $a'', a''', a^{IV} \dots$ &c. But for the member a' it is

$$\tan v = \frac{1}{\frac{w-1}{3w-a'}}$$

which after reduction gives

$$\tan v = \frac{3w - a'}{3w^2 - 1 - wa'},$$

the form as we assumed it.⁴⁷

§ 20. Now that we have found

$$\tan v = \frac{A - ma^{n-1}}{B - pa^{n-1}}$$

⁴⁵In §. 19-21, a number of occurrences of n have been replaced by $n-1$. This was considered the best means to fall back on correct results. Speiser also made amendments in 1948, but different ones [96]. Contrary to the text edited in 1948, I did not have to alter other sections such as §. 33 and §. 34.

⁴⁶The original article had mistakenly written

$$a^n = \frac{1}{(2n+1)w - a^{n+1}},$$

but this had no consequences, as the correct fractions were used in §.22.

⁴⁷It is actually already true for a , since $\tan v = \frac{1}{w-a}$ is also of that form.

$$\tan v = \frac{A(2n+1)w - m - A \cdot a^n}{B(2n+1)w - p - B \cdot a^n},$$

let us substitute also for a^n its value

$$a^n = \frac{1}{(2n+3)w - a^{n+1}},$$

and we will have

$$\tan v = \frac{[A(2n+1)w - m] \cdot (2n+3)w - A - [A(2n+1)w - m] \cdot a^{n+1}}{[B(2n+1)w - p] \cdot (2n+3)w - B - [B(2n+1)w - p] \cdot a^{n+1}}$$

§. 21. So, by letting in each of these three values of $\tan v$, the members a^{n-1} , a^n , a^{n+1} equal to zero, we will obtain the general form of the fractions that we have to find.⁴⁸

$$\begin{aligned} & \frac{A}{B}, \\ & \frac{A(2n+1)w - m}{B(2n+1)w - p}, \\ & \frac{[A(2n+1)w - m] \cdot (2n+3)w - A}{[B(2n+1)w - p] \cdot (2n+3)w - B}. \end{aligned}$$

These three fractions being for the omission of a^{n-1} , a^n , a^{n+1} , they come in sequence, and it is easy to see *that the third can be obtained using the previous two, so that its numerator and its denominator can be computed separately.*⁴⁹ This follows because the numerator of the second fraction must be multiplied by the quotient corresponding to a^n , and from the product one subtracts the numerator from the first fraction. The remainder will be the numerator from the third fraction. Its denominator is obtained similarly using the denominators of the two previous fractions.

§. 22. *Now, in order to obtain the fractions themselves*, it suffices to write the quotients in three columns, with the numerators and denominators of the first two fractions (§. 17.) and the following numerators and denominators will be obtained by the simple operation we have just indicated. Here is the

⁴⁸Note incidentally that the fraction preceding $\frac{A}{B}$ is $\frac{m}{p}$. This will be used again in §.33.

⁴⁹By using these fractions, Lambert dispenses with the values of m and p .

model⁵⁰

Quotients	numerators	denominators
	1	w
$5w$	$3w$	$3w^2 - 1$
$7w$	$15w^2 - 1$	$15w^3 - 6w$
$9w$	$105w^3 - 10w$	$105w^4 - 45w^2 + 1$
$11w$	$945w^4 - 105w^2 + 1$	$945w^5 - 420w^3 + 15w$
&c.	$10395w^5 - 1260w^3 + 21w$ &c.	$10395w^6 - 4725w^4 + 210w^2 - 1$ &c.

This gives the fractions

$$\frac{1}{w}, \frac{3w}{3w^2 - 1}, \frac{15w^2 - 1}{15w^3 - 6w}, \frac{105w^3 - 10w}{105w^4 - 45w^2 + 1} \text{ \&c.}$$

of which each one is expressing the tangent of v better than those which precede it.

§. 23. However, although by means of the rule we have just given (§. 21.), each of these fractions can be obtained from the two which precede it immediately, *it will be convenient, in order to avoid a kind of induction, to give and to prove the general expression.* Let us first observe that the coefficients of each vertical column obey a very simple law in that its factors are partly figured numbers and partly odd numbers. Here they are resolved

Fraction	Quotient	Denominator
1st		w
2nd	$5w$	$3 \cdot w^2 - 1 \cdot 1$
3rd	$7w$	$3 \cdot 5 \cdot w^3 - 2 \cdot 3w$
4th	$9w$	$3 \cdot 5 \cdot 7w^4 - 3 \cdot 3 \cdot 5w^2 + 1 \cdot 1$
5th	$11w$	$3 \cdot \dots \cdot 9w^5 - 4 \cdot 3 \cdot 5 \cdot 7w^3 + 3 \cdot 5w$
6th	$13w$	$3 \cdot \dots \cdot 11w^6 - 5 \cdot 3 \cdot \dots \cdot 9w^4 + 6 \cdot 5 \cdot 7w^2 - 1 \cdot 1$
7th	$15w$	$3 \cdot \dots \cdot 13w^7 - 6 \cdot 3 \cdot \dots \cdot 11w^5 + 10 \cdot 5 \cdot 7 \cdot 9w^3 - 4 \cdot 7w$
&c.	&c.	&c.

⁵⁰The quotients represent the values of $2n + 3$ for $n = 1, 2, \dots$. So we have

$$\begin{aligned} 5w \times 3w - 1 &= 15w^2 - 1 \\ 5w \times (3w^2 - 1) - w &= 15w^3 - 6w \\ 7w \times (15w^2 - 1) - 3w &= 105w^3 - 10w, \text{ etc.} \end{aligned}$$

Fraction	Quotient	Numerator
1st		1
2nd	$5w$	$3 \cdot w$
3rd	$7w$	$3 \cdot 5 \cdot w^2 - 1 \cdot 1$
4th	$9w$	$3 \cdot 5 \cdot 7w^3 - 2 \cdot 5w$
5th	$11w$	$3 \cdots 9w^4 - 3 \cdot 5 \cdot 7w^2 + 1 \cdot 1$
6th	$13w$	$3 \cdots 11w^5 - 4 \cdot 5 \cdot 7 \cdot 9w^3 + 3 \cdot 7w$
7th	$15w$	$3 \cdots 13w^7 - 5 \cdot 5 \cdots 11w^4 + 6 \cdot 7 \cdot 9w^2 - 1 \cdot 1$
&c.	&c.	&c.

§. 24. *This observation makes it easier for us to find the general expression for any of the fractions.* Let the n -th fraction be given, and we will have its Denominator

$$\begin{aligned}
&= w^n [1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)] - \frac{w^{n-2}}{2} \cdot [(2n-2) \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3)] \\
&+ \frac{w^{n-4}}{2 \cdot 3 \cdot 4} \cdot [(2n-4) \cdot (2n-6) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-5)] \\
&- \frac{w^{n-6}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot [(2n-6) \cdot (2n-8) \cdot (2n-10) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-7)] \\
&+ \frac{w^{n-8}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdot [(2n-8)(2n-10)(2n-12)(2n-14) \cdots 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-9)] \\
&- \&c.
\end{aligned}$$

Numerator

$$\begin{aligned}
&= w^{n-1} [1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)] - \frac{w^{n-3}}{2 \cdot 3} \cdot [(2n-4) \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3)] \\
&+ \frac{w^{n-5}}{2 \cdot 3 \cdot 4 \cdot 5} \cdot [(2n-6) \cdot (2n-8) \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-5)] \\
&- \frac{w^{n-7}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \cdot [(2n-8) \cdot (2n-10) \cdot (2n-12) \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-7)] \\
&+ \frac{w^{n-9}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \cdot [(2n-10)(2n-12)(2n-14)(2n-16) \cdots 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-9)] \\
&- \&c.
\end{aligned}$$

It therefore only remains to prove the universality of these expressions.

§. 25. This will be done in that by admitting this form for the n -th fraction, we deduce the forms for the $(n-1)$ -th and $(n-2)$ -th, by substituting $(n-1)$, $(n-2)$ for n . Then, one proceeds in accordance with the rule of §. 21. by deducing both the denominator and the numerator of the n -th

fraction, from those of the two preceding fractions as we have found them using the first operation. And by that, we must reproduce the form of the n -th fraction, as we have given it. It is clear that this procedure leads to establish that if two fractions which are in immediate sequence have this form, the one which follows them will also have this form, and that consequently, since the fractions from the previous table, which are the first ones, have this form, it will follow that all the following ones will also have this form.

§. 26. So, if in order to abridge this proof, we want to restrict ourselves to the general term, it will nevertheless be necessary to compute separately the one of the numerator and the one of the denominator, be it only to simplify the computation. Indeed, both will be computed using the same rule (§. 21.). Let us begin with the *denominator*, and in taking the m -th term of its general expression for the n -th fraction, it will also be necessary to take the m -th term for the $(n - 1)$ -th fraction, but we will only take the $(m - 1)$ -th term for the $(n - 2)$ -th fraction. We see that we must do so with respect to the dimensions or exponents of the letter w .

§. 27. Now, the m -th term of the n -th fraction for the denominator is

$$M = \frac{w^{n-2m+2} \cdot [(2n - 2m + 2) \cdot (2n - 2m) \cdot (2n - 2m - 2) \cdots (2n - 4m + 6)] \cdot [1 \cdot 3 \cdot 5 \cdots (2n - 2m + 1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2m - 2)}$$

from which, by substituting $(n - 1)$ for n , we find the m -th term of the $(n - 1)$ -th fraction

$$M' = \frac{w^{n-2m+1} \cdot [(2n - 2m) \cdot (2n - 2m - 2) \cdots (2n - 4m + 4)] \cdot [1 \cdot 3 \cdot 5 \cdots (2n - 2m - 1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2m - 2)}$$

And by substituting $(n - 2)$ for n , and $(m - 1)$ for m , we find the $(m - 1)$ -th term of the $(n - 2)$ -th fraction.⁵¹

$$-M'' = \frac{w^{n-2m+2} \cdot [(2n - 2m) \cdot (2n - 2m - 2) \cdots (2n - 4m + 6)] \cdot [1 \cdot 3 \cdot 5 \cdots (2n - 2m - 1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2m - 4)}$$

But, by the rule of §. 21. we must have⁵²

$$M = (2n - 1)w \cdot M' - M''$$

⁵¹For consistency, I assume that M , M' , and M'' include the signs. The negative sign of M'' was absent from the original article. M and M' have the same sign, and M'' is of opposite sign.

⁵²That is in fact $M = (2n - 1)w \cdot M' + (-M'')$.

and consequently we will be able to remove from these three expressions all their common factors by setting them = P . From that we will have

$$\begin{aligned} +M &= \frac{P \cdot w \cdot (2n - 2m + 2) \cdot (2n - 2m + 1)}{(2m - 2) \cdot (2m - 3)} \\ +M' &= \frac{P \cdot (2n - 4m + 4)}{(2m - 2) \cdot (2m - 3)} \\ -M'' &= P \cdot w. \end{aligned}$$

By setting

$$\frac{P}{(2m - 2) \cdot (2m - 3)} = Q,$$

we will have⁵³

$$\begin{aligned} +M &= Qw \cdot (2n - 2m + 2) \cdot (2n - 2m + 1) \\ +M' &= Q \cdot (2n - 4m + 4) \\ -M'' &= Qw \cdot (2m - 2) \cdot (2m - 3). \end{aligned}$$

From that, by multiplying, we will have

$$\begin{aligned} (2n - 1)wM' &= Qw \cdot (4n^2 - 8mn + 6n + 4m - 4) \\ -M'' &= Qw \cdot (4m^2 - 10m + 6) \end{aligned}$$

therefore

$$(2n - 1)wM' - M'' = Qw(4n^2 - 8nm + 6n + 4m^2 - 6m + 2).$$

But we also have

$$M = Qw \cdot (2n - 2m + 2)(2n - 2m + 1) = Qw(4n^2 - 8nm + 6n + 4m^2 - 6m + 2).$$

Since these two values are the same, it follows that

$$M = (2n - 1)w \cdot M' - M'',$$

and that consequently the form, which we gave to the general term is such as it should be.⁵⁴

§. 28. Let us now move to the *numerator*. The m -th term of the numerator of the n -th fraction must be

$$+N = \frac{w^{n-2m+1} \cdot [(2n - 2m) \cdot (2n - 2m - 2) \cdots (2n - 4m + 4)] \cdot [1 \cdot 3 \cdot 5 \cdots (2n - 2m + 1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2m - 1)}$$

⁵³There were errors in the original expressions.

⁵⁴In fact, the above proof is applicable only to the case $m > 1$. The case $m = 1$ is easily checked separately and in that case $M'' = 0$.

from which, by substituting $(n - 1)$ for n , we find the same m -th term as for the $(n - 1)$ -th fraction,

$$+N' = \frac{w^{n-2m} \cdot [(2n - 2m - 2) \cdot (2n - 2m - 4) \cdots (2n - 4m + 2)] \cdot [1 \cdot 3 \cdot 5 \cdots (2n - 2m - 1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2m - 1)}$$

And by substituting $(n - 2)$, $(m - 1)$, for n , m , we will have the $(m - 1)$ -th term of the $(n - 2)$ -th fraction,

$$-N'' = \frac{w^{n-2m+1} \cdot [(2n - 2m - 2) \cdot (2n - 2m - 4) \cdots (2n - 4m + 4)] \cdot [1 \cdot 3 \cdot 5 \cdots (2n - 2m - 1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2m - 3)}$$

Therefore, setting the common factors to these three expressions = P , we will have

$$\begin{aligned} +N &= \frac{Pw \cdot (2n - 2m) \cdot (2n - 2m + 1)}{(2m - 1) \cdot (2m - 2)} \\ +N' &= \frac{P \cdot (2n - 4m + 2)}{(2m - 1) \cdot (2m - 2)} \\ -N'' &= Pw, \end{aligned}$$

and by setting $P = Q \cdot (2m - 1) \cdot (2m - 2)$, we will have

$$\begin{aligned} +N &= Qw \cdot (2n - 2m) \cdot (2n - 2m + 1) \\ +N' &= Q \cdot (2n - 4m + 2) \\ -N'' &= Qw \cdot (2m - 1) \cdot (2m - 2) \end{aligned}$$

But we must have

$$N = (2n - 1)w \cdot N' - N'',$$

therefore, by substituting the values found, we will have

$$\begin{aligned} (2n - 1)wN' &= Qw \cdot (4nn - 8nm + 2n + 4m - 2) \\ -N'' &= Qw(4m^2 - 6m + 2), \end{aligned}$$

hence

$$N = (2n - 1)wN' - N'' = Qw(4n^2 - 8nm + 2n + 4m^2 - 2m).$$

But the same value results from

$$N = (2n - 2m) \cdot (2n - 2m + 1) \cdot Qw.$$

It follows from that that the form of the general term is as it should be.⁵⁵

§. 29. Let us take again the general expressions given in §. 24. and let us divide the one of the denominator by its first term, and we will have the series

$$1 - \frac{w^{-2}}{2} \cdot \frac{2n-2}{2n-1} + \frac{w^{-4}}{2 \cdot 3 \cdot 4} \cdot \frac{(2n-4) \cdot (2n-6)}{(2n-1) \cdot (2n-3)} - \frac{w^{-6}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{(2n-6) \cdot (2n-8) \cdot (2n-10)}{(2n-1)(2n-3)(2n-5)} \\ + \frac{w^{-8}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdot \frac{(2n-8) \cdot (2n-10) \cdot (2n-12) \cdot (2n-14)}{(2n-1) \cdot (2n-3) \cdot (2n-5) \cdot (2n-7)} - \&c.$$

which, by substituting⁵⁶ $v = w^{-1}$, and by setting $n = \infty$, gives

$$1 - \frac{v^2}{2} + \frac{v^4}{2 \cdot 3 \cdot 4} - \frac{v^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c.$$

which is the cosine of v , and consequently the denominator of which we made use (§. 5.) to find the quotients w , $3w$ &c.

§. 30. Let us moreover divide the general expression of the numerator (§. 24.) by the same first term of the denominator, and we will have the series

$$w^{-1} - \frac{w^{-3}}{2 \cdot 3} \cdot \frac{2n-4}{2n-1} + \frac{w^{-5}}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{(2n-6) \cdot (2n-8)}{(2n-1) \cdot (2n-3)} \\ - \frac{w^{-7}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \cdot \frac{(2n-8) \cdot (2n-10) \cdot (2n-12)}{(2n-1) \cdot (2n-3) \cdot (2n-5)} \\ + \&c.$$

which gives for $n = \infty$ the series

$$v - \frac{1}{2 \cdot 3} v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} v^5 - \&c.$$

which is $= \sin v$, and consequently the numerator, which was used in §. 5.

§. 31. It also follows, *that, no matter how big the first term of the two general formulæ (§. 24.) is, the second term, and even more the following ones, will not only be smaller, but even smaller than the $\frac{1}{2}$, $\frac{1}{2 \cdot 3}$, $\frac{1}{2 \cdot 3 \cdot 4}$ &c. part of the first term.* But, by substituting for n successively 1, 2, 3, 4 &c. *ad infinitum*, the first term, being the product of as many of odd numbers $1 \cdot 3 \cdot 5 \cdot 7$ &c. will increase more than any increasing geometric sequence; it is also clear that, although the 2nd, 4th, 6th &c. terms are subtractive, this does not

⁵⁵Like in §. 27., this proof is only applicable to the case $m > 1$, and the case $m = 1$ can be checked separately.

⁵⁶We had $w = 1/v$, see §. 16.

prevent the sum of the terms to increase faster than any increasing geometric sequence.⁵⁷ And this is what I observe here, because I will make use of it in the sequel of this Memoir. Here is a first one, that presents itself.

§. 32. The object is to find the law, according to which the fractions

$$\frac{1}{w}, \frac{3w}{3w^2 - 1}, \frac{15w^2 - 1}{15w^3 - 6w}, \&c.$$

approach the value of the tangent. To this effect, we will only need to subtract each one from the one which follows, and the remainders will be

$$\frac{1}{w \cdot (3w^2 - 1)}, \frac{1}{(3w^2 - 1) \cdot (15w^3 - 6w)}, \&c.$$

These remainders show how much each fraction is greater than the one preceding it. Let us show in general that all the numerators are = 1, and that all the denominators are the product of those of the two fractions of which these remainders give the difference.

§. 33. To this effect, we will take again the three general formulæ given at §. 21. and which are

$$\begin{aligned} & \frac{A}{B}, \\ & \frac{A(2n+1)w - m}{B(2n+1)w - p}, \\ & \frac{[A(2n+1)w - m](2n+3)w - A}{[B(2n+1)w - p](2n+3)w - B}. \end{aligned}$$

But, subtracting the first from the second, the remainder will be

$$= \frac{Ap - Bm}{B \cdot [B(2n+1)w - p]}.$$

But the numerator of this remainder is the same which results from the subtraction

$$\frac{A}{B} - \frac{m}{p} = \frac{Ap - Bm}{B \cdot p}.$$

$\frac{m}{p}$ being the fraction preceding the fraction $\frac{A}{B}$, it is clear that the numerator of all these remainders is the same, and that the denominator is the product

⁵⁷This does easily follow because of Lambert's implicit assumption $w \geq 1$, already mentioned above.

of those of the fractions, of which the remainders give the difference. So, when starting with any of the fractions $\frac{m}{p}$, the remainders will be⁵⁸

$$\frac{1}{p \cdot B}, \frac{1}{B[B(2n+1)w-p]} \text{ \&c.}$$

§. 34. Let us now observe *that all these remainders being added to the first fraction, which is taken as the base, the sum will always express the tangent of v , so that in general we will have*⁵⁹

$$\tan v = \frac{m}{p} + \frac{1}{p \cdot B} + \frac{1}{B \cdot [B(2n+1)w-p]} + \text{\&c.}$$

and consequently

$$\tan v = \frac{1}{w} + \frac{1}{w(3w^2-1)} + \frac{1}{(3w^2-1) \cdot (15w^3-6w)} + \text{\&c.}$$

$$\tan v = \frac{3w}{3w^2-1} + \frac{1}{(3w^2-1)(15w^3-6w)} + \text{\&c.}$$

$$\tan v = \frac{15w^2-1}{15w^3-6w} + \frac{1}{(15w^2-6w) \cdot (105w^4-45w^2+1)} + \text{\&c.}$$

\&c.

We see therefore from what we have said (§. 31.) *that all these sequences are more convergent than is any decreasing geometric progression.*⁶⁰ Let for instance $v = w = 1$, and the tangent of this arc will be⁶¹ $= 1,55740772 \dots$

$$= 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 9} + \frac{1}{9 \cdot 61} + \frac{1}{61 \cdot 540} + \frac{1}{540 \cdot 5879} + \frac{1}{5879 \cdot 75887} + \frac{1}{75887 \cdot 1132426} + \text{\&c.}$$

And for every arc $v < 1$, we will have an even more convergent sequence.

⁵⁸This follows, because the first remainder $\left(\frac{1}{w(3w^2-1)}\right)$ has the numerator 1 and the product of the two denominators as denominator.

⁵⁹If the fractions approximating the tangent are $u_1 = \frac{1}{w}$, $u_2 = \frac{3w}{3w^2-1}$, etc., Lambert takes the limit of $u_n = (u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + \dots + (u_{p+1} - u_p) + u_p$ where u_p is the initial term $\frac{m}{p}$. (The initial term need not be $\frac{1}{w}$, but could be $\frac{3w}{3w^2-1}$, $\frac{15w^2-1}{15w^3-6w}$, etc.)

⁶⁰As a consequence of the expressions of §. 24, the factors $3, 3 \times 15, 15 \times 105, \dots$, grow faster than a geometric progression, but this is also true for $w, w(3w^2-1), (3w^2-1) \cdot (15w^3-6w)$, etc., for $w \geq 1$.

⁶¹This is $\tan(1)$. The development was missing the third term and some denominators were incorrect. These denominators can easily be computed: $2 \times 5 - 1 = 9, 9 \times 7 - 2 = 61, 61 \times 9 - 9 = 540, 540 \times 11 - 61 = 5879, 5879 \times 13 - 540 = 75887, 75887 \times 15 - 5879 = 1132426$, etc.

§. 35. Let now $w = \omega : \varphi$, $v = \varphi : \omega$, such that φ , ω are integers, mutually prime. We will only have to substitute these values, and we will have⁶²

$$\tan\left(\frac{\varphi}{\omega}\right) = \frac{\varphi}{\omega - \frac{\varphi\varphi}{3\omega - \frac{\varphi\varphi}{5\omega - \frac{\varphi\varphi}{7\omega - \frac{\varphi\varphi}{9\omega - \&c.}}}}$$

§. 36. Then the fractions approaching the value of $\tan \frac{\varphi}{\omega}$ will be⁶³

$$\frac{\varphi}{\omega}, \frac{3\omega\varphi}{3\omega^2 - \varphi^2}, \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\varphi^2\omega}, \frac{105\omega^3\varphi - 10\omega\varphi^3}{105\omega^4 - 45\omega^2\varphi^2 + \varphi^4}, \&c.$$

so that if any two of these immediately succeeding fractions are

$$\frac{m}{p},$$

$$\frac{A}{B},$$

the one which will follow will be⁶⁴

$$\frac{A(2n+1)\omega - m\varphi^2}{B(2n+1)\omega - p\varphi^2}.$$

§. 37. Finally the differences of these fractions will be

$$\frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)}, \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)}, \&c.$$

and

$$\tan \frac{\varphi}{\omega} = \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c.$$

⁶²This follows from §. 14.

⁶³According to §. 22.

⁶⁴This is easily shown given that the new values m' , p' are m and p multiplied by φ^n and the new values A' and B' are A and B multiplied by φ^{n+1} (φ is $1 \times \varphi$, $3\omega\varphi$ is $3\omega \times \varphi^2$, $15\omega^2\varphi - \varphi^3$ is $(15\omega^2 - 1) \times \varphi^3$, etc.). The old sequence is $\frac{m}{p}$, $\frac{A}{B}$, $\frac{A(2n+1)\omega - m}{B(2n+1)\omega - p}$ (see §. 21), and it becomes $\frac{m\varphi^n}{p\varphi^n}$, $\frac{A\varphi^{n+1}}{B\varphi^{n+1}}$, $\frac{A(2n+1)\omega\varphi^{n+1} - m\varphi^{n+2}}{B(2n+1)\omega\varphi^{n+1} - p\varphi^{n+2}}$, which is $\frac{m'}{p'}$, $\frac{A'}{B'}$, $\frac{A'(2n+1)\omega - m'\varphi^2}{B'(2n+1)\omega - p'\varphi^2}$.

And I claim that this tangent will never be commensurable to the radius, whatever the integer numbers ω , φ .

§. 38. In order to prove this theorem, we set

$$\tan \frac{\varphi}{\omega} = \frac{M}{P},$$

such that M , P are quantities expressed in some way, possibly even by decimal expansions, which is always possible, even in the case that M , P are integers, as they could be both multiplied by some irrational quantity. We could also assume, if wished, that $M = \sin \frac{\varphi}{\omega}$, $P = \cos \frac{\varphi}{\omega}$, as we did above (§. 5.). And it is clear that, even if $\tan \frac{\varphi}{\omega}$ were rational, this would not always be the case for $\sin \frac{\varphi}{\omega}$ and $\cos \frac{\varphi}{\omega}$.

§. 39. But the fraction

$$\frac{M}{P}$$

being an exact expression of the tangent of $\frac{\varphi}{\omega}$, it must give all the quotients w , $3w$, $5w$ &c. which in the present case are⁶⁵

$$+\frac{\omega}{\varphi}, -\frac{3\omega}{\varphi}, +\frac{5\omega}{\varphi}, -\frac{7\omega}{\varphi}, +\&c.$$

§. 40. Then, if $\tan \frac{\varphi}{\omega}$ is rational, it is clear that M will be to P as an integer number μ to an integer number π , so that if μ , π , are mutually prime, we will have

$$M : \mu = P : \pi = D,$$

and D will be the greatest common divisor⁶⁶ of M , P . And since reciprocally we have

$$\begin{aligned} M : D &= \mu, \\ P : D &= \pi, \end{aligned}$$

it is clear that if M , P are assumed to be irrational quantities, their greatest common divisor will likewise be an irrational quantity, and be as smaller than the integers⁶⁷ μ , π , are large.

§. 41. These are therefore *the two assumptions which will have to be shown incompatible*.⁶⁸ Let us first divide P by M , and the quotient⁶⁹ must

⁶⁵See §. 14.

⁶⁶The assumption is therefore that $D > 0$. D is not necessarily an integer. This will lead to a contradiction in §. 49.

⁶⁷The original memoir had "quotients."

⁶⁸Namely, that $\frac{\varphi}{\omega}$ and $\tan \frac{\varphi}{\omega}$ are both rational. In what follows, Lambert will define a series of remainders R' , R'' , R''' , etc., such that D divides all of them. R' is defined in §. 41, R'' in §. 42 and the others in §. 43.

⁶⁹ P corresponds to B in §. 6, M to A , and the quotient to $Q' = \frac{1}{v} = w = \frac{\omega}{\varphi}$.

be $= \omega : \varphi$. But since $\omega : \varphi$ is a fraction of integers,⁷⁰ let us divide φP by M , and the quotient ω will be φ times $\omega : \varphi$. It is clear that we will be able to divide it by φ , whenever it is desired. Here, it will not be required, since it will be sufficient for us that it is an integer. Having thus, by dividing φP by M , obtained the quotient ω , let the remainder $= R'$. This remainder will likewise be equal to φ times what it would have been,⁷¹ and this is what we will take into account. Now, since we have $P : D = \pi$, an integer, we will also have $\varphi P : D = \varphi\pi$, an integer. Finally, $R' : D$ will also be an integer. Indeed, since

$$\varphi P = \omega M + R',$$

we have

$$\frac{\varphi P}{D} = \frac{\omega M}{D} + \frac{R'}{D}.$$

But

$$\begin{aligned}\varphi P : D &= \varphi\pi, \\ \omega M : D &= \omega\mu,\end{aligned}$$

therefore

$$\varphi\pi = \omega\mu + \frac{R'}{D},$$

which gives

$$\frac{R'}{D} = \varphi\pi - \omega\mu = \text{integer number},$$

which we will set $= r'$, so that

$$\frac{R'}{D} = r'.$$

Thus the remainder of the first division will still have the divisor D , which is the greatest common divisor of M, P .

⁷⁰*nombre rompu* in the original text. So, we assume here that $\frac{\omega}{\varphi}$ is a rational. $\frac{M}{P}$ is a fraction, but not necessarily a fraction of integers. However, we assume that μ and π are integers in order to obtain a contradiction.

⁷¹that is, what it would have been if one had written $P = \frac{\omega}{\varphi}M + \dots$.

§. 42. Let us now move to the second division. The remainder R' being φ times what it would be if we had divided P , instead of φP , by M , this must be taken into account for this second division, by dividing φM , instead of M , by R' , in order to obtain the second quotient,⁷² which is⁷³ $3\omega : \varphi$. But, in order to avoid here too the broken quotient,⁷⁴ let us divide $\varphi^2 M$ by R' , so as to obtain the quotient 3ω , an integer. Let the remainder = R'' , and we will have⁷⁵

$$\varphi^2 M = 3\omega R' + R'',$$

thus dividing by D ,

$$\frac{\varphi^2 M}{D} = \frac{3\omega R'}{D} + \frac{R''}{D}.$$

But we have

$$\begin{aligned} \frac{\varphi^2 M}{D} &= \varphi^2 m = \text{integer number,} \\ \frac{3\omega R'}{D} &= 3\omega r' = \text{integer number,} \end{aligned}$$

therefore

$$\varphi^2 m = 3\omega r' + \frac{R''}{D},$$

which gives

$$\frac{R''}{D} = \varphi^2 m - 3\omega r' = \text{integer number,}$$

which we will set = r'' , so that we have

$$\frac{R''}{D} = r''.$$

⁷²We will therefore obtain the same sequence of quotients $\frac{\omega}{\varphi}$, $3\frac{\omega}{\varphi}$, $5\frac{\omega}{\varphi}$, etc.

⁷³In fact, the second quotient is $-3\omega : \varphi$, but until §. 46, Lambert only divides by positive values. This will be adapted in §. 46. Here, ignoring the signs changes the values of R' , but not the result of this paragraph.

⁷⁴That is, a non integer quotient.

⁷⁵Consequently, if the remainders obtained from the division of P by M are R'_0 , R''_0 , R'''_0 , etc., we actually have $R' = \varphi R'_0$, $R'' = \varphi^2 R''_0$, $R''' = \varphi^3 R'''_0$, etc., $R^n = \varphi^n R^n_0$. $M = 3\frac{\omega}{\varphi} R'_0 + R''_0 \Rightarrow \varphi^2 M = 3\omega R' + R''$.

Consequently, the greatest common divisor of M , P , R' , also divides the second remainder R'' .

§. 43. Let the following remainders⁷⁶ ... R''' , R^{IV} ... R^n , R^{n+1} , R^{n+2} ... , corresponding to the φ -uple quotients⁷⁷ ... 5ω , 7ω , ... $(2n-1)\omega$, $(2n+1)\omega$, $(2n+3)\omega$, ... , and we must prove in general that if any two remainders R^n , R^{n+1} , which are in immediate sequence, have D as their divisor, the next remainder R^{n+2} will also have it, so that if we set⁷⁸

$$\begin{aligned} R^n : D &= r^n, \\ R^{n+1} : D &= r^{n+1}, \end{aligned}$$

r^n , r^{n+1} are integers, we will also have

$$R^{n+2} : D = r^{n+2},$$

an integer. Here is the proof.

§. 44. When dividing $\varphi^2 R^n$ by R^{n+1} , the quotient will be $(2n+3)\omega =$ integer number, and the remainder being $= R^{n+2}$, we will have⁷⁹

$$\varphi^2 R^n = (2n+3)\omega \cdot R^{n+1} + R^{n+2},$$

therefore by dividing by D ,

$$\frac{\varphi^2 \cdot R^n}{D} = \frac{(2n+3)\omega \cdot R^{n+1}}{D} + \frac{R^{n+2}}{D}$$

But we have

$$\begin{aligned} \frac{\varphi^2 R^n}{D} &= \varphi^2 r^n = \text{integer number}, \\ \frac{(2n+3)\omega \cdot R^{n+1}}{D} &= (2n+3)\omega r^{n+1} = \text{integer number}, \end{aligned}$$

⁷⁶As mentioned above, having named the normal remainders R'_0 , R''_0 , etc., we have (ignoring the signs) $P = \frac{\omega}{\varphi}M + R'_0$, $M = 3\frac{\omega}{\varphi}R'_0 + R''_0$, $R'_0 = 5\frac{\omega}{\varphi}R''_0 + R'''_0$, $R''_0 = 7\frac{\omega}{\varphi}R'''_0 + R^{IV}_0$, etc., and setting $R^n = \varphi^n R^n_0$, we obtain easily $\varphi^2 R^n = (2n+3)\omega R^{n+1} + R^{n+2}$. Indeed, writing $\varphi P = \omega M + \varphi R'_0$, $\varphi^2 M = 3\omega(\varphi R'_0) + (\varphi^2 R''_0)$, $\varphi^2 R'_0 = 5\omega(\varphi R''_0) + (\varphi^2 R'''_0)$, ... , $\varphi^2 R^n_0 = (2n+3)\omega(\varphi R^{n+1}_0) + (\varphi^2 R^{n+2}_0)$, we obtain $R^n/\varphi^{n-2} = (2n+3)\omega R^{n+1}/\varphi^n + R^{n+2}/\varphi^n$.

⁷⁷that is, the quotients $5\frac{\omega}{\varphi}$, multiplied by φ , $7\frac{\omega}{\varphi}$ multiplied by φ , etc.

⁷⁸The original memoir had $R^{n+2} : D = r^{n+1}$.

⁷⁹The original memoir had $\varphi^2 R^n = (2n+1)\omega \cdot R^{n+1} + R^{n+2}$ and I have replaced all occurrences of $2n+1$ by $2n+3$ in this section. These corrections were not made by Speiser in 1948 [96].

therefore

$$\varphi^2 r^n = (2n + 3)\omega \cdot r^{n+1} + \frac{R^{n+2}}{D},$$

which gives

$$\frac{R^{n+2}}{D} = \varphi^2 \cdot r^n - (2n + 3)\omega \cdot r^{n+1} = \text{integer number} = r^{n+2}.$$

And this is what had to be proven.

§. 45. We have seen that r' , r'' are integers (§. 41. 42.) and therefore also r''' , r^{iv} , ... r^n ... &c. will be integers. So all the remainders R' , R'' , R''' , ..., R^n , ... &c. will have D as a common divisor. Let us find the value of these remainders expressed by M , P .

§. 46. For that effect, each division results in an equation, that is

$$\begin{aligned} R' &= \varphi P - \omega M, \\ R'' &= \varphi^2 M - 3\omega \cdot R', \\ R''' &= \varphi^2 R' - 5\omega \cdot R'', \\ &\text{\&c.} \end{aligned}$$

But we should observe that, in the present case, the quotients ω , 3ω , 5ω &c. are alternatively positive and negative, and that the signs of the remainders appear in the sequence $- - + +$. From that, these equations become⁸⁰

$$\begin{aligned} R' &= \omega M - \varphi P, \\ R'' &= 3\omega R' - \varphi^2 M, \\ R''' &= 5\omega R'' - \varphi^2 R', \\ R^{iv} &= 7\omega R''' - \varphi^2 R'', \\ &\text{\&c.} \end{aligned}$$

And in general

$$R^{n+2} = (2n + 3)\omega \cdot R^{n+1} - \varphi^2 R^n.$$

From that we see that each remainder can be obtained from the two previous ones, in the same way as the numerators and the denominators of the fractions approaching the value of $\tan \frac{\varphi}{\omega}$. (§. 36.)

⁸⁰So, now Lambert is changing the definition of R' , R'' , R''' , R^{iv} , ..., in the middle of his demonstration... The sign of R' is inverted and all the other expressions follow accordingly. Below, I have replaced Lambert's $2n - 1$ by $2n + 3$.

§. 47. Doing therefore the substitutions that these equations indicate, in order to express all these remainders by M , P , we will have

$$\begin{aligned} R' &= \omega M - \varphi P, \\ R'' &= (3\omega^2 - \varphi^2)M - 3\omega\varphi \cdot P, \\ R''' &= (15\omega^3 - 6\omega\varphi^2)M - (15\omega^2\varphi - \varphi^3)P, \\ &\&c. \end{aligned}$$

And these coefficients of M , P , being the numerators and the denominators of the fractions obtained above for the $\tan \frac{\varphi}{\omega}$, (§. 36.)⁸¹ it is clear that we will have

$$\begin{aligned} \frac{M}{P} - \frac{\varphi}{\omega} &= \frac{R'}{\omega P}, \\ \frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} &= \frac{R''}{(3\omega^2 - \varphi^2) \cdot P}, \\ \frac{M}{P} - \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\omega\varphi^2} &= \frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P}, \\ &\&c. \end{aligned}$$

§. 48. But we have⁸²

$$\frac{M}{P} = \tan \frac{\varphi}{\omega}.$$

Therefore (§. 37. 34.)

$$\begin{aligned} \frac{M}{P} - \frac{\varphi}{\omega} &= \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c. \\ \frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} &= \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c. \end{aligned}$$

Hence

$$\begin{aligned} \frac{R'}{\omega P} &= \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c. \\ \frac{R''}{(3\omega^2 - \varphi^2)P} &= \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c. \\ \frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P} &= \frac{\varphi^7}{(15\omega^3 - 6\omega\varphi^2) \cdot (105\omega^4 - 45\omega^2\varphi^2 + \varphi^4)} + \&c. \\ &\&c. \end{aligned}$$

⁸¹It is easy to see that the expressions for R' , R'' , R''' , etc., enable a formal manipulation of both numerators and denominators as seen in §. 22, but this is not proven by Lambert.

⁸²The memoir had $\frac{M}{P} = \tan \varphi$.

Consequently, all the remainders are obtained using the sequence of the differences (§. 37.)

$$\begin{aligned} \tan \frac{\varphi}{\omega} &= \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2)(15\omega^3 - 6\omega\varphi^2)} \\ &\quad + \frac{\varphi^7}{(15\omega^3 - 6\omega\varphi^2)(105\omega^4 - 45\omega^2\varphi^2 + \varphi^4)} \\ &\quad + \&c. \end{aligned}$$

omitting 1, 2, 3, 4 &c. of the first terms, and multiplying the sum of the following ones by the first factor of the denominator of the first retained number, and by P .⁸³

§. 49. But this sequence of differences is more convergent than any decreasing geometric progression (§. 34. 35.). Consequently⁸⁴ the remainders R' , R'' , R''' &c. decrease in such a way that they eventually become smaller than any assignable quantity. And since each of these remainders, having D as a common divisor, is a multiple of D , it follows that this common divisor D is smaller than any assignable quantity, which means $D = 0$, and therefore $(M : P)$ is a quantity incommensurable to unity, viz. an irrational one.⁸⁵

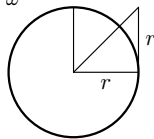
§. 50. Consequently *whenever a circle arc = $\frac{\varphi}{\omega}$ will be commensurable to the radius = 1, or rational, the tangent of this arc will be a quantity incommensurable to the radius, or irrational. And reciprocally, no rational tangent is that of a rational arc.*

§. 51. But the tangent of 45° being rational, in that it is equal to the radius,⁸⁶ it follows that the arc of 45° , and therefore also the arcs of 90, 180,

⁸³In other words, R''' , for instance, is obtained by omitting the first three terms of $\tan \frac{\varphi}{\omega}$, keeping $\frac{\varphi^7}{(15\omega^3 \dots)} + \&c.$ and multiplying by $15\omega^3 - 6\omega\varphi^2$ and by P .

⁸⁴Lambert does not clearly prove that the remainders R' , R'' , etc., tend to 0. This seems to have been observed first by Gauss [123, 124] who provided a more rigorous proof. Gauss noted that the sequence R' , R'' , etc., may be initially increasing but that it will eventually decrease. As part of his proof, Gauss considers the values of $\frac{\varphi}{\omega}$, $\frac{3\omega^2 - \varphi^2}{\varphi^2}$, $\frac{15\omega^3 - 6\omega\varphi^2}{\varphi^3}$, etc., and shows that it increases faster than a geometrical series, even when $\omega/\varphi < 1$. However, the case $\omega/\varphi > 1$ is sufficient in Lambert's proof. This gap has also been rediscovered more recently by Baltus [7, p. 11] who also provided a fix for this "gap," not aware of Pringsheim's articles cited above (possibly because Baltus based himself on Wallisser [150] who also omitted one of Pringsheim's articles).

⁸⁵This follows, because $D = 0$ contradicts the assumption of §. 40 that such a D exists, that is, that $\tan \frac{\varphi}{\omega}$ is rational.



⁸⁶ $\tan 45^\circ = r$:

360 degrees, is incommensurable to the radius. Thus *the circumference of the circle is not to the diameter like a integer number to an integer number*. We now have this theorem in the form of a corollary of another infinitely more universal theorem.

§. 52. Indeed, it is precisely this absolute universality, of which we may be surprised. In addition to showing us how much the circular quantities are transcendental,⁸⁷ it also shows us *that the rational tangents and the rational arcs are not spread over the whole circumference of the circle as if they were thrown randomly, but that there must be a certain order, and that this order prevents them from ever meeting*.⁸⁸ This order is worthy, without contest, to be examined in more detail. Let us therefore see how far it will be possible to determine its laws. This will be the result of the following theorems.

§. 53. First, we know that, *two tangents being rational, the tangent of the sum and that of the difference of their arcs are also rational*. Indeed, we have⁸⁹

$$\begin{aligned}\tan(\omega + \varphi) &= \frac{t\omega + t\varphi}{1 - t\omega \cdot t\varphi}, \\ \tan(\omega - \varphi) &= \frac{t\omega - t\varphi}{1 + t\omega \cdot t\varphi}.\end{aligned}$$

§. 54. From that it follows *that if a tangent is rational, the tangent of any multiple of its arc will also be rational*.

§. 55. But on the contrary, *a tangent being irrational,⁹⁰ no aliquot part of its arc will have a rational tangent*. Indeed, the given arc being a multiple of each of its aliquot parts, it is clear that its tangent would be rational, if the tangent of one of its aliquot parts were rational (§. 54.).

§. 56. *If the tangent of each of two commensurable arcs is rational, the tangent of the greatest common measure⁹¹ of these two arcs will likewise be rational*. Let ω , φ , be the two given arcs. Since they are commensurable, ω will be to φ like an integer number m to an integer number n . Let these numbers m , n , be mutually prime, and the unit will be their greatest common measure. We thus set

$$\begin{aligned}\omega &= m\psi, \\ \varphi &= n\psi,\end{aligned}$$

⁸⁷Here, Lambert means that there are many non rational tangents.

⁸⁸In other words, they will not lie both on a line drawn from the origin.

⁸⁹ $\tan(\omega + \varphi) = \frac{\tan \omega + \tan \varphi}{1 - \tan \omega \cdot \tan \varphi}$.

⁹⁰The memoir had "rational."

⁹¹that is, the greatest common divisor.

and the arc ψ will be the greatest common measure of the arcs ω, φ . I claim that the $\tan \psi$ will be rational. Let $m > n$, and subtracting n from m as many times as possible, let the last remainder = r , all the $\tan(m - n)\psi = t(\omega - \varphi)$, $\tan(m - 2n)\psi = t(\omega - 2\varphi)$, &c. $\tan r\psi$, will be rational⁹² (§. 53.). Subtract r from n as many times as possible, let the last remainder = r' . Then subtract r' from r as many times as possible, let the last remainder be r'' &c. And by continuing like that, you will reach a remainder = 1, the numbers m, n being mutually prime. (*Euclid*. Pr. I. Book. VII.)⁹³ But by §. 53 all the tangents

$$\begin{array}{llll} t(m - n)\psi, & t(m - 2n)\psi & \dots\dots\dots & tr \psi, \\ t(n - r)\psi, & t(n - 2r)\psi & \dots\dots\dots & tr' \psi, \\ t(r - r')\psi, & t(n - 2r')\psi & \dots\dots\dots & tr''\psi, \\ & \&c. & \\ \dots\dots\dots & & & t \psi, \end{array}$$

will be rational. Therefore &c.

§. 57. Since all these tangents can be obtained from $\tan \omega, \tan \varphi$, without their arcs being known (§. 53.), it is clear that if any two rational tangents are given, we will find if their arcs are commensurable.⁹⁴ If the arcs are not commensurable, the process would be endless.

§. 58. *Two aliquot parts of a given arc having rational tangents, I claim that the tangent of the greatest common measure of these two aliquot parts will likewise be rational.* This theorem immediately follows from the previous one (§. 56.). One has merely to remember that two arcs ω, φ , which are aliquot parts of an arc A , are commensurable.⁹⁵

§. 59. *Similarly, if any number of aliquot parts of an arc A have rational tangents, the tangent of the arc which is the greatest common measure of these aliquot parts will likewise be rational.* Let two of these aliquot parts be ω, φ , and let their greatest common measure = ψ , and the tangent ψ will be rational (§. 56. 58.). But ψ being an aliquot part of the arcs ω, φ , which are aliquot parts of the arc A , it is clear that ψ will be aliquot part of the arc A , and that in place of the arcs ω, φ , we can substitute ψ , by comparing ψ to one of the other aliquot parts of the given arc A . One can then find again their greatest common measure, of which the tangent will likewise be rational. &c.

⁹²The previous sentence had a typo.

⁹³See Heath 1908, volume 2, p. 296.

⁹⁴We start with $x = \tan \omega$ and $y = \tan \varphi$ which are rational and compute $\tan(\omega - \varphi)$, $\tan(\omega - 2\varphi)$, etc., only using x and y , and stopping before the result is negative. Then we have $\tan(r\psi)$, if ω and φ are commensurable. This process goes on until one of the tangents becomes equal to 0. If it does, the angles are commensurable, since we have applied Euclid's algorithm on the angles by working on the exact values of the tangents.

⁹⁵If $A = m\omega = n\varphi$, then $\frac{\omega}{\varphi} = \frac{n}{m}$.

§. 60. Let us call *primary tangent*⁹⁶ any rational tangent which is that of an arc of which no aliquot part has a rational tangent.

§. 61. Such is for instance the tangent of 45° . Indeed, let n be any integer, every $\tan(45 : n)^\circ$ will be one of the roots of the equation

$$0 = 1 - nx - n \cdot \frac{n-1}{2}x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot x^4 \\ - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5}x^5 - \&c.$$

whose coefficients are the same as those of *Newton's* binomial formula,⁹⁷ and whose signs change according to the sequence $- - + +$. But, for every integer n , these coefficients are integers, and every⁹⁸

$$\tan\left(\frac{45^\circ}{n}\right) < 1.$$

Therefore, if one or more of the $\tan(45^\circ : n)$ were rational, it would be a rational *fraction* < 1 , and if it were the case, not all the coefficients would be integers.⁹⁹ But they are. Therefore $\&c.$ ¹⁰⁰

§. 62. *Any primary tangent being given, only the multiples of its arc have rational tangents, with the exception of all the other arcs which are commensurable with it.* Let $\tan \omega$ be primary, and m, n , being mutually prime integers,¹⁰¹ assume that $\tan\left(\frac{m}{n}\omega\right)$ is rational. But the arc $\left(\frac{\omega}{n}\right)$ being the greatest common measure of the arcs ω , and $\left(\frac{m\omega}{n}\right)$,¹⁰² the tangent of $\frac{\omega}{n}$ will

⁹⁶The values of the arcs could be represented as trees, and when an arc is that of a primary tangent, none of the children has a rational tangent. For instance, none of the tangents $\tan 1^\circ$, $\tan 3^\circ$, $\tan 5^\circ$, $\tan 9^\circ$ are rational if (see below) $\tan 45^\circ$ is a primary tangent, but also none of $\tan 4.5^\circ$, $\tan 2.25^\circ$, etc. or any other angle obtained by dividing 45° by an integer greater than 1.

⁹⁷This formula can be obtained as follows. We have $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$, $\tan(3x) = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$, etc., which have the general pattern

$$\tan(n\alpha) = \frac{nT - \frac{n(n-1)(n-2)}{3!}T^3 + \frac{n(n-1)\cdots(n-4)}{5!}T^5 - \dots}{1 - \frac{n(n-1)}{2!}T^2 + \frac{n(n-1)\cdots(n-3)}{4!}T^4 - \dots}$$

with $T = \tan \alpha$. This formula can be proven by induction. Lambert's equation is obtained by writing $\tan(n\alpha) = 1$.

⁹⁸ n is assumed > 1 .

⁹⁹Lambert's statement is not obvious and should be clarified.

¹⁰⁰Therefore none of the $\tan\left(\frac{45^\circ}{n}\right)$ for $n > 1$ are rational.

¹⁰¹ n is implicitly assumed > 1 .

¹⁰²If we assume $\omega = \alpha p$ with $p < n$ and $\frac{m}{n}\omega = \alpha q$, then it is easy to show that $p \geq n$, a contradiction.

be rational (§. 56.). But $\frac{\omega}{n}$ being an aliquot part of ω , $\tan \omega$ would not be primary. This going against the assumption, it is clear that no $\tan\left(\frac{m}{n}\omega\right)$ could be rational. There thus only remain the multiples of ω , whose tangents will be rational (§. 54.). This is the reason why these kinds of tangents deserve the name of primary ones. They resemble in some way the prime numbers, in that only their multiples are integers, &c.

§. 63. *Two primary tangents being given, I claim that their arcs are incommensurable.* Indeed, let $\tan \omega$, $\tan \varphi$ be primary, and let us assume that the arcs ω , φ are commensurable. They will thus be like an integer m to an integer n . Therefore

$$\varphi = \frac{m\omega}{n}.$$

Consequently¹⁰³ (§. 62.) $\frac{\omega}{n}$, aliquot part of ω , as well as $\frac{\varphi}{m}$, aliquot part of φ , will have rational tangents. Thus $t\varphi$, $t\omega$, will not be primary. This going against the assumption, it is clear that the arcs ω , φ , can not be commensurable.

§. 64. *Therefore all the arcs of primary tangents are incommensurable.* Indeed, by the previous theorem, any two such arcs are incommensurable.

§. 65. *Any rational and non primary tangent being given, I claim that its arc will be a multiple of the arc of a primary tangent.* Indeed, this tangent, although it is rational, it is not primary and this can only be so because there are aliquot parts of its arc whose tangents are rational. Let these aliquot parts be $\frac{\omega}{m}$, $\frac{\omega}{n}$, $\frac{\omega}{p}$, $\frac{\omega}{q}$ &c. whose number is assumed to be finite.¹⁰⁴ But, since we take all of them, the one which is the common measure of all the others must also be among them, whereas by §. 59. its tangent is likewise rational. Assume it is $\frac{\omega}{r}$, I claim that $\tan \frac{\omega}{r}$ is primary. For, if it were not primary, the tangents of some of the aliquot parts of $\left(\frac{\omega}{r}\right)$ would be rational. But since these aliquot parts of $\left(\frac{\omega}{r}\right)$ are also aliquot parts of the given arc ω , it is clear that they would already be included in the aliquot parts $\frac{\omega}{m}$, $\frac{\omega}{n}$, $\frac{\omega}{p}$... $\frac{\omega}{r}$, and that consequently $\frac{\omega}{r}$ would likewise be their greatest common measure. Thus $\frac{\omega}{r}$ would be a measure of its aliquot parts. This is absurd, it is clear that $\tan \frac{\omega}{r}$ is primary. But ω is a multiple of $\frac{\omega}{r}$. Therefore &c.

§. 67. We now have *all rational tangents sorted in certain classes.* They are either themselves primary, or they stem, so to say, in direct line from a primary tangent, because only the multiples of the arcs of primary tangents have rational tangents (§. 62.). But, if there were only one primary tangent, all the rational tangents would derive from it, and all their arcs would be

¹⁰³By §. 56, the greatest common measure of ω and $\frac{m}{n}\omega$, namely $\frac{\omega}{n}$ will be rational. And also $\frac{\varphi}{m}$ and this is a contradiction.

¹⁰⁴Lambert does not prove that the number of aliquot parts with rational tangents is finite.

commensurable to each other. But we are far from having only one primary tangent. Such a tangent should be smaller than any assignable quantity. In order to prove it, let us assume that it has a finite value = $\tan \varphi$. And it is clear¹⁰⁵ that there will be rational tangents which are smaller than $\tan \varphi$. If these tangents are primary, $\tan \varphi$ will not be the only primary one. If they are not primary, they derive from one or more primary tangents, in that their arcs will be multiples of the primary tangents (§. 65.).¹⁰⁶ Hence, there is more than one, more than 2, 3, 4 &c. primary tangents. And as long as their number will be assumed to be finite, we will find likewise that there are more. Here is another way to find an infinite number of them.

§. 67. Let $t\omega$, $t\varphi$ be two primary tangents. First, they will be rational, and their arcs will be incommensurable between each other (§. 64.). Let m , n , be any mutually prime numbers, and $(m\omega + n\varphi)$ will be an arc incommensurable both to ω and to φ . But its tangent will be rational (§. 62. 53.). But the arc $(m\omega + n\varphi)$ being multiple neither of ω , nor of φ , the $\tan(m\omega + n\varphi)$ will either be primary itself, or it will derive from a primary tangent, necessarily different from $t\omega$, $t\varphi$. But, by varying the numbers m , n , in all possible ways, so that they are always mutually prime, we will find as many arcs $(m\omega + n\varphi)$, incommensurable with each other and incommensurable with the arcs ω , φ , and which consequently are neither multiples of each other, nor of ω , φ . Therefore their tangents, which are all rational, will derive from as many primary tangents, which are different from each other.

§. 68. This is therefore what infinitely restricts the possibility of finding a rational arc whose tangent is likewise rational. For, the arcs of all primary tangents being incommensurable to each other, it follows that, if it were possible to find a primary tangent whose arc were commensurable to the radius, it would be the only one, since the arcs of all other primary tangents would necessarily be incommensurable with the radius. But, as a consequence of what we saw above, even this sole one cannot have its arc rational.¹⁰⁷

§. 69. The tangent of the angle of 45° being primary (§. 61.) and being located in the trigonometrical tables, I will observe moreover as a corollary that it is the only primary tangent,¹⁰⁸ and at the same time the only rational tangent which is found there.¹⁰⁹ The reason is that all the arcs of which the tangents are given in these tables are commensurable to each other, without there being other multiples of 45° besides the angle of 90° whose tangent is

¹⁰⁵Any rational number is the tangent of an arc, and we can choose a rational which is smaller than $\tan \varphi$.

¹⁰⁶but §. 65 has not completely been proven.

¹⁰⁷This should be better explained.

¹⁰⁸That is, if the table is restricted to $[0, 90^\circ]$.

¹⁰⁹See §. 62.

infinite.

§. 70. I will also observe that, if the cosine of any angle ω is rational, the cosine of any multiple is likewise rational.¹¹⁰ This circumstance entails that the same reasoning exposed regarding tangents can be applied, with minor change, to the cosines. We will have *prime cosines* as we had *prime tangents*, and the arcs of prime cosines will likewise be incommensurable between each other; consequently, if it were possible to find a prime cosine whose arc were rational, it would be the only one that could be found, since for that reason the arcs of all other prime cosines would be irrational.

§. 71. The same is not true for sines, because if a given $\sin \omega$ is rational, in general only the $\sin 3\omega$, $\sin 5\omega$, $\sin 7\omega$ &c. are rational,¹¹¹ but the $\sin 2\omega$, $\sin 4\omega$, $\sin 6\omega$ &c. are not always rational, except if $\cos \omega$ is also rational, so that if we wish here too to find *prime sines*, it will be necessary to proceed in a different way than that used for tangents.

§. 72. But, without stopping on this matter, I will return to the continued fraction obtained previously¹¹²

$$\tan v = \frac{1}{w - \frac{1}{3w - \frac{1}{5w - \frac{1}{7w - \frac{1}{9w - 1 \text{ \&c.}}}}}}$$

We have seen that all the fractions

$$\frac{1}{w}, \frac{3w}{3w^2 - 1}, \frac{15w^2 - 1}{15w^3 - 6w}, \text{ \&c.}$$

that it produces, only approach the value of the tangent of v by default,¹¹³ in that they are all smaller than this tangent. But, since it must be possible to find similar fractions which, although they approach the value of $\tan v$, do so by excess, I started to search for them. I will content myself to give here again the continued fraction which contains alternatively the one and

¹¹⁰ $\cos(na)$ is a polynomial function of $\cos a$.

¹¹¹Only when n is odd does $\sin(n\omega)$ expand to a polynomial in $\sin \omega$. Otherwise it is a polynomial in $\sin \omega$ times $\cos \omega$.

¹¹²See §. 16.

¹¹³The fact that the value of the tangent is approached by default, hence that all these fractions are positive, was not proven by Lambert.

the others.¹¹⁴ Here it is¹¹⁵

$$\cot v = \frac{1}{0 + \frac{1}{(w-1) + \frac{1}{1 + \frac{1}{(3w-2) + \frac{1}{1 + \frac{1}{(5w-2) + \frac{1}{1 + \dots}}}}}}}$$

This fraction goes on for ever, so that the quotients are

$$0, (w-1), 1, (3w-2), 1, (5w-2), 1, (7w-2), 1, (9w-2), \dots, 1, ((2n+1)w-2), 1 \text{ \&c.}$$

And the fractions approaching the value of $\tan v$ are

$$\frac{1}{w-1}, \frac{1}{w}, \frac{3w-1}{3w^2-w-1}, \frac{3w}{3w^2-1}, \frac{15w^2-3w-1}{15w^3-3w^2-6w+1}, \frac{15w^2-1}{15w^3-6w}, \text{ \&c.}$$

The first, 3rd, 5th, 7th &c. are greater than $\tan v$, and the 2nd, 4th, 6th &c., are smaller, and the same as the ones we found above (§. 22.). I will not stop to give the proof,¹¹⁶ since this continued fraction can be obtained in the same way as we found the one that we used up to now, and which is a lot simpler. I will therefore only observe that the first quotient being here = 0, it will suffice, in order to cancel it, to inverse the fraction so that it expresses the tangent¹¹⁷ of v , since

$$\cot v = \frac{1}{\tan v}.$$

¹¹⁴That is, the fractions by default and those by excess.

¹¹⁵The memoir had $\tan v = \dots$.

¹¹⁶It should still be proven.

¹¹⁷The memoir had cotangent.

Consequently we will have

$$\tan v = \frac{1}{(w-1) + \frac{1}{1 + \frac{1}{(3w-2) + \frac{1}{1 + \frac{1}{(5w-2) + \frac{1}{1 + \frac{1}{(7w-2) + \frac{1}{1 + \dots}}}}}}}}}$$

§. 73. Let us now compare the *transcendental circular quantities to the logarithmic quantities* which correspond to them. Let e be the number whose hyperbolic¹¹⁸ logarithm = 1. And it is known that if in the two series that we used above (§. 4.)

$$\begin{aligned}\sin v &= v - \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}v^7 + \&c. \\ \cos v &= 1 - \frac{1}{2}v^2 + \frac{1}{2 \cdot 3 \cdot 4}v^4 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}v^6 + \&c.\end{aligned}$$

all the signs are taken positive, these equations become¹¹⁹

$$\begin{aligned}\frac{e^v - e^{-v}}{2} &= v + \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}v^7 + \&c. \\ \frac{e^v + e^{-v}}{2} &= 1 + \frac{1}{2}v^2 + \frac{1}{2 \cdot 3 \cdot 4}v^4 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}v^6 + \&c.\end{aligned}$$

But, by handling these last two series in the same way as we handled the first two (§. 4 and following) the operation will only differ by the signs, which in the present case are all positive. As one can convince himself or herself

¹¹⁸The “hyperbolic logarithm” is the natural (or neperian) logarithm. On Lambert’s work on hyperbolic functions, see also his article published in 1770 [93] and Barnett’s study [9].

¹¹⁹ $\sinh(v) = \frac{e^v - e^{-v}}{2}$ and $\cosh(v) = \frac{e^v + e^{-v}}{2}$.

easily, I will not give the detail.¹²⁰ We will therefore have

$$\frac{e^v - e^{-v}}{e^v + e^{-v}} = \frac{1}{1 : v + 1} \frac{1}{3 : v + 1} \frac{1}{5 : v + 1} \frac{1}{7 : v + 1} \frac{1}{9 : v + 1} \frac{1}{11 : v + 1} \frac{1}{13 : v + 1} \&c.$$

§. 74. And since we have

$$\frac{e^v - e^{-v}}{e^v + e^{-v}} = \frac{e^{2v} - 1}{e^{2v} + 1},$$

we see that by making $2v = x$, we will have

$$\frac{e^x - 1}{e^x + 1} = \frac{1}{2 : x + 1} \frac{1}{6 : x + 1} \frac{1}{10 : x + 1} \frac{1}{14 : x + 1} \frac{1}{18 : x + 1} \&c.$$

from which we obtain¹²¹

$$\frac{e^x + 1}{2} = \frac{1}{1 - 1} \frac{1}{2 : x + 1} \frac{1}{6 : x + 1} \frac{1}{10 : x + 1} \frac{1}{14 : x + 1} \&c.$$

or

$$\frac{e^x - 1}{2} = \frac{1}{(2 : x) - 1 + 1} \frac{1}{6 : x + 1} \frac{1}{10 : x + 1} \frac{1}{14 : x + 1} \frac{1}{18 : x + 1} \&c.$$

¹²⁰A new proof should be given.

¹²¹ $\frac{e^x + 1}{2} = \frac{1}{1 - \frac{1}{e^x + 1}}$ and $\frac{e^x - 1}{2} = \frac{1}{\frac{e^x + 1}{e^x - 1} - 1}$.

It is clear that these expressions lead to similar consequences than those that we deduced above from the formula

$$\tan v = \frac{1}{w - 1} \frac{1}{\frac{3w - 1}{5w - \&c.}}$$

It will be found again here that v & e^v , as well as x & e^x will never be rational quantities at the same time. I will thus not stop to give a repeated deduction of it. We should instead interpret the formulæ that we have just exposed. I therefore note that they should have, with respect to the equilateral hyperbola,¹²² a quite analogous meaning than the one that had the fraction

$$\tan v = \frac{1}{w - 1} \frac{1}{3w - \&c.}$$

with respect to the circle. For, in addition of knowing that by letting $u = v\sqrt{-1}$, the expressions

$$\begin{aligned} e^u + e^{-u}, \\ e^u - e^{-u}, \end{aligned}$$

produce the circular quantities¹²³

$$\begin{aligned} e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} &= 2 \cos v, \\ e^{v\sqrt{-1}} - e^{-v\sqrt{-1}} &= 2 \sin v \cdot \sqrt{-1}, \end{aligned}$$

Mr. *de Foncenex* has also shown in a very simple & direct manner, how this affinity is obtained by comparing together the circle & the equilateral¹²⁴ hyperbola having a common center & a common diameter. See *Miscell. Societ. Taurin.* Tom. I. p. 128 and following.¹²⁵

§. 75. But the matter is here to find out how far this affinity can be obtained independently of the imaginary quantities. Let therefore C be the center, CH the axis, CA the half-diameter¹²⁶ of the equilateral hyperbola

¹²²The equation of the equilateral hyperbola considered here is $x^2 - y^2 = 1$.

¹²³ $\frac{e^{iv} + e^{-iv}}{2} = \cos v$ and $\frac{e^{iv} - e^{-iv}}{2i} = \sin v$.

¹²⁴The equilateral hyperbola has its two asymptotes at right angle.

¹²⁵François Daviet de Foncenex, "Réflexions sur les quantités imaginaires", *Miscellanea Philosophico-Mathematica Societatis Privatae Taurinensis*, 1759, Tomus Primus, 113–146 [33].

¹²⁶ CA is the semi-major axis.

AMG & of the circle *AND*, *CF* the asymptote, *AB* perpendicular to the axis, & at the same time the common tangent to the circle & the hyperbola. Let the two infinitely close lines *CM*, *Cm* drawn from the center *C*, & from the intersection points *M*, *m*, *N*, *n*, let the ordinates *MP*, *mp*, *NQ*, *nq* be obtained from perpendiculars. Finally, let the radius *AC* = 1. Set the angle *MCA* = φ , & let¹²⁷

for the hyperbola		for the circle
the abscissa <i>CP</i> = ξ <i>CQ</i> = x
the ordinate <i>PM</i> = η <i>QN</i> = y
the segment <i>AMCA</i> = $u : 2$ <i>ANCA</i> = $v : 2$
<p>& we will have</p>		
$\tan \varphi = \frac{\eta}{\xi}$ $\tan \varphi = \frac{y}{x}$,
$1 + \eta\eta = \xi\xi = \eta\eta \cdot \cot^2 \varphi$ $1 - yy = xx = yy \cdot \cot^2 \varphi$,
$\xi\xi - 1 = \eta\eta = \xi\xi \cdot \tan^2 \varphi$ $1 - xx = yy = xx \tan^2 \varphi$,
$CM^2 = \xi^2 + \eta^2 = \xi^2(1 + \tan^2 \varphi)$		$CN^2 = x^2 + y^2 = x^2(1 + t^2 \varphi)$,
$= \frac{1+t^2 \varphi}{1-t^2 \varphi}$		$= \frac{1+t^2 \varphi}{1+t^2 \varphi} = 1$.
<p>Hence</p>		
$+du = d\varphi \cdot \left(\frac{1+t^2 \varphi}{1-t^2 \varphi} \right) = \frac{dt \varphi}{1-t^2 \varphi}$		$+dv = d\varphi = \frac{dt \varphi}{1+t^2 \varphi}$,
$+d\xi = \frac{t\varphi \cdot dt \varphi}{(1-t^2 \varphi)^{3:2}}$		$-dx = \frac{t\varphi \cdot dt \varphi}{(1+t^2 \varphi)^{3:2}}$,
$+d\eta = \frac{dt \varphi}{(1-t^2 \varphi)^{3:2}}$		$+dy = \frac{dt \varphi}{(1+t^2 \varphi)^{3:2}}$,
$\xi = \frac{1}{\sqrt{1-t^2 \varphi}}$		$x = \frac{1}{\sqrt{1+t^2 \varphi}}$,
$\eta = \frac{t\varphi}{\sqrt{1-t^2 \varphi}}$		$y = \frac{t\varphi}{\sqrt{1+t^2 \varphi}}$,
<p>Hence</p>		
$+d\xi : du = \eta$		$-dx : dv = y$,
$+d\eta : du = \xi$		$+dy : dv = x$,

¹²⁷In the sequel of this section, the original text had several typos, and some exponents were missing. I have also rewritten $\tan \varphi^2$ as $\tan^2 \varphi$ for clarity. u is twice the surface of *AMCA* and v is twice the surface of *ANCA*, hence the divisions by 2. We also have $v = \varphi$. “Segment” denotes a sector. The equation of the hyperbola is $\xi^2 - \eta^2 = 1$. It is easy to see that $1 + \eta^2 = \xi^2 = \frac{\eta^2}{\tan^2 \varphi}$ and therefore $\eta^2 = \frac{\tan^2 \varphi}{1 - \tan^2 \varphi}$ and $\xi^2 = \frac{1}{1 - \tan^2 \varphi}$. We have $d(t\varphi) = t(\varphi + d\varphi) - t(\varphi) = \tan(d\varphi) \times (1 + \tan(\varphi + d\varphi) \tan \varphi) = d\varphi(1 + t^2 \varphi) = \frac{d\varphi}{\cos^2 \varphi}$. The half of du is a triangle of base *CM* and height *CMd* φ , hence $du = d\varphi \cdot CM^2$. Similarly, $dv = d\varphi$, but the sector’s surface is only $dv/2$. When developing $(\xi + d\xi)^2 - (\eta + d\eta)^2 = 1$, we obtain (at the first order) $\xi d\xi = \eta d\eta$. Then, $d\varphi = \arctan\left(\frac{\eta + d\eta}{\xi + d\xi}\right) - \arctan\left(\frac{\eta}{\xi}\right)$. Therefore $t(d\varphi) = \frac{\frac{\eta + d\eta}{\xi + d\xi} - \frac{\eta}{\xi}}{1 + \frac{\eta + d\eta}{\xi + d\xi} \times \frac{\eta}{\xi}} = \frac{\frac{d\eta}{\xi} - \frac{\eta}{\xi^2} d\xi}{1 + \frac{\eta^2}{\xi^2}} = \frac{\frac{1}{\eta} - \frac{\eta}{\xi^2}}{1 + \frac{\eta^2}{\xi^2}} d\xi$ and $d\xi = \frac{1 + \frac{\eta^2}{\xi^2}}{\frac{1}{\eta} - \frac{\eta}{\xi^2}} t(d\varphi) = \frac{1 + t^2 \varphi}{\eta(1 - t^2 \varphi)} d\varphi = \frac{1 + t^2 \varphi}{1 - t^2 \varphi} \times \frac{t\varphi}{\sqrt{1 - t^2 \varphi}} d\varphi = \frac{1 + t^2 \varphi}{(1 - t^2 \varphi)^{3/2}} t\varphi \times \frac{d(t\varphi)}{1 + t^2 \varphi} = \frac{t\varphi \times d(t\varphi)}{(1 - t^2 \varphi)^{3/2}}$. And from $d\xi$ it follows that $d\eta = \frac{\xi}{\eta} d\xi = \frac{d(t\varphi)}{(1 - t^2 \varphi)^{3/2}}$. Similarly, since $x^2 + y^2 = 1$, we have $dy = -\frac{x}{y} dx = \frac{d(t\varphi)}{(1 + t^2 \varphi)^{3/2}}$.

$$+d\xi = d\eta \cdot \tan \varphi \dots\dots\dots | \quad -dx : dy = \tan \varphi.$$

§. 76. Since the angle φ is the same for the hyperbola & for the circle, it follows from the last two equations that we have

$$\tan \varphi = d\xi : d\eta = -dx : dy = \eta : \xi = y : x.$$

The angles Mmp , Nnq , are therefore equal. As a result

$$Mm : Nn = d\xi : -dx = d\eta : dy.$$

And the characteristic triangles $Mm\mu$, $Nn\nu$, are similar. Finally, since we have¹²⁸ $Cnq = Cmp$, & $Nnq = Mmp$, we will have¹²⁹ $Cnq + Nnq = Cmp + Mmp = 90^\circ$. By drawing the normal mV , we will have $Vmq + Mmq = 90^\circ$, hence¹³⁰ $Vmp = Cmp$. The normal mV prolonged to the axis AC is therefore equal to Cm , in the same way that in the circle the normal Cn is equal to Cn . These are consequently the foundations of everything which is real in the comparisons which have been made between the circle & the hyperbola.

§. 77. Next, if for the hyperbola we want to express ξ , η , using u , it will be found easily that by employing infinite sequences their form must be

$$\begin{aligned} \xi &= 1 + Au^2 + Bu^4 + Cu^6 + \&c. \\ \eta &= au + bu^3 + cu^5 + du^7 + \&c. \end{aligned}$$

For, by letting $u = 0$, we have $\xi = 1$, $\eta = 0$. Moreover, by taking u infinitely small, ξ will increase like u^2 , & η will increase like u , because the angle at A is a right angle, & the osculating radius of the hyperbola at A is $= AC$. Finally, by taking u negative, all the values of ξ will be the same as for the positive values of u , from which it follows that the abscissa ξ must be expressed by even dimensions of u . And by taking u negative, the values of η will be the same, but negative. η must be expressed by odd dimensions of u . There therefore only remains to determine the coefficients. It is for this purpose that we will use the two formulæ found above

$$\begin{aligned} d\xi : du &= \eta, \\ d\eta : du &= \xi. \end{aligned}$$

We will therefore have, by differentiating the first sequence

$$d\xi : du = 2Au + 4Bu^3 + 6Cu^5 + \dots + \mu \cdot Mu^{\mu-1}$$

¹²⁸Namely the angles \widehat{Cnq} , \widehat{Cmp} , etc.

¹²⁹because $\widehat{CnN} = 90^\circ$.

¹³⁰The original article had $Vmq = Cmq$.

which must be $= \eta$, hence

$$d\xi : du = au + bu^3 + cu^5 + \dots + m \cdot u^{\mu-1}$$

Consequently, comparing the terms

$$2A = a,$$

$$4B = b,$$

$$6C = c,$$

&c.

$$\mu M = m.$$

But, by differentiating η , we must also have $d\eta : du = \xi$, hence

$$\begin{aligned} d\eta : du &= a + 3bu^2 + 5cu^4 + \dots (\mu - 1) \cdot mu^{\mu-2} \\ &= 1 + Au^2 + Bu^4 + \dots L \cdot u^{\mu-2} \end{aligned}$$

So, comparing the terms

$$a = 1,$$

$$3b = A,$$

$$5c = B,$$

&c.

$$(\mu - 1)m = L.$$

Using these equations, we obtain

$$a = 1,$$

$$A = \frac{1}{2}a = \frac{1}{2},$$

$$b = \frac{1}{3}A = \frac{1}{2 \cdot 3},$$

$$B = \frac{1}{4}b = \frac{1}{2 \cdot 3 \cdot 4},$$

$$c = \frac{1}{5}B = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5},$$

$$C = \frac{1}{6}c = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6},$$

&c.

$$m = \frac{1}{(\mu - 1)}L = \frac{1}{2 \cdot 3 \cdot 4 \cdot \dots (\mu - 1)},$$

$$M = \frac{1}{\mu} \cdot m = \frac{1}{2 \cdot 3 \cdot 4 \cdot \dots \mu}.$$

We therefore will have

$$\xi = 1 + \frac{1}{2}u^2 + \frac{1}{2 \cdot 3 \cdot 4}u^4 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}u^6 + \&c.$$

$$\eta = u + \frac{1}{2 \cdot 3}u^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}u^5 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}u^7 + \&c.$$

Here we have therefore the abscissa ξ , & the ordinate η , expressed by the letter u , which is twice the area of the hyperbolic segment $AMCA$. But we know that if instead of u we take v which is twice the circular segment $ANCA$,¹³¹ the abscissa x , & the ordinate y , both circulary, are

$$x = 1 - \frac{1}{2}v^2 + \frac{1}{2 \cdot 3 \cdot 4}v^4 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}v^6 + \&c.$$

$$y = v - \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}v^7 + \&c.$$

two sequences, which for the form only differ from the two previous ones by the alternating change of signs.

§. 78. And since we have (§. 73.)

$$\frac{e^u + e^{-u}}{2} = 1 + \frac{1}{2}u^2 + \frac{1}{2 \cdot 3 \cdot 4}u^4 + \&c.$$

$$\frac{e^u - e^{-u}}{2} = u + \frac{1}{2 \cdot 3}u^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}u^5 + \&c.$$

it is clear that we will have

$$\xi = \frac{e^u + e^{-u}}{2},$$

$$\eta = \frac{e^u - e^{-u}}{2},$$

& that consequently these quantities express the abscissa $\xi = CP$, & the ordinate $\eta = PM$ of the hyperbola.¹³²

§. 79. And since $\eta : \xi = \tan \varphi$, it is clear that we also have

$$\tan \varphi = \frac{e^u - e^{-u}}{e^u + e^{-u}},$$

¹³¹If v is twice the circular segment (that is, the area of) $ANCA$, this area is equal to $\varphi/2$, we actually have $v = \varphi$ and Lambert gives the familiar expressions of $\sin x$ and $\cos x$.

¹³²Consequently, we have $\xi = \cosh(u)$, $\eta = \sinh(u)$, and $\tan \varphi = \frac{\eta}{\xi} = \tanh(u)$.

and hence from §. 81.¹³³

$$\tan \varphi = \frac{1}{1 : u + 1} \cfrac{3 : u + 1}{5 : u + 1} \cfrac{7 : u + 1}{9 : u + \&c.}$$

And since the same tangent is also¹³⁴

$$\tan v = \tan \varphi = \frac{1}{1 : v - 1} \cfrac{3 : v - 1}{5 : v - 1} \cfrac{7 : v - 1}{9 : v - \&c.}$$

we see that this tangent is obtained from these two continued fractions, which for the form do only differ by the signs: when the first is used, we only employ $u = 2AMCA$, instead of $v = 2ANCA$ to obtain the same tangent by means of the second one. Now we have the analogy which was sought independently from the imaginary quantities, & without using them.

§. 80. Now we can draw in very clear terms the consequence that *the area of the hyperbolic sector AMCA, as well as that of the corresponding circular sector ANCA, will be an irrational quantity, incommensurable with respect to the square of the radius AC, whenever the angle φ , which is that formed by each of the two sectors at the center C, will have a rational tangent, & that conversely this tangent will be irrational whenever one of these two sectors will be a rational quantity.*

§. 81. There is a quite analogous consequence to draw regarding the continued fraction (§. 74.)

$$\frac{e^u + 1}{2} = \frac{1}{1 - 1} \cfrac{2 : u + 1}{6 : u + 1} \cfrac{10 : u + 1}{14 : u + 1} \cfrac{18 : u + 1}{\&c.}$$

¹³³Actually, perhaps from §. 73.

¹³⁴See §. 7. Note that since $\tan \varphi = \tanh(u)$, we have $\tanh(u) = \tan(v)$.

which gets transformed into

$$\frac{e^u + 1}{2} = \frac{1}{1 + \frac{1}{-2 : u + 1} + \frac{1}{-6 : u + 1} + \frac{1}{-10 : u + 1} + \dots}$$

& from which we obtain, for negative values of u

$$\frac{e^{-u} + 1}{2} = \frac{1}{1 + \frac{1}{2 : u + 1} + \frac{1}{6 : u + 1} + \frac{1}{10 : u + 1} + \frac{1}{14 : u + 1} + \dots}$$

These fractions show us *how much the irrationality of the number*¹³⁵

$$e = 2,71828182845904523536028 \dots$$

is transcendental,¹³⁶ in that none of its powers,¹³⁷ nor any of its roots is rational. Indeed u & e^u can not be rational quantities at the same time. But since u is the hyperbolic logarithm of e^u , it follows that *any rational hyperbolic logarithm is that of an irrational number, & that conversely any rational number has an irrational hyperbolic logarithm.*

§. 82. But let us still examine what e^u & e^{-u} mean in the figure. Going back to this purpose to §. 78., we find the two formulæ

$$\xi = \frac{e^u + e^{-u}}{2},$$

$$\eta = \frac{e^u - e^{-u}}{2},$$

hence, taking the sum & the difference, we have

$$e^u = \xi + \eta,$$

$$e^{-u} = \xi - \eta.$$

¹³⁵Euler published a proof of the irrationality of e in 1744 (“De fractionibus continuis dissertatio,” *Commentarii academiae scientiarum imperialis Petropolitanae*, IX, p. 98-137) [44].

¹³⁶Euler was probably the first to define a transcendental number as a number which is not a solution of a polynomial equation with integer coefficients. Lambert was convinced of the transcendence of e but did not actually prove it.

¹³⁷“dignités” in French.

But the asymptotes CF , CS , making between themselves a right angle, which is divided by the axis CH in two equal parts, we have

$$\begin{aligned}\xi &= CP = PS = PR, \\ \eta &= PM,\end{aligned}$$

hence

$$\begin{aligned}\xi + \eta &= SM, \\ \xi - \eta &= MR,\end{aligned}$$

& and therefore

$$\begin{aligned}e^u &= SM, \\ e^{-u} &= MR,\end{aligned}$$

from which we see at the same time that we have

$$e^u \cdot e^{-u} = SM \cdot MR = 1.$$

We can see moreover that, since we have

$$\begin{aligned}e^u &= SM, \\ e^{-u} &= MR, \\ AB &= 1,\end{aligned}$$

we will have, taking the logarithms,¹³⁸

$$u = \log \frac{SM}{AB} = \log \frac{AB}{MR}.$$

And since u , e^u , cannot be rational at the same time,¹³⁹ we see that the same is true for the area of the sector $AMCA = \frac{1}{2}u$, & of the ordinates SM , MR .

§. 83. We moreover have (§. 75.) the differential

$$du = \frac{d \tan \varphi}{1 - t^2 \varphi},$$

whose integral happens to be

$$2u = \log \frac{1 + t\varphi}{1 - t\varphi} = \log \tan(45^\circ + \varphi) = 1. \tan SCM,$$

¹³⁸hyperbolic logarithms.

¹³⁹See §. 74.

or

$$2u = -\log \frac{1-t\varphi}{1+t\varphi} = -1. \tan(45^\circ - \varphi) = -1. \tan RCM.$$

Let us retain the first of these formulæ

$$2u = \log \left(\frac{1+t\varphi}{1-t\varphi} \right),$$

& it will enable us to find again for hyperbolic sectors what we found to be *first tangent* for circular sectors. Here is how.

§. 84. Let us first consider that the hyperbolic sector *AMCA* increases with the angle $\varphi = MCA$, in such a way that it becomes infinite when $\varphi = 45^\circ$. It is thus clear that when one of these sectors is given, it is possible to find others, which are either any multiples of it, & any parts, or which exceed it by any amount. Now, to each of these sectors corresponds an angle *MCP*, by which it is formed, & the tangent of this angle being $= \varphi$, the sector $= \frac{1}{2}u$, we have seen that it is

$$2u = \log \frac{1+t\varphi}{1-t\varphi}.$$

§. 85. Let therefore be three sectors $\frac{1}{2}u, \frac{1}{2}u', \frac{1}{2}u''$, such that the third is the sum of the first two. Let moreover $\varphi, \varphi', \varphi''$ be the corresponding angles. We will have

$$\begin{aligned} 2u &= \log \frac{1+t\varphi}{1-t\varphi}, \\ 2u' &= \log \frac{1+t\varphi'}{1-t\varphi'}, \\ 2u'' &= \log \frac{1+t\varphi''}{1-t\varphi''}. \end{aligned}$$

Since we must have

$$\frac{1}{2}u'' = \frac{1}{2}u' + \frac{1}{2}u,$$

we also have

$$\log \frac{1+t\varphi''}{1-t\varphi''} = \log \frac{1+t\varphi'}{1-t\varphi'} + \log \frac{1+t\varphi}{1-t\varphi},$$

which gives

$$\frac{1+t\varphi''}{1-t\varphi''} = \frac{1+t\varphi'}{1-t\varphi'} \cdot \frac{1+t\varphi}{1-t\varphi},$$

from which it follows

$$t\varphi'' = \frac{t\varphi + t\varphi'}{1 + t\varphi \cdot t\varphi'},$$

& reciprocally for the difference

$$t\varphi' = \frac{t\varphi'' - t\varphi}{1 - t\varphi \cdot t\varphi''}.$$

These two formulæ only differ by their signs from those which are found for the sectors, or the circular arcs, & they too let us conclude that *if the tangents which correspond to two hyperbolic sectors are rational, then the tangents which correspond to the sector which is equal to the sum & the difference of these two sectors will likewise be rational.*

§. 86. This proposition alone is sufficient to show that everything that was said above (§. 52. . . 71.) for the circle will also apply to the hyperbola. We can merely use an abridged expression, and name *tangent of some hyperbolic sector ACMA*, the tangent of the angle *ACM* which is = *AT*, the radius *AC* being set = 1. Then, we should observe that all the sectors considered here must have the axis *AC* as their common beginning, as do all the *MCAM*, *mCAm* sectors. So, *e.g.* since the *mCM* sector does not meet the axis, it must be replaced by another one which is equal to it, & which is contiguous to the *AC* axis, when we want to obtain the angle φ & the corresponding tangent. It is obvious that this remark was not necessary in the case of the circle, because each diameter of the circle can be viewed as the axis.

§. 87. It is in that sense that I will say that *the hyperbola has an infinity of prime tangents*, that *the sectors of all these prime tangents are incommensurable between themselves & to the unit*, that *the tangent of a sector being prime, only the multiples of this sector are rational*: That *every rational tangent is either prime itself, or its sector is a multiple of a sector whose tangent is prime.* &c. Since the proof of these theorems would only be a repetition of those that I gave for the circle, I will omit them, even more so that I only give these theorems in order to show again in that matter the analogy between the circle & the equilateral hyperbola.

§. 88. Let us still compare together the circular sector *ANCA*, & the hyperbolic sector *AMCA*. Mr. *de Foncenex*, in the memoir mentioned above (§. 74.) showed that, by employing the imaginary quantities, these two sectors happen to be in the ratio of 1 to $\sqrt{-1}$, which is purely imaginary. But let us see what is the real ratio? This is what we will find by expressing one of these sectors in terms of the other. To this effect we will use the two

sequences¹⁴⁰

$$v = t\varphi - \frac{1}{3}t^3\varphi + \frac{1}{5}t^5\varphi - \frac{1}{7}t^7\varphi + \&c.$$

$$t\varphi = u - \frac{1}{3}u^3 + \frac{2}{15}u^5 - \frac{17}{315}u^7 + \&c.$$

which are easily found using the differential formulæ given above (§. 75.).¹⁴¹ By substituting the value of the second of these sequences within the first, we will have, after reduction,¹⁴²

$$v = u - \frac{2}{3}u^3 + \frac{2}{3}u^5 - \frac{244}{315}u^7 + \&c.$$

& reciprocally¹⁴³

$$u = v + \frac{2}{3}v^3 + \frac{2}{3}v^5 + \frac{244}{315}v^7 + \&c.$$

These two sequences¹⁴⁴ only differ with respect to the signs, the coefficients & the exponents being the same. If in the first of these sequences we set

$$u = v\sqrt{-1},$$

we find

$$v = \sqrt{-1} \cdot \left(v + \frac{2}{3}v^3 + \frac{2}{3}v^5 + \frac{244}{315}v^7 + \&c. \right)$$

which means that

$$v = u\sqrt{-1}.$$

Hence, by means of an imaginary hyperbolic sector, we find an imaginary circular sector, & reciprocally.

§. 89. All that I have shown on the circular & logarithmic transcendental quantities seems to be based on principles much more universal, but which

¹⁴⁰In the expression of v , exponents have been moved for clarity. The first expression is obtained by integrating $dv = \frac{d(t\varphi)}{1+t^2\varphi} = d(t\varphi)(1-t^2\varphi+t^4\varphi-t^6\varphi+\dots)$. The second member of the second equation is equal to $\tanh(u)$ (the original article wrote $t\varphi = v - \frac{1}{3}u^3 + \dots$).

¹⁴¹How was the expression of $t\varphi$ found? By division of η and ξ given in §. 78?

¹⁴²As observed in an earlier note, this amounts to

$$v = \arctan(\tanh(u)) \text{ and } u = \operatorname{artanh}(\tan(v)).$$

¹⁴³This is correct, but how did Lambert obtain it?

¹⁴⁴The sequences of numerators and denominators have been submitted to the On-Line Encyclopedia of Integer Sequences (OEIS) as sequences A335257 and A335258.

are not yet enough developed. Here is however what will help to give an idea of it. It is not sufficient to have found that these transcendental quantities are irrational, that is incommensurable to the unit. This property is not unique to them. For, in addition to irrational quantities which may be obtained by chance, & which for that reason are hardly subject to analysis, there are still an infinity of others which are named *algebraic*: & such are all the *radical irrational quantities*, like $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$ &c. $\sqrt{2 + \sqrt{3}}$ &c. & all the *irrational roots of algebraic equations*, such as *e.g.* those of the equations

$$\begin{aligned} 0 &= xx - 4x + 1, \\ 0 &= x^3 - 5x + 1, \\ &\text{\&c.} \end{aligned}$$

I will name the ones & the others *radical irrational quantities*, & here is the theorem, which I believe can be proven.

§. 90. I say thus *that no circular & logarithmic transcendental quantity can be expressed by any irrational radical quantity, which refers to the same unit, & in which there enters no transcendental quantity*. The proof of this theorem appears to rest on the fact that the transcendental quantities depend on

$$e^x,$$

where the exponent varies, whereas the radical quantities assume constant exponents. Thus *e.g.* the arc of a circle being rational or commensurable to the radius, its tangent, which we found to be irrational, cannot be a square root of any rational quantity. For, let the proposed arc = ω , & let us make $\tan \omega = \sqrt{a}$, we will have¹⁴⁵

$$t^2 \omega = \frac{\sin^2 \omega}{\cos^2 \omega} = \frac{1 - \cos 2\omega}{1 + \cos 2\omega} = a,$$

from which it follows that

$$\cos 2\omega = \frac{1 - a}{1 + a};$$

but this quantity being rational, it follows that the arc 2ω is irrational, which contradicts the assumption, it is clear that by making $\tan \omega = \sqrt{a}$, the quantity a cannot be rational, & that consequently the tangent of any rational arc is not the square root of any rational quantity.

§. 91. This theorem being once proven in all its universality, it will follow that since the circumference of the circle cannot be expressed by any radical

¹⁴⁵I have moved the exponents for clarity and replaced f by \sin .

quantity, nor by any rational quantity, there will be no means to determinate it by some geometric construction. For everything that can be constructed geometrically corresponds to the rational & radical quantities; & we are even far from being able to construct the latter ones without any constraints. It is clear that it will also be the case for all the arcs of circles whose length or the two extreme points are given, either by rational quantities, or by radical quantities. For, if the length of the arc is given, it will be necessary to find the two extreme points, by using the chord, the sine, the tangent, or some other straight line which, in order to be constructed, will always depend or be reducible to one of the lines I have mentioned. But the length of the arc being given by rational or radical quantities, these lines will be transcendental, & for that very reason irreducible to any rational or radical quantity. It will also be the case if the two extreme points of the arc are given, I mean by rational or radical quantities. For, in that case, the length of the arc will be a transcendental quantity: this means irreducible to any rational or radical quantity, & consequently it does not admit of any geometrical construction.

Chapter 3

Legendre's proof (1794)

The following is a translation from pages 296-304 of Legendre's Éléments de géométrie published in 1794 [98].¹

NOTE VI. Where it is proven that the ratio of the circumference to the diameter can not be expressed in integer numbers.

A proof of this proposition is already known, it was given by Lambert in the Memoirs of Berlin, year 1761; but since this proof is lengthy and difficult to follow, I have tried to shorten it and to simplify it. Here is the result of my researches.

Consider the infinite series²

$$1 + \frac{a}{z} + \frac{1}{2} \cdot \frac{a^2}{z \cdot (z+1)} + \frac{1}{2 \cdot 3} \cdot \frac{a^3}{z \cdot (z+1) \cdot (z+2)} + , \text{ etc.}$$

and assume that $\varphi(z)$ is its sum.³ If one replaces z by $z+1$, $\varphi(z+1)$ will likewise be the sum of the series

$$1 + \frac{a}{z+1} + \frac{1}{2} \cdot \frac{a^2}{(z+1) \cdot (z+2)} + \frac{1}{2 \cdot 3} \cdot \frac{a^3}{(z+1) \cdot (z+2) \cdot (z+3)} + , \text{ etc.}$$

Let us subtract these two series term by term one from the other, and we will have $\varphi(z) - \varphi(z+1)$ for the sum of the remainder, which will be

$$\frac{a}{z \cdot (z+1)} + \frac{a^2}{z \cdot (z+1) \cdot (z+2)} + \frac{1}{2} \cdot \frac{a^3}{z \cdot (z+1) \cdot (z+2) \cdot (z+3)} + , \text{ etc.}$$

¹Translated by Denis Roegel, 18 June 2020. Legendre's note was also translated in German by Rudio [131]. My translation is not based on the earlier one published in 1828 [99].

²I have added parentheses for intelligibility.

³Legendre used the notation $\varphi : z$ for our $\varphi(z)$. I have adapted all the occurrences of ' $\varphi :$ ' and ' $\psi :$ ' in this translation.

But this remainder can be put in the form

$$\frac{a}{z \cdot (z+1)} \cdot \left(1 + \frac{a}{z+2} + \frac{1}{2} \cdot \frac{a^2}{(z+2) \cdot (z+3)} + , \text{etc.} \right);$$

and then it reduces to $\frac{a}{z \cdot (z+1)} \varphi(z+2)$. Therefore we will generally have

$$\varphi(z) - \varphi(z+1) = \frac{a}{z \cdot (z+1)} \varphi(z+2).$$

Let us divide this equation by $\varphi(z+1)$; and, in order to simplify the result, let ψ be a new function of z such that $\psi(z) = \frac{a}{z} \cdot \frac{\varphi(z+1)}{\varphi(z)}$; then instead of $\frac{\varphi(z)}{\varphi(z+1)}$, we can put $\frac{a}{z\psi(z)}$, and $\frac{(z+1)\psi(z+1)}{a}$ instead of $\frac{\varphi(z+2)}{\varphi(z+1)}$. Once the substitution is done, we have $\psi(z) = \frac{a}{z+\psi(z+1)}$. But by putting in succession in this equation $z+1$, $z+2$, etc., instead of z , we will obtain $\psi(z+1) = \frac{a}{z+1+\psi(z+2)}$, $\psi(z+2) = \frac{a}{z+2+\psi(z+3)}$, etc. Thus the value of $\psi(z)$ can be expressed as follows as a continued fraction:

$$\psi(z) = \frac{a}{z + \frac{a}{z+1 + \frac{a}{z+2 + \dots}, \text{etc.}}}$$

Reciprocally this continued fraction, extended to the infinite, has as its sum $\psi(z)$, or its equivalent $\frac{a}{z} \cdot \frac{\varphi(z+1)}{\varphi(z)}$, and this sum, developed in ordinary series, is

$$\frac{a}{z} \cdot \frac{1 + \frac{a}{z+1} + \frac{1}{2} \cdot \frac{a^2}{(z+1) \cdot (z+2)} + , \text{etc.}}{1 + \frac{a}{z} + \frac{1}{2} \cdot \frac{a^2}{z \cdot (z+1)} + , \text{etc.}}$$

Let now be $z = \frac{1}{2}$, the continued fraction will be

$$\psi(z) = \frac{2a}{1 + \frac{4a}{3 + \frac{4a}{5 + \dots}, \text{etc.}}}$$

so that its numerators, except the first, will be equal to $4a$, and its denominators will form the sequence of odd numbers 1, 3, 5, 7, etc. The value of this continued fraction can therefore also be expressed by

$$2a \cdot \frac{1 + \frac{4a}{2 \cdot 3} + \frac{16a^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{64a^3}{2 \cdot 3 \cdot \dots \cdot 7} + , \text{etc.}}{1 + \frac{4a}{2} + \frac{16a^2}{2 \cdot 3 \cdot 4} + \frac{64a^3}{2 \cdot 3 \cdot \dots \cdot 6} + , \text{etc.}}$$

But these series are known, and it is known that by denoting by e the number whose hyperbolic logarithm is 1, the previous expression reduces to $\frac{e^{2\sqrt{a}} - e^{-2\sqrt{a}}}{e^{2\sqrt{a}} + e^{-2\sqrt{a}}} \cdot \sqrt{a}$; so that in general we will have

$$\frac{e^{2\sqrt{a}} - e^{-2\sqrt{a}}}{e^{2\sqrt{a}} + e^{-2\sqrt{a}}} \cdot 2\sqrt{a} = \frac{4a}{1 + \frac{4a}{3 + \frac{4a}{5 + \dots}}, \text{ etc.}}$$

From these two main formulæ follow depending on whether a is positive or negative. Let first be $4a = x^2$, we will have

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{x}{1 + \frac{x^2}{3 + \frac{x^2}{5 + \dots}}, \text{ etc.}}$$

Let then be $4a = -x^2$, and because $\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \sqrt{-1} \cdot \tan x$, we will have

$$\tan x = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \dots}}, \text{ etc.}}$$

The latter is the formula will be the basis to our proof. But first we must prove the two following lemmas.

LEMMA 1. *Given a continued fraction extended to the infinite,*

$$\frac{m}{n + \frac{m'}{n' + \frac{m''}{n'' + \dots}}, \text{ etc.}}$$

in which all the numbers $m, n, m', n', \text{ etc.}$ are positive or negative integers; if we assume that the composing fractions $\frac{m}{n}, \frac{m'}{n'}, \frac{m''}{n''}, \text{ etc.}$ are all smaller than the unit, I claim that the total value of the continued fraction will necessarily be an irrational number.

First I claim that this value will be smaller than the unit.⁴ Indeed, without reducing the generality of the continued fraction, we can assume that all

⁴In what follows, the composing fractions are assumed to be smaller than 1 *in absolute value*.

the denominators $n, n', n'',$ etc. are positive: but, if we take only one term of the proposed sequence, we will have by hypothesis $\frac{m}{n} < 1$. If we take the first two, because of $\frac{m'}{n'} < 1$, it is clear that $n + \frac{m'}{n'}$ is larger than $n - 1$: but m is smaller than n ; and since they are both integers, m will also be smaller than $n + \frac{m'}{n'}$. Hence the value which results from the two terms

$$\frac{\frac{m}{n}}{n + \frac{m'}{n'}}$$

is smaller than the unit. Let us compute three terms of the proposed continued fraction; and first, following what we have seen, the value of the component

$$\frac{\frac{m'}{n'}}{n' + \frac{m''}{n''}}$$

will be smaller than the unit. Let us call this value ω , and it is clear that $\frac{m}{n+\omega}$ will be even smaller than unit: therefore what results from the three terms

$$\frac{\frac{m}{n}}{n + \frac{m'}{n' + \frac{m''}{n''}}}$$

is smaller than the unit. Pursuing the same reasoning, one will see that, whatever the number of terms which are computed of the proposed continued fraction, the value which will result is smaller than the unit; therefore the total value of this fraction extended to the infinite is also smaller than the unit. It could be equal to the unit only in the sole case where the proposed fraction would be of the form

$$\frac{\frac{m}{m+1 - \frac{m'}{m'+1 - \frac{m''}{m''+1 - \dots}}}}{m''+1 - \dots}, \text{ etc.}$$

In all the other cases it will be smaller.

That being set, if one denies that the value of the proposed continued fraction is equal to an irrational number, let us assume it is equal to a rational number, and let this number be $\frac{B}{A}$, B and A being some integers; we will

thus have

$$\frac{B}{A} = \frac{m}{n + \frac{m'}{n' + \frac{m''}{n'' + \dots}}}, \text{ etc.}$$

Let C, D, E , etc. be indeterminate values such that we have⁵

$$\frac{C}{B} = \frac{m'}{n' + \frac{m''}{n'' + \frac{m'''}{n'' + \dots}}}, \text{ etc.}$$

$$\frac{D}{C} = \frac{m''}{n'' + \frac{m'''}{n'' + \frac{m^{IV}}{n^{IV} + \dots}}}, \text{ etc.}$$

and so on to the infinite. These different continued fractions having all their terms smaller than the unit, their values or sums $\frac{B}{A}, \frac{C}{B}, \frac{D}{C}, \frac{E}{D}$, etc. will be smaller than the unit, following what we have just proven, and therefore we will have⁶ $B < A, C < B, D < C$, etc.; in such a way that the sequence A, B, C, D, E , etc. is decreasing to the infinite. But the sequence of continued fractions considered here gives

$$\frac{B}{A} = \frac{m}{n + \frac{C}{B}} \quad ; \text{ from which it follows } C = mA - nB,$$

$$\frac{C}{B} = \frac{m'}{n' + \frac{D}{C}} \quad ; \text{ from which it follows } D = m'B - n'C,$$

$$\frac{D}{C} = \frac{m''}{n'' + \frac{E}{D}} \quad ; \text{ from which it follows } E = m''C - n''D,$$

etc. etc.

And since the first two numbers A and B are integers by assumption, it follows that all the others C, D, E , etc., which until now were indeterminate, are also integer numbers. But a contradiction follows from the fact that an infinite sequence A, B, C, D, E , etc. is both decreasing and composed of

⁵These expressions are valid, because C, D, E , etc., are not assumed to be integers. C is determined by the choice of B , D is determined by the choice of C , and so on.

⁶all these orders are implicitly in absolute value: $|B| < |A|$, etc.

integer numbers; indeed as a matter of fact none of the numbers A, B, C, D, E , etc. can be zero, since the proposed continued fraction extends to the infinite, and that thus the sums represented by $\frac{B}{A}, \frac{C}{B}, \frac{D}{C}$, etc. must always be something.⁷ Therefore the assumption that the sum of the proposed continued fraction is equal to a rational quantity $\frac{B}{A}$ could not be maintained; therefore this sum is necessarily an irrational number.

LEMMA II. *The same things being set, if the component fractions $\frac{m}{n}, \frac{m'}{n'}, \frac{m''}{n''}$, etc. are of any size at the beginning of the sequence, but if after a certain interval they are constantly smaller than the unit, I claim that the proposed continued fraction, still assuming it extends to the infinite, will have an irrational value. Indeed if starting with $\frac{m'''}{n'''}$, for instance, all the fractions $\frac{m'''}{n'''}, \frac{m^{IV}}{n^{IV}}, \frac{m^V}{n^V}$, etc. at the infinite, are smaller than the unit; then, according to lemma I, the continued fraction*

$$\frac{m'''}{n'''} + \frac{m^{IV}}{n^{IV} + \frac{m^V}{n^V + \dots}}, \text{ etc.}$$

will have an irrational value. Let us call this value ω , and the proposed continued fraction will become

$$\frac{m}{n + \frac{m'}{n' + \frac{m''}{n'' + \omega}}}.$$

But if one does in succession

$$\frac{m''}{n'' + \omega} = \omega', \quad \frac{m'}{n' + \omega'} = \omega'', \quad \frac{m}{n + \omega''} = \omega''',$$

it is clear that, because ω is irrational, that all the quantities $\omega', \omega'', \omega'''$, will likewise be. But the last one ω''' is equal to the proposed continued fraction; thus its value is irrational.

We can now, coming back to our subject, prove this general proposition.

THEOREM.

⁷that is, they must have some non-zero value.

If an arc is commensurable with the radius, its tangent will be incommensurable with that same radius.

Indeed, let the radius = 1, and the arc $x = \frac{m}{n}$, m and n being integer numbers, the formula obtained above will give, when the substitution is performed,

$$\tan \frac{m}{n} = \frac{m}{n - \frac{m^2}{3n - \frac{m^2}{5n - \frac{m^2}{7n - \dots}}}}, \text{ etc.}$$

But this continued fraction is in the case of lemma II; for it is clear that the denominators $3n, 5n, 7n$, etc. increasing continually whereas the numerator m^2 remains of the same size, the composing fractions will or will soon become smaller than the unit; consequently the value of $\tan \frac{m}{n}$ is irrational; therefore, *if the arc is commensurable with the radius, its tangent will be incommensurable.*

From these results as a very immediate consequence the proposition which is the object of this note. Let π be the half circumference whose radius is 1; if π were rational, the arc $\frac{\pi}{4}$ would also be, and consequently its tangent should be irrational: but it is known instead that the tangent of the arc $\frac{\pi}{4}$ is equal to the radius 1; thus π can not be rational. Therefore *the ratio of the circumference to the diameter is an irrational number.*

It is likely that the number π does not even belong to the algebraic irrationals, that is that it cannot be the root of an algebraic equation of a finite number of terms whose coefficients are rational: but it seems very difficult to prove this proposition rigorously; we can only show that the square of π is again an irrational number.

Indeed if in the continued fraction which expresses $\tan x$, one does $x = \pi$, because of $\tan \pi = 0$, one must have⁸

$$0 = 3 - \frac{\pi^2}{5 - \frac{\pi^2}{7 - \frac{\pi^2}{9 - \dots}}}, \text{ etc.}$$

But if π^2 were rational, and if we had $\pi^2 = \frac{m}{n}$, m and n being integers, this

⁸From the expression of $\tan x$, one actually finds that $1 - \frac{\pi^2}{3 - \frac{\pi^2}{\dots}}$ must be infinite, and hence that $3 - \frac{\pi^2}{\dots}$ must be equal to 0.

would entail

$$3 = \frac{m}{5n - \frac{m}{7n - \frac{m}{9n - \frac{m}{11n - \text{etc.}}}}}$$

But it is visible that this continued fraction is still in the case of lemma II; its value is therefore irrational and could not be equal to the number 3. Therefore *the square of the ratio of the circumference to the diameter is an irrational number.*

Chapter 4

Modern proofs

In this chapter, I give the main modern shorter proofs of the irrationality of π I know of, and allude to a few others which have not been included, either because they are longer, or because they would go beyond the limits I set myself for this work.

4.1 Gauss (1850s?)

In his Nachlass [55], Carl Friedrich Gauss (1777-1855) considers $m, n > 0$ and the series

$$P = 1 - \frac{1}{2} \cdot \frac{m^2}{n^2} + \frac{1}{2 \cdot 4} \cdot \frac{1}{1 \cdot 3} \cdot \frac{m^4}{n^4} - \frac{1}{2 \cdot 4 \cdot 6} \cdot \frac{1}{1 \cdot 3 \cdot 5} \cdot \frac{m^6}{n^6} + \text{etc.}$$

and for $\theta \geq 1$

$$P_\theta = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2\theta - 1)} \cdot \frac{m^{2\theta-1}}{n^\theta} - \frac{1}{2} \cdot \frac{1}{1 \cdot 3 \cdot 5 \cdots (2\theta + 1)} \cdot \frac{m^{2\theta+1}}{n^{\theta+2}} \\ + \frac{1}{2 \cdot 4} \cdot \frac{1}{1 \cdot 3 \cdot 5 \cdots (2\theta + 3)} \cdot \frac{m^{2\theta+3}}{n^{\theta+4}} - \text{etc.}$$

which are all convergent. Although Gauss does not explicit it, we have $P = \cos \frac{m}{n}$ and $P_1 = \sin \frac{m}{n}$, as observed by Pringsheim [123].

Gauss's reasoning is actually based on §. 46 in Lambert's memoir and he observes that $P_\theta, P_{\theta+1}, P_{\theta+2}, \text{etc.}$, represents a sequence of decreasing values, as long as $2\theta + 1 \geq \frac{m^2}{n} + \frac{m^2}{2n^2}$. Then, if $\frac{P_1}{P} = \tan \frac{m}{n}$ is rational, P and P_1 are proportional to integers, and so are P_θ for $\theta > 1$, because $P_{\theta+2} = (2\theta + 1)nP_{\theta+1} - m^2P_\theta$. We would then have an infinite decreasing sequence of integers which is impossible.

Gauss thus concludes that $\tan \frac{m}{n}$ cannot be rational if m and n are integers.

Pringsheim [123] adds some comments on Gauss's derivation, stressing that Gauss must have been very familiar with Lambert's work, something which was confirmed in Pringsheim's second note.

In the note published shortly afterwards [124],¹ Pringsheim was able to locate an unpublished manuscript by Gauss, dated 1850. The derivation published from the Nachlass may also be dated from the 1850s.

Gauss's observations concern §. 48 and §. 49 in Lambert's memoir, who assumes, without saying so, that $w > 1$. In his manuscript, Gauss proves that the limit of R' , R'' , etc., is 0, which was not clearly proved by Lambert.

4.2 Glaisher (1872)

In a note published in 1872 [56], J.W.L. Glaisher (1848-1928) gave a concise version of Lambert's proof of the irrationality of π , but he also gave an interesting derivation of Lambert's continued fraction for $\tan v$ using differential equations.

Glaisher considers the equation² $y = \cos(\sqrt{2x})$ which satisfies the differential equation

$$y + y' + 2xy'' = 0$$

It is easy to see that by differentiating this equation i times, that one obtains

$$y^{(i)} + (2i + 1)y^{(i+1)} + 2xy^{(i+2)} = 0$$

And it follows that

$$\begin{aligned} \frac{y'}{y} &= \frac{-1}{1 + 2x\frac{y''}{y'}} \\ \frac{y''}{y'} &= \frac{-1}{3 + 2x\frac{y'''}{y''}} \\ &\dots\dots \\ \frac{y^{(i+1)}}{y^{(i)}} &= \frac{-1}{(2i + 1) + 2x\frac{y^{(i+2)}}{y^{(i+1)}}} \end{aligned}$$

¹For some reason, this second note was forgotten in Wallisser's account [150] who cites only the first note by Pringsheim.

²I have slightly simplified the definition of y , as it was including two constants one of which can be taken equal to 0 and the other to 1.

and therefore

$$\frac{y'}{y} = -\frac{1}{\sqrt{2x}} \tan(\sqrt{2x}) = \frac{-1}{1 - \frac{2x}{3 - \frac{2x}{5 - \text{etc.}}}}$$

Setting $\sqrt{2x} = v$, one then obtains

$$\tan v = \frac{v}{1 - \frac{v^2}{3 - \frac{v^2}{5 - \text{etc.}}}}$$

4.3 Hermite (1873)

No significant new proof of the irrationality of π seems to have been published between Legendre and Hermite, the work of Gauss having not been published.

In an excerpt of a letter to the German mathematician Carl Borchardt (1817-1880) [67], the French mathematician Charles Hermite (1822-1901) gave a new proof of the irrationality of π . I give here a sketch of this proof, using ideas by Juhel [86]. Hermite considers the equations

$$A_n = Q_n(x) \sin x - P_n(x) \cos x \quad (4.1)$$

and refers for definitions of A_n , Q_n and P_n to a letter sent by him to Paul Gordan (1837-1912) [66]. There, Hermite sets $A_0(x) = \sin x$, and successively

$$\begin{aligned} A_1(x) &= \int_0^x A_0(t)t dt = \sin x - x \cos x \\ A_2(x) &= \int_0^x A_1(t)t dt = (3 - x^2) \sin x - 3x \cos x, \\ A_3(x) &= \int_0^x A_2(t)t dt = (15 - 6x^2) \sin x - (15x - x^3) \cos x, \\ &\text{etc.} \end{aligned}$$

In fact, we have $Q_0(x) = Q_1(x) = 1$, $P_0(x) = 0$, $P_1(x) = x$, and

$$\begin{aligned} Q_n(x) &= (2n - 1)Q_{n-1}(x) - x^2Q_{n-2}(x) \\ P_n(x) &= (2n - 1)P_{n-1}(x) - x^2P_{n-2}(x) \end{aligned}$$

It is possible to show that

$$A_n(x) = \frac{x^{2n+1}}{2 \cdot 4 \cdots 2n} \int_0^1 (1 - z^2)^n \cos(xz) dz$$

This expression is in turn related to the Bessel function $J_{n+1/2}(x)$.

In any case, assuming n even, $Q_n(x)$ is a polynomial in x^2 and if we let $x = \frac{\pi}{2}$ with $\frac{\pi^2}{4} = \frac{a}{b}$, then $Q_n(x) = \frac{N}{a^{n/2}}$ where N is an integer.

Then

$$N = \frac{\left(\frac{b}{a}\right)^{\frac{1}{2}} \left(\frac{b}{\sqrt{a}}\right)^n}{2 \cdot 4 \cdots 2n} \int_0^1 (1 - z^2)^n \cos\left(\frac{\pi}{2}z\right) dz, \quad (4.2)$$

This leads to a contradiction, because the right-hand side of this equation tends to 0, without ever being equal to 0, and N is an integer.

Therefore π^2 is not rational, and also not π .

Cartwright's proof (see below) is a simplification of Hermite's proof.

4.4 Hermite (1882)

In his 1882 lecture [68, p. 68-69], Hermite dispenses with the integrals and adapted his earlier proof:

Hermite sets

$$\begin{aligned} X &= \frac{\sin x}{x} \\ X_1 &= -\frac{1}{x}X' = \frac{1}{x^3}(\sin x - x \cos x) \\ X_2 &= -\frac{1}{x}X_1' = \frac{1}{x^5}[(3 - x^2)\sin x - 3x \cos x] \\ X_3 &= -\frac{1}{x}X_2' = \frac{1}{x^7}[(15 - 6x^2)\sin x - (15x - x^3)\cos x] \\ &\dots\dots \\ X_{n+1} &= -\frac{1}{x}X_n' \end{aligned}$$

It is now already clear that $X_n = \frac{1}{x^{2n+1}}A_n$ with the 1873 definition of A_n . We have

$$X_n = \frac{1}{x^{2n+1}} [\Phi(x) \sin x - \Phi_1(x) \cos x]$$

where $\Phi(x)$ and $\Phi_1(x)$ are of degree n and $n - 1$, or $n - 1$ and n , depending whether n is even or odd.

But Hermite goes on and writes

$$\begin{aligned} X &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \\ X_n &= \frac{1}{1.3.5 \dots (2n+1)} \left[1 - \frac{x^2}{2(2n+3)} + \frac{x^4}{2.4(2n+3)(2n+5)} - \dots \right] \\ &= \frac{1}{1.3.5 \dots (2n+1)} S \end{aligned}$$

which is easily proved by induction.

By grouping two adjoining terms, S can be rewritten as follows:

$$S = \left[1 - \frac{x^2}{2(2n+3)} \right] + \frac{x^4}{2.4(2n+3)(2n+5)} \left[1 - \frac{x^2}{6(2n+7)} \right] + \dots$$

This series is strictly positive if $1 - \frac{x^2}{2(2n+3)} > 0$ and this condition is met if $x = \frac{\pi}{2}$ and $n \geq 0$.

Let us now assume that $\frac{\pi}{2} = \frac{b}{a}$, with a and b integers. We therefore have

$$\Phi(b/a) = \left(\frac{\pi}{2}\right)^{2n+1} X_n = \frac{A}{a^n}$$

where A is a positive integer (as a polynomial in b of degree $\leq n$). And therefore

$$A = \frac{a^n \left(\frac{\pi}{2}\right)^{2n+1}}{1.3.5 \dots (2n+1)} S$$

And since the right-hand side tends to 0 but always remains positive, we obtain a contradiction, and π must be irrational.

Hermite observes that the result still holds if we take $\left(\frac{\pi}{2}\right)^2 = \frac{b}{a}$, because $\Phi(x)$ only contains even powers.

4.5 Hobson (1918)

In his *Treatise on plane trigonometry* (1918 and probably earlier editions) [72, p. 374-375], Hobson gave a proof of the irrationality of π which is a clear adaptation of Legendre's proof where a is restricted to the case < 0 .

Laczkovich uses the same line of thought, but refers to Gauss [53]. Hobson's function below is a hypergeometric function [40, 59], and Gauss derives some continued fractions from hypergeometric functions, but not exactly the case below.

Hobson considers the function

$$f(c) = 1 - \frac{x^2}{1 \cdot c} + \frac{x^4}{1 \cdot 2 \cdot c(c+1)} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} + \dots$$

Then

$$f(c+1) - f(c) = \frac{x^2}{c(c+1)} f(c+2)$$

and therefore

$$\frac{f(c)}{f(c+1)} = 1 - \frac{x^2}{c(c+1)} \frac{f(c+2)}{f(c+1)}$$

and $\frac{f(c+1)}{f(c)}$ can be expressed as the continued fraction

$$\frac{1}{1 - \frac{x^2/[c(c+1)]}{1 - \frac{x^2/[(c+1)(c+2)]}{1 - \frac{x^2/[(c+2)(c+3)]}{1 - \text{etc.}}}}}$$

Now, if $c = \frac{1}{2}$ and replacing x by $x/2$, the series $f(c)$ becomes

$$f(c) = \cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \dots$$

and $f(c+1)$ becomes $\frac{\sin x}{x}$. Consequently (for details, see [91])

$$\frac{\tan x}{x} = \frac{1}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \text{etc.}}}}}$$

For $x = \frac{\pi}{4}$, Hobson observes that the right-hand side is irrational, referring to Chrystal [29], who himself used Legendre's lemma and this proves the irrationality of π .

4.6 Popken (1940)

Jan Popken (1905-1970) had the purpose of freeing Lambert's proof from the theory of continued fractions and therefore made it simpler. His proof

is actually related to Hermite's proof, as he observed later [120]. In his first article [115], he proceeds as follows. He first proves a lemma about the existence of polynomials $p_h(x^{-1})$ and $q_h(x^{-1})$ with integral coefficients and of degree at most $2h$ (with $h \geq 0$), such that

$$p_h(x^{-1}) \sin x + q_h(x^{-1}) \cos x = (-2)^h \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2) \cdots (n+h)}{(2n+2h+1)!} x^{2n+1} \quad (4.3)$$

This is clear for $h = 0$:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \quad (4.4)$$

In his later publication [117], Popken also gives examples of such polynomials for $h = 1$ and $h = 2$:

$$\begin{aligned} -x^{-2} \sin x + x^{-1} \cos x &= -2 \left(\frac{1 \cdot x}{3!} - \frac{2 \cdot x^3}{5!} + \frac{3 \cdot x^5}{7!} - \frac{4 \cdot x^7}{9!} + \cdots \right) \\ (3x^{-4} - x^{-2}) \sin x - 3x^{-3} \cos x &= (-2)^2 \left(\frac{1 \cdot 2x}{5!} - \frac{2 \cdot 3x^3}{7!} + \frac{3 \cdot 4x^5}{9!} - \cdots \right) \end{aligned}$$

One can notice that these expressions have a simple relationship with Hermite's A_n .

Now, assuming that π is rational and setting $\frac{\pi}{4} = \frac{a}{b}$, and applying the lemma with $x = \frac{a}{b}$, we obtain for $h \geq 0$:

$$\left\{ p_h \left(\frac{b}{a} \right) + q_h \left(\frac{b}{a} \right) \right\} \frac{\sqrt{2}}{2} = (-2)^h \left(\frac{h!}{(2h+1)!} \frac{\pi}{4} - \frac{2 \cdot 3 \cdots (h+1)}{(2h+3)!} \left(\frac{\pi}{4} \right)^3 + \cdots \right) \quad (4.5)$$

In the alternating series between brackets, the terms decrease and tend to 0.

Consequently

$$2^h \left(\frac{h!}{(2h+1)!} \frac{\pi}{4} - \frac{2 \cdot 3 \cdots (h+1)}{(2h+3)!} \left(\frac{\pi}{4} \right)^3 \right) < \frac{\sqrt{2}}{2} \left| p_h \left(\frac{b}{a} \right) + q_h \left(\frac{b}{a} \right) \right| < 2^h \frac{h!}{(2h+1)!} \frac{\pi}{4} < \frac{2^h}{h!}, \quad (4.6)$$

and

$$0 < \left| a^{2h} p_h \left(\frac{b}{a} \right) + a^{2h} q_h \left(\frac{b}{a} \right) \right| < \sqrt{2} \frac{2^h |a|^{2h}}{h!} \quad (4.7)$$

The assumptions on the polynomials p_h and q_h entail that $|a^{2h} p_h \left(\frac{b}{a} \right) + a^{2h} q_h \left(\frac{b}{a} \right)|$ is an integer, for each $h \geq 0$. But since $\lim_{h \rightarrow \infty} \sqrt{2} \frac{2^h |a|^{2h}}{h!} = 0$, this is impossible. Hence, π cannot be rational.

4.7 Cartwright (1945)

The following proof was given as an exercise by Mary Cartwright (1900-1998) in Cambridge in 1945, but it has an earlier origin and is obviously related to Hermite's first proof. It was published by Jeffreys in 1957 [79, p. 231] and 1973 [80, p. 268].

Cartwright considers the integrals

$$I_n = \int_{-1}^1 (1-x^2)^n \cos(\alpha x) dx$$

Integrating by parts twice gives (for $n \geq 2$)

$$\alpha^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2}$$

Setting $J_n = \alpha^{2n+1}I_n$, it becomes

$$J_n = 2n(2n-1)J_{n-1} - 4n(n-1)\alpha^2 J_{n-2}.$$

We also have $J_0 = 2 \sin \alpha$ and $J_1 = -4\alpha \cos \alpha + 4 \sin \alpha$. Therefore for all n

$$J_n = \alpha^{2n+1}I_n = n!(P_n \sin \alpha + Q_n \cos \alpha)$$

where P_n, Q_n are polynomials in α of degree $\leq 2n$ and with integral coefficients depending on n .

Now, let $\alpha = \frac{\pi}{2} = \frac{b}{a}$ where a and b are integers. Then

$$\frac{b^{2n+1}}{n!}I_n = P_n a^{2n+1}.$$

The right side is an integer. But $0 < I_n < 2$ since

$$0 < (1-x^2)^n \cos\left(\frac{1}{2}\pi x\right) < 1 \text{ for } -1 < x < 1$$

and $b^{2n+1}/n! \rightarrow 0$ as $n \rightarrow \infty$. Hence, for sufficiently large n

$$0 < b^{2n+1}I_n/n! < 1$$

and there would be an integer between 0 and 1. This shows that $\frac{\pi}{2}$ cannot be a rational.

Cartwright's proof was reprinted in several other places, such as in Earl's *Towards Higher Mathematics* [41, p. 383]. Boros and Noll [20, p. 117-118]

also give a slight variant where they take $\pi = \frac{a}{b}$ instead of $\frac{\pi}{2} = \frac{b}{a}$ in the above proof. They do however incorrectly attribute their proof to Niven.

In 2004, Ghislain Dupont [39] published another variant of Cartwright's proof (without referring to it), first defining

$$I_n = \frac{1}{n!} \int_{-\pi/2}^{\pi/2} \left(\frac{\pi^2}{4} - t^2 \right)^n dt$$

He observes that $I_n > 0$ and $\lim_{n \rightarrow \infty} r^n I_n = 0$ for any integer $r > 0$. He then proves by induction that I_n is a polynomial in π^2 of degree at most n and with integer coefficients:

$$I_n = P_n(\pi^2)$$

Assuming then that $\pi^2 = r/s$ with r and s integers, it follows that $s^n I_n$ is an integer whose limit is 0 when $n \rightarrow \infty$, which is impossible since $I_n > 0$. Hence π^2 is irrational, and so must be π . The same proof was borrowed by Oliveira [36], without any sources.

Lord [102] may have some additional information, but I haven't been able to obtain his article.

4.8 Niven (1947)

In 1947, Ivan Niven (1915-1999) published a simple proof that π is irrational [109]. This was also based on a technique introduced by Hermite. Assuming that $\pi = a/b$, Niven defined the two polynomials

$$f(x) = \frac{x^n(a - bx)^n}{n!},$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \dots + (-1)^n f^{(2n)}(x),$$

For $j < n$, $f^{(j)}(x) = x^{n-j} \times \dots$ and $f^{(j)}(0) = 0$.

For $n \leq j \leq 2n$, we have

$$f(x) = \frac{x^n}{n!} \left[\dots + \binom{n}{j-n} a^{2n-j} (-bx)^{j-n} + \dots \right]$$

and consequently

$$f^{(j)}(x) = \dots + \frac{j!}{n!} \binom{n}{j-n} a^{2n-j} (-b)^{j-n} + \dots$$

$\frac{j!}{n!}$, $\binom{n}{j-n}$ and $a^{2n-j} (-b)^{j-n}$ are integers, hence $f^{(j)}(0)$ has an integral value.

And since $f(x) = f(a/b - x)$, we have $f(\pi) = 0$ and as $f^{(j)}(x) = (-1)^j f^{(j)}(a/b - x)$, $f^{(j)}(\pi)$ is also an integer.

We also have

$$\frac{d}{dx} \{F'(x) \sin x - F(x) \cos x\} = F''(x) \sin x + F'(x) \cos x = f(x) \sin x$$

because $f^{(2n+2)}(x) = 0$.

Consequently

$$\int_0^\pi f(x) \sin x \, dx = [F'(x) \sin x - F(x) \cos x]_0^\pi = F(\pi) + F(0) \quad (4.8)$$

which is equal to an integer, because of the above results.

But, because of the definition of $f(x)$, we have, for $0 < x < \pi$,

$$0 < f(x) \sin x < \frac{\pi^n a^n}{n!}$$

Therefore the integral (4.8) is positive, but can be made arbitrarily small for n sufficiently large. It follows that (4.8) is false, and with it the assumption that π is rational.

Niven's proof can actually be slightly adapted to prove that π^2 is irrational, which then also entails the irrationality of π .

Bourbaki's treatise [22] has an exercise for proving the irrationality of π . It is in fact Niven's proof, but taking instead of $f(x) = \frac{x^n(a-bx)^n}{n!}$ the function $h(x) = \frac{x^n(a/b-x)^n}{n!} = f(x)/b^n$.

In his book on irrational numbers [110], Niven reproduces his previous proof of the irrationality of π , but also gives another proof where he deduces the irrationality of π from the fact that $\cos r$ is irrational for any rational number $r \neq 0$. Since $\cos \pi = -1$, π is necessarily irrational.

Hardy and Wright [64], Remmert [128], Spivak [141], as well as Eymard and Lafon [46, p. 129-130] reproduced slight adaptations of Niven's proof where

$$f(x) = \frac{x^n(1-x)^n}{n!}.$$

x then varies between 0 and 1 instead of between 0 and π .

Ribenboim [129] and Beukers [19] also gave Niven's proof.

Jones published an insightful motivation of Niven's proof [81, 84] which I will not describe here, but that the curious reader might want to study. Other interesting studies are those of Conrad [30] and again of Jones [83] and Zhou and Markov [155, 153, 154].

Parks [113] generalized Niven's proof and proved for instance that if $0 < |r| \leq \pi$ and if $\cos(r)$ and $\sin(r)$ are rational, then r is irrational. He also proved that if r is positive and rational, $r \neq 1$, then $\ln(r)$ is irrational.

Koksma [90] has also adapted Niven's proof, this time in order to prove the irrationality of $e^{p/q}$ for a rational $p/q > 0$.

4.9 Popken (1948)

In 1948, Popken gave a simpler proof [119].

Popken first obtains the expressions for R' , R'' , etc. (see §. 7 in Lambert's memoir):

$$\begin{aligned}\cos v &= w \sin v + R' \\ \sin v &= -3wR' + R'' \\ R' &= 5wR'' + R''' \\ &\dots\dots \\ R^n &= (-1)^{n+1}(2n+3)wR^{n+1} + R^{n+2}\end{aligned}$$

and in general, as mentioned earlier (in a note to Lambert's §. 8):

$$R^n = (-1)^{n(n+1)/2} 2^n \sum_{m=0}^{\infty} (-1)^m \frac{(n+m)!}{m!(2n+2m+1)!} v^{n+2m+1} \quad (4.9)$$

Popken then assumes that v is rational and that the denominator of $w = 1/v$ is a . He also assumes that $\tan v = q/p$, so that $\cos v = p\rho$ and $\sin v = q\rho$, for $\rho > 0$.

Now, from the above, it follows that

$$R'(v) = p\rho - wq\rho$$

so that $\frac{R'}{\rho}$ is a fraction with denominator a . Likewise, $\frac{R''}{\rho}$ is a fraction with denominator a^2 , etc. In general, $\frac{R^n}{\rho}$ is a fraction with denominator a^n , so that $\frac{R^n a^n}{\rho}$ is an integer.

From equation (4.9), it follows that

$$|R^n| < \frac{A^n}{n!}$$

where A is a positive integer not depending on n .

For n sufficiently large, we therefore have $\frac{R^n a^n}{\rho} < 1$, and consequently $R^n = 0$.

However if for a given value of n we have both R^n and R^{n+1} equal to 0, then it follows from the construction of R^n that $R^{n-1} = 0$, $R^{n-2} = 0$, and eventually $\sin v = 0$ and $\cos v = 0$.

Now, for $v = \pi$, $\cos v = -1$ and $\sin v = 0$, therefore we should have also $\cos v = 0$ which is a contradiction. Therefore, π can not be rational.

4.10 Breusch (1954)

In 1954, Breusch [24] gave yet another relatively elementary proof. He began by assuming that $\pi = a/b$ with a and b integers. Then, setting $N = 2a = 2b\pi$, one immediately has $\sin N = 0$, $\cos N = 1$ and $\cos(N/2) = \pm 1$ (depending on the parity of b).

Breusch then defined the series

$$A_m(x) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^m \frac{x^{2k+1}}{(2k+1)!}$$

for each integer $m \geq 0$. It is easy to see that $A_0 = \sin x$ and that

$$A_{m+1}(x) = x \frac{dA_m}{dx}$$

and to prove by induction that

$$A_m(x) = P_m(x) \cos x + Q_m(x) \sin x$$

where $P_m(x)$ and $Q_m(x)$ are polynomials with integer coefficients.

Consequently, $A_m(N) = P_m(N)$ is an integer for every $m \geq 0$.

Then for an integer $t > 0$, Breusch defines the series

$$\begin{aligned} B_t(N) &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1-t-1)(2k+1-t-2) \cdots (2k+1-2t)}{(2k+1)!} N^{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^t - b_1(2k+1)^{t-1} + \cdots \pm b_t}{(2k+1)!} N^{2k+1} \\ &= A_t(N) - b_1 A_{t-1}(N) + \cdots \pm b_t A_0(N). \end{aligned}$$

The values of b_i are integers depending on t and in particular $b_t = (t+1)(t+2) \cdots 2t$.

Consequently $B_t(N)$ is an integer too.

Breusch now separates the sum $B_t(N)$ in three parts:

$$B_t(N) = \sum_{k=0}^{\lfloor (t-1)/2 \rfloor} + \sum_{k=\lfloor (t+1)/2 \rfloor}^{t-1} + \sum_{k=t}^{\infty}$$

In all three sums, the numerator of each fraction is a product of t consecutive integers, and is therefore divisible by $t!$. In particular, in the first sum, since $2k + 1 \leq t$, the numerator is also divisible by $(2k + 1)!$ and each term of the first sum is an integer. In the second sum, the second factor of the first numerator,

$$(t + 2 - (t + 1)) \overbrace{(t + 2 - (t + 2))}^{=0} \cdots (t + 2 - 2t)$$

the fourth factor of the second numerator,

$$(t + 4 - (t + 1))(t + 4 - (t + 2))(t + 4 - (t + 3)) \overbrace{(t + 4 - (t + 4))}^{=0} \cdots (t + 4 - 2t)$$

the sixth factor of the third numerator, etc., as well as the penultimate factor of the last numerator

$$(2t - 1 - (t + 1)) \cdots \overbrace{(2t - 1 - (2t - 1))}^{=0} (2t - 1 - 2t)$$

are equal to 0. Therefore, the second sum is equal to 0.

Therefore, since $B_t(N)$ is an integer, the third sum $\sum_{k=t}^{\infty}$ must also be an integer.

Now, since

$$(2k + 1 - t - 1)(2k + 1 - t - 2) \cdots (2k + 1 - 2t) = \frac{(2k - t)!}{(2k - 2t)!}$$

the third sum is

$$\begin{aligned} & \sum_{k=t}^{\infty} (-1)^k \frac{(2k - t)!}{(2k + 1)!(2k - 2t)!} N^{2k+1} \\ &= (-1)^t \frac{t!}{(2t + 1)!} N^{2t+1} \\ & \left(1 - \frac{(t + 1)(t + 2)}{(2t + 2)(2t + 3)} \frac{N^2}{2!} + \frac{(t + 1)(t + 2)(t + 3)(t + 4)}{(2t + 2)(2t + 3)(2t + 4)(2t + 5)} \frac{N^4}{4!} - \cdots \right) \end{aligned}$$

If $S(t)$ is the sum between parentheses, we clearly have

$$|S(t)| < 1 + N + \frac{N^2}{2!} + \cdots = e^N$$

and therefore

$$\left| \sum_{k=t}^{\infty} (-1)^k \frac{(2k - t)!}{(2k + 1)!(2k - 2t)!} N^{2k+1} \right| < \frac{t!}{(2t + 1)!} N^{2t+1} e^N < \frac{t!}{t!t^{t+1}} N^{2t+1} e^N < \left(\frac{N^2}{t} \right)^{t+1} e^N$$

When t is greater than some integer t_0 , $\left(\frac{N^2}{t}\right)^{t+1} e^N < 1$. Therefore, the third sum above, which is an integer, must be $S(t) = 0$ for every integer $t > t_0$.

However, this is impossible because

$$\lim_{t \rightarrow \infty} S(t) = 1 - \frac{1}{2^2} \cdot \frac{N^2}{2!} + \frac{1}{2^4} \cdot \frac{N^4}{4!} - \dots = \cos(N/2) = \pm 1.$$

Therefore the initial assumption that $\pi = a/b$ is not valid and π is irrational.

4.11 Laczkovich (1997)

In 1997, Laczkovich [91] published a new proof of the irrationality of π , in that he simplified a result by Popken [119]. Laczkovich vaguely refers to Gauss [53], but an exact reference is not given. Laczkovich may however have been inspired by the way Gauss obtains a continued fraction from the ratio of two hypergeometric functions. In any case, as already mentioned, it should be noted that Laczkovich's proof is similar to Hobson's proof shown above.

Laczkovich starts with the family of series

$$f_k(x) = 1 - \frac{x^2}{k} + \frac{x^4}{k(k+1) \cdot 2!} - \frac{x^6}{k(k+1)(k+2) \cdot 3!} + \dots$$

$f_k(x)$ converges for every x and $k \neq 0, -1, -2, \dots$

It is easy to see that $f_{1/2}(x) = \cos(2x)$ and $f_{3/2}(x) = \frac{\sin(2x)}{2x}$.

Likewise, we have

$$\frac{x^2}{k(k+1)} f_{k+2}(x) = f_{k+1}(x) - f_k(x)$$

for the same values of x and k as above.

Now, Laczkovich uses an argument by Popken [119] and proves the theorem

Theorem. *If $x \neq 0$ and x^2 is rational, then $f_k(x) \neq 0$ and $f_{k+1}(x)/f_k(x)$ is irrational for every $k \in \mathbb{Q}$, $k \neq 0, -1, -2, \dots$*

Laczkovich's proof is by infinite descent but I refer the reader to the original article for details. The proof itself is rather straightforward.

The irrationality of π^2 (and therefore of π) follows from the observation $f_{1/2}(\pi/4) = \cos(\pi/2) = 0$.

Finally Laczkovich derives the continued fraction

$$\tan x = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \text{etc.}}}}}$$

in much the same way as Hobson, but with more detail.

4.12 Other proofs

The summary above is not meant to be complete. There are certainly other proofs of the irrationality of π , some of them mere variants of proofs I have described, and some of them using different ideas. Without going into the details of the proofs, let me mention a few:

- In 2001, Huylebrouck [74] provided a uniform treatment for the irrationality proofs of π , $\ln 2$, $\zeta(2)$ and $\zeta(3)$. His proof of the irrationality of π involves the integral

$$\int_0^1 P_n(x) \sin(\pi x) dx$$

where P_n is a polynomial defined by

$$P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^n(1-x)^n)$$

Huylebrouck claims that these polynomials are the so-called Legendre polynomials, but they are actually a variant of them.

For more on this proof and the related ones, I refer the reader to Huylebrouck's article.

- Another proof of interest is that of Angell [2]. In order to prove the irrationality of $\tan r$ for a rational $r > 0$, Angell studies the theory of generalized continued fraction, and borrows some ideas from Chrystal [29]. Again, the interested reader should consult Angell's proof. Bott also derived the irrationality of π in a similar way [21].

4.13 Irrationality of powers of π and other results

The methods used to prove the irrationality of π have often been adapted to obtain other results. For instance, observing that there were no separate proofs of the irrationality of π^n for $n > 2$, other than those deriving the irrationality of π^n from the transcendence of π , Jones obtained such proofs for e^n and π^n [82].

Iwamoto [77] extended Niven's proof to π^2 and Inkeri [75] proved the irrationality of π^2 as a direct application of Hermite's method.

Beukers has also proven the irrationality of $\zeta(2) = \frac{\pi^2}{6}$ [18] and his proof is summarized by Eymard and Lafon [46, p. 136-138].

In 1975 and 1976, Novák [111] and Inkeri [76] generalized some of Niven's results. In 1977, Murty and Murty [107] also obtained more general results from which Niven's proof and others are corollaries.

Other generalizations are those of Schneider [133] and Desbrow [38].

4.14 Transcendence

As mentioned earlier, the quadrature of the circle would only be possible if π were algebraic. But the fact that π is not algebraic, hence is transcendental, was proved by Lindemann in 1882 [101].

For more on the transcendence of π , see Sylvester [145, 144], Hessenberg [69], and more recent expositions by Juhel [86] and Jacob [78]. Fritsch has also commemorated the hundredth anniversary of Lindemann's result [51].

More recently, Lian and Zhang [100] gave a simple proof of the transcendence of the trigonometric functions. Milla [104] has also recently published a new proof of the transcendence of π .

Bibliography

The following list gives the main references related to the proof of the irrationality of π . Only some of these references have been used in my review, and the others are only given as possible useful leads for the interested reader. Some of the references contain links to other works that I have not cited. Note that the capitalization of titles has been normalized.

- [1] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Berlin: Springer, 2018.
- [2] David Angell. Irrationality and transcendence, 2007. [class notes, University of New South Wales].
- [3] Archimedes. Measurement of a circle. In Thomas Little Heath, editor, *The works of Archimedes*, pages 91–98. Cambridge: University Press, 1897.
- [4] Jörg Arndt and Christoph Haenel. *Pi-Unleashed*. Berlin: Springer, 2001. [also *Pi : Algorithmen, Computer, Arithmetik*, Springer-Verlag, 2013].
- [5] Alexander Aycock. Euler und die analytische Theorie der Kettenbrüche. *Mathematische Semesterberichte*, 62:143–158, 2015.
- [6] David H. Bailey and Jonathan M. Borwein. *Pi: The next generation — A sourcebook on the recent history of pi and its computation*. Springer, 2016.
- [7] Christopher Baltus. Continued fractions and the first proofs that π is irrational. In Catherine Bonan-Hamada, Phil Gustafson, and Lyle Cochran, editors, *Communications in the Analytic Theory of Continued Fractions*, volume 11, pages 5–24. Grand Junction: Mesa State College, 2003.

- [8] Christopher Baltus. The gap in Lambert's proof (1767) that pi is irrational. In *Abstracts of Papers Presented to the American Mathematical Society*. American Mathematical Society, 2004. [not seen].
- [9] Janet Heine Barnett. Enter, stage center: The early drama of the hyperbolic functions. *Mathematics Magazine*, 77(1):15–30, February 2004.
- [10] Friedrich Ludwig Bauer. Lamberts Kettenbruch. *Informatik-Spektrum*, 28:303–309, 2005.
- [11] Friedrich Ludwig Bauer. Johann Heinrich Lambert (1728 bis 1777). *Akademie Aktuell*, 1:12–15, 2006.
- [12] Friedrich Ludwig Bauer. Gregory-Leibniz und Euler : Arcus-Cotangens-Relationen (Anlässlich des 300. Geburtstags von Leonhard Euler). *Informatik-Spektrum*, 30:444–451, 2007.
- [13] Friedrich Ludwig Bauer. Kettenbruch-Phänomene : Teil I : Periodische Kettenbrüche. *Informatik-Spektrum*, 32:54–64, 2009.
- [14] Friedrich Ludwig Bauer. Kettenbruch-Phänomene : Teil II : Wechselwegnahme für transzendente Zahlen. *Informatik-Spektrum*, 32:168–174, 2009.
- [15] Friedrich Ludwig Bauer and Christoph Haenel. *Übersehene numerische Aspekte in der Geschichte der Kettenbrüche*. München: Bayerische Akademie der Wissenschaften, 2007.
- [16] Petr Beckmann. *A history of π* . Boulder: Golem Press, 1970. [many editions].
- [17] Lennart Berggren, Jonathan Borwein, and Peter Borwein. *Pi: A source book*. New York: Springer-Verlag, 2004. [third edition; contains an excerpt from Lambert's original proof, p. 129-140 (sections 1 to 16), a partial translation, p. 141-146 (sections 37 to 51), and Struik's discussion of the proof [143], as well as Niven's proof of the irrationality, 1947, p. 276 [109]].
- [18] Frits Beukers. A note on the irrationality of $\zeta(2)$ and $\zeta(3)$. *Bulletin of the London Mathematical Society*, 11(3):268–272, October 1979. [reprinted in Berggren et al. [17]].

- [19] Frits Beukers. A rational approach to π . *Nieuw Archief voor Wiskunde*, 5/1(4):372–379, December 2000.
- [20] George Boros and Victor Moll. *Irresistible integrals: Symbolics, analysis and experiments in the evaluation of integrals*. Cambridge: Cambridge University Press, 2004.
- [21] Cliff Bott. Lambert’s irrationality proofs for e and π , 2016. [7 pages, <https://someclassicalmaths.files.wordpress.com/2016/05/lamberts-irrationality-proofs-rev1.pdf>].
- [22] Nicolas Bourbaki. *Fonctions d’une variable réelle (théorie élémentaire), chapitres I, II, III*. Paris: Hermann, 1949. [see pp. 137-138 for a proof of the irrationality of π].
- [23] Conrad Brandenberger. Die arithmetischen Theorien der Irrationalzahlen. *Schweizerische pädagogische Zeitschrift*, 13(4):188–206, 1903.
- [24] Robert Breusch. A proof of the irrationality of π . *The American Mathematical Monthly*, 61(9):631–632, November 1954.
- [25] Claude Brezinski. *History of continued fractions and Padé approximants*. Berlin: Springer, 1991.
- [26] Ludolf van Ceulen. *Vanden circkel daer in gheleert werdt te vinden de naeste proportie des circkels-diameter teghen synen omloop etc.* Delft: Andriesz, 1596.
- [27] Ludolf van Ceulen. *Arithmetische en geometrische fondamenten*. Leiden: Joost van Colster and Jacob Marcus, 1615.
- [28] Ludolf van Ceulen. *Vanden circkel daer in gheleert werdt te vinden de naeste proportie des circkels-diameter teghen synen omloop etc.* Leyden: Jacob Marcus, 1615.
- [29] George Chrystal. *Algebra: An elementary text-book for the higher classes of secondary schools and for colleges*, volume 2. Edinburgh: Adam and Charles Black, 1889.
- [30] Keith Conrad. Irrationality of π and e , 2005. <https://kconrad.math.uconn.edu/blurbs/analysis/irrational.pdf>.
- [31] Julian Lowell Coolidge. The number e . *The American Mathematical Monthly*, 57(9):591–602, November 1950.

- [32] Davide Crippa. *The impossibility of squaring the circle in the 17th century: A debate among Gregory, Huygens and Leibniz*. Springer, 2019.
- [33] François Daviet de Foncenex. Réflexions sur les quantités imaginaires. *Miscellanea Philosophico-Mathematica Societatis Privatae Taurinensis*, 1:113–146, 1759.
- [34] Raymond H. Daw. Johann Heinrich Lambert (1728-1777). *Journal of the Institute of Actuaries (1886-1994)*, 107(3):345–363, September 1980. [Lambert’s contributions to actuarial science].
- [35] Augustus De Morgan. Quadrature of the circle. In Charles Knight, editor, *The English Cyclopædia, section “Arts and sciences”*, volume 6, pages 868–874. London: Bradbury, Evans, & Co., 1861.
- [36] Fernando Neres de Oliveira. Uma prova elementar da irracionalidade de π . *C.Q.D. — Revista Eletrônica Paulista de Matemática*, 13:31–45, December 2018.
- [37] Jean-Paul Delahaye. *Le fascinant nombre π* . Paris: Belin, 1997.
- [38] Darrell Desbrow. On the irrationality of π^2 . *American Mathematical Monthly*, 97(10):903–906, December 1990.
- [39] Ghislain Dupont. Irrationalité de Pi, 2004. [actually proves the irrationality of π^2 which entails that of π].
- [40] Jacques Dutka. The early history of the hypergeometric function. *Archive for History of Exact Sciences*, 31(1):15–34, 1984.
- [41] Richard Earl. *Towards higher mathematics: a companion*. Cambridge: Cambridge University Press, 2017. [Cartwright’s proof appears on p. 383].
- [42] Peter Nikolaus Caspar Egen. *Handbuch der allgemeinen Arithmetik*, volume 1. Berlin: Duncker and Humblot, 1846.
- [43] Euclid. *The thirteen books of Euclid’s elements*. Cambridge: University Press, 1908. [edited by Thomas Little Heath].
- [44] Leonhard Euler. De fractionibus continuis dissertatio. *Commentarii academiae scientiarum imperialis Petropolitanae*, IX:98–137, 1744. [English translation by Myra F. Wyman and Bostwick F. Wyman in *Mathematical Systems Theory*, volume 18, p. 295-328, 1985].

- [45] Leonhard Euler. *Introductio in analysin infinitorum*. Lausanne: Marc-Michel Bousquet, 1748. [2 volumes].
- [46] Pierre Eymard and Jean-Pierre Lafon. *Autour du nombre π* . Paris: Hermann, 1999. [English translation: “The Number π ,” American Mathematical Society, 2004].
- [47] Johann Albert Eytelwein. *Grundlehren der höhern Analysis*, volume 1. Berlin: G. Reimer, 1824.
- [48] Thomas Fantet de Lagny. Mémoire sur la quadrature du cercle, et sur la mesure de tout arc, tout secteur, et tout segment donné. In *Histoire de l’Académie Royale des Sciences. Année M.DCCXIX, Mémoires de mathématiques et de physique, tirés des registres de l’Académie Royale des Sciences*, pages 135–145. Paris: Imprimerie royale, 1721.
- [49] Giovanni Ferraro. *The rise and development of the theory of series up to the early 1820s*. Springer, 2008.
- [50] Johann Heinrich Samuel Formey. Éloge de M. Lambert. *Nouveaux mémoires de l’Académie Royale des Sciences et Belles-Lettres (Berlin)*, 1778:72–90, 1780. [English translation in [138]].
- [51] Rudolf Fritsch. Die Quadratur des Kreises — unmöglich ! *Umbits*, 12(7):4–7, December 1982.
- [52] Carl Friedrich Gauß. Determinatio seriei nostrae per aequationem differentialem secundi ordinis. In *Werke*, volume 3, pages 207–230. Göttingen: Königliche Gesellschaft der Wissenschaften, 1866. [sequel of [53], part of Gauß’s Nachlass; German translation in [54]].
- [53] Carl Friedrich Gauß. Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$. In *Werke*, volume 3, pages 123–162. Göttingen: Königliche Gesellschaft der Wissenschaften, 1866. [originally published in *Commentationes societatis regiae scientiarum Göttingensis recentiores*, volume 2, 1813; German translation in [54]; the second part is [52]].
- [54] Carl Friedrich Gauß. *Allgemeine Untersuchungen über die unendliche Reihe* $1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \text{u. s. w.}$. Berlin: Julius Springer, 1888. [German translation by Heinrich Simon of [53] and [52]].

- [55] Carl Friedrich Gauß. Beweis der Irrationalität der Tangenten rationaler Bögen in einer neuen Gestalt. In *Werke*, volume 8, pages 27–29. Leipzig: B. G. Teubner, 1900. [with a note by Robert Fricke].
- [56] James Whitbread Lee Glaisher. On Lambert’s proof of the irrationality of π , and on the irrationality of certain other quantities. In *Report of the forty-first meeting of the British association for the advancement of science held at Edinburgh in August 1871, Notices and abstracts of miscellaneous communications to the sections*, pages 12–16, 1872.
- [57] James Whitbread Lee Glaisher. Remarks on the calculation of π . *The Messenger of Mathematics*, 2:119–128, 1873.
- [58] James Whitbread Lee Glaisher. On the quadrature of the circle, A.D. 1580-1630. *The Messenger of Mathematics*, 3:27–46, 1874.
- [59] Ronald Lewis Graham, Donald Ervin Knuth, and Oren Patashnik. *Concrete mathematics*. Reading, Ma.: Addison-Wesley, 1994. [second edition].
- [60] Jeremy John Gray and Laura Tilling. Johann Heinrich Lambert, mathematician and scientist, 1728-1777. *Historia Mathematica*, 5:13–41, 1978.
- [61] David Bierens de Haan. Notice sur quelques quadrateurs du cercle dans les Pays-Bas. *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche*, 7:99–140, March 1874. [on van Ceulen and Snell’s publications of the values of π].
- [62] David Bierens de Haan. On Ludolff van Ceulen’s 35-decimal value of π and on some of his works. *The Messenger of Mathematics*, 3:24–26, 1874.
- [63] Ernst Hairer and Gerhard Wanner. *Analysis by its history*. Springer, 2008.
- [64] Godfrey Harold Hardy and Edward Maitland Wright. *An introduction to the theory of numbers*. Oxford: Oxford University Press, 1979. [fifth edition].
- [65] Julian Havil. *The irrationals: a story of the numbers you can’t count on*. Princeton: Princeton University Press, 2012.

- [66] Charles Hermite. Extrait d'une lettre de Monsieur Ch. Hermite à Monsieur Paul Gordan. *Journal für die reine und angewandte Mathematik*, 76:303–311, 1873.
- [67] Charles Hermite. Extrait d'une lettre de Mr. Ch. Hermite à Mr. Borchardt. *Journal für die reine und angewandte Mathematik*, 76:342–344, 1873.
- [68] Charles Hermite. *Cours de M. Hermite, rédigé en 1882 par M. Andoyer, élève à l'École normale*. Paris: A. Hermann, 1887. [3rd edition, <https://gallica.bnf.fr/ark:/12148/bpt6k97981084>].
- [69] Gerhard Hessenberg. *Transzendenz von e und π* . Leipzig: B. G. Teubner, 1912.
- [70] Meier Hirsch. *Sammlung geometrischer Aufgaben*, volume 1. Berlin: Heinrich Frölich, 1805.
- [71] Ernest William Hobson. “*Squaring the circle*”: *A history of the problem*. Cambridge: University Press, 1913.
- [72] Ernest William Hobson. *A treatise on plane trigonometry*. Cambridge: University Press, 1918. [4th edition].
- [73] Daniel Huber. *Johann Heinrich Lambert nach seinem Leben und Wirken aus Anlaß der zu seinem Andenken begangenen Secularfeier in drei Abhandlungen dargestellt*. Basel: Schweighauser, 1829.
- [74] Dirk Huylebrouck. Similarities in irrationality proofs for π , $\ln 2$, $\zeta(2)$, and $\zeta(3)$. *The American Mathematical Monthly*, 108(3):222–231, March 2001.
- [75] Kustaa Adolf Inkeri. The irrationality of π^2 . *Nordisk Matematisk Tidskrift*, 8(1):11–16, 1960.
- [76] Kustaa Adolf Inkeri. On the irrationality of certain values of elementary transcendental functions. *Nieuw Archief Voor Wiskunde*, 24:226–230, 1976.
- [77] Yosikazu Iwamoto. A proof that π^2 is irrational. *J. Osaka Institute of Science and Technology*, 1:147–148, 1949. [not seen].
- [78] Marie Jacob. *La quadrature du cercle*. Paris: Fayard, 2006.

- [79] Harold Jeffreys. *Scientific inference*. Cambridge: University Press, 1957. [2nd edition].
- [80] Harold Jeffreys. *Scientific inference*. Cambridge: University Press, 1973. [3rd edition].
- [81] Timothy W. Jones. Discovering and proving that π is irrational. *American Mathematical Monthly*, 117(6):553–557, June-July 2010. [see [154] for some notes].
- [82] Timothy W. Jones. The powers of π are irrational, 2010. [viXra:1102.0058, 19 October 2010, 17 pages].
- [83] Timothy W. Jones. Euler’s identity, Leibniz tables, and the irrationality of pi. *The College Mathematics Journal*, 43(5):361–364, November 2012.
- [84] Timothy W. Jones. Discovering and proving that π is irrational, 2nd edition, 2017. [11 pages].
- [85] William Jones. *Synopsis Palmariorum Matheseos*. London, 1706. [the earliest known appearance of π is on page 243].
- [86] Alain Juhel. Irrationalité et transcendance : Etat des lieux avant Hermite, 1994. [conférence aux journées Padé (Lycée Faidherbe, Lille, 1994)]
http://www.mathouriste.eu/Hermite/irrat_av_Hermite.pdf.
- [87] Alain Juhel. Lambert et l’irrationalité de π . In Alexandre Moatti, editor, *Regards sur les textes fondateurs de la science*, volume 1, pages 138–157. Paris: Cassini, 2010. [English translation: <http://www.bibnum.education.fr/sites/default/files/24-lambert-analysis.pdf>].
- [88] Victor Joseph Katz. The calculus of the trigonometric functions. *Historia mathematica*, 14:311–324, 1987.
- [89] Wilbur Richard Knorr. Archimedes and the measurement of the circle: A new interpretation. *Archive for History of Exact Sciences*, 15(2):115–140, 1976.
- [90] Jurjen Ferdinand Koksma. On Niven’s proof that π is irrational. *Nieuw Archief voor Wiskunde*, (II. S.) 23:39, 1949.

- [91] Miklós Laczkovich. On Lambert's proof of the irrationality of π . *American Mathematical Monthly*, 104(5):439–443, 1997.
- [92] Johann Heinrich Lambert. Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques. *Histoire de l'Académie Royale des Sciences et Belles-Lettres (Berlin)*, XVII (1761):265–322, 1768. [partially reproduced in Berggren et al. [17], and Struik [143]].
- [93] Johann Heinrich Lambert. Observations trigonométriques. *Histoire de l'Académie Royale des Sciences et Belles-Lettres (Berlin)*, XXIV (1768):327–354, 1770.
- [94] Johann Heinrich Lambert. Verwandlung der Brüche. In *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, volume 2, pages 54–132. Berlin: Realschule, 1770.
- [95] Johann Heinrich Lambert. Vorläufige Kenntnisse für die, so die Quadratur und Rectification des Circuls suchen. In *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, volume 2, pages 140–169. Berlin: Realschule, 1770. [written in 1766; also in Lambert, *Opera Mathematica*, volume 1, p. 194–212 (not checked), reprinted by Rudio [131]].
- [96] Johann Heinrich Lambert. Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques. In *Opera Mathematica*, volume 2, pages 112–159. Zurich: Orell Füssli Verlag, 1948. [edited by Andreas Speiser].
- [97] Henri Lebesgue. *Leçons sur les constructions géométriques : professées au Collège de France en 1940-1941*. Paris: Gauthier-Villars, 1950.
- [98] Adrien-Marie Legendre. *Éléments de géométrie, avec des notes*. Paris: Firmin Didot, 1794. [proof of irrationality of π^2 on page 304; English translation in [99]].
- [99] Adrien-Marie Legendre. *Elements of geometry and trigonometry; with notes*. New York: James Ryan, 1828. [proof of irrationality of π^2 on page 256; translation of [98]; earlier English translations (1819 and 1825) do not seem to contain the proof of irrationality].
- [100] Yuanyuan Lian and Kai Zhang. A simple proof of the transcendence of the trigonometric functions, 2019. [arXiv:1912.05348v1].

- [101] Ferdinand von Lindemann. Über die Zahl π . *Mathematische Annalen*, 20:213–225, 1882.
- [102] Nick Lord. Three footnotes to Cartwright’s proof that π is irrational. *The Mathematical Gazette*, 103(558):514–517, November 2019. [not seen].
- [103] Adriaan Metius. *Arithmeticae et Geometriae practica*. Franeker: Rombertus Doyema, 1611. [not seen].
- [104] Lorenz Milla. The transcendence of π and the squaring of the circle, 2020. [arXiv:2003.14035v3, also in German].
- [105] Ulysses Grant Mitchell and Mary Strain. The number e . *Osiris*, 1:476–496, January 1936.
- [106] Jean-Étienne Montucla. *Histoire des recherches sur la quadrature du cercle, avec une addition concernant les problèmes de la duplication du cube et de la trisection de l’angle*. Paris: Bachelier père et fils, 1831. [second edition, with additions probably written by Sylvestre François Lacroix; Lambert’s work is cited on page 278; the first edition was published in 1754].
- [107] Maruti Ram Pedaprolu Murty and Vijaya Kumar Murty. Irrational numbers arising from certain differential equations. *Canadian Mathematical Bulletin*, 20(1):117–120, March 1977.
- [108] Isaac Newton. *Analysis per quantitatum series, fluxiones, ac differentias: cum enumeratione linearum tertii ordinis*. London: Pearson, 1711.
- [109] Ivan Morton Niven. A simple proof that π is irrational. *Bulletin of the American Mathematical Society*, 53(6):509, 1947. [reprinted in Berggren et al. [17], p. 276].
- [110] Ivan Morton Niven. *Irrational Numbers*, volume 11 of *The Carus Mathematical Monographs*. Mathematical Association of America, 1956.
- [111] Břetislav Novák. A remark to a paper of J. F. Koksma. *Nieuw Archief Voor Wiskunde*, 23:195–197, 1975.
- [112] Raymond Oberlé et al., editors. *Colloque international et interdisciplinaire Jean-Henri Lambert (Mulhouse, 26-30 septembre 1977)*. Paris: Éditions Ophrys, 1979.

- [113] Alan E. Parks. π , e , and other irrational numbers. *The American Mathematical Monthly*, 93(9):722–723, November 1986.
- [114] *Petit Archimède, numéro spécial π* . Amiens: Le Petit Archimède, 1980. [supplement to issue 64-65 of *Le Petit Archimède*, May 1980; see pages 124-127 which are based on Lebesgue's exposition].
- [115] Jan Popken. On Lambert's proof for the irrationality of π . *Proceedings van de Koninklijke Nederlandse Akademie van Wetenschappen*, 43:712–714, 1940. [apparently reprinted as [116]].
- [116] Jan Popken. On Lambert's proof for the irrationality of π . *Indagationes Mathematicae*, 2:279–281, 1940. [seems to be a reprint of [115]].
- [117] Jan Popken. Over de onmeetbaarheid van π . *Euclides (Groningen)*, 17(5):217–227, 1941.
- [118] Jan Popken. Over de irrationaliteit van de tangens van een rational getal. *Norsk Matematisk Tidsskrift*, 26:66–70, 1944. [not seen].
- [119] Jan Popken. Over de irrationaliteit van π . Technical Report ZW 1948-014, Stichting Mathematisch Centrum, Amsterdam, 1948.
- [120] Jan Popken. Remark on my paper "On Lambert's proof for the irrationality of π ". *Proceedings van de Koninklijke Nederlandse Akademie van Wetenschappen*, 52:504, 1949. [reprinted in *Indagationes Mathematicae*, volume 11, 1949, p. 164].
- [121] Alfred Pringsheim. Ueber die Convergenz unendlicher Kettenbrüche. *Sitzungsberichte der mathematisch-physikalischen Classe der k.b. Akademie der Wissenschaften zu München*, 28 (1898):295–324, 1899.
- [122] Alfred Pringsheim. Ueber die ersten Beweise der Irrationalität von e und π . *Sitzungsberichte der mathematisch-physikalischen Classe der k.b. Akademie der Wissenschaften zu München*, 28 (1898):325–337, 1899.
- [123] Alfred Pringsheim. Kritisch-historische Bemerkungen zur Funktionentheorie. V. Über einen Gaußschen Beweis der Irrationalität von $\tan x$ bei rationalem x . *Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München*, 1932(3):193–200, 1933.

- [124] Alfred Pringsheim. Kritisch-historische Bemerkungen zur Funktionentheorie. VI. Aus Gauß' Nachlaß. *Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München*, 1933(1):61–70, 1933.
- [125] Eugène Prouhet. Note sur les arcs de cercle dont la tangente est rationnelle. *Journal de mathématiques pures et appliquées*, II(1):215–222, 1856.
- [126] Eugène Prouhet. Sur le rapport d'Adrien Metius. *Bulletin de bibliographie, d'histoire et de biographie mathématiques*, 8:6–8, 1862.
- [127] Sur la quadrature du cercle. *Nouvelles annales de mathématiques*, 12:298–302, 1853.
- [128] Reinhold Remmert. What is π ? In Heinz-Dieter Ebbinghaus, Hans Hermes, Friedrich Hirzebruch, Max Koecher, Klaus Mainzer, Jürgen Neukirch, Alexander Prestel, and Reinhold Remmert, editors, *Numbers*, pages 123–153. New York: Springer, 1991.
- [129] Paulo Ribenboim. *My numbers, my friends: Popular lectures on number theory*. New York: Springer, 2000.
- [130] Patricia Rothman. The man who invented the concept of pi: William Jones and his circle. *History Today*, 59(7):24–30, 2009.
- [131] Ferdinand Rudio. *Archimedes, Huygens, Lambert, Legendre: Vier Abhandlungen über die Kreismessung*. Leipzig, 1892. [contains a reprint of [95]].
- [132] Charles Edward Sandifer. Who proved, that e is irrational?, 2006. <http://eulerarchive.maa.org/hedi/index.html>.
- [133] Theodor Schneider. Über die Irrationalität von π . *Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München*, 1954(5):99–101, 1955.
- [134] Johann Schulz. *Sehr leichte und kurze Entwicklung einiger der wichtigsten mathematischen Theorien*. Königsberg: Friedrich Nicolovius, 1803.
- [135] Christoph J. Scriba. Johann Heinrich Lambert. In Charles Coulston Gillispie, editor, *Dictionary of Scientific Biography*, volume 7, pages 595–600. Charles Scribner's Sons, 1973.

- [136] Michel Serfati. L'affaire Lambert. In *Actes de l'université d'été sur l'histoire des mathématiques, Toulouse, 6-12 juillet 1986*, pages 381–416. Toulouse: IREM, 1987.
- [137] Michel Serfati. *Quadrature du cercle, fractions continues et autres contes : sur l'histoire des nombres irrationnels et transcendants aux XVIII^e et XIX^e siècles*, volume 4 of *Fragments d'histoire des mathématiques*. Paris: Association des Professeurs de Mathématiques de l'Enseignement Public (APMEP), 1992. [brochure A.P.M.E.P. numéro 86].
- [138] Oscar Sheynin. *Portraits: Leonhard Euler, Daniel Bernoulli, Johann-Heinrich Lambert*. Berlin, 2010. [contains translations of several biographical notes, including those of Formey [50] and Wolf [151]].
- [139] Willebrord Snell. *Cyclometricus*. Leiden: Elzevir, 1621.
- [140] Jonathan Sondow. A geometric proof that e is irrational and a new measure of its irrationality. *The American Mathematical Monthly*, 113(7):637–641, Aug-Sep 2006.
- [141] Michael Spivak. *Calculus*. Houston: Publish or Perish, Inc., 1994. [third edition].
- [142] Max Steck. *Bibliographia Lambertiana : Ein Führer durch das gedruckte und ungedruckte Schriftum und den wissenschaftlichen Briefwechsel von Johann Heinrich Lambert 1728-1777*. Hildesheim: H. A. Gerstenberg, 1970. [see p. 25 for the 1768 Memoir].
- [143] Dirk Jan Struik. *A source book in mathematics, 1200-1800*. Cambridge: Harvard University Press, 1969. [an excerpt of Lambert's article is translated in pages 369-374 (sections 37 to 51); reproduced in Berggren et al. [17]].
- [144] James Joseph Sylvester. Preuve que π ne peut pas être racine d'une équation algébrique à coefficient entiers. In Henry Frederick Baker, editor, *The Collected Mathematical Papers of James Joseph Sylvester*, volume 4, pages 682–686. Cambridge: University Press, 1912. [originally published in 1890].
- [145] James Joseph Sylvester. Sur le rapport de la circonférence au diamètre. In Henry Frederick Baker, editor, *The Collected Mathematical Papers of James Joseph Sylvester*, volume 4, pages

- 680–681. Cambridge: University Press, 1912. [originally published in 1890].
- [146] Ephraim Salomon Unger. *Die Lehre von dem Kreise*. Leipzig: F. A. Brockhaus, 1829.
- [147] Theodor Vahlen. *Konstruktionen und Approximationen in systematischer Darstellung : eine Ergänzung der niederen, eine Vorstufe zur höheren Geometrie*. Leipzig: B. G. Teubner, 1911. [see p. 319, Irrationality of π and π^2].
- [148] Tullio Viola. Lambert et l’analyse mathématique. In Oberlé et al. [112], pages 237–250.
- [149] Jules Vuillemin. La démonstration de l’irrationalité de π chez Leibniz, Lambert et Kant. *Revue Philosophique de la France et de l’Étranger*, 151:417–431, 1961.
- [150] Rolf Wallisser. On Lambert’s proof of the irrationality of π . In Franz Halter-Koch and Robert F. Tichy, editors, *Algebraic Number Theory and Diophantine Analysis*, pages 521–530. Berlin: Walter de Gruyter, 2000.
- [151] Rudolf Wolf. Joh. Heinrich Lambert von Mühlhausen. 1728-1777. In *Biographien zur Kulturgeschichte der Schweiz. Dritter Cyclus*, pages 317–356. Zürich: Orell, Füßli & Comp., 1860. [English translation in [138]].
- [152] Adolf-Andreï Pavlovitch Youshkevitch. Lambert et Léonhard Euler. In Oberlé et al. [112], pages 211–223.
- [153] Li Zhou. Irrationality proofs à la Hermite. *The Mathematical Gazette*, 95(53):407–413, November 2011.
- [154] Li Zhou. On “Discovering and proving that π is irrational”, 2011. [comments on Jones’ paper [81], 7 pages].
- [155] Li Zhou and Lubomir Markov. Recurrent proofs of the irrationality of certain trigonometric values. *American Mathematical Monthly*, 117(4):360–362, April 2010.