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# On unitriangular basic sets for symmetric and alternating groups

*Olivier Brunat, Jean-Baptiste Gramain and Nicolas Jacon*

**Abstract**

We study the modular representation theory of the symmetric and alternating groups. One of the most natural ways to label the irreducible representations of a given group or algebra in the modular case is to show the unitriangularity of the decomposition matrices, that is, the existence of a unitriangular basic set. We study several ways to obtain such sets in the general case of a symmetric algebra. We apply our results to the symmetric groups and to their Hecke algebras and thus obtain new ways to label the simple modules for these objects. Finally, we show that these sets do not always exist in the case of the alternating groups by studying two explicit cases in characteristic 3.

## 1 Introduction

One of the major problems in the representation theory of finite groups is to find a natural classification of the set of simple modules, in particular in the modular case. This problem may be attacked using the notion of unitriangular basic sets in the wider context of symmetric algebras. Let $A$ be an integral domain and let $\mathcal{H}$ be an associative symmetric $A$-algebra, finitely generated and free over $A$ (see [9, Def. 7.1.1]). A standard example of such structure is the group algebra of a finite group. Another example is the Iwahori-Hecke algebra of a finite Weyl group. Let $\theta : A \to L$ be a ring homomorphism into a field $L$ such that $L$ is the field of fractions of $\theta(A)$. We obtain an $L$-algebra $L\mathcal{H} := L \otimes_A \mathcal{H}$, where $L$ is regarded as an $A$-module via $\theta$. We assume that $L\mathcal{H}$ is split. Our problem is then to describe the set $\mathrm{Irr}(L\mathcal{H})$ of isomorphism classes of simple $L\mathcal{H}$-modules. Note that when $\mathcal{H}$ is a group algebra of a finite group and $L$ is a sufficiently large field of characteristic $p$ dividing the order of the group, this problem coincides with that of finding the irreducible modular representations of the finite group.

A useful tool in this setting is the decomposition matrix. Let $K$ be the field of fractions of $A$ and write $K\mathcal{H} := K \otimes_A \mathcal{H}$. We assume that $A$ is integrally closed in $K$ and that $K\mathcal{H}$ is split semisimple. Assume that we get a classification of the simple $K\mathcal{H}$-modules

$$\mathrm{Irr}(K\mathcal{H}) = \{V^\lambda \mid \lambda \in \Lambda\} \tag{1}$$

for some labeling set $\Lambda$. The decomposition map is then the map (see [9, Theorem 7.4.3])

$$d_\theta : R_0(K\mathcal{H}) \to R_0(L\mathcal{H})$$

between the associated Grothendieck groups of finite dimensional modules. In particular, for all $\lambda \in \Lambda$ and $M \in \mathrm{Irr}(L\mathcal{H})$, there are uniquely determined non-negative integers $d_{\lambda,M}$ such that

$$d_\theta([V^\lambda]) = \sum_{M \in \mathrm{Irr}(L\mathcal{H})} d_{\lambda,M}[M]. \tag{2}$$

We then get the corresponding *decomposition matrix* $(d_{\lambda,M})_{\lambda\in\Lambda, M\in\mathrm{Irr}(L\mathcal{H})}$, which controls the representation theory of $L\mathcal{H}$.

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**Definition 1.** A *unitriangular basic set for* $(\mathcal{H}, \theta)$ is the datum of a triplet $(B, \leq, \Psi)$ where $B \subset \Lambda$, $\leq$ is a total order defined on $\Lambda$ and $\Psi$ is a bijective map:

$$\Psi : B \to \mathrm{Irr}(L\mathcal{H})$$

satisfying:

1. for all $\lambda \in B$, we have $d_{\lambda,\Psi(\lambda)} = 1$,
2. for all $M \in \mathrm{Irr}(L\mathcal{H})$ and $\lambda \in \Lambda$, we have $d_{\lambda,M} = 0$ unless $\lambda \leq \Psi^{-1}(M)$.

The existence of such a set can be helpful in order to solve our main problem. Indeed, we first remark that

$$\mathrm{Irr}(L\mathcal{H}) = \{\Psi(\lambda) \mid \lambda \in B\},$$

and the unitriangularity implies that the rows (labeled by $\Lambda$) and columns (labeled by $\mathrm{Irr}(L\mathcal{H})$) of the associated decomposition matrix can be ordered to get a unitriangular shape as follows:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots & & \vdots \\ * & & & 1 & & \vdots \\ * & & & & & 1 \\ * & & & & & * \\ \vdots & & & & & \vdots \end{bmatrix}.$$

This then gives a natural and canonical way to label the simple modules of $L\mathcal{H}$ by the set $B$. Thus, it is an important problem to construct, if possible, unitriangular basic sets associated to a given symmetric algebra. In this paper, we will mainly focus on the case where $\mathcal{H}$ is the group algebra of the alternating or symmetric group, or the Hecke algebra of the symmetric group. In the first two cases, the field $L$ will be a field of positive characteristic and the decomposition matrix given above corresponds to the usual "$p$-decomposition matrix" for a finite group.

In [3], the first two authors have already shown the existence of a weak version of unitriangular basic sets (simply called *basic sets*) for alternating groups. It was expected that this result may be strengthened by exhibiting unitriangular basic sets. In Section 3, we will in fact show that this is not possible in general. This result is surprising and unexpected, especially when we compare with the case of the symmetric groups where it is well known that unitriangular basic sets do exist [11]. As usual, the representations of the alternating groups can be obtained from those of the symmetric groups using Clifford theory. We will recall several aspects of this theory in relation with these unitriangular basic sets, and in particular deduce a criterion for the existence of unitriangular basic sets for alternating groups. Then, using this, we deduce two counterexamples for the existence of unitriangular basic sets for $n = 18$ and $n = 19$ and $p = 3$ by two different methods Furthermore, the definition of unitriangular basic set may be refined using the well known partition of the simple modules into $p$-blocks. In the cases of the symmetric and alternating groups, we can attach to each $p$-block a non-negative integer called its *$p$-weight*. The structure of two $p$-blocks with the same $p$-weight is very similar. For example, Chuang and Rouquier [4] proved that two such $p$-blocks of symmetric groups are derived equivalent, and, in [3], a weaker result for $p$-blocks of the alternating groups with the same weight is shown. We can then expect that two $p$-blocks of the alternating groups with the same $p$-weight should have the same unitriangularity property together. But we will show that this is not the case.

Despite our counterexamples for the alternating groups, one can ask in which cases and for which types of blocks unitriangular basic sets exist. To attack this problem, our strategy consists in focusing on the symmetric group, where we will be interested in the existence of unitriangular basic sets of a special kind.

There is also another (connected) motivation to search for these sets. It is well known that the ordinary irreducible representations of the symmetric group $\mathfrak{S}_n$ are naturally parametrized by the set $\Pi^1(n)$ of partitions of $n$, that is, the set of non-increasing sequences of non-negative integers of total sum $n$. In addition, these representations, or equivalently the simple $K\mathfrak{S}_n$-modules (where $K$ is a field of characteristic 0), can be constructed explicitly thanks to the Specht module theory (for example) and we get nice formulae for their dimensions. For $\lambda \in \Pi^1(n)$, we write $S^\lambda$ for the corresponding simple $K\mathfrak{S}_n$-module. We have

$$\mathrm{Irr}(K\mathfrak{S}_n) = \{S^\lambda \mid \lambda \in \Pi^1(n)\}. \tag{3}$$

When $L$ is a field of characteristic $p > 0$, there is also a canonical way to label the simple $L\mathfrak{S}_n$-modules by a certain subset $\mathrm{Reg}_p(n)$ of partitions, called the *$p$-regular partitions*. These are the partitions in $\Pi^1(n)$ which do not have $p$ or more parts of the same positive size. A complete set of simple $L\mathfrak{S}_n$-modules is then obtained as certain non zero quotients of the Specht modules:

$$\mathrm{Irr}(L\mathfrak{S}_n) = \{D^\lambda \mid \lambda \in \mathrm{Reg}_p(n)\}.$$

In characteristic zero, the one-dimensional representations of the symmetric group are naturally labeled by the partition $(n)$ (for the trivial representation) and its conjugate partition $(\underbrace{1,\ldots,1}_{n \text{ times}})$ (for the sign representation). In positive characteristic $p > 2$, the trivial representation is still labeled by the partition $(n)$ but the partition $(\underbrace{1,\ldots,1}_{n \text{ times}})$ is not $p$-regular in general so it cannot label the sign representation. In fact, the problem of finding which $p$-regular partition labels the sign representation is a particular case of a problem raised by Mullineux in [15]. The problem is the following. If $\lambda$ is a partition, then it is well known that tensoring the simple module $S^\lambda$ by the sign representation leads to a simple module isomorphic to $S^{\lambda'}$ where $\lambda'$ is the conjugate of $\lambda$. This construction still makes sense in positive characteristic for the simple modules $D^\lambda$ indexed by the $p$-regular partitions. However, the analogue of the conjugate partition is here more difficult to obtain. In fact, we now know that it can be described by several non-trivial recursive algorithms [12, 5, 16] (the first one having been conjectured by Mullineux [15]).

It is a natural question to ask if one can obtain a classification of the simple modules for which the tensor product by the sign representation is easier to describe. One of our results will be to give an answer to this problem using the theory of unitriangular basic sets.

Let us now explain in detail the organization of the paper. First, in Section 2, we consider the general situation where $G$ is a finite group, $H$ is an index two subgroup of $G$ and $p$ is an odd prime number dividing the order of $G$. We denote by $\varepsilon : G \to \{-1, 1\}$ the linear character of $G$ obtained by inflating the faithful linear character of $G/H$ to $G$, and by $\sigma : H \to H$, $g \mapsto xgx^{-1}$ an automorphism of $H$, where $x \in G$ is a fixed element with $x \notin H$. In this context, we will give relations between unitriangular $p$-basic sets of $G$ and of $H$. More precisely, we will show that if $\mathfrak{B}$ is a union of $p$-blocks that covers a $\sigma$-stable union $\mathfrak{b}$ of $p$-blocks of $H$ such that $\mathfrak{b}$ has a unitriangular $p$-basic set $b$, then there is a unitriangular $p$-basic set $(B, \leq, \Theta)$ of $\mathfrak{B}$ satisfying the following two conditions:

(i) The set $B$ is $\varepsilon$-stable.

(ii) The map $\Theta : B \to \mathrm{IBr}_p(\mathfrak{B})$ is $\varepsilon$-equivariant, where $\mathrm{IBr}_p(\mathfrak{B})$ is the set of irreducible Brauer characters of $\mathfrak{B}$.

Furthermore, a unitriangular $p$-basic set of $\mathfrak{B}$ that satisfies these properties restricts to a unitriangular $p$-basic set of $\mathfrak{b}$. See Theorem 12, Theorem 18 and Remark 20.

In Section 3, we study in detail the case of the symmetric and alternating groups. We begin by applying our previous results to this case. Then, in §3.2, we give two counterexamples for the existence of a unitriangular basic set for the alternating groups, by showing that the principal 3-blocks of $\mathfrak{A}_{18}$ and $\mathfrak{A}_{19}$ have no unitriangular 3-basic set.

The aim of the last section is to give a general procedure to produce unitriangular basic sets in the case of the symmetric and alternating groups when this is possible. In fact, we consider in Section 4 this

problem in the more general setting of symmetric algebras. We propose a procedure to obtain from a given unitriangular basic set, new such sets with nice additional properties. This part is quite elementary but should be of independent interest. In Section 5, we give some applications of the above result to the symmetric group and to a well known unitriangular basic set for this group, or more generally to its Hecke algebra. Even these resulting new unitriangular $p$-basic sets are not completely stable by tensoring by the sign of $\mathfrak{S}_n$ in general, but they *almost* are. We then give a way to modify these sets to obtain $p$-basic sets satisfying conditions (i) and (ii) above. Then, finally, we apply these results to prove that any $p$-block of $\mathfrak{A}_n$ with an odd $p$-weight has a unitriangular $p$-basic set. We also study explicitly an example which shows that "unitriangularity" is not an invariant of the $p$-weight of the $p$-blocks of the alternating groups.

## 2 Groups with an index two subgroup

In this section, we study the relations between the basic sets of a group with those of an index two subgroup. Of course, we can have in mind the symmetric and the alternating group as a fundamental example. Theorem 12 shows how one can obtain a unitriangular basic set for $H$ from a particular unitriangular basic set for $G$. Theorem 15 gives a necessary condition for the existence of a unitriangular basic set for $H$. Both results will be crucial in the rest of the paper.

### 2.1 Setting

Let $G$ be a finite group and $p$ be a prime number dividing $|G|$. Assume that $A$ is a valuation ring such that $p$ belongs to its maximal ideal. Suppose that its field of fractions $K$ is a splitting field for $G$ containing $A$. Consider the canonical map $\theta : A \to L$, where $L$ is the residue field of $A$. In this case, $(K, A, L)$ is a modular system for $G$ and $p$ large enough in the following. To any $LG$-module $M$, we can associate its Brauer character $\varphi_M$. It is an $A$-valued class function which vanishes on the set of $p$-singular elements of $G$. We write $\mathrm{IBr}_p(G)$ for the set of Brauer characters of irreducible $LG$-modules, and $\mathrm{Irr}(G)$ for the set of (ordinary) irreducible characters of $G$. We assume that $\Lambda$ is an indexing set for $\mathrm{Irr}(G)$, so that we have $\mathrm{Irr}(G) = \{\chi_\lambda \mid \lambda \in \Lambda\}$. Furthermore, for any $\varphi \in K\,\mathrm{Irr}(G)$, we define

$$\widehat{\varphi}(g) = \begin{cases} \varphi(g) & \text{if } g \text{ is a } p\text{-regular element,} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Recall that $\widehat{\chi} \in \mathbb{N}\,\mathrm{IBr}_p(G)$ for all $\chi \in \mathrm{Irr}(G)$, and that

$$\widehat{\chi}_\lambda = \sum_{M \in \mathrm{Irr}(L\mathcal{H})} d_{\lambda,M}\,\varphi_M. \tag{5}$$

The numbers $d_{\lambda,M}$ are the $p$-decomposition numbers, and the associated matrix $(d_{\lambda,M})_{\lambda\in\Lambda, M\in\mathrm{Irr}(L\mathcal{H})}$ is the decomposition matrix. The number $d_{\lambda,M}$ is also denoted by $d_{\chi,\varphi}$ when $\chi$ is the irreducible character of $G$ labeled by $\lambda$ and $\varphi$ is the Brauer character of the simple $L\mathcal{H}$-module $M$.

Now, for $\varphi \in \mathrm{IBr}_p(G)$, the projective indecomposable character corresponding to $\varphi$ is the ordinary character $\Phi_\varphi$ defined by

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi,\varphi}\chi. \tag{6}$$

Write $\mathrm{IPr}_p(G)$ for the set of projective indecomposable characters of $G$. The set $\mathbb{N}\,\mathrm{IPr}_p(G)$ is the set of projective characters of $G$. On the other hand, recall that $\mathbb{Z}\,\mathrm{IPr}_p(G)$ is the set of generalized characters of $G$ vanishing on the set of $p$-singular elements. Recall that $\mathrm{IPr}_p(G)$ is the dual basis of $\mathrm{IBr}_p(G)$ with respect to the hermitian scalar product $\langle\,,\,\rangle_G$ on $K\,\mathrm{Irr}(G)$, that is, the unique $K$-basis of the subspace of $K\,\mathrm{Irr}(G)$ vanishing on $p$-singular elements such that $\langle \Phi_\varphi, \vartheta\rangle_G = \delta_{\varphi,\vartheta}$ for all $\varphi$, $\vartheta \in \mathrm{IBr}_p(G)$.

**Lemma 2.** *We keep the notation as above. For any $\sigma \in \mathrm{Aut}(G)$ and $\varphi \in \mathrm{IBr}_p(G)$, we have*

$${}^{\sigma}\Phi_\varphi = \Phi_{{}^{\sigma}\varphi}.$$

*Proof.* Assume that we have $\sigma \in \mathrm{Aut}(G)$. By Equation (5) applied to ${}^{\sigma}\widehat{\chi}$ and $\widehat{\chi}$, and using that ${}^{\sigma}\widehat{\chi} = \widehat{{}^{\sigma}\chi}$, we obtain

$$\sum_{\varphi \in \mathrm{IBr}_p(G)} d_{\chi,\varphi}\, {}^{\sigma}\varphi = {}^{\sigma}\widehat{\chi} = \widehat{{}^{\sigma}\chi} = \sum_{\varphi \in \mathrm{IBr}_p(G)} d_{{}^{\sigma}\chi,\varphi}\, \varphi = \sum_{\varphi \in \mathrm{IBr}_p(G)} d_{{}^{\sigma}\chi,{}^{\sigma}\varphi}\, {}^{\sigma}\varphi.$$

Hence, the uniqueness of the coefficients in a basis gives

$$d_{\chi,\varphi} = d_{{}^{\sigma}\chi,{}^{\sigma}\varphi} \tag{7}$$

for all $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}_p(G)$. Furthermore, for $\varphi \in \mathrm{IBr}_p(G)$, Equation (7) gives

$${}^{\sigma}\Phi_{\varphi} = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi,\varphi}\, {}^{\sigma}\chi = \sum_{\chi \in \mathrm{Irr}(G)} d_{{}^{\sigma}\chi,{}^{\sigma}\varphi}\, {}^{\sigma}\chi = \Phi_{{}^{\sigma}\varphi},$$

as required. □

The *p-blocks* of ordinary characters of $G$ are the equivalence classes of the following equivalence relation on $\mathrm{Irr}(G)$: for $\chi$, $\psi \in \mathrm{Irr}(G)$, the characters $\chi$ and $\psi$ are in relation if and only if there are $\varphi_0, \ldots, \varphi_r \in \mathrm{IBr}_p(G)$ such that $\chi \in \Phi_{\varphi_0}$, $\psi \in \Phi_{\varphi_r}$ and, for each $0 \leq i \leq r-1$,

$$\langle \Phi_{\varphi_i}, \Phi_{\varphi_{i+1}} \rangle_G \neq 0.$$

The set of Brauer characters appearing in those PIMs form a $p$-block of Brauer characters. In the following, a $p$-block of $G$ can refer to a subset of $\mathrm{Irr}(G)$ or of $\mathrm{IBr}_p(G)$ according to the context. It will be denoted by $\mathrm{Irr}(\mathfrak{B})$ or $\mathrm{IBr}_p(\mathfrak{B})$, or sometimes only by $\mathfrak{B}$ if there is no possible confusion.

**Lemma 3.** *Let $\sigma \in \mathrm{Aut}(G)$ and $p$ be a prime number. Let $\mathfrak{B}$ be a $p$-block of $G$. Then*

$${}^{\sigma}\mathrm{Irr}(\mathfrak{B}) = \{{}^{\sigma}\chi \mid \chi \in \mathrm{Irr}(\mathfrak{B})\} \quad \textit{and} \quad {}^{\sigma}\mathrm{IBr}_p(\mathfrak{B}) = \{{}^{\sigma}\varphi \mid \varphi \in \mathrm{IBr}_p(\mathfrak{B})\}$$

*is a $p$-block of $G$, denoted by ${}^{\sigma}\mathfrak{B}$.*

*Proof.* First, we remark that, for any class functions $\alpha$ and $\beta$ of $G$, we have

$$\langle \alpha, \beta \rangle_G = \langle {}^{\sigma}\alpha, {}^{\sigma}\beta \rangle_G.$$

Now, $\chi$ and $\psi$ are in $\mathrm{Irr}(\mathfrak{B})$ if and only if there are $\varphi_0, \ldots, \varphi_r \in \mathrm{IBr}_p(G)$ such that $\langle \Phi_{\varphi_i}, \Phi_{\varphi_{i+1}} \rangle_G \neq 0$. Since

$$\langle \Phi_{\varphi_i}, \Phi_{\varphi_{i+1}} \rangle_G = \langle {}^{\sigma}\Phi_{\varphi_i}, {}^{\sigma}\Phi_{\varphi_{i+1}} \rangle_G = \langle \Phi_{{}^{\sigma}\varphi_i}, \Phi_{{}^{\sigma}\varphi_{i+1}} \rangle_G,$$

the result follows. □

**Definition 4.** With the above notation, a subset $B \subseteq \mathrm{Irr}(\mathfrak{B})$ of a $p$-block $\mathfrak{B}$ is a *$p$-basic set* of $\mathfrak{B}$ if the set $\{\widehat{\chi} \mid \chi \in B\}$ is a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module $\mathbb{Z}\,\mathrm{IBr}_p(\mathfrak{B})$.

*Remark* 5. Since the set $\{\widehat{\chi} \mid \chi \in \mathrm{Irr}(\mathfrak{B})\}$ generates over $\mathbb{Z}$ the module $\mathbb{Z}\,\mathrm{IBr}_p(\mathfrak{B})$, a subset $B \subseteq \mathrm{Irr}(\mathfrak{B})$ is a $p$-basic set of the $p$-block $\mathfrak{B}$ if and only if

- For any $\chi \in \mathrm{Irr}(\mathfrak{B})$, $\widehat{\chi}$ is a $\mathbb{Z}$-linear combination of the set $\widehat{B} = \{\widehat{\psi} \mid \psi \in B\}$.
- The family $\widehat{B}$ is free.

## 2.2 Decomposition matrix for index two subgroups

In this section, we keep the notation of the last section and we assume that $p$ is odd. The following result can be seen as an analogue of Clifford theory for the projective indecomposable modules.

**Proposition 6.** *Let $G$ be a finite group and $H$ be a subgroup of $G$ of index 2. Let $p$ be an odd prime number dividing $|G|$. Write $\varepsilon : G \to \{-1, 1\}$ for the surjective morphism with kernel $H$ induced by the canonical projection $G \to G/H$. Then*

$$\forall \varphi \in \mathrm{IBr}_p(G),\ \varepsilon \otimes \Phi_\varphi = \Phi_{\varepsilon\otimes\varphi}. \tag{8}$$

*Moreover,*

1. *If $\varepsilon \otimes \varphi \neq \varphi$, then $\mathrm{Res}_H^G(\Phi_\varphi) = \mathrm{Res}_H^G(\Phi_{\varepsilon\otimes\varphi}) \in \mathrm{IPr}_p(H)$.*
2. *If $\varepsilon \otimes \varphi = \varphi$, then $\Phi_\varphi$ splits into two projective indecomposable characters of $H$.*

*All projective indecomposable characters of $H$ are obtained by this process.*

*Proof.* First, we remark that $\varepsilon \otimes \varphi \in \mathrm{IBr}_p(G)$ for all $\varphi \in \mathrm{IBr}_p(G)$. Furthermore, for $\varphi$, $\vartheta \in \mathrm{IBr}_p(G)$ we have

$$\begin{aligned}
\langle \varepsilon \otimes \Phi_\varphi, \varepsilon \otimes \vartheta \rangle_G &= \frac{1}{|G|} \sum_{g\in G} \overline{\varepsilon(g)\Phi_\varphi(g)}\varepsilon(g)\vartheta(g) \\
&= \frac{1}{|G|} \sum_{g\in G} \varepsilon(g)^2 \overline{\Phi_\varphi(g)}\vartheta(g) \\
&= \langle \Phi_\varphi, \vartheta \rangle_G \\
&= \delta_{\varphi,\vartheta} \\
&= \delta_{\varepsilon\otimes\varphi,\varepsilon\otimes\vartheta} \\
&= \langle \Phi_{\varepsilon\otimes\varphi}, \varepsilon \otimes \vartheta \rangle_G.
\end{aligned}$$

Hence, by uniqueness of the dual basis, we deduce that

$$\varepsilon \otimes \Phi_\varphi = \Phi_{\varepsilon\otimes\varphi}.$$

Now, since $p$ is odd and $G/H$ is cyclic of order prime to $p$, Clifford's theory for Brauer characters [10, Theorem 9.18] can be applied. We have

- If $\varepsilon \otimes \varphi \neq \varphi$, then $\mathrm{Res}_H^G(\varphi) = \mathrm{Res}_H^G(\varepsilon \otimes \varphi) \in \mathrm{IBr}_p(H)$. To simplify the notation, we still denote by $\varphi$ the restriction of $\varphi$ to $H$. [10, Theorem 9.8] also gives that

$$\mathrm{Ind}_H^G(\varphi) = \varphi + \varepsilon \otimes \varphi. \tag{9}$$

- If $\varepsilon \otimes \varphi = \varphi$, then $\mathrm{Res}_H^G(\varphi) = \varphi^+ + \varphi^-$ with $\varphi^\pm \in \mathrm{IBr}_p(H)$, and [10, Theorem 9.8] again gives that

$$\mathrm{Ind}_H^G(\varphi^+) = \mathrm{Ind}_H^G(\varphi^-) = \varphi. \tag{10}$$

Write $\mathrm{IPr}_p(H) = \{\Psi_\alpha \mid \alpha \in \mathrm{IBr}_p(H)\}$. Let $\varphi \in \mathrm{IBr}_p(G)$. Since $\mathrm{Res}_H^G(\Phi_\varphi)$ vanishes on $p$-singular elements of $H$, it is a generalized projective character of $H$. Thus, there are integers $a_\alpha$ for $\alpha \in \mathrm{IBr}_p(H)$ such that

$$\mathrm{Res}_H^G(\Phi_\varphi) = \sum_{\alpha\in\mathrm{IBr}_p(H)} a_\alpha \Psi_\alpha.$$

Frobenius reciprocity gives that

$$a_\alpha = \langle \mathrm{Res}_H^G(\Phi_\varphi), \alpha \rangle_H = \langle \Phi_\varphi, \mathrm{Ind}_H^G(\alpha) \rangle_G,$$

which is zero except for $\alpha = \mathrm{Res}_H^G(\varphi) = \mathrm{Res}_H^G(\varepsilon\otimes\varphi)$ when $\varepsilon\otimes\varphi \neq \varphi$ by (9), and for $\alpha = \varphi^\pm$ when $\varepsilon\otimes\varphi = \varphi$ by (10). In these last cases, $a_\alpha = 1$. Hence, we obtain

$$\mathrm{Res}_H^G(\Phi_\varphi) = \mathrm{Res}_H^G(\varepsilon\otimes\Phi_\varphi) = \Psi_\varphi \quad \text{if } \varepsilon\otimes\varphi \neq \varphi,$$

and

$$\mathrm{Res}_H^G(\Phi_\varphi) = \Psi_{\varphi^+} + \Psi_{\varphi^-} \quad \text{if } \varepsilon\otimes\varphi = \varphi,$$

as required. Finally, every projective indecomposable character of $H$ is obtained by this process, because if $\Psi_\alpha \in \mathrm{IPr}_p(H)$, then there is $\varphi \in \mathrm{IBr}_p(G)$ such that $\langle \Psi_\alpha, \mathrm{Res}_H^G(\Phi_\varphi)\rangle_H = \langle \mathrm{Ind}_H^G(\Psi_\alpha), \Phi_\varphi\rangle_G \neq 0$ since $\mathrm{Ind}_H^G(\Psi_\alpha) \in \mathbb{Z}\,\mathrm{IPr}_p(G)$. The result follows. □

*Remark* 7. Note that, by Clifford theory, the constituents $\varphi^\pm \in \mathrm{IBr}_p(H)$ of an $\varepsilon$-stable Brauer character $\varphi \in \mathrm{IBr}_p(G)$ are $G$-conjugate, that is

$$\varphi^+ = {}^\sigma\varphi^-$$

for some automorphism $\sigma \in \mathrm{Aut}(H)$ induced by an inner automorphism of $G$. In particular, Lemma 2 implies that

$${}^\sigma\Phi_{\varphi^+} = \Phi_{\varphi^-}. \tag{11}$$

By (4), we have $\varepsilon\otimes\widehat{\varphi} = \widehat{\varepsilon\otimes\varphi}$ for all class functions $\varphi$ on $G$. Hence (5) gives $d_{\varepsilon\otimes\chi,\varepsilon\otimes\varphi} = d_{\chi,\varphi}$ for any $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}_p(G)$. It follows that $\varepsilon$ acts on the $p$-blocks of $G$. Let $\mathfrak{B}$ be a $p$-block of $G$. We write $\mathrm{Res}_H^G(\mathfrak{B})$ for the set of constituents of the $\mathrm{Res}_H^G(\chi)$ for $\chi \in \mathfrak{B}$. The last proposition gives information on the decomposition matrix of $\mathrm{Res}_H^G(\mathfrak{B})$ from that of $\mathfrak{B}$ as follows.

**Proposition 8.** *Let $\mathfrak{B}$ be a $p$-block of $G$. Write $D_\mathfrak{B}$ for the decomposition matrix of $\mathfrak{B}$.*

1. *If $\mathfrak{B} \neq \varepsilon(\mathfrak{B})$, then $b = \mathrm{Res}_H^G(\mathfrak{B}) = \mathrm{Res}_H^G(\varepsilon(\mathfrak{B}))$ is a $p$-block of $H$, and the restriction of $D_\mathfrak{B}$ to $H$ (which is equal to that of $D_{\varepsilon(\mathfrak{B})}$) is the decomposition matrix of $b$.*

2. *If $\mathfrak{B} = \varepsilon(\mathfrak{B})$, then $\mathrm{Res}_H^G(\mathfrak{B})$ is the sum of at most two $p$-blocks of $H$. Moreover, if there is $\alpha \in \mathfrak{B}$ such that $\alpha \neq \varepsilon\otimes\alpha$, then it is a single $p$-block whose decomposition matrix has columns $\Psi_\varphi$ for $\varphi \in \mathfrak{B}$ such that $\varphi \neq \varepsilon\otimes\varphi$ and $\Psi_{\varphi^+}$ and $\Psi_{\varphi^-}$ otherwise.*

*Proof.* Assume $\mathfrak{B} \neq \varepsilon(\mathfrak{B})$. Then $\mathfrak{B}\cap\varepsilon(\mathfrak{B}) = \emptyset$ and $\mathfrak{B}$ contains no $\varepsilon$-stable character. The columns of $D_\mathfrak{B}$ and $D_{\varepsilon(\mathfrak{B})}$ are the ordinary constituents of $\Phi_\varphi$ and $\Phi_{\varepsilon\otimes\varphi}$ for $\varphi \in \mathfrak{B}$. Thanks to (8), the columns of $D_\mathfrak{B}$ and $D_{\varepsilon(\mathfrak{B})}$ are interchanged by $\varepsilon$, and part 1 of Proposition 6 implies that $\mathfrak{b} = \mathrm{Res}_H^G(\mathfrak{B})$ is a single $p$-block of $H$ and that the restriction of $D_\mathfrak{B}$ to $H$ is the decomposition matrix of $\mathfrak{b}$.

Assume now $\mathfrak{B} = \varepsilon(\mathfrak{B})$. Let $\alpha$ and $\beta$ be in $\mathfrak{B}$. Then there are $\alpha_0, \ldots, \alpha_r \in \mathfrak{B}$ with $\alpha_0 = \alpha$, $\alpha_r = \beta$ and such that $\langle \Phi_{\alpha_i}, \Phi_{\alpha_{i+1}}\rangle_G \neq 0$ for $0 \leq i \leq r-1$. Furthermore, we derive from the proof of Proposition 6 that $\mathrm{Ind}_H^G(\Psi_\beta) = \Phi_\beta + \varepsilon\otimes\Phi_\beta$ if $\beta \neq \varepsilon\otimes\beta$, and $\mathrm{Ind}_H^G(\Psi_{\beta^+}) = \mathrm{Ind}_H^G(\Psi_{\beta^-}) = \Phi_\beta$ otherwise. Suppose that $\alpha_i = \alpha_i\otimes\varepsilon$ and $\alpha_{i+1} = \alpha_{i+1}\otimes\varepsilon$. Then

$$\langle \Psi_{\alpha_i^+} + \Psi_{\alpha_i^-}, \Psi_{\alpha_{i+1}^\pm}\rangle_H = \langle \mathrm{Res}_H(\Phi_{\alpha_i}), \Psi_{\alpha_{i+1}^\pm}\rangle_H = \langle \Phi_{\alpha_i}, \mathrm{Ind}_H^G(\Psi_{\alpha_{i+1}^\pm})\rangle_G = \langle \Phi_{\alpha_i}, \Phi_{\alpha_{i+1}}\rangle_G \neq 0.$$

It follows that either $\langle \Psi_{\alpha_i^+}, \Psi_{\alpha_{i+1}^\pm}\rangle_H \neq 0$ or $\langle \Psi_{\alpha_i^-}, \Psi_{\alpha_{i+1}^\pm}\rangle_H \neq 0$. Assume $\alpha_i \neq \varepsilon\alpha_i$. If $\alpha_{i+1} = \varepsilon\alpha_{i+1}$, then

$$\langle \Psi_{\alpha_i}, \Psi_{\alpha_{i+1}^\pm}\rangle_H = \langle \mathrm{Res}_H^G(\Phi_{\alpha_i}), \Psi_{\alpha_{i+1}^\pm}\rangle_H = \langle \Phi_{\alpha_i}, \mathrm{Ind}(\Psi_{\alpha_{i+1}^\pm})\rangle_G = \langle \Phi_{\alpha_i}, \Phi_{\alpha_{i+1}}\rangle_G \neq 0.$$

If $\alpha_{i+1} \neq \varepsilon\alpha_{i+1}$, then

$$\langle \Psi_{\alpha_i}, \Psi_{\alpha_{i+1}}\rangle_H = \langle \mathrm{Res}_H^G(\Phi_{\alpha_i}), \Psi_{\alpha_{i+1}}\rangle_H = \langle \Phi_{\alpha_i}, \mathrm{Ind}(\Psi_{\alpha_{i+1}})\rangle_G = \langle \Phi_{\alpha_i}, \Phi_{\alpha_{i+1}} + \varepsilon\Phi_{\alpha_{i+1}}\rangle_G \neq 0.$$

Now, assume there is $\alpha \in \mathfrak{B}$ such that $\alpha \neq \varepsilon\otimes\alpha$. Let $\beta \in \mathfrak{B}$. Then there are $\alpha_1, \ldots, \alpha_r \in \mathfrak{B}$ as above, and the previous computations show that the constituents of $\mathrm{Res}_H^G(\alpha_i)$ and of $\mathrm{Res}_H^G(\alpha_{i+1})$ for any $0 \leq i \leq r-1$ are in the same $p$-block of $H$. Hence, $\mathrm{Res}_H^G(\mathfrak{B})$ is a $p$-block of $H$, and the result follows. □

## 2.3 Relation between unitriangular $p$-basic sets of $G$ and $H$

We now make the following assumption

**Hypothesis 9.** Let $G$ be a finite group and $H$ be a normal subgroup of index 2. Let $x \in G \backslash H$. Denote by $\sigma$ the automorphism of $H$ induced by conjugation by $x$, and $\varepsilon$ the linear character of $G$ induced by the canonical morphism $G \to G/H$. Let $p$ be an odd prime number and $\mathfrak{b}$ be a union of $p$-blocks of $H$ covered by a union $\mathfrak{B}$ of $p$-blocks of $G$.

Let $\mathfrak{B}$ be a $p$-block of $G$. For any $p$-basic set $B$ of $\mathfrak{B}$, we denote by $\mathrm{Res}_H^G(B)$ the set of constituents of the $\mathrm{Res}_H^G(\chi)$ for $\chi \in B$.

**Proposition 10.** *With the notation as above, if $B$ is an $\varepsilon$-stable $p$-basic set of a union of $p$-blocks $\mathfrak{B}$ of $G$ then*

$$|\,\mathrm{IBr}_p(\mathrm{Res}_H^G(\mathfrak{B})| = |\,\mathrm{Res}_H^G(B)|.$$

*Proof.* We follow step by step the proof of [3, Proposition 6.1]. This is given for the symmetric group $\mathfrak{S}_n$, the sign character $\varepsilon$ and the full decomposition matrix of $\mathrm{Irr}(\mathfrak{S}_n)$ but the argument is analogous for a group $G$ with index two subgroup $H$ and a $p$-block $\mathfrak{B}$ with $\varepsilon$-stable $p$-basic set $B$. We obtain

$$|\,\mathrm{Fix}_\varepsilon(B)| = |\,\mathrm{Fix}_\varepsilon(\mathrm{IBr}_p(\mathfrak{B}))|, \tag{12}$$

where $\mathrm{Fix}_\varepsilon(B))$ is the subset of $\varepsilon$-fixed characters of $B$. Now, since $p$ is odd, Clifford's modular theory [10, Theorem 9.18] implies that each Brauer character of $\mathrm{Fix}_\varepsilon(\mathrm{IBr}(\mathfrak{B}))$ splits into two irreducible Brauer characters of $\mathrm{Res}_H^G(\mathrm{IBr}(\mathfrak{B}))$, while the others restrict irreducibly to $H$. Using that every irreducible character $\chi$ of $B$ also splits into two or one irreducible character(s) of $\mathrm{Res}_H^G(B)$ depending on whether $\chi \in \mathrm{Fix}_\varepsilon(B)$ or not, the result follows. □

*Remark* 11. Let $B$ be an $\varepsilon$-stable $p$-basic set of a $p$-block $\mathfrak{B}$ of $G$. Even though $\mathrm{Res}_H^G(B)$ is not a $p$-basic set of $\mathrm{Res}_H^G(\mathfrak{B})$ in general, Proposition 10 asserts however that we can derive from $B$ the number of Brauer characters of $\mathrm{Res}_H^G(\mathfrak{B})$.

The following result is one of the main results of this paper.

**Theorem 12.** *We assume that Hypothesis 9 holds. We suppose that $\mathfrak{B}$ has an $\varepsilon$-stable unitriangular $p$-basic set $(B, \leq, \Theta)$ such that $\Theta : B \to \mathrm{IBr}_p(\mathfrak{B})$ is $\varepsilon$-equivariant. Then $b = \mathrm{Res}_H^G(B)$ is a unitriangular $p$-basic set of $\mathfrak{b}$.*

*Proof.* We consider a subset $A = \{\chi_1, \ldots, \chi_t\}$ of $B$ that contains all $\varepsilon$-stable characters of $B$, and only one of $\chi$ and $\varepsilon \otimes \chi$ when $\chi$ is a non $\varepsilon$-stable character of $B$. By Clifford theory, each character of $A$ is above a character of $b$. Furthermore, we suppose that the characters are chosen such that

$$\chi_1 \leq \chi_2 \leq \cdots \leq \chi_s \quad \text{and} \quad \varepsilon \otimes \chi_i \leq \chi_i \text{ if } i \text{ is such that } \varepsilon \otimes \chi_i \neq \chi_i.$$

Since $B$ is $\varepsilon$-stable, Condition (ii) and Equation (8) give that $\chi_i$ is $\varepsilon$-stable if and only if $\Phi_{\Theta(\chi_i)}$ is. If $\chi_i$ is $\varepsilon$-stable, then we denote by $\varphi_i^\pm$ the constituents of $\mathrm{Res}_H^G(\Theta(\chi_i))$. If $\chi_i \neq \varepsilon \otimes \chi_i$, then $\Theta(\chi_i) \neq \varepsilon \otimes \Theta(\chi_i)$ by Condition (ii), and we write $\varphi_i = \mathrm{Res}_H^G(\Theta(\chi_i)) \in \mathrm{IBr}_p(\mathfrak{b})$. Now, we order $\mathrm{Res}_H^G(B)$ such that if $\psi$ and $\psi'$ are constituents of $\mathrm{Res}_H^G(\chi_i)$ and $\mathrm{Res}_H^G(\chi_j)$ with $i \leq j$, then $\psi \leq \psi'$. Suppose that $\chi_i \in B$ is $\varepsilon$-stable. Write $\psi_i^+$ and $\psi_i^-$ for the constituents of $\mathrm{Res}_H^G(\chi_i)$. We have

$$\begin{aligned} d_{\psi_i^\pm, \varphi_i^+} + d_{\psi_i^\pm, \varphi_i^-} &= \langle \psi_i^\pm, \Phi_{\varphi_i^+} \rangle_H + \langle \psi_i^\pm, \Phi_{\varphi_i} \rangle_H \\ &= \langle \psi_i^\pm, \mathrm{Res}_H^G(\Phi_{\Theta(\chi_i)}) \rangle_H \\ &= \langle \chi_i, \Phi_{\Theta(\chi_i)} \rangle_G \\ &= 1. \end{aligned}$$

Hence $\{d_{\psi_i^\pm,\varphi_i^+}, d_{\psi_i^\pm,\varphi_i^-}\} = \{0,1\}$, because decomposition numbers are non-negative integers. We assume that the labeling of $\psi_i^\pm$ is chosen such that $d_{\psi_i^+,\varphi_i^+} = 1$ and $d_{\psi_i^+,\varphi_i^-} = 0$. Furthermore, (7) implies that $d_{\psi_i^-,\varphi_i^-} = 1$ and $d_{\psi_i^-,\varphi_i^+} = 0$. With these choices, we define $\psi_i^+ \leq \psi_i^-$. Finally, we set

$$\begin{array}{rccc} \Psi: & \mathrm{Res}_H^G(B) & \longrightarrow & \mathrm{IBr}_p(\mathfrak{b}), \\ & \psi_i^\pm & \longmapsto & \varphi_i^\pm \\ & \psi_i & \longmapsto & \varphi_i \end{array}.$$

Assume that $1 \leq i \leq j \leq t$.

- Suppose that $\psi_i$ and $\psi_j$ are $\sigma$-stable. Then

$$\begin{array}{rcl} d_{\psi_i,\varphi_j} & = & \langle \psi_i, \Phi_{\varphi_j}\rangle_H \\ & = & \langle \mathrm{Ind}_H^G(\psi_i), \Phi_{\Theta(\chi_j)}\rangle_G \\ & = & \langle \chi_i + \varepsilon\otimes\chi_i, \Phi_{\Theta(\chi_j)}\rangle_G \\ & = & d_{\chi_i,\Theta(\chi_j)} + d_{\varepsilon\otimes\chi_i,\Theta(\chi_j)}. \end{array} \tag{13}$$

  However, $\varepsilon\otimes\chi_i \leq \chi_i \leq \chi_j$. If $i<j$ then $d_{\chi_i,\Theta(\chi_j)} = 0 = d_{\varepsilon\otimes\chi_i,\Theta(\chi_j)}$, and $d_{\psi_i,\varphi_j} = 0$. If $i=j$, then (13) gives $d_{\psi_i,\varphi_i} = 1$ because $d_{\chi_i,\Theta(\chi_i)} = 1$, and $d_{\varepsilon\otimes\chi_i,\Theta(\chi_i)} = 0$ since $\varepsilon\otimes\chi_i \leq \chi_i$.

- Suppose that $\psi_i$ is $\sigma$-stable and $j$ labels $\psi_j^+$ and $\psi_j^-$. Then

$$\begin{aligned} d_{\psi_i,\varphi_j^+} + d_{\psi_i,\varphi_j^-} &= \langle \psi_i, \Phi_{\varphi_j^+}\rangle_H + \langle \psi_i, \Phi_{\varphi_j^-}\rangle_H = \langle \psi_i, \Phi_{\varphi_j^+} + \Phi_{\varphi_j^-}\rangle_H = \langle \psi_i, \mathrm{Res}_H^G(\Phi_{\Theta(\chi_j)})\rangle_H \\ &= \langle \mathrm{Ind}_H^G(\psi_i), \Phi_{\Theta(\chi_j)}\rangle_G = \langle \varepsilon\otimes\chi_i + \chi_i, \Phi_{\Theta(\chi_j)}\rangle_G \\ &= d_{\chi_i,\Theta(\chi_j)} + d_{\varepsilon\otimes\chi_i,\Theta(\chi_j)} \\ &= 0 \end{aligned}$$

  since $\varepsilon\otimes\chi_i \leq \chi_i < \chi_j$.

- Suppose that $i$ and $j$ label non $\sigma$-stable characters. Assume $i<j$. The same computation as above gives

$$\begin{aligned} d_{\psi_i^\pm,\varphi_j^+} + d_{\psi_i^\pm,\varphi_j^-} &= \langle \chi_i, \Phi_{\Theta(\chi_j)}\rangle_G \\ &= d_{\chi_i,\Theta(\chi_j)} \\ &= 0. \end{aligned}$$

  If $i=j$, then $d_{\psi_i^+,\varphi_i^+} = 1 = d_{\psi_i^-,\varphi_i^-}$ and $d_{\psi_i^+,\varphi_i^-} = 0$ by construction.

- Suppose that $i$ labels two characters and $\chi_j$ is $\varepsilon$-stable. Then

$$\begin{aligned} d_{\psi_i^\pm,\varphi_j} &= \langle \psi_i^\pm, \mathrm{Res}_H^G(\Phi_{\Theta(\chi_j)})\rangle_H = \langle \mathrm{Ind}_H^G(\psi_i^\pm), \Phi_{\Theta(\chi_j)}\rangle_G \\ &= \langle \chi_i, \Phi_{\Theta(\chi_j)}\rangle_G \\ &= d_{\chi_i,\Theta(\chi_j)} \\ &= 0. \end{aligned}$$

This proves the result. □

We consider the set $\mathcal{T}$ of $\varepsilon$-stable irreducible characters $\chi$ of $G$ such that the constituents $\chi^+$ and $\chi^-$ of their restriction to $H$ satisfies $\widehat{\chi}^+ \neq \widehat{\chi}^-$.

*Remark* 13. Note that, if $B$ is a unitriangular $p$-basic set of $G$ which satisfies the assumptions of Theorem 12, then the $\sigma$-stable characters of $B$ lie in $\mathcal{T}$. Indeed, since $\mathrm{Res}_H^G(B)$ is a $p$-basic set of $H$, for any $\varepsilon$-stable character $\chi$ of $B$, by Remark 5, the family $(\widehat{\chi}^+, \widehat{\chi}^-)$ is free and, in particular, $\widehat{\chi}^+ \neq \widehat{\chi}^-$.

**Example 14.** Let $G = \mathfrak{S}_6$ and $H = \mathfrak{A}_6$. Write sgn for the sign character of $G$. The principal 3-block $\mathfrak{B}_0$ of $G$ has 9 irreducible characters and 5 Brauer characters. It has a unitriangular 3-basic set $(B, \leq, \Theta)$ with $B = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$, such that $\chi_2 = \mathrm{sgn} \otimes \chi_1$, $\chi_4 = \mathrm{sgn} \otimes \chi_3$, and $\chi_5$ is $\varepsilon$-stable and lies in $\mathcal{T}$. The restriction of the 3-decomposition matrix to $B$ is

$$\begin{array}{c} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We remark that $\Theta$ is $\varepsilon$-equivariant. We set $\widetilde{\varphi}_i = \Theta(\chi_i)$. Furthermore, we write $\psi_i = \mathrm{Res}_H^G(\chi_i)$ for $1 \leq i \leq 4$ and $\mathrm{Res}_H^G(\chi_5) = \psi_5^+ + \psi_5^-$. Then the set $A$ appearing in the proof of Theorem 12 is $A = \{\chi_2, \chi_4, \chi_5\}$. Set $\varphi_2 = \mathrm{Res}_H^G(\widetilde{\varphi}_1)$, $\varphi_4 = \mathrm{Res}_H^G(\widetilde{\varphi}_4)$ and $\mathrm{Res}_H^G(\widetilde{\varphi}_5) = \varphi_5^+ + \varphi_5^-$. Using the fact that $\langle \psi_5^+, \Phi_{\varphi_i} \rangle_H = \langle \psi_5^-, \Phi_{\varphi_i} \rangle_H = d_{\chi_5, \Phi_{\widetilde{\varphi}_i}}$ for $i \in \{2, 4\}$ (see for example the last computation in the proof of Theorem 12), we deduce from Theorem 12 that the restriction to $\{\psi_2, \psi_4, \psi_5^+, \psi_5^-\}$ of the 3-decomposition matrix of the principal block of $\mathfrak{A}_6$ is

$$\begin{array}{c} \psi_2 \\ \psi_4 \\ \psi_5^+ \\ \psi_5^- \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Now, we assume that Hypothesis 9 holds and that $\mathfrak{b}$ has a unitriangular $p$-basic set $(b, \leq, \Psi)$. We write $\mathcal{R}_{\mathfrak{b}}$ for the set of Brauer characters $\widetilde{\varphi} \in \mathrm{IBr}_p(G)$ such that the restriction of $\varphi$ to $H$ splits into the sum of two Brauer characters of $\mathfrak{b}$. Set

$$\mathcal{S}_{\mathfrak{b}} = \{\varphi, {}^\sigma\varphi \mid \widetilde{\varphi} \in \mathcal{R}_{\mathfrak{b}}\} \quad \text{and} \quad \mathcal{C}_b = \{\Psi^{-1}(\varphi) \mid \varphi \in \mathcal{S}_{\mathfrak{b}}\}.$$

**Notation 15.** Assume Hypothesis 9 is satisfied, and that $\mathfrak{b}$ has a unitriangular $p$-basic set $(b, \leq, \Psi)$. For any $\varphi \in \mathcal{S}_{\mathfrak{b}}$, we write $E(\varphi)$ and $E(\varphi)^\star$ for the irreducible characters of $b$ such that

$$\{E(\varphi), E(\varphi)^\star\} = \{\Psi^{-1}(\varphi), \Psi^{-1}({}^\sigma\varphi)\} \quad \text{and} \quad E(\varphi) \leq E(\varphi)^\star. \tag{14}$$

Note that this construction depends on the basic set $b$.

**Theorem 16.** *Assume Hypothesis 9 holds and that $\mathfrak{b}$ has a unitriangular $p$-basic set $(b, \leq, \Psi)$. Then, for any $\varphi \in \mathcal{S}_{\mathfrak{b}}$, we have*

$${}^\sigma\widehat{E}(\varphi) \neq \widehat{E}(\varphi) \quad \textit{and} \quad E(\varphi) \leq {}^\sigma E(\varphi). \tag{15}$$

*Furthermore, if we write*

$$\chi = \mathrm{Ind}_H^G(E(\varphi)) \quad \textit{and} \quad \vartheta = \mathrm{Ind}_H^G(\varphi),$$

*then $\chi \in \mathrm{Irr}(G)$ and*

$$d_{\chi, \vartheta} = 1.$$

*Proof.* First, we remark that ${}^\sigma\mathfrak{b} \cap \mathfrak{b} \neq \emptyset$ since $\varphi$ and ${}^\sigma\varphi$ lie in $\mathfrak{b}$. Hence, ${}^\sigma\mathfrak{b} = \mathfrak{b}$. Furthermore, $\Phi_\varphi$ and $\Phi_{{}^\sigma\varphi}$ are distinct by Lemma 2 since $\varphi \neq {}^\sigma\varphi$.

By assumption, $b$ is a unitriangular $p$-basic set of $\mathfrak{b}$ and $d_{E(\varphi),\varphi} = 1$. Hence, Equation (7) gives

$$d_{{}^\sigma E(\varphi),{}^\sigma\varphi} = 1 \neq 0,$$

and $E(\varphi)^\star \leq {}^\sigma E(\varphi)$, where $E(\varphi)^\star$ is defined in Notation 15. Then (14) implies that $E(\varphi) \leq {}^\sigma E(\varphi)$. On the other hand, by the choice of $E(\varphi)$, we also have $d_{E(\varphi),{}^\sigma\varphi} = 0$. Hence $\widehat{E}(\varphi) \neq {}^\sigma\widehat{E}(\varphi)$, otherwise $d_{E(\varphi),{}^\sigma\varphi} = d_{{}^\sigma E(\varphi),{}^\sigma\varphi} = 1$. In particular, $E(\varphi) \neq {}^\sigma E(\varphi)$, and $\chi$ is an irreducible character of $G$ which satisfies $\chi = \varepsilon \otimes \chi$ by Clifford theory. Write

$$\widehat{\chi} = \sum_{\beta \in \mathrm{IBr}_p(G)} d_{\chi,\beta}\, \beta.$$

By Clifford theory, restricting this relation to $H$, we obtain

$$\widehat{E}(\varphi) + {}^\sigma\widehat{E}(\varphi) = \sum_{\beta = \varepsilon\otimes\beta \in \mathrm{IBr}_p(G)} d_{\chi,\beta}\,(\beta^+ + \beta^-) + \sum_{\beta \neq \varepsilon\otimes\beta} (d_{\chi,\beta} + d_{\chi,\varepsilon\otimes\beta})\beta,$$

and the following two identities:

$$\widehat{E}(\varphi) = \sum_{\beta = \varepsilon\otimes\beta} (d_{E(\varphi),\beta^+}\beta^+ + d_{E(\varphi),\beta^-}\beta^-) + \sum_{\beta \neq \varepsilon\beta} d_{E(\varphi),\beta}\,\beta,$$
$${}^\sigma\widehat{E}(\varphi) = \sum_{\beta = \varepsilon\otimes\beta} (d_{{}^\sigma E(\varphi),\beta^+}\beta^+ + d_{{}^\sigma E(\varphi),\beta^-}\beta^-) + \sum_{\beta \neq \varepsilon\beta} d_{{}^\sigma E(\varphi),\beta}\,\beta.$$

By uniqueness of the coefficients in a basis, for all $\beta \in \mathrm{IBr}_p(G)$ such that $\beta = \varepsilon \otimes \beta$, we obtain

$$d_{\chi,\beta} = d_{E(\varphi),\beta^+} + d_{{}^\sigma E(\varphi),\beta^+} \quad \text{and} \quad d_{\chi,\beta} = d_{E(\varphi),\beta^-} + d_{{}^\sigma E(\varphi),\beta^-}. \tag{16}$$

In particular, $d_{\chi,\vartheta} = d_{E(\varphi),\varphi} + d_{{}^\sigma E(\varphi),\varphi}$. Assume that $d_{\chi,\vartheta} > 1$. Then Equation (7) gives

$$\begin{array}{rcl} d_{E(\varphi),{}^\sigma\varphi} & = & d_{{}^\sigma E(\varphi),\varphi} \\ & = & d_{\chi,\vartheta} - d_{E(\varphi),\vartheta} \\ & = & d_{\chi,\vartheta} - 1 \\ & > & 0, \end{array}$$

which is a contradiction. The result follows. □

For any subsets $A \subseteq \mathrm{Irr}(H)$ and $B \subseteq \mathrm{IBr}_p(H)$, we write $D_{A,B}$ for the restriction of the $p$-decomposition matrix of $H$ to $A \times B$.

**Proposition 17.** *Assume Hypothesis 9 holds, and that $\mathfrak{b}$ has a unitriangular $p$-basic set $(b, \leq, \Psi)$ Let $D$ be the $p$-decomposition matrix of $G$. Then there is a subset $\mathcal{T}_b$ of $\mathcal{T}$ so that $D_{\mathcal{T}_b,\mathcal{R}_\mathfrak{b}}$ is unitriangularisable.*

*Proof.* We label $\mathcal{R}_\mathfrak{b} = \{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_r\}$ so that

$$E(\varphi_1) \leq E(\varphi_2) \leq \cdots \leq E(\varphi_r).$$

For any $1 \leq i \leq r$, set

$$\chi_i = \mathrm{Ind}_H^G(E(\varphi_i)).$$

Then $\chi_i \in \mathcal{T}$ by Theorem 16. Set

$$\mathcal{T}_b = \{\chi_1, \ldots, \chi_r\}. \tag{17}$$

We remark that the elements of $\mathcal{T}_b$ are pairwise distinct. Indeed, if $\chi_i = \chi_j$ for $i \neq j$, then $E(\varphi_i) = {}^\sigma E(\varphi_j)$. However, Theorem 16 gives that $E(\varphi_i) \leq {}^\sigma E(\varphi_i) = E(\varphi_j)$ and $E(\varphi_j) \leq {}^\sigma E(\varphi_j) = E(\varphi_i)$. Hence, $E(\varphi_i) = E(\varphi_j) = {}^\sigma E(\varphi_i)$, which is a contradiction. In particular, $|\mathcal{T}_b| = r = |\mathcal{R}_b|$.

Now, we define an order on $\mathcal{T}_b$ by setting $\chi_i \leq \chi_j$ whenever $i \leq j$, and we set

$$\Theta : \mathcal{T}_b \longrightarrow \mathcal{R}_\mathfrak{b},\ \chi_i \longmapsto \widetilde{\varphi}_i.$$

The map $\Theta$ is well-defined and surjective by construction. Hence, it is bijective by cardinality. Now, we prove that $D_{\mathcal{T}_b,\mathcal{R}_\mathfrak{b}}$ is unitriangular with respect to the datum $(\leq, \Theta)$. First, by Theorem 16, we have, for all $1 \leq i \leq r$

$$d_{\chi_i,\Theta(\chi_i)} = d_{\mathrm{Ind}_H^G(E(\varphi_i)),\mathrm{Ind}_H^G(\varphi_i)} = 1.$$

Assume $i \leq j$ (in particular $\chi_i \leq \chi_j$). Then

$$\begin{aligned}
d_{\chi_i,\widetilde{\varphi}_j} &= \langle \chi_i, \Phi_{\widetilde{\varphi_j}} \rangle_G \\
&= \langle E(\varphi_i), \mathrm{Res}_H^G(\Phi_{\widetilde{\varphi}_j}) \rangle_H \quad \text{(by Frobenius Reciprocity)} \\
&= \langle E(\varphi_i), \Phi_{\varphi_j} + \Phi_{{}^\sigma\varphi_j} \rangle_H \quad \text{(by Proposition 6)} \\
&= \langle E(\varphi_i), \Phi_{\varphi_j} \rangle_H + \langle E(\varphi_i), \Phi_{{}^\sigma\varphi_j} \rangle_H \\
&= \langle E(\varphi_i), \Phi_{\Psi(E(\varphi_j))} \rangle_H + \langle E(\varphi_i), \Phi_{{}^\sigma\Psi(E(\varphi_j))} \rangle_H \\
&= d_{E(\varphi_i),\Psi(E(\varphi_j))} + d_{E(\varphi_i),\Psi(E(\varphi_j)^*)}.
\end{aligned}$$

However, $E(\varphi_i) \leq E(\varphi_j) \leq E(\varphi_j)^*$, hence

$$d_{E(\varphi_i),\Psi(E(\varphi_j))} = 0 \quad \text{and} \quad d_{E(\varphi_i),\Psi(E(\varphi_j)^*)} = 0.$$

This proves that $d_{\chi_i,\widetilde{\varphi}_j} = 0$ whenever $\chi_i \leq \chi_j$, as required. □

**Theorem 18.** *Let $G$ be a finite group and $H$ be an index 2 subgroup of $G$. Let $p$ be an odd prime number. Assume Hypothesis 9 is satisfied, and that $\mathfrak{b}$ has a unitriangular $p$-basic set $(b, \leq, \Psi)$. Let $\mathfrak{B}$ be the union of all $p$-blocks of $G$ covering $\mathfrak{b}$. Then $\mathfrak{B}$ has a unitriangular $p$-basic set.*

*Proof.* We keep Notation 15, and define $E_b = \mathrm{Irr}(\mathfrak{b}) \backslash \{E(\varphi)^* \mid \widetilde{\varphi} \in \mathcal{R}_\mathfrak{b}\}$. Suppose that $E_b = \{\psi_1, \ldots, \psi_s\}$ with

$$\psi_1 \leq \psi_2 \leq \cdots \leq \psi_s.$$

Note that, for all $1 \leq i \leq s$, we have ${}^\sigma\psi_i = \psi_i$ if and only if $\Phi_{\Psi(\psi_i)} = {}^\sigma\Phi_{\Psi(\psi_i)}$. For any $1 \leq i \leq s$, we write $\widetilde{\varphi}_i^+$ and $\widetilde{\varphi}_i^-$ for the constituents of $\mathrm{Ind}_H^G(\Psi(\psi_i))$ if $\psi_i$ is $\sigma$-stable, and $\widetilde{\varphi}_i = \mathrm{Ind}_H^G(\Phi(\psi_i))$ if $\psi_i$ is not $\sigma$-stable. Moreover, we also write

$$\mathrm{Res}_H^G(\widetilde{\varphi_i}^+) = \varphi_i \quad \text{and} \quad \mathrm{Res}_H^G(\widetilde{\varphi}_i) = \varphi_i + {}^\sigma\varphi_i.$$

Assume that $\mathrm{Ind}_H^G(\psi_i)$ has two constituents, $\chi_i^+$ and $\chi_i^-$. Then

$$\begin{aligned}
d^\pm_{\chi_i^+,\widetilde{\varphi}_i^\pm} + d^\pm_{\chi_i^-,\widetilde{\varphi}_i^\pm} &= \langle \mathrm{Ind}_H^G(\psi_i), \Phi_{\widetilde{\varphi}_i^\pm} \rangle_G \\
&= \langle \psi_i, \mathrm{Res}_H^G(\Phi_{\widetilde{\varphi}_i^\pm}) \rangle_H \\
&= \langle \psi_i, \Phi_{\varphi_i} \rangle_H \\
&= 1.
\end{aligned} \tag{18}$$

Since these are non-negative integers, one has to be equal to 1 and the other to 0. We then choose the labeling such that $d_{\chi_i^+,\widetilde{\varphi}_i^+} = 1$ and $d_{\chi_i^-,\widetilde{\varphi}_i^-} = 0$. Furthermore, since the characters of $\{\chi_i^+, \chi_i^-\}$ and of the set $\{\Phi_{\widetilde{\varphi}_i^+}, \Phi_{\widetilde{\varphi}_i^-}\}$ are interchanged by $\varepsilon$, we deduce that $d_{\chi_i^+,\widetilde{\varphi}_i^-} = 0$ and $d_{\chi_i^-,\widetilde{\varphi}_i^-} = 1$.

We define $B$ as the set of constituents of the $\mathrm{Ind}_H^G(\psi_i)$'s for $1 \leq i \leq s$. We order $B$ so that the natural order of the indices is respected and $\chi_i^+ \leq \chi_i^-$. Finally, we set

$$\begin{array}{rccc}
\Theta : & B & \longrightarrow & \mathrm{IBr}_p(\mathfrak{B}), \\
& \chi_i^\pm & \longmapsto & \widetilde{\varphi}_i^\pm \\
& \chi_i & \longmapsto & \widetilde{\varphi}_i
\end{array}. \tag{19}$$

Now, we prove that $B$ is a unitriangular $p$-basic set of $\mathfrak{B}$ with respect to $(\leq, \Theta)$. We already checked in Proposition 17 that if $\chi_i$ and $\chi_j$ are $\varepsilon$-stable and such that $\chi_i \leq \chi_j$, then $d_{\chi_i,\Theta(\chi_i)} = 1$ and $d_{\chi_i,\Theta(\chi_j)} = 0$. Suppose $1 \leq i < j \leq s$.

- Assume that $i$ labels two non $\varepsilon$-stable characters $\chi_i^+$ and $\chi_i^-$. If $\chi_j$ is $\varepsilon$-stable, then so is $\widetilde{\varphi}_j$ and

$$\begin{aligned} d_{\chi_i^+,\widetilde{\varphi}_j} + d_{\chi_i^-,\widetilde{\varphi}_j} &= \langle \mathrm{Ind}_H^G(\psi_i), \Phi_{\widetilde{\varphi}_j}\rangle_G \\ &= \langle \psi_i, \mathrm{Res}_H^G(\Phi_{\widetilde{\varphi}_j})\rangle_H \\ &= \langle \psi_i, \Phi_{\varphi_j}\rangle_H + \langle \psi_i, \Phi_{\sigma\varphi_j}\rangle_H \\ &= 0, \end{aligned}$$

  because $\psi_j = E(\varphi_j)$, whence $\psi_i \leq E(\varphi_j) \leq E(\varphi_j)^*$. It follows that $d_{\chi_i^+,\widetilde{\varphi}_j} = 0 = d_{\chi_i^-,\widetilde{\varphi}_j}$ since both are non-negative integers. If $j$ labels two non $\varepsilon$-stable characters, then $\mathrm{Res}_H^G \Phi_{\widetilde{\varphi}_j^+} = \mathrm{Res}_H^G \Phi_{\widetilde{\varphi}_j^-} = \Phi_{\varphi_j}$. Hence, an analogue computation as above shows that

$$\begin{aligned} d^{\pm}_{\chi_i^+,\widetilde{\varphi}_j^{\pm}} + d^{\pm}_{\chi_i^-,\widetilde{\varphi}_j^{\pm}} &= \langle \mathrm{Ind}_H^G(\psi_i), \Phi_{\widetilde{\varphi}_j^{\pm}}\rangle_G \\ &= \langle \psi_i, \mathrm{Res}_H^G(\Phi_{\widetilde{\varphi}_j^{\pm}})\rangle_H \\ &= \langle \psi_i, \Phi_{\varphi_j}\rangle_H \\ &= 0 \end{aligned}$$

  because $\psi_i \leq \psi_j$. Finally, by the choice of our labeling (see the discussion after (18)), we have $d_{\chi_i^{\pm},\Phi_{\widetilde{\varphi}_i^{\pm}}} = 1$ and $d_{\chi_i^+,\Phi_{\widetilde{\varphi}_i^-}} = 0$.

- Assume $\chi_i$ is $\varepsilon$-stable and $j$ labels two non $\varepsilon$-stable characters. Then

$$\begin{aligned} d_{\chi_i,\widetilde{\varphi}_j^{\pm}} &= \langle \chi_i, \Phi_{\widetilde{\varphi}_j^{\pm}}\rangle_G \\ &= \langle \mathrm{Ind}_H^G(\psi_i), \Phi_{\widetilde{\varphi}_j^{\pm}}\rangle_G \\ &= \langle \psi_i, \mathrm{Res}_H^G(\Phi_{\widetilde{\varphi}_j^{\pm}})\rangle_H \\ &= \langle \psi_i, \Phi_{\varphi_j}\rangle_H = d_{\psi_i,\varphi_j} \\ &= 0. \end{aligned}$$

This proves the result. □

**Example 19.** Consider the unitriangular 3-basic set $(b, \leq, \Psi)$ of the principal block $\mathfrak{b}_0$ of $\mathfrak{A}_6$ obtained in Example 14, where $b = \{\psi_2, \psi_4, \psi_5^+, \psi_5^-\}$. We have $\mathcal{R}_{\mathfrak{b}_0} = \{\chi_5\}$ and

$$E(\varphi_5^+) = E(\varphi_5^-) = \psi_5^+ \quad \text{and} \quad E(\varphi_5^+)^* = E(\varphi_5^-)^* = \psi_5^-.$$

Applying Theorem 18, we obtain a unitriangular 3-basic set of $\mathfrak{B}_0$ with decomposition matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that, even though we deduce the existence of a unitriangular 3-basic set of $\mathfrak{B}_0$, Theorem 18 does not give its complete associated decomposition matrix. Since $d_{\psi_4,\Phi_{\varphi_2}} = 1$, we only know that the missing values are either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

*Remark* 20. The $p$-basic set $\widetilde{b}$ of $\mathfrak{B}$ constructed from $b$ in Theorem 18 is $\varepsilon$-stable, and its $\varepsilon$-stable characters lie in $\mathcal{T}$ by construction. Furthermore, the bijection $\Theta$ is $\varepsilon$-equivariant by (19). In particular, $\widetilde{b}$ satisfies the assumptions of Theorem 12. Hence, if $\mathfrak{b}$ has a unitriangular $p$-basic set, then $\mathfrak{b}$ has a unitriangular $p$-basic set that comes from the restriction of an $\varepsilon$-stable $p$-basic set of $\mathfrak{B}$. However, note that in general

$$\mathrm{Res}_H^G(\widetilde{b}) \neq b.$$

**Example 21.** The principal 3-block $\mathfrak{b}_0$ of $\mathfrak{A}_6$ has an $\varepsilon$-stable character $\psi_6$ of degree 5 whose modular restriction with respect to the labeling of $\mathrm{IBr}_3(\mathfrak{b}_0)$ given in Example 14 is

$$[0\ 1\ 1\ 1].$$

In particular, $b' = \{\psi_2, \psi_4, \psi_5^+, \psi_6\}$ is also a unitriangular 3-basic set of $\mathfrak{b}_0$. Note that, when we apply Theorem 18 with $b'$, we obtain the same unitriangular basic set of $\mathfrak{B}_0$ as that of Example 19.

# 3 Consequences for the $p$-basic sets of the alternating groups

## 3.1 Ordinary and modular representations

Let $p$ be a prime number and $n$ be a positive integer. For this section, we refer to [11] for more details. For any $\lambda \in \Pi^1(n)$ we denote by $\chi_\lambda$ the character of the simple module $S^\lambda$ given in (3). We write $\lambda'$ for the conjugate partition of $\lambda$ and recall that $S^{\lambda'} = \mathrm{sgn} \otimes S^\lambda$, where sgn is the sign character of $\mathfrak{S}_n$. Furthermore, the irreducible characters of $\mathfrak{A}_n$ can be described from those of $\mathfrak{S}_n$ as follows. If $\lambda \neq \lambda'$, then $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\chi_\lambda) = \mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\mathrm{sgn} \otimes \chi_\lambda)$ is irreducible and denoted by $\rho_\lambda$. If $\lambda = \lambda'$, then $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\chi_\lambda)$ splits into two irreducible characters $\rho_\lambda^+$ and $\rho_\lambda^-$ of $\mathfrak{A}_n$. These two class functions take the same values everywhere, except on the elements with cycle structure given by the partition $\overline{\lambda}$, whose parts are the diagonal hook lengths of $\lambda$. These elements form a single conjugacy class in $\mathfrak{S}_n$ which splits into two classes $\overline{\lambda}_-$ and $\overline{\lambda}_+$ of $\mathfrak{A}_n$, and, following [11, Theorem 2.5.13], the notation can be chosen such that

$$\rho_\lambda^\pm(t_{\overline{\lambda}_+}) = x_\lambda \pm y_\lambda \quad \text{and} \quad \rho_\lambda^\pm(t_{\overline{\lambda}_-}) = x_\lambda \mp y_\lambda,$$

where $t_{\overline{\lambda}_+}$ (resp. $t_{\overline{\lambda}_-}$) is a representative of the class $\overline{\lambda}_+$ (resp. $\overline{\lambda}_-$) of $\mathfrak{A}_n$, and, if $\overline{\lambda} = (d_1, d_2, \ldots, d_k)$, then

$$x_\lambda = \frac{1}{2}(-1)^{(n-k)/2} \quad \text{and} \quad y_\lambda = \frac{1}{2}\sqrt{(-1)^{(n-k)/2} d_1 \cdots d_k}. \tag{20}$$

**Lemma 22.** *Let $p$ be an odd prime number. The set $\mathcal{T}$ defined on page 9 for $\mathfrak{A}_n$ is*

$$\mathcal{T} = \{\chi_\lambda \mid \lambda \in \mathcal{G}\},$$

*where $\mathcal{G}$ is the set of self-conjugate partitions none of whose diagonal hooks has length divisible by $p$.*

*Proof.* For any self-conjugate partition $\lambda$, if $p$ divides a part of $\overline{\lambda}$, then the corresponding class is $p$-singular. In particular, $\widehat{\rho}_\lambda^\pm(t_{\overline{\lambda}_-}) = 0 = \widehat{\rho}_\lambda^\pm(t_{\overline{\lambda}_+})$, and $\widehat{\rho}_\lambda^+ = \widehat{\rho}_\lambda^-$. Thus, $\chi_\lambda \notin \mathcal{T}$. Now, if $\lambda \in \mathcal{G}$, then $\overline{\lambda}_+$ and $\overline{\lambda}_-$ label $p$-regular classes of $\mathfrak{A}_n$. The values of $\widehat{\rho}_\lambda^+$ and $\widehat{\rho}_\lambda^-$ on these classes are distinct by (20), and the result follows. □

By [11], the simple $\mathfrak{S}_n$-modules in characteristic $p$ are labeled by the set of $p$-regular partitions $\mathcal{R}_n^p$ of $n$. Write $D^\mu$ for the simple $\mathfrak{S}_n$-module labeled by $\mu \in \mathcal{R}_n^p$ and $\varphi_\mu$ for its Brauer character. For any $p$-regular partition $\mu$, the Mullineux partition $m(\mu)$ associated to $\mu$ is the $p$-regular partition such that

$$\mathrm{sgn} \otimes D^\mu \simeq D^{m(\mu)}.$$

We denote by $\mathcal{M}_n^p$ the set of $p$-regular partitions $\mu$ such that $m(\mu) = \mu$. Since $p$ is odd, and following the notation of §2.2, we then have

$$\mathrm{IBr}_p(\mathfrak{A}_n) = \{\varphi_\mu \mid \mu \neq m(\mu)\} \cup \{\varphi_\mu^\pm \mid \mu \in \mathcal{M}_n^p\}. \tag{21}$$

We write $\sigma$ for conjugation by the transposition (1 2). In particular, for any self-conjugate partition $\lambda \in \Pi_n^1$ and $\mu \in \mathcal{M}_n^p$,

$$^{\sigma}\rho_\lambda^{\pm} = \rho_\lambda^{\mp} \quad \text{and} \quad {}^{\sigma}\varphi_\mu^{\pm} = \varphi_\mu^{\mp}.$$

Now, we recall the construction of the $p$-blocks of $\mathfrak{S}_n$ and $\mathfrak{A}_n$. To any partition $\lambda$ of $n$, we can associate its $p$-core $\lambda_{(p)}$ and its $p$-quotient $\lambda^{(p)} = (\lambda^1, \dots, \lambda^p) \in \Pi^p(w)$, where $\Pi^p(n)$ is the set of $p$-multipartitions of $n$ and $w = (n - |\lambda_{(p)}|)/p$. Note that the map

$$\lambda \mapsto (\lambda_{(p)}, \lambda^{(p)}) \tag{22}$$

is bijective. Recall that (see for example [3, Lemma 3.1])

$$(\lambda')_{(p)} = \lambda'_{(p)} \quad \text{and} \quad (\lambda')^{(p)} = \left((\lambda^p)', (\lambda^{p-1})', \dots, (\lambda^1)'\right). \tag{23}$$

Furthermore, the Nakayama Conjecture [11, §6.1] asserts that two irreducible characters $\chi_\lambda$ and $\chi_\mu$ lie in the same $p$-blocks of $\mathfrak{S}_n$ if and only if $\lambda_{(p)} = \mu_{(p)}$. It follows that the $p$-block of $\mathfrak{S}_n$ are parametrized by the $p$-cores of partitions of $n$. In particular, by (22), the irreducible characters lying in a $p$-block of $\mathfrak{S}_n$ corresponding to a $p$-core $\gamma$ are labeled by the set $\Pi^p(w)$, where $w = (n - |\gamma|)/p$ is called the *$p$-weight* of the block.

The $p$-blocks of $\mathfrak{A}_n$ can then be parametrized as follows. Let $\gamma$ be the $p$-core of a partition of $n$. Write $\mathfrak{B}_\gamma$ for the $p$-block of weight $w$ of $\mathfrak{S}_n$ corresponding to $\gamma$, and $\mathfrak{b}_\gamma = \mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\mathfrak{B}_\gamma)$. If $\gamma \neq \gamma'$ then $\mathfrak{b}_\gamma = \mathfrak{b}_{\gamma'}$ is a $p$-block of $\mathfrak{A}_n$. If $\gamma = \gamma'$ and $w > 0$, then the character $\chi_\lambda$ of $\mathfrak{S}_n$ such that $\lambda_{(p)} = \gamma$ and $\lambda^{(p)} = ((w), \emptyset, \dots, \emptyset)$ is not $\varepsilon$-stable by (23). Hence $\mathfrak{b}_\gamma$ is again a single $p$-block of $\mathfrak{A}_n$ by Proposition 8. Finally, when $w = 0$, the block $\mathfrak{B}_\gamma$ contains a unique character whose restriction to $\mathfrak{A}_n$ splits into two irreducible characters of $\mathfrak{A}_n$, which give two $p$-blocks of $\mathfrak{A}_n$ with defect zero.

*Remark* 23. The set $\mathcal{T}$ of Lemma 22 is described in [3, Lemma 3.4] in terms of the bijection (22). It is the set $\mathcal{C}$ of self-conjugate partitions $\lambda \in \Pi_n^1$ whose $p$-quotient's $(p+1)/2$-th part is the empty partition (or, equivalently, whose diagonal hook lengths are prime to $p$).

**Lemma 24.** *Let $p$ be an odd prime number. Let $\gamma$ be a symmetric $p$-core of $n$ labeling a $p$-block of $\mathfrak{S}_n$ with positive $p$-weight.*

*(i) If $\mathfrak{b}_\gamma$ has a $p$-basic set $b$, then the non $\sigma$-stable characters of $b$ are labeled by the set $\mathcal{C}_\gamma$ of partitions of $\mathcal{C}$ with $p$-core $\gamma$. Moreover, any $\lambda \in \mathcal{C}_\gamma$ labels at least one non $\sigma$-stable character of $b$.*

*(ii) If $\mathfrak{B}_\gamma$ has an $\varepsilon$-stable $p$-basic set $B$ which restricts to a $p$-basic set of $\mathfrak{b}_\gamma$, then the set of $\varepsilon$-stable characters of $B$ is labeled by $\mathcal{C}_\gamma$.*

*Proof.* Let $b$ be a $p$-basic set of $\mathfrak{b}_\gamma$. The non $\sigma$-stable characters of $b$ have to be labeled by partitions of $\mathcal{C}_\gamma$ by Lemma 22. Let $\lambda \in \mathcal{C}_\gamma$. Assume that $b$ contains neither $\rho_\lambda^+$ nor $\rho_\lambda^-$. Since $b$ is a $p$-basic set of $\mathfrak{b}_\gamma$, by Remark 5 there are $\psi_1, \dots, \psi_r \in b$ and $a_1, \dots, a_r \in \mathbb{Z}$ such that

$$\widehat{\rho}_\lambda^+ = \sum_{j=1}^{r} a_j \widehat{\psi}_j. \tag{24}$$

By Lemma 22, $t_{\overline{\lambda}_+}$ and $t_{\overline{\lambda}_-}$ are $p$-regular elements. On the other hand, for any symmetric partition $\mu \neq \lambda$, $\rho_\mu^+(t_{\overline{\lambda}_\pm}) = \rho_\mu^-(t_{\overline{\lambda}_\pm})$. Now, using that $\rho_\lambda^\pm \notin b$ and evaluating equality (24) on $t_{\overline{\lambda}_\pm}$, we obtain

$$\widehat{\rho}_\lambda^+(t_{\overline{\lambda}_+}) = \sum_{j=1}^{r} a_j \widehat{\psi}_j(t_{\overline{\lambda}_+}) = \sum_{j=1}^{r} a_j \widehat{\psi}_j(t_{\overline{\lambda}_-}) = \widehat{\rho}_\lambda^+(t_{\overline{\lambda}_-}),$$

which is a contradiction. Hence, either $\rho_\lambda^+$ or $\rho_\lambda^-$ lies in $b$.

Suppose now there is an $\varepsilon$-stable $p$-basic set $B$ of $\mathfrak{B}_\gamma$ such that $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(B) = b$. Again by Lemma 22, the $\varepsilon$-stable characters of $B$ are in $\mathcal{T}$, that is, are labeled by partitions of $\mathcal{C}_\gamma$. But at least one character of $b$ is labeled by a partition of $\mathcal{C}_\gamma$ and its induced character to $\mathfrak{S}_n$ is in $B$. The result follows. □

*Remark* 25. (i) Note that if $b$ is the restriction of an $\varepsilon$-stable $p$-basic set of $\mathfrak{B}_\gamma$, then $b$ contains both $\rho_\lambda^+$ and $\rho_\lambda^-$ for all $\lambda \in \mathcal{C}_\gamma$. The converse is not true, as we see in Remark 5 and in Example 21.

(ii) Assume $\mathfrak{b}_\gamma$ has a unitriangular $p$-basic set. Then for any $\widetilde{\varphi} \in \mathcal{R}_\mathfrak{b}$, there exists a unique $\lambda \in \mathcal{C}_\gamma$ such that either $E(\varphi) = \rho_\lambda^+$ or $E(\varphi) = \rho_\lambda^-$. In particular, the set $\mathcal{T}_b$ of Proposition 17 constructed in (17) is labeled by $\mathcal{C}_\gamma$.

**Example 26.** We return to Example 14. We have $\gamma = \emptyset$, $\mathfrak{B}_0$ has 3-weight 2, and the characters $\chi_1, \ldots, \chi_5$ are labeled by the partitions $(6)$, $(1^6)$, $(5,1)$, $(2,1^4)$ and $(3,2,1)$, respectively. The 3-quotient of $(3,2,1)$ is $((1),\emptyset,(1))$. In particular, $(3,2,1)$ lies in $\mathcal{T}$, and is the only self-conjugate partition of 6 of weight 2 in $\mathcal{T}$.

## 3.2 Counterexamples for alternating groups

Now, we will prove that, for $p = 3$, the alternating groups $\mathfrak{A}_{18}$ and $\mathfrak{A}_{19}$ have no unitriangular 3-basic set.

### 3.2.1 The case of $\mathfrak{A}_{18}$ and $p = 3$

In this section, we consider the principal block $\mathfrak{b}_0$ of $\mathfrak{A}_{18}$ for $p = 3$. Assume $\mathfrak{b}_0$ has a unitriangular 3-basic set $(b, \leq, \Psi)$. Let $\mathfrak{B}_0$ be the principal block of $\mathfrak{S}_{18}$. Note that this is the unique 3-block of $\mathfrak{S}_{18}$ that covers $\mathfrak{b}_0$. Furthermore, $\mathfrak{B}_0$ has exactly three PIMs $\Phi_{\mu^1}$, $\Phi_{\mu^2}$ and $\Phi_{\mu^3}$ fixed under sgn, where $\mu^1$, $\mu^2$ and $\mu^3$ are the 3-regular partitions of $\mathcal{M}_{18}^3$ given by

$$\mu^1 = (10, 4^2), \quad \mu^2 = (9, 4^2, 1) \quad \text{and} \quad \mu^3 = (7, 5, 2^2, 1^2).$$

Since $\mathfrak{b}_0$ has a unitriangular $p$-basic set, by Lemma 24 and Remark 25, the set $\mathcal{T}_b$ defined in (17) is the set of characters of $\mathfrak{S}_{18}$ labeled by $\mathcal{C}_0 = \{\lambda^1, \lambda^2, \lambda^3\}$, where

$$\lambda^1 = (9, 2, 1^7), \quad \lambda^2 = (7, 4, 2^2, 1^3) \quad \text{and} \quad \lambda^3 = (6, 5, 2^3, 1).$$

Note that $(\lambda^1)^{(3)} = ((1^3), \emptyset, (3))$, $(\lambda^2)^{(3)} = ((3), \emptyset, (1^3))$ and $(\lambda^3)^{(3)} = ((2,1), \emptyset, (2,1))$.

Now, using Chevie [6], we obtain that the part of the decomposition matrix of $\mathfrak{B}_0$ corresponding to these characters is

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \tag{25}$$

where the lines are labeled (from top to bottom) by $\lambda^1$, $\lambda^2$ and $\lambda^3$ and the columns (from left to right) by $\mu^1$, $\mu^2$ and $\mu^3$. Note that $D$ is not unitriangularisable (because $D$ has only two 0 entries). Thus, by Proposition 17, $\mathfrak{b}_0$ has no unitriangular 3-basic set.

### 3.2.2 The case of $\mathfrak{A}_{19}$ and $p = 3$

In this section, we show that, in the case of the alternating group $\mathfrak{A}_{19}$, for $p = 3$, there cannot exist a unitriangular $p$-basic set.

To see that, we need the theory of Fock spaces. We briefly recall how this theory appears in our context and we refer to [8, Ch. 6] for details. Let $v$ be an indeterminate and define $\mathcal{F}_v$ to be the $\mathbb{C}(v)$ vector space with basis given by all integer partitions:

$$\mathcal{F}_v := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\lambda \in \Pi^1(n)} \mathbb{C}(v)\lambda.$$

The module $\mathcal{F}_v$ is in fact an integrable module over the affine quantum group of affine type $A$. The action of the divided powers of the Chevalley operators $E_i$ and $F_i$ can be explicitly given via a combinatorial formula which we do not recall here. We get the following well-known proposition whose proof can be found for example in [13, Prop 2.3].

**Proposition 27.** *Assume that there exist $k \in \mathbb{N}$, $(i_1, \dots, i_k) \in (\mathbb{Z}/p\mathbb{Z})^k$ and $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that*

$$F_{i_1}^{(a_1)} \dots F_{i_k}^{(a_k)}.\emptyset = \lambda + \sum_{\mu \neq \lambda} a_{\lambda,\mu}(v)\mu$$

*with $a_{\mu,\lambda} \in v\mathbb{N}v$ for all $\mu \neq \lambda$, then $\lambda$ is $p$-regular and we have, for all $\lambda \neq \mu$, that $a_{\mu,\lambda}(1) = d_{\lambda,\mu}$.*

Now, to each $p$-regular partition $\lambda$, by [8, §3.5.11], one can attach a certain sequence of elements in $\mathbb{Z}/p\mathbb{Z}$:

$$\eta_p(\lambda) := \underbrace{i_1, \dots, i_1}_{a_1 \text{ times}}, \dots, \underbrace{i_k, \dots, i_k}_{a_k \text{ times}},$$

where $a_1 + \dots + a_k = n$. This sequence is the analogue of the sequence obtained using the *ladder method* (but this is not identical.) From this sequence, we define an element of the Fock space which is the action of a certain divided power of the Chevalley operators $F_i$ on the empty partition:

$$A(\lambda) := F_{i_1}^{(a_1)} \dots F_{i_k}^{(a_k)}.\emptyset \in \mathcal{F}_v.$$

Let us now consider the case $n = 19$ and $p = 3$, and the 3-regular partition $\lambda = (10, 4, 4, 1)$. First, one can check that $m(10, 4, 4, 1) = (10, 4, 4, 1)$. Now, we have:

$$\eta_p(\lambda) = (0, 2, 1, 1, 0, 2, 2, 1, 0, 0, 0, 1, 2, 2, 0, 1, 1, 2, 0),$$

which allows the definition of

$$A(10, 4, 4, 1) := F_0 F_2 F_1^{(2)} F_0 F_2^{(2)} F_1 F_0^{(3)} F_1 F_2^{(2)} F_0 F_1^{(2)} F_2 F_0.\emptyset.$$

This element may be computed using the file "arikidec" in the directory "contr"' in the package Chevie [7] of gap3 (the function is called "aliste"). We get that:

$$\begin{aligned}
A(10,4,4,1) \quad = \quad & (10,4,1,1) + v(9,5,4,1) + v(8,5,4,1,1) + v(10,4,3,2) + v^2(9,5,3,2) + v(8,5,3,3) + 2v^2(8,5,3,2,1) \\
& +2v^2(8,4,3,2,2) + (v^3+v)(7,5,4,3) + v^2(7,4,4,4) + (v^4+2v^2)(7,5,4,2,1) + (v^3+v)(7,4,4,2,2) \\
& +v(7,6,5,1) + v^2(6,6,6,1) + v^2(7,6,4,1,1) + v^2(7,6,3,3) + v^3(7,6,3,2,1) + v^3(6,6,3,2,2) \\
& +v^2(7,5,5,2) + v^3(6,5,4,4) + (v^4+2v^2)(6,5,4,2,2) + v(10,3,3,2,1) + v(9,3,3,2,2) \\
& +v^3(8,3,3,3,2) + v^2(10,3,3,1,1,1) + 2v^2(9,3,3,1,1,1,1) + v^3(8,5,3,1,1,1) + 2v^3(8,4,3,1,1,1,1) \\
& +(v^4+v^2)(7,5,3,2,1,1) + 2v^2(7,4,3,2,2,1)3v^3(7,4,3,2,1,1,1) + 2v^3(7,3,3,3,2,1)+ \\
& v^4(7,3,3,3,1,1,1) + v^3(5,5,5,2,2) + v^3(5,5,4,4,1) + (v^4+v^2)(5,5,4,3,2) + v^3(7,5,4,1,1,1)+ \\
& 2v^3(6,5,4,1,1,1,1) + v^4(5,5,5,1,1,1,1)2v^3,(6,5,3,2,2,1) + 2v^4,(6,5,3,2,1,1,1)+ \\
& (2v^4+v^2)(5,5,3,3,2,1) + (v^5+v^3)(5,5,3,3,1,1,1) + v^3(5,4,4,4,2) + (v^4+v^2)(6,4,3,2,2,1,1)+ \\
& (2v^4+v^2)(5,4,3,3,2,1,1) + v^3,(5,5,3,2,2,2) + v^3(5,4,3,2,2,2,1) + v^3(4,4,4,4,2,1)+ \\
& (v^5+v^3)(4,4,4,3,2,1,1) + v^4(4,4,4,2,2,2,1) + v^2(7,3,3,3,3) + v^3(5,5,3,3,3) + v^3(6,3,3,3,2,1,1)+ \\
& v^4(5,3,3,3,2,2,1) + v^4(4,4,3,3,3,1,1) + v^4(4,3,3,3,3,3) + v^5(4,3,3,3,3,2,1) + v(10,3,2,1,1,1,1)+ \\
& v^2(8,5,2,1,1,1,1) + (v^4+v^2)(8,3,3,1,1,1,1,1) + v^2(10,2,2,2,1,1,1) + v^3(9,2,2,2,2,1,1), \\
& v^3(8,2,2,2,2,1,1,1) + v^2(7,5,2,2,1,1,1) + v^4(7,4,2,2,2,1,1) + 2v^3(7,3,3,2,1,1,1,1)+ \\
& v^4(7,3,2,2,2,1,1,1) + v^2(10,3,1,1,1,1,1,1) + v^3(8,5,1,1,1,1,1,1) + v^3(10,2,1,1,1,1,1,1,1)+ \\
& v^3(9,2,1,1,1,1,1,1,1,1) + v^4(8,2,2,1,1,1,1,1,1,1) + v^3(7,5,1,1,1,1,1,1,1) + v^4(7,4,1,1,1,1,1,1,1,1)+ \\
& 2v^4(7,3,3,1,1,1,1,1,1) + v^5(7,3,2,1,1,1,1,1,1,1) + v^2(7,4,4,1,1,1,1) + v^3(5,5,4,1,1,1,1,1)+ \\
& 2v^4(5,5,3,2,1,1,1,1) + 2v^4(5,4,3,2,2,1,1,1) + v^4(4,4,4,4,1,1,1) + v^5(4,4,4,2,2,1,1,1)+ \\
& v^3(6,3,3,2,2,1,1,1) + v^5(5,3,3,3,2,1,1,1) + v^4(6,3,3,1,1,1,1,1,1,1) + v^5(5,5,3,1,1,1,1,1,1)+ \\
& v^5(5,4,3,1,1,1,1,1,1,1) + v^4(4,4,3,2,2,1,1,1,1) + v^5(4,4,3,2,1,1,1,1,1,1) + v^5(4,3,3,3,2,1,1,1,1)+ \\
& v^6(4,3,3,3,1,1,1,1,1,1)
\end{aligned}$$

One can see that it satisfies the hypotheses of Proposition 27. Now, for $n = 19$, we have three partitions in $\mathcal{C}_{(1)}$ which are $\mu^1 := (6, 5, 3, 2, 2, 1)$, $\mu^2 = (7, 4, 3, 2, 1, 1, 1)$ and $\mu^3 = (10, 1, 1, 1, 1, 1, 1, 1, 1)$. We thus have $d_{\lambda,\mu^1} = 2$, $d_{\lambda,\mu^2} = 3$ and $d_{\lambda,\mu^3} = 0$. Thus, by Proposition 17, there is no unitriangular $p$-basic set for $\mathfrak{A}_{19}$ and $p = 3$.

## 4 Unitriangular basic sets for symmetric algebras

This section concerns the existence and the construction of unitriangular basic sets in the context of symmetric algebras (see the definition in [9] - this thus covers the particular case of finite groups). Given a unitriangular basic set and an involution on the algebra, we show how one can construct a new unitriangular basic set which enjoys a nice "stability" property with respect to the involution.

### 4.1 Setting

In this section and the following one, we will make two hypotheses.

**Hypothesis 28.** Let $A$ be an integral domain, $L$ be a field, and let $\theta : A \to L$ be a ring homomorphism. Let $\mathcal{H}$ be an associative symmetric $A$-algebra.

1. We assume that $(B, \leq, \Psi)$ is a unitriangular basic set for $(\mathcal{H}, \theta)$ with respect to the bijective map $\Psi : B \to \mathrm{Irr}(L\mathcal{H})$.
2. We assume that we have an algebra automorphism $\phi \in \mathrm{Aut}(\mathcal{H})$ such that $\phi^2 = \mathrm{Id}_{\mathcal{H}}$.

From this, we will show how one can construct another unitriangular basic set, different from $B$ in general, and which enjoys nice properties with respect to the automorphism $\phi$. To do this, we first remark the following.

- If $N$ is a simple $K\mathcal{H}$-module (resp. $L\mathcal{H}$-module), then we can twist this module by $\phi$ to obtain a new simple $KH$-module (resp. $L\mathcal{H}$-module) $N^{\phi}$. For $\lambda \in \Lambda$, we denote by $\lambda'$ the element of $\Lambda$ such that $(V^{\lambda})^{\phi} \simeq V^{\lambda'}$. For $\lambda \in B$, we denote by $m(\lambda)$ the element of $B$ such that $\Psi(\lambda)^{\phi} \simeq \Psi(m(\lambda))$.
- By the definition of decomposition maps, there is a compatibility with the action by automorphisms on modules, and we thus have, for all $\lambda \in \Lambda$ and $\mu \in B$:

$$d_{\lambda,\Psi(\mu)} = d_{\lambda',\Psi(m(\mu))}.$$

We will need the following two elementary results coming from the above hypotheses (the first one being an analogue of [2, Prop 4.2] in our context).

**Lemma 29.** *For all $\lambda \in B$, we have $m(\lambda)' \leq \lambda$ and $\lambda' \leq m(\lambda)$.*

*Proof.* For $\lambda \in B$, we have that $d_{m(\lambda),\Psi(m(\mu))} = 1$ and, by the above property:

$$d_{m(\lambda),\Psi(m(\lambda))} = d_{m(\lambda)',\Psi(\lambda)}.$$

By the definition of unitriangular basic set, this means that $m(\lambda)' \leq \lambda$. Now, to conclude, note that $m(\lambda)$ is in $B$ and that $m^2$ is the identity. □

The following result follows from a direct application of Lemma 29.

**Lemma 30.** *Let $\lambda \in B$.*

1. *If $\lambda = m(\lambda)$, then we have $\lambda \geq \lambda'$.*
2. *If $\lambda = \lambda'$, then we have $m(\lambda) \geq \lambda$.*

## 4.2 Constructing new unitriangular basic sets

We keep the notation of the previous section. We first define a new total order on the set $\Lambda$ by setting $\lambda \preceq \mu$ if:

- $\lambda = \mu$,
- or $\lambda = \mu'$ and $\mu \leq \mu'$,
- or if
$$\mathrm{Max}_{\leq}(\lambda, \lambda') < \mathrm{Max}_{\leq}(\mu, \mu').$$

We now define a map:

$$\begin{array}{rccc} \Theta : & B & \to & \Lambda \\ & \lambda & \mapsto & \mathrm{Max}_{\preceq}\{\lambda, m(\lambda)'\}. \end{array}$$

In other words, by Lemma 29, we have:

$$\Theta(\lambda) = \begin{cases} \lambda & \text{if } m(\lambda) \leq \lambda, \\ m(\lambda)' & \text{if } m(\lambda) > \lambda. \end{cases}$$

**Proposition 31.** *The map $\Theta$ is injective.*

*Proof.* Assume that $(\lambda, \mu) \in B^2$ satisfies $\Theta(\lambda) = \Theta(\mu)$. If $\Theta(\lambda) = \lambda$ and $\Theta(\mu) = \mu$, or if $\Theta(\lambda) = m(\lambda)'$ and $\Theta(\mu) = m(\mu)'$, we directly obtain that $\lambda = \mu$, so we may assume that we have $\Theta(\mu) = m(\mu)'$ so that $\mu < m(\mu)$, and $\Theta(\lambda) = \lambda$ so that $\lambda \geq m(\lambda)$. We thus have $m(\mu)' = \lambda$. We show that this case is not possible.

Indeed, by Lemma 29, we have $\mu \geq m(\mu)'$ so $m(\mu) > m(\mu)'$. On the other hand, as $\lambda \geq m(\lambda)$, we have $m(\mu)' \geq m(m(\mu)')$. Now, by Lemma 29, since $\lambda = m(\mu)'$ is in $B$, we have $m(m(\mu)') \geq m(\mu)$ and thus $m(\mu) \leq m(\mu)'$, which is a contradiction. We must thus have $\lambda = \mu$, and $\Theta$ is injective. □

From this, we can prove the main result of this section.

**Theorem 32.** *Under the Hypothesis 28, set $\widetilde{B} := \Theta(B)$ and*

$$\widetilde{\Psi} : \widetilde{B} \to \mathrm{Irr}(L\mathcal{H})$$

*such that, for all $\lambda \in \widetilde{B}$, we have $\widetilde{\Psi}(\lambda) = \Psi(\Theta^{-1}(\lambda))$. Then $(\widetilde{B}, \preceq, \widetilde{\Psi})$ is a unitriangular basic set for $(\mathcal{H}, \theta)$.*

*Proof.* Assume that $\lambda \in B$.

- Assume that $\Theta(\lambda) = \mathrm{Max}_{\preceq}(\lambda, m(\lambda)') = \lambda$. This means that we have
$$\mathrm{Max}_{\leq}(\lambda, \lambda') \geq \mathrm{Max}_{\leq}(m(\lambda), m(\lambda)').$$
As $\lambda$ is in $B$, we already know that $d_{\lambda, \Psi(\lambda)} = 1$. In addition, we have $d_{\mu, \Psi(\lambda)} = 0$ unless $\mu \leq \lambda$. We also have $d_{\nu, \Psi(m(\lambda))} = 0$ unless $\nu \leq m(\lambda)$.

  Assume that $d_{\mu, \widetilde{\Psi}(\lambda)} \neq 0$. We then have $\widetilde{\Psi}(\lambda) = \Psi(\lambda)$, so $d_{\mu, \Psi(\lambda)} \neq 0$. We need to show that $\mu \preceq \lambda$. We have $\mu \leq \lambda$, and thus $\mu \leq \mathrm{Max}_{\leq}(\lambda, \lambda')$. We also have $d_{\mu', \Psi(m(\lambda))} \neq 0$, and thus $\mu' \leq m(\lambda)$. But we have $m(\lambda) \leq \mathrm{Max}_{\leq}(\lambda, \lambda')$. Thus the result follows in this case.

- Assume on the other hand that
$$\Theta(\lambda) = \mathrm{Max}_{\preceq}(\lambda, m(\lambda)') = m(\lambda)'.$$
This means that we have $\mathrm{Max}_{\leq}(\lambda, \lambda') \leq \mathrm{Max}_{\leq}(m(\lambda), m(\lambda)')$. We already know $d_{m(\lambda), \Psi(m(\lambda))} = 1$ because $m(\lambda)$ is in $B$, and thus that $d_{m(\lambda)'\lambda} = 1$.

  Assume that $d_{\mu, \widetilde{\Psi}(\Theta(\lambda))} \neq 0$. We thus have $d_{\mu, \Psi(\lambda)} \neq 0$, and then we need to show that $\mu \preceq \Theta(\lambda)$, or in other words that $\mu \preceq m(\lambda)$. We must have $\mu \leq \lambda$, which implies that $\mu \leq \mathrm{Max}_{\leq}(\lambda, \lambda') \leq \mathrm{Max}_{\leq}(m(\lambda), m(\lambda)')$. On the other hand, we have $d_{\mu', \Psi(m(\lambda))} \neq 0$, and thus $\mu' \leq m(\lambda)$, which implies that $\mu' \leq \mathrm{Max}_{\leq}(m(\lambda), m(\lambda)')$. This means that $\mu \preceq m(\lambda)$, whence the result.

This proves the claim. □

Let us now give some consequences of the above theorem. First, we get a new classification of the set of simple modules for $L\mathcal{H}$:

$$\mathrm{Irr}(L\mathcal{H}) := \{\widetilde{\Psi}(\lambda) \mid \lambda \in \widetilde{B}\}.$$

For $\lambda \in \widetilde{B}$, we denote by $\widetilde{m}(\lambda)$ the element of $\widetilde{B}$ such that $\widetilde{\Psi}(\lambda)^\phi \simeq \widetilde{\Psi}(\widetilde{m}(\lambda))$.

- Assume that $M \in \mathrm{Irr}(L\mathcal{H})$ is such that $M^\phi \simeq M$. Set $\lambda := \Phi^{-1}(M)$. Then $m(\lambda) = \lambda$, and thus $\Theta(\lambda) = \lambda$.
- Assume that $M \in \mathrm{Irr}(L\mathcal{H})$ is such that $M^\phi$ is not isomorphic to $M$. Set $\lambda := \Phi^{-1}(M)$. Then $\Theta(m(\lambda)) = \Theta(\lambda)'$.

As a conclusion, if $\lambda \in \widetilde{B}$, then we have

$$\widetilde{m}(\lambda) = \begin{cases} \lambda & \text{if } \widetilde{\Psi}(\lambda)^\phi \simeq \widetilde{\Psi}(\lambda) \ (\iff m(\lambda) = \lambda), \\ \lambda' & \text{otherwise.} \end{cases}$$

Note that, in the latter case, we have $\lambda' \neq \lambda$ by Lemma 30, but that we have also $\lambda' \neq \lambda$ in the first case in general.

Thus, the basic set $\widetilde{B}$ allows a parametrization of the set of simple $L\mathcal{H}$-modules where the twist by the involution is more convenient to understand on the parametrization. In terms of $B$, we have:

$$\widetilde{B} = \{\lambda \mid \lambda \in B,\ m(\lambda) < \lambda\} \cup \{\lambda' \mid \lambda \in B,\ m(\lambda) < \lambda\} \cup \{\lambda \mid \lambda \in B,\ m(\lambda) = \lambda\}.$$

We present a number of applications below.

## 5 Consequences for symmetric groups and their Hecke algebras

The aim of this section is to apply the previous section to the case of the Hecke algebra of type $A$, and to the case of the symmetric and alternating groups.

### 5.1 General case

We now apply our result to the case of the Hecke algebra of the symmetric group. This will be a first application of Theorem 32.

Let $n$ be a positive integer. We write $\mathcal{H}$ for the Hecke algebra of the symmetric group $\mathfrak{S}_n$ over a commutative ring $R$ with unit, and let $q \in R$ be invertible. We have a presentation of $\mathcal{H}$ by:

- generators: $T_s$, where $s \in S := \{s_1, \ldots, s_{n-1}\}$;
- relations: $T_{s_i}T_{s_j} = T_{s_j}T_{s_i}$ if $|i-j| > 1$, $T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}}$ for all $i \in \{1, \ldots, n-2\}$, and the relation $(T_i - q)(T_i + 1) = 0$ for all $i = 1, \ldots, n-1$.

We assume that we have a specialization $\theta : R \to L$, where $L$ is a field. We denote

$$e := \min(i \in \mathbb{N} \mid 1 + \theta(q) + \ldots + \theta(q)^{i-1} = 0),$$

and assume that $e \in \mathbb{N}_{>1}$. In this case, we are in the setting of the previous section.

- It is known that the $\mathcal{H}$-simple modules are naturally indexed by the set $\Lambda$ of all partitions of rank $n$, that is, non increasing sequences $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers of total sum $n$.
- $B$ is the set $\mathcal{R}_n^e$ of $e$-regular partitions, that is, the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ of $n$ such that there exist no $i \in \mathbb{N}$ such that $\lambda_i = \ldots = \lambda_{i+e-1} \neq 0$.

- $\leq$ is any refinement of the dominance order on partitions; for example, take $\leq$ to be the lexicographic order on partitions.
- $\phi$ is the algebra automorphism sending $T_i$ to $-T_i + (q-1)$ for all $i = 1, \ldots, n-1$.
- For $\lambda \in \Lambda$, it is well-known that $\lambda'$ is the usual conjugate partition of $\lambda$, that is, the partition obtained by reflecting the Young diagram of the partition $\lambda$ along its main diagonal.
- Finally, $m$ is known as the Mullineux involution, which we still denote by $m$. It can be computed using different recursive algorithms which we don't recall here [2, 12, 15].

Applying the main result of the previous section thus yields the following result:

**Proposition 33.** *$(\mathcal{H}, \theta)$ admits a unitriangular basic set $(\widetilde{B}, \preceq, \widetilde{\Psi})$, where $\preceq$ is defined from $\leq$ in (4.2), $\widetilde{\Psi} = \Psi \circ \Theta^{-1}$, and*

$$\widetilde{B} = \{\lambda \in \mathcal{R}_n^p \mid m(\lambda) < \lambda\} \sqcup \{\lambda' \mid \lambda \in \mathcal{R}_n^p,\ m(\lambda) < \lambda\} \sqcup \{\lambda \in \mathcal{R}_n^p \mid m(\lambda) = \lambda\}.$$

We thus obtain a new classification of the set of simple modules. Comparing to the usual classification given by the $e$-regular partitions, the main advantage of this classification is that the twist by the automorphism is easy to read on it.

- Assume that $e = 2$. In this case, we have, for all 2-regular partitions, $m(\lambda) = \lambda$, because the involution $m$ is the identity. This happens for example when $q = 1$ and $L$ is a field of characteristic 2. In this case, the Hecke algebra is nothing but the group algebra of the symmetric group over this field. We see in this case that $B = \widetilde{B}$. However, the orders $\leq$ and $\preceq$ are of course different.
- The most interesting case is the case where $e \neq 2$. Indeed, the sign representation is labeled by the partition $(1, \ldots, 1)$, and this label lies in $\widetilde{B}$ as in the semisimple case. However, if we want to obtain an explicit description of $\widetilde{B}$, we need to compute the Mullineux involution for each $e$-regular partition, which is a long, recursive and non-trivial problem. It would thus be desirable to describe the $e$-regular partitions satisfying $m(\lambda) \leq \lambda$ and $m(\lambda) = \lambda$ without computing $m(\lambda)$. We hope to come back to this problem in future work.

The above result can in particular be applied to the case where $q = 1$ and $R$ is a field of characteristic $p > 2$, so that $e = p$. The algebra $H$ is then nothing but the group algebra of the symmetric group over a field of characteristic $p$. It is now natural to ask if the unitriangular basic set satisfies the hypothesis of Theorem 12. It should not be the case, otherwise we would always have a unitriangular basic set for the alternating groups, and we have already seen that this is not the case. The problem comes from the fact that our basic set is not stable with respect to the sign representation. In fact, from Lemma 30, we deduce:

- if $\lambda \neq m(\lambda)$, then both $\lambda$ and $\lambda'$ are in the basic set $\widetilde{B}$, and we do have $\widetilde{\Psi}(\lambda') = \varepsilon \otimes \widetilde{\Psi}(\lambda)$;
- if $\lambda = m(\lambda)$, then $\lambda'$ is not in $\widetilde{B}$.

**Example 34.** Take $p = 3$ and $n = 5$. Then the set of 3-regular partitions of $n$ is:

$$B = \{(2,2,1), (3,1,1), (3,2), (4,1), (5)\}.$$

Here we consider the lexicographic order on partitions. For each of these partitions, we compute the image under the Mullineux involution to find our new basic set $\widetilde{B}$ (computed with respect to the lexicographic order). We have $m(2,2,1) = (4,1) > (2,2,1)$, so that $(4,1) \in \widetilde{B}$ and $(2,1,1,1) = (4,1)' \in \widetilde{B}$. We have $m(3,1,1) = (3,1,1) \in \widetilde{B}$. And $m(3,2) = (5) > (3,2)$, so that $(5) \in \widetilde{B}$ and $(1,1,1,1,1) = (5)' \in \widetilde{B}$. We get:

$$\widetilde{B} = \{(5), (1,1,1,1,1), (3,1,1), (4,1), (2,1,1,1)\}.$$

We can see that, in this particular case, the unitriangular basic set is stable with respect to the sign and $\varepsilon$-equivariant.

**Example 35.** Take $p = 3$ and $n = 8$. Again using the lexicographic order, we get:

$$\widetilde{B} = \{(4,2,1,1),(2,2,2,1,1),(2,2,1,1,1,1),(3,1,1,1,1,1),(3,2,1,1,1),(2,1,1,1,1,1,1),(1,1,1,1,1,1,1,1),$$
$$(5,2,1),(5,3),(6,1,1),(6,2),(7,1),(8)\}.$$

Note that the conjugate of $(4,2,1,1)$ is not in the basic set. Thus, the unitriangular basic set isn't stable with respect to tensoring by the sign character.

*Remark* 36. In this section, we have considered the lexicographic order for our examples, but we can in fact choose any total order which is compatible with the dominance order. For example, we can take the following one. For $(\lambda, \mu) \in \Lambda^2$

$$\lambda \leq' \mu \iff \lambda' \geq \mu',$$

where $\geq$ is the lexicographic order. Quite remarkably, this order gives a different basic set in general. For example, the partition $\lambda = (6,2,2,1,1)$ is in the basic set associated with the lexicographic order when $p = 3$, because this is a 3-regular partition and we have $m(\lambda) = (5,3,2,2)$. So we have $m(\lambda) \leq \lambda$. However we see that $m(\lambda) \geq' \lambda$ so that $(5,3,2,2)$ (and its conjugate) are in the unitriangular basic set associated with $\leq'$, while $(6,2,2,1,1)$ is not.

Of course, all of the above results make sense if we restrict ourselves to blocks of the symmetric group. Let $\gamma$ be a $p$-core and assume that we have constructed our unitriangular basic set $(\widetilde{B}_\gamma, \preceq, \widetilde{\Psi})$ with respect to an order associated to the block $\mathfrak{B}_\gamma$ of the symmetric group. Then we have :

$$\widetilde{B}_\gamma = \widetilde{B}^1_\gamma \sqcup \widetilde{B}^2_\gamma,$$

where

$$\widetilde{B}^1_\gamma = \{\lambda \in \mathcal{R}^p_n \mid m(\lambda) < \lambda\} \sqcup \{\lambda' \mid \lambda \in \mathcal{R}^p_n,\ m(\lambda) < \lambda\}$$
$$\text{and} \quad \widetilde{B}^2_\gamma = \{\lambda \in \mathcal{R}^p_n \mid m(\lambda) = \lambda\}.$$

In the final two subsections, we will discuss two favorable cases which allow the construction of a unitriangular basic set respecting the hypotheses of Theorem 12:

- Subsection 5.2 concerns a case where $\widetilde{B}^2_\gamma = \emptyset$.
- In Subsection 5.3, we study a case where there exists a bijection

  $$\rho : \widetilde{B}^2_\gamma \to \mathcal{G}_\gamma,$$

  where $\mathcal{G}_\gamma$ is the set of self-conjugate partitions whose diagonal $p$-hooks have length not divisible by $p$, satisfying the following two properties:

  - For all $\mu \in \widetilde{B}^2_\gamma$, we have $d_{\rho(\mu),\mu} = 1$.
  - For all $\mu \in \widetilde{B}^2_\gamma$ and for all $\lambda \in \widetilde{B}_\gamma$ such that $\rho(\mu) \prec \lambda \prec \mu$, we have $d_{\rho(\mu),\lambda} = 0$.

  Then, in this case, by [1], it follows from the form of the decomposition matrix that $(\widetilde{B}'_\gamma, \preceq, \widetilde{\Psi}')$ is a unitriangular $p$-basic set for $\mathfrak{B}_\gamma$, where

  $$\widetilde{B}'_\gamma = \widetilde{B}^1_\gamma \sqcup \rho(\widetilde{B}^2_\gamma),$$

  and, for all $\lambda \in \widetilde{B}'_\gamma$, we have:

  $$\widetilde{\Psi}'(\lambda) = \begin{cases} \widetilde{\Psi}(\lambda) & \text{if } \lambda \in \widetilde{B}^1_\gamma, \\ \widetilde{\Psi}(\rho^{-1}(\lambda)) & \text{if } \lambda \in \rho(\widetilde{B}^2_\gamma). \end{cases}$$

  From this, one can deduce a unitriangular $p$-basic set for $\mathfrak{b}_\gamma$ by Theorem 12.

Of course, the above unitriangular basic set $\widetilde{B}$ satisfies the assumptions of Theorem 12 if no self-Mullineux $e$-regular partition exists. This is exactly what happens for the blocks considered in the next subsection. There is another way to obtain the desired unitriangular basic set which is explained in [1].

## 5.2 Blocks with odd weight

**Theorem 37.** *Let $p$ be an odd prime number. If $\mathfrak{b}_\gamma$ is a $p$-block of $\mathfrak{A}_n$ with an odd $p$-weight, then $\mathfrak{b}_\gamma$ has a unitriangular $p$-basic set.*

*Proof.* Let $\mathfrak{B}_\gamma$ be the $p$-block of $\mathfrak{S}_n$ covering $\mathfrak{b}_\gamma$. We only have to consider the case where $\mathfrak{B}_\gamma$ is $\varepsilon$-stable. First, we remark that

$$\mathfrak{B}_\gamma \cap \mathcal{T} = \emptyset.$$

Indeed, any character $\chi_\lambda \in \mathfrak{B}_\gamma \cap \mathcal{T}$ satisfies $\lambda_{(p)} = \gamma$ and $\lambda^{(p)} \in \mathcal{C}_\gamma$. In particular, if $w$ is the $p$-weight of $\mathfrak{B}_\gamma$, then

$$w = \sum_{i=1}^{p} |\lambda^i| = 2 \sum_{i=1}^{(p-1)/2} |\lambda^i|,$$

which contradicts the hypothesis that $w$ is odd. If then $(\widetilde{B}_\gamma, \preceq, \widetilde{\Psi})$ is a unitriangular basic set constructed from the unitriangular $p$-basic set indexed by the $p$-regular partitions (using for example the lexicographic order), then $\widetilde{B}_\gamma = \widetilde{B}^1_\gamma$, whence, by the above discussion, this unitriangular $p$-basic set restricts to a unitriangular basic set of $\mathfrak{b}_\gamma$, as required. □

*Remark* 38. The condition of Theorem 37 is not necessary. Indeed, we see in Example 14 that the principal 3-block of $\mathfrak{A}_6$, which has 3-weight 2, has a unitriangular 3-basic set.

## 5.3 The case of $\mathfrak{S}_{23}$ and $p = 3$

We now consider the case $n = 23$ and $p = 3$, and the $p$-block associated to the 3-core $(3, 1, 1)$. We also consider the order on partitions given in Remark 36 (this is thus not the lexicographic order).

The unitriangular basic set of the block of the symmetric group with core $(3, 1, 1)$ in Proposition 33 contains 65 characters:

- 31 characters labeled by the 3-regular partitions $\lambda$ such that $\lambda > m(\lambda)$,
- 31 characters labeled by the conjugates of the above partitions,
- 3 characters labeled by the partitions which are stable with respect to the Mullineux involution; these are $(12, 6, 5)$, $(10, 4, 4, 3, 1, 1)$ and $(9, 6, 3, 3, 1, 1)$.

We denote by $\widetilde{B}^1_{(3,1,1)}$ the partitions labeling the characters of the first two sets of characters above, and by $\widetilde{B}^2_{(3,1,1)}$ the partitions labeling the three characters of the last set, so that

$$\widetilde{B}^2_{(3,1,1)} = \{(12, 6, 5), (10, 4, 4, 3, 1, 1), (9, 6, 3, 3, 1, 1)\}.$$

We now focus on the elements of $\mathrm{IPr}_p(G)$ labeled by these last three partitions. As in section 3.2.2, we consider the following element of the Fock space $\mathcal{F}_v$:

$$A(9, 6, 3, 3, 1, 1) = F_2 F_1^{(2)} F_0^2 F_2^{(3)} F_1 F_0^2 F_2 F_1^{(2)} F_2 F_0^{(2)} F_1 F_2^{(2)} F_0 F_1 F_2 F_0.\emptyset,$$

which gives the following element of the Fock space:

$$\begin{aligned}
A(9,6,3,3,1,1) &= (9,6,3,3,1) + v(8,7,3,3,1,1) + v(9,5,4,3,1,1) + v^2(8,5,5,3,1,1) + v^2(7,7,4,3,1,1)) \\
&+ v(9,4,4,3,2,1) + (v^3+v)(8,4,4,3,2,2) + v^2(9,4,3,3,3,1) + v^2(8,4,3,3,3,2) + v^3(7,4,4,3,3,2) \\
&+ v^4+v^2)(8,4,4,3,1,1,1,1) + v^2(6,6,5,3,2,1) + (v^4+v^2)(6,6,4,3,2,2) + v^3(6,6,3,3,3,2) \\
&+ v^4(6,5,4,3,3,2) + v^3(6,6,5,3,1,1,1) + (v^5+v^3)(6,6,4,3,1,1,1,1) + v^3(6,4,4,3,3,2,1) \\
&+ v(9,6,2,2,2,2) + v^2(8,7,2,2,2,2) + v^2(9,4,4,2,2,2) + v^3(9,4,2,2,2,2,2) + v^3(8,4,2,2,2,2,2,1) \\
&+ v^3(6,6,5,2,2,2) + v^4(6,6,2,2,2,2,2,1) + v^4(6,4,4,3,2,2,2) + v^5(6,4,4,2,2,2,2,1) \\
&+ v^3(8,7,2,2,1,1,1,1) + v^2(9,5,2,2,1,1,1,1,1) + v^3(7,7,2,2,1,1,1,1,1) + v^3(9,4,4,2,1,1,1,1)+ \\
&+ v^3(8,4,3,3,1,1,1,1,1) + v^4(7,4,4,3,1,1,1,1,1) + v^4(9,4,2,2,2,1,1,1,1) + v^4(8,4,2,2,2,2,1,1,1) \\
&+ v^4(6,6,3,3,1,1,1,1,1) + v^4(6,5,5,2,1,1,1,1,1) + v^5(6,5,4,3,1,1,1,1,1) + v^5(6,6,2,2,2,2,1,1,1) \\
&+ v^6(6,4,4,2,2,2,1,1,1) + v^2(9,4,4,3,1,1,1) + v^3(6,5,5,3,3,1) + v^4(6,4,4,3,3,1,1,1) \\
&+ v^3(7,6,5,3,1,1) + v^2(9,6,2,2,1,1,1,1) + v^3(9,4,3,2,1,1,1,1,1) + v^3(9,4,3,2,1,1,1,1,1) \\
&+ v^2(9,6,2,2,1,1,1,1) + v^4(6,6,5,2,1,1,1,1) + v^5(6,4,4,3,2,1,1,1,1) + v^3(7,6,5,3,1,1).
\end{aligned}$$

The assumptions of Proposition 27 are satisfied. This means in particular that the decomposition number $d_{(6,5,5,3,3,1),(9,6,3,3,1,1)}$ is 1. Note that $(6,5,5,3,3,1)$ is in $\mathcal{G}_{(3,1,1)}$.

Now let us consider

$$A(10,4,4,3,1,1) = F_1F_2f_1^3F_0F_3^{(4)}F_1F_0^{(2)}F_1^{(2)}F_2^{(2)}F_0^{(2)}F_1^{(2)}F_2F_0.\emptyset,$$

which gives the following element of the Fock space:

$$\begin{aligned}
A(10,4,4,3,1,1) &= (10,4,4,3,1,1) + v(9,5,4,3,1,1) + v^2(9,4,4,4,1,1) + v^3(9,4,4,3,2,1) + v^4(9,4,4,3,1,1,1) \\
&+ v(7,6,5,3,1,1) + v^2(6,6,6,3,1,1) + v^3(6,6,5,4,1,1) + v^4(6,6,5,3,2,1) + v^5(6,6,5,3,1,1,1) \\
&+ v(7,4,4,3,3,2) + v^2(6,5,4,3,3,2) + v^3(6,4,4,4,3,2) + v^4(6,4,4,3,3,3) + v^5(6,4,4,3,3,2,1) \\
&+ v(10,4,2,2,1,1,1,1,1) + v^2(9,5,2,2,1,1,1,1,1) + v^3(9,4,3,2,1,1,1,1,1) + v^4(9,4,2,2,2,1,1,1,1) \\
&+ v^5(9,4,2,2,1,1,1,1,1,1) + v^2(7,4,4,3,1,1,1,1,1) + v^3(6,5,4,3,1,1,1,1,1) + v^5(6,4,4,3,2,1,1,1,1) \\
&+ v^5(6,4,4,3,2,1,1,1,1) + v^6(6,4,4,3,1,1,1,1,1) + v^5(6,4,4,3,2,1,1,1,1).
\end{aligned}$$

Again, the assumptions of Proposition 27 are satisfied. This means in particular that the decomposition number $d_{(9,4,3,2,1,1,1,1,1),(10,4,4,3,1,1)}$ is 1 and $(9,4,3,2,1,1,1,1,1)$ is in $\mathcal{G}_{(3,1,1)}$.

Now, let us consider the partition $(12,6,5)$ and the partition $(12,1,1,1,1,1,1,1,1,1,1,1,1)$ which is in $\mathcal{G}_{(3,1,1)}$. Note that the regularization of this last partition is exactly given by $(12,6,5)$, so that the associated decomposition number $d_{(12,1,1,1,1,1,1,1,1,1,1,1,1),(12,6,5)}$ is also 1.

Now, we will use the same argument as the one used in [1], and already explained in §5.1. Let $\mu$ be one of the above three 3-regular partitions fixed by the Mullineux involution. For each of them, we have found an element $\rho(\mu) := \nu \in C_{(3,1,1)}$ such that $d_{\nu,\widetilde{\Psi}(\mu)} = 1$. Assume that, for all $\lambda$ in $\widetilde{B}$ such that $\nu \prec \lambda \prec \mu$, we have $d_{\nu,\widetilde{\Psi}(\lambda)} = 0$. Here, we have $\rho(12,6,5) = (12,1,1,1,1,1,1,1,1,1,1,1,1)$, $\rho(9,6,3,3,1,1) = (6,5,5,3,3,1)$ and $\rho(10,4,4,3,1,1) = (9,4,3,2,1,1,1,1,1)$. Then, by [1], we have that

$$(\widetilde{B}_1 \sqcup \{(9,4,3,2,1,1,1,1,1),(12,1,1,1,1,1,1,1,1,1,1,1,1),(6,5,5,3,3,1)\}, \preceq, \widetilde{\Psi}')$$

gives a unitriangular basic set for the block which satisfies the assumption of Theorem 12. Indeed, in the three cases, the decomposition number $d_{\nu,\widetilde{\Psi}(\lambda)}$ is $d_{\nu,\Psi(\lambda)}$ if $\lambda$ is $p$-regular and $\lambda \geq m(\lambda)$, and $d_{\nu,\Psi(m(\lambda'))} = d_{\nu,\Psi(\lambda')}$ otherwise. It is zero if $\lambda$ does not dominate $\nu$ or $\lambda'$ does not dominate $\nu$. This property is always satisfied, which can be seen using a direct computation, except in the case where $\nu = (6,5,5,3,3,1)$, $\mu = (9,6,3,3,1,1)$ and $\lambda = (10,4,4,3,1,1)$. But we know the decomposition number $d_{\nu,\Psi(\lambda)}$ (see above) in this case, and we see that it is zero.

It thus gives a unitriangular basic set for the associated block of the group $\mathfrak{A}_{23}$. Furthermore, by [4], the blocks of the symmetric group with the same weights are all derived equivalent. The above result together with the example studied in §3.2.1 show that the set of conditions of Theorem 12 is not an invariant under this equivalence.

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