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► To cite this version:

Toufic Arwadi, W Youssef. STABILIZATION OF BRESSE SYSTEM WITH THERMODIFFUSION EFFECTS. 2020. <hal-02983848>

HAL Id: hal-02983848

<https://hal.science/hal-02983848v1>

Preprint submitted on 30 Oct 2020

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STABILIZATION OF BRESSE SYSTEM WITH THERMODIFFUSION EFFECTS

T. EL ARWADI AND W. YOUSSEF

ABSTRACT. We study the stabilization of the the Bresse beam with thermodiffusion effects. An exponential stability is obtained in the case of equal speed of propagation. Otherwise, a polynomial stability is proved. The techniques used is the frequency domain method.

0.1. Introduction.

$$\left\{ \begin{array}{l} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - l \left(k_0 + \frac{\beta_2^2}{\varrho_2} \right) (\omega_x - l\varphi) \\ \quad - l\gamma_2 \theta_2 - l\beta_2 C_2 + \mu \varphi_t = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) - \gamma_1 \theta_{1x} - \beta_1 C_{1x} = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) - \gamma_1 \theta_{2x} - \beta_1 C_{2x} = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_3 \theta_{1t} - \varpi_1 C_{1t} - m_1 \theta_{1xx} - \gamma_1 \psi_{tx} = 0 \quad \text{in } (0, L) \times (0, \infty), \\ C_{1t} - h_1(\beta_1 \psi_x + \varrho_1 C_1 - \varpi_1 \theta_1)_{xx} = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_4 \theta_{2t} - \varpi_1 C_{2t} - m_2 \theta_{2xx} - \gamma_2 (\omega_x - l\varphi)_t = 0 \quad \text{in } (0, L) \times (0, \infty), \\ C_{2t} - h_2(\beta_2 (\omega_x - l\varphi) + \varrho_2 C_2 - \varpi_2 \theta_2)_{xx} = 0 \quad \text{in } (0, L) \times (0, \infty) \end{array} \right. \quad (0.1)$$

$$P_1 = \beta_1 \psi_x + \varrho_1 C_1 - \varpi_1 \theta_1, \quad P_2 = \beta_2 (\omega_x - l\varphi) + \varrho_2 C_2 - \varpi_2 \theta_2.$$

Date: October 30, 2020.

Key words and phrases. Bresse; thermodiffusion effects.

Substituting C_1 and C_2 , (0.1) leads to

$$\left\{ \begin{array}{l} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(\omega_x - l\varphi) - l\gamma_{21}\theta_2 - l\gamma_{22}P_2 \\ \quad + \mu\varphi_t = 0 \text{ in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - \alpha_1 \psi_{xx} + k(\varphi_x + \psi + lw) - \gamma_{11}\theta_{1x} - \gamma_{12}P_{1x} = 0 \text{ in } (0, L) \times (0, \infty), \\ \rho_1 \omega_{tt} - \alpha_2(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + lw) - \gamma_{21}\theta_{2x} - \gamma_{22}P_{2x} = 0 \text{ in } (0, L) \times (0, \infty), \\ c_1 \theta_{1t} + d_1 P_{1t} - m_1 \theta_{1xx} - \gamma_{11}\psi_{tx} = 0 \text{ in } (0, L) \times (0, \infty), \\ d_1 \theta_{1t} + r_1 P_{1t} - h_1 P_{1xx} - \gamma_{12}\psi_{tx} = 0 \text{ in } (0, L) \times (0, \infty), \\ c_2 \theta_{2t} + d_2 P_{2t} - m_2 \theta_{2xx} - \gamma_{21}(\omega_x - l\varphi)_t = 0 \text{ in } (0, L) \times (0, \infty), \\ d_2 \theta_{2t} + r_2 P_{2t} - h_2 P_{2xx} - \gamma_{22}(\omega_x - l\varphi)_t = 0 \text{ in } (0, L) \times (0, \infty), \end{array} \right. \quad (0.2)$$

where

$$\begin{aligned} \alpha_1 &= b - \frac{\beta_1^2}{\varrho_1}, \gamma_{11} = \gamma_1 + \beta_1 \frac{\varpi_1}{\varrho_1}, \gamma_{12} = \frac{\beta_1}{\varrho_1}, c_1 = \rho_3 + \frac{\varpi_1^2}{\varrho_1}, d_1 = \frac{\varpi_1}{\varrho_1}, r_1 = \frac{1}{\varrho_1}, \\ \alpha_2 &= k_0 - \frac{\beta_2^2}{\varrho_2}, \gamma_{21} = \gamma_2 + \beta_2 \frac{\varpi_2}{\varrho_2}, \gamma_{22} = \frac{\beta_2}{\varrho_2}, c_2 = \rho_4 + \frac{\varpi_2^2}{\varrho_2}, d_2 = \frac{\varpi_2}{\varrho_2}, r_2 = \frac{1}{\varrho_2}. \end{aligned}$$

The initial conditions for this system are given by:

$$\begin{aligned} \omega(0, \cdot) &= \omega_0, \omega_t(0, \cdot) = \omega_1, \varphi(0, \cdot) = \varphi_0, \varphi_t(0, \cdot) = \varphi_1, \\ \psi(0, \cdot) &= \psi_0, \psi_t(0, \cdot) = \psi_1 \quad \text{on } (0, L). \end{aligned} \quad (0.3)$$

Two type of boundary conditions are considered:

$$\begin{aligned} \varphi(t, x) &= \psi(t, x) = \omega(t, x) = \theta_1(t, x) = P_1(t, x) \\ &= \theta_2(t, x) = P_2(t, x) = 0, \quad x = 0, L \quad (\text{Dirichlet}) \end{aligned} \quad (0.4)$$

and

$$\begin{aligned} \varphi(t, x) &= \psi_x(t, x) = \omega_x(t, x) = \theta_1(t, x) = P_1(t, x) \\ &= \theta_2(t, x) = P_2(t, x) = 0, \quad x = 0, L \quad (\text{Neuman}). \end{aligned} \quad (0.5)$$

This work we generalize the study of the Timoshenko beam with thermodiffusion and we and treat the proposed open problem by the authors [1]. Indeed, they showed, without assuming the well-known equal wave speeds condition, the lack of exponential stability for the Neumann problem, meanwhile one linear frictional damping is strong enough to guarantee the exponential stability for the Dirichlet problem. Then, they introduced a finite element approximation and they proved that the associated discrete energy decays.

In this work, to establish the exponential stability we use the theorem introduced by J. Prüss [4] and F. L. Huang [3]:

Theorem 0.1. *A \mathcal{C}^0 semigroup $e^{t\mathcal{A}}$ of contractions on a Hilbert space \mathcal{H} is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad \sup_{\lambda \in \mathbb{R}} \|(i\lambda - \mathcal{A})^{-1}\| < +\infty. \quad (0.6)$$

However, our result on the polynomial stability, it is based on the Borichev and Tomilov theorem [2].

Theorem 0.2. *Let \mathcal{A} be the generator of \mathcal{C}^0 -semigroup of contractions on a Hilbert space X . Then,*

$$i\mathbb{R} \subseteq \rho(\mathcal{A}), \quad \|(i\lambda I - \mathcal{A})^{-1}\| < C|\lambda|^s, \quad (0.7)$$

for some $s > 0$ and for all $\lambda \in \mathbb{R}$ if and only if we have

$$\|S(t)W^0\|_X \leq \frac{C}{t^{1/s}} \|W^0\|_{D(\mathcal{A})}, \quad (0.8)$$

and for some constant $C > 0$ and for all $W_0 \in D(\mathcal{A})$.

1. WELL-POSEDNESS

Consider the tow following Hilbert spaces:

$$\mathcal{H}_1 = (H_0^1(0, L))^3 \times (L^2(0, L))^7,$$

$$\mathcal{H}_2 = (H_0^1(0, L)) \times (H_*^1(0, L))^2 \times L^2(\Omega) \times (L_*^2(\Omega))^2 \times (L^2(\Omega))^4,$$

where $H_*^1(0, L)$ and $L_*^2(0, L)$ are given by

$$H_*^1(0, L) = \left\{ f \in H^1(0, L); \int_0^L f(x)dx = 0 \right\}$$

and

$$L_*^2(0, L) = \left\{ f \in L^2(0, L); \int_0^L f(x)dx = 0 \right\}.$$

Now, let us define the following linear unbounded operators

$$\mathcal{A}_j : \mathcal{H}_j \longrightarrow \mathcal{H}_j \quad (j = 1, 2)$$

by

$$D(\mathcal{A}_1) = (H_0^1(0, L) \cap H^2(0, L))^3 \times (H_0^1(0, L))^7,$$

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} (\varphi, \psi, \omega, u, v, z, \theta_1, P_1, \theta_2, P_2) \in \mathcal{H}_2; \varphi \in H_0^1(0, L) \cap H^2(0, L), \\ \psi, \omega \in H_*^1(0, L) \cap H^2(0, L), \psi_x, \omega_x \in H_0^1(0, L), u \in H_0^1(0, L), \\ v, z \in H_*^1(0, L), (\theta_1, P_1, \theta_2, P_2) \in (H_0^1(0, L))^4 \end{array} \right\},$$

and

$$\mathcal{A}_j W = \begin{pmatrix} u \\ v \\ z \\ \frac{k}{\rho_1}(\varphi_{xx} + \psi_x + l\omega_x) + \frac{lk_0}{\rho_1}(\omega_x - l\varphi) + l\gamma_{21}\theta_2 + l\gamma_{22}P_2 \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + l\omega) + \gamma_{11}\theta_{1x} + \gamma_{12}P_{1x} \\ \frac{k_0}{\rho_1}(\omega_{xx} - l\varphi_x) - \frac{lk}{\rho_1}(\varphi_x + \psi + l\omega) + \gamma_{21}\theta_{2x} + \gamma_{22}P_{2x} \\ \delta^{-1} \left[f_1 m_1 \theta_{1xx} - f_2 h_1 P_{1xx} + l_1 v_x^n \right] \\ \delta^{-1} \left[-f_2 m_1 \theta_{1xx} + f_1 h_1 P_{1xx} + l_2 v_x^n \right] \\ \delta^{-1} \left[g_1 m_2 \theta_{2xx} - g_2 h_2 P_{2xx} + h_1 (z_x^n - l u^n) \right] \\ \delta^{-1} \left[-g_2 m_2 \theta_{2xx} + g_1 h_2 P_{2xx} + h_2 (z_x^n - l u^n) \right] \end{pmatrix}.$$

For $W = (\varphi, \psi, \omega, u, v, z, \theta_1, P_1, \theta_2, P_2)$, using a classical semigroup theorem, we show easily that the system

$$\begin{cases} W'(t) = \mathcal{A}_j W(t), & t > 0 \\ W(0) = W^0 \in D(\mathcal{A}), \end{cases} \quad (1.9)$$

is well-posed for $j = 1, 2$, where $j = 1, 2$ corresponds to the boundary conditions (0.4) and (0.5) respectively. Here and in the followig, we set that $l \neq \frac{n\pi}{L}$ for all $n \in \mathbb{N}^8$ when $i = 2$.

2. EXPONENTIAL STABILITY

Let us start by proving the first member in (0.6). Indeed, $D(\mathcal{A})$ has a compact embedding into \mathcal{H} , then \mathcal{A}^{-1} is compact in \mathcal{H} . Hence, it is sufficient to check that \mathcal{A} has no pure imaginary eigenvalue. Then, suppose that there exists $\lambda \in \mathbb{R}^*$ such that $i\lambda$ is an eigenvalue and $W = (\varphi, \psi, \omega, u, v, z, \theta_1, P_1, \theta_2, P_2)$ be the normalized eigenfunction, i.e.

$$\mathcal{A}W = i\lambda W. \quad (2.10)$$

Therefore, we have

$$u(x) = i\lambda\varphi(x), \quad x \in (0, L), \quad (2.11)$$

$$v(x) = i\lambda\psi(x), \quad x \in (0, L), \quad (2.12)$$

$$z(x) = i\lambda\omega(x), \quad x \in (0, L), \quad (2.13)$$

$$\lambda^2\varphi + \frac{k}{\rho_1}(\varphi_{xx} + \psi_x + l\omega_x) + \frac{lk_0}{\rho_1}(\omega_x - l\varphi) + l\frac{\gamma_{21}}{\rho_1}\theta_2 + l\frac{\gamma_{22}}{\rho_1}P_2 = 0, \quad x \in (0, L) \quad (2.14)$$

$$\lambda^2\psi + \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + l\omega) + \frac{\gamma_{11}}{\rho_2}\theta_{1x} + \frac{\gamma_{12}}{\rho_2}P_{1x} = 0, \quad x \in (0, L) \quad (2.15)$$

$$\lambda^2\omega + \frac{k_0}{\rho_1}(\omega_{xx} - l\varphi_x) - \frac{lk}{\rho_1}(\varphi_x + \psi + l\omega) + \frac{\gamma_{21}}{\rho_1}\theta_{2x} + \frac{\gamma_{22}}{\rho_1}P_{2x} = 0, \quad x \in (0, L) \quad (2.16)$$

$$-i\lambda\theta_1(x) + \delta^{-1}\left[f_1m_1\theta_{1xx} - f_2h_1P_{1xx} + l_1v_x^n\right] = 0, \quad x \in (0, L) \quad (2.17)$$

$$-i\lambda P_1(x) + \delta^{-1}\left[-f_2m_1\theta_{1xx} + f_1h_1P_{1xx} + l_2v_x^n\right] = 0, \quad x \in (0, L) \quad (2.18)$$

$$-i\lambda\theta_2(x) + \delta^{-1}\left[g_1m_2\theta_{2xx} - g_2h_2P_{2xx} + h_1(z_x^n - lu^n)\right] = 0, \quad x \in (0, L) \quad (2.19)$$

$$-i\lambda P_2(x) + \delta^{-1}\left[-g_2m_2\theta_{2xx} + g_1h_2P_{2xx} + h_2(z_x^n - lu^n)\right] = 0, \quad x \in (0, L) \quad (2.20)$$

By (2.10), we have

$$\operatorname{Re}\langle \mathcal{A}W, W \rangle_{\mathcal{H}_j} = \operatorname{Re}(i\lambda\|W\|_{\mathcal{H}}^2) = -\|\theta_{1x}\|^2 - \|P_{1x}\|^2 - \|\theta_{2x}\|^2 - \|P_{2x}\|^2 = 0. \quad (2.21)$$

Therefore, by straightforward calculation we conclude that $W = 0$. This is a contradiction.

It remains to complete the proof of the exponential stability by applying the theorem 1.2 by establishing that

$$\lim_{\lambda \rightarrow \infty} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty. \quad (2.22)$$

Let us decompose this proof into several steps for clarification.

Step 1. Let us argue by contradiction. Then, we suppose that the condition (3.5)

is false. There exists a real sequence (λ_n) and a sequence

$$W^n = (\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta_1^n, P_1^n, \theta_2^n, P_2^n) \in D(\mathcal{A}_j)$$

such that $|\lambda_n| \rightarrow +\infty$,

$$\|W^n\|_{\mathcal{H}_j} = 1 \quad j = 1, 2, \quad (2.23)$$

and

$$(i\lambda_n - \mathcal{A}_j)W^n = (f_n^1, f_n^2, \dots, f_n^{10}) \rightarrow 0 \quad \text{dans } \mathcal{H}_j \quad j = 1, 2. \quad (2.24)$$

Recall that

$$\begin{aligned} \|W^n\|_{\mathcal{H}_j}^2 &= \rho_1 \|u^n\|^2 + \rho_2 \|v^n\|^2 + \rho_1 \|z^n\|^2 + b \|\psi_x^n\| + k \|\varphi_x^n + \psi^n + l\omega^n\|^2 \\ &\quad + k_0 \|\omega_x^n - l\varphi^n\|^2 + \int_0^L \Lambda \begin{pmatrix} \theta_1^n \\ P_1^n \\ \theta_2^n \\ P_2^n \end{pmatrix} \cdot \begin{pmatrix} \theta_1^n \\ P_1^n \\ \theta_2^n \\ P_2^n \end{pmatrix} dx, \quad j = 1, 2, \end{aligned}$$

where

$$\Lambda = \begin{pmatrix} c_1 & d_1 & 0 & 0 \\ d_1 & r_1 & 0 & 0 \\ 0 & 0 & c_2 & d_2 \\ 0 & 0 & d_2 & r_2 \end{pmatrix}$$

is a definite-positive matrix.

Then, thanks to (3.4), we have

$$i\lambda_n \varphi^n - u^n = f_n^1 \rightarrow 0 \quad \text{in } H_0^1(0, L), \quad (2.25)$$

$$i\lambda_n \psi^n - v^n = f_n^2 \rightarrow 0 \quad \text{in } \begin{cases} H_0^1(0, L) \text{ pour } j = 1 \\ H_*^1(0, L) \text{ pour } j = 2 \end{cases}, \quad (2.26)$$

$$i\lambda_n \omega^n - z^n = f_n^3 \rightarrow 0 \quad \text{in } \begin{cases} H_0^1(0, L) \text{ pour } j = 1 \\ H_*^1(0, L) \text{ pour } j = 2 \end{cases}. \quad (2.27)$$

We also have the following convergence in $L^2(0, L)$:

$$\begin{cases} i\lambda_n u^n - \frac{k}{\rho_1}(\varphi_{xx}^n + \psi_x^n + l\omega_x^n) - \frac{lk_0}{\rho_1}(\omega_x^n - l\varphi^n) + l\gamma_{21}\theta_2 + l\gamma_{22}P_2 \\ = f_n^4 \rightarrow 0, \end{cases} \quad (2.28)$$

$$i\lambda_n v^n - \frac{b}{\rho_2}\psi_{xx}^n + \frac{k}{\rho_2}(\varphi_x^n + \psi^n + l\omega^n) + \frac{\gamma_{11}}{\rho_2}\theta_{1x}^n + \frac{\gamma_{12}}{\rho_2}P_{1x}^n = f_n^5 \rightarrow 0, \quad (2.29)$$

$$i\lambda_n z^n - \frac{k_0}{\rho_1}(\omega_{xx}^n - l\varphi_x^n) + \frac{lk}{\rho_1}(\varphi_x^n + \psi^n + l\omega^n) + \frac{\gamma_{21}}{\rho_1}\theta_{2x}^n + \frac{\gamma_{22}}{\rho_1}P_{2x}^n = f_n^6 \rightarrow 0, \quad (2.30)$$

$$i\lambda_n \theta_1^n - \delta^{-1}(f_1 m_1 \theta_{1xx}^n - f_2 h_1 P_{1xx}^n + l_1 v_x^n) = f_n^7 \rightarrow 0, \quad (2.31)$$

$$i\lambda_n P_1^n - \delta^{-1}(-f_2 m_1 \theta_{1xx} + f_1 h_1 P_{1xx} + l_2 v_x^n) = f_n^8 \rightarrow 0, \quad (2.32)$$

$$i\lambda_n \theta_2^n - \delta^{-1}(g_1 m_2 \theta_{2xx} - g_2 h_2 P_{2xx} + h_1(z_x^n - l u^n)) = f_n^9 \rightarrow 0, \quad (2.33)$$

and

$$i\lambda_n P_2^n - \delta^{-1}(-g_2 m_2 \theta_{2xx} + g_1 h_2 P_{2xx} + h_2(z_x^n - l u^n)) = f_n^{10} \rightarrow 0. \quad (2.34)$$

Due to (3.4), we have

$$\begin{aligned} \text{Re}\langle (i\lambda_n - \mathcal{A}_j)W^n, W^n \rangle_{\mathcal{H}_j} &= -\text{Re}\langle \mathcal{A}_j U^n, U^n \rangle_{\mathcal{H}_j} \\ &= \|\theta_{1x}^n\|^2 + \|P_{1x}^n\|^2 + \|\theta_{2x}^n\|^2 + \|P_{2x}^n\|^2 \quad j = 1, 2. \end{aligned} \quad (2.35)$$

Therefore, due to (2.24) et (3.16), on a

$$\theta_{1x}^n, \theta_{2x}^n, P_{1x}^n, P_{2x}^n \longrightarrow 0, \quad \text{in } L^2(0, L). \quad (2.36)$$

Moreover, due to the boundedness of u^n , v^n , and z^n and the fact that $\lambda_n \longrightarrow 0$, we get from the dividing of (2.25), (2.26), and (2.27) by λ_n , respectively,

$$\varphi^n, \psi^n, \omega^n \longrightarrow 0 \quad \text{in } L^2(0, L). \quad (2.37)$$

Now, eliminating v^n in (2.29) by (2.26), taking the inner product of the resulting equation with $\frac{\psi_x^n}{\lambda_n}$ in $L^2(0, L)$, and then integrating by parts, we obtain

$$\begin{aligned} i\langle \theta_1^n, \psi_x^n \rangle + \delta^{-1} f_1 m_1 \langle \theta_{1x}^n, \frac{\psi_{xx}^n}{\lambda_n} \rangle - \delta^{-1} f_1 m_1 \left[\frac{1}{\lambda_n} \theta_{1x}^n(x) \psi_x^n(x) \right]_0^L \\ - \delta^{-1} f_2 h_1 \langle P_{1x}^n, \frac{\psi_{xx}^n}{\lambda_n} \rangle + \delta^{-1} f_2 h_1 \left[\frac{1}{\lambda_n} P_{1x}^n(x) \psi_x^n(x) \right]_0^L + i l_1 \|\psi_x^n\|^2 \longrightarrow 0. \end{aligned} \quad (2.38)$$

Using (2.36), the first term in (2.38) converges to zero since ψ_x^n is bounded by (2.23). On the other hand, dividing (2.29) by λ_n , we conclude that $\frac{\psi_{xx}^n}{\lambda_n}$ is bounded because all the other terms in the resulting equation are bounded due to (2.23). Analogously, the fourth term tends to zero due to (2.36). The boundary terms vanish in the case when $j = 2$. However, in the case when $j = 1$, using the Gagliardo-Nirenberg inequality, for $x = 0, L$ we have

$$\left| \frac{1}{\lambda_n} \theta_{1x}^n(x) \psi_x^n(x) \right| \leq K \|\theta_{1x}^n\|^{\frac{1}{2}} \left\| \frac{\theta_{1xx}^n}{\lambda_n} \right\|^{\frac{1}{2}} \|P_{1x}^n\|^{\frac{1}{2}} \left\| \frac{P_{1xx}^n}{\lambda_n} \right\|^{\frac{1}{2}}$$

for some $K > 0$. Dividing (2.31) and (2.32) by λ_n , we obtain the boundedness of $\frac{\theta_{1xx}^n}{\lambda_n}$ and $\frac{P_{1xx}^n}{\lambda_n}$. So, (3.8) implies that $\frac{1}{\lambda_n} \theta_{1x}^n(x) \psi_x^n(x)$ converges to zero for $x = 0, L$. Repeating same arguments leads to $\frac{1}{\lambda_n} P_{1x}^n(x) \psi_x^n(x)$ also converges to zero for $x = 0, L$. Consequently, in the two cases ($j = 1$ or $j = 2$), the expression (2.38) implies that

$$\psi_x^n \longrightarrow 0 \quad \text{in } L^2(0, L). \quad (2.39)$$

Repeating the same procedures by eliminating z^n in (2.30) by (3.8), we obtain

$$\omega_x^n - l\varphi^n \longrightarrow 0 \quad \text{in } L^2(0, L). \quad (2.40)$$

Step 2. In the case when $j = 1$, we shall prove that

$$\psi_x^n(0), \psi_x^n(L) \longrightarrow 0. \quad (2.41)$$

We start by multiplying (2.29) by $\frac{v^n}{\lambda_n}$. Then taking the inner product of the resulting equation with in $L^2(0, L)$, integrating by parts, and using the fact that $v^n \in H_0^1(0, L)$, we obtain

$$\begin{aligned} i\|v^n\|^2 + \frac{b}{\rho_2} \langle \psi_x^n, \frac{v_x^n}{\lambda_n} \rangle + \frac{k}{\rho_2} \langle \varphi_x^n + \psi^n + l\omega^n, \frac{v^n}{\lambda_n} \rangle + \frac{\gamma_{11}}{\rho_2} \langle \theta_{1x}^n, \frac{v^n}{\lambda_n} \rangle \\ + \frac{\gamma_{12}}{\rho_2} \langle P_{1x}^n, \frac{v^n}{\lambda_n} \rangle \longrightarrow 0, \end{aligned} \quad (2.42)$$

By (2.26) and (2.39), $\frac{v_x^n}{\lambda_n} \rightarrow 0$. Moreover, $\frac{v^n}{\lambda_n} \rightarrow 0$ since v^n is bounded and $\lambda_n \rightarrow +\infty$. Thus, due to the boundedness of the terms ψ_x^n , $\varphi_x^n + \psi^n + l\omega^n$ and due to (2.36), the last four terms in (2.42) converges to zero. Thereby,

$$v^n \rightarrow 0 \quad \text{in } L^2(0, L). \quad (2.43)$$

Therefore, thanks to (2.26), we get

$$\lambda_n \psi^n \rightarrow 0 \quad \text{in } L^2(0, L). \quad (2.44)$$

Now, let us eliminate v^n in (2.29) by (2.26), then multiply by $(L-x)\overline{\psi_x^n}$ and taking the real part of the resulting equation, we obtain

$$\begin{aligned} & \operatorname{Re} \left\{ -\langle \lambda_n^2 \psi^n, (L-x)\overline{\psi_x^n} \rangle - \frac{b}{\rho_2} \langle \psi_{xx}^n, (L-x)\overline{\psi_x^n} \rangle \right. \\ & \quad + \frac{k}{\rho_2} \langle (\varphi_x^n + \psi^n + l\omega^n), (L-x)\overline{\psi_x^n} \rangle + \frac{\gamma_{11}}{\rho_2} \langle \theta_{1x}^n, (L-x)\overline{\psi_x^n} \rangle \\ & \quad \left. + \frac{\gamma_{12}}{\rho_2} \langle P_{1x}^n, (L-x)\overline{\psi_x^n} \rangle \right\} \\ & = \operatorname{Re} \left\{ \langle f_n^5, (L-x)\overline{\psi_x^n} \rangle + i \langle \lambda_n f_n^2, (L-x)\overline{\psi_x^n} \rangle \right\}. \end{aligned} \quad (2.45)$$

Next, we have

$$\begin{aligned} \operatorname{Re} \left\{ \langle \lambda_n f_n^2, (L-x)\overline{\psi_x^n} \rangle \right\} &= \operatorname{Re} \left\{ -\langle \lambda_n f_{n_x}^2, (L-x)\overline{\psi^n} \rangle + \langle \lambda_n f_n^2, \overline{\psi^n} \rangle \right. \\ & \quad \left. + \left[\lambda_n f_n^2 (L-x)\overline{\psi^n} \right]_0^L \right\} \rightarrow 0 \end{aligned} \quad (2.46)$$

because $f_n^2, f_{n_x}^2 \rightarrow 0$ in $L^2(0, L)$, (ψ^n) , $(\lambda_n \psi^n)$ are bounded and $\psi^n|_{0,L} = 0$.

Also, we have

$$\operatorname{Re} \left\{ \langle (L-x)f_n^5, \overline{\psi_x^n} \rangle \right\} \rightarrow 0 \quad (2.47)$$

because $f_n^5 \rightarrow 0$ in $L^2(0, L)$ and (ψ_x^n) is bounded.

Consequently, using (2.46) and (2.47) in (2.45), we conclude

$$\left\{ \begin{aligned} & -\frac{1}{2} \|\lambda_n \psi^n\|^2 - \frac{b}{2\rho_2} \|\psi_x^n\|^2 + \frac{bL}{2\rho_2} |\psi_x^n(0)|^2 \\ & + \frac{k}{\rho_2} \operatorname{Re} \left\{ \langle (L-x)(\varphi_x^n + \psi^n + l\omega^n), \overline{\psi_x^n} \rangle + \frac{\gamma_{11}}{\rho_2} \langle \theta_{1x}^n, (L-x)\overline{\psi_x^n} \rangle \right. \\ & \quad \left. + \frac{\gamma_{12}}{\rho_2} \langle P_{1x}^n, (L-x)\overline{\psi_x^n} \rangle \right\} \end{aligned} \right\} \rightarrow 0. \quad (2.48)$$

Thanks to (2.39), the last three terms in (2.48) converge to zero. Therefore, using (2.44) and (2.39) in (2.48) gives

$$\psi_x^n(0) \rightarrow 0.$$

Repeating the same arguments by multiplying (2.29) by $x\overline{\psi_x^n}$, we show that

$$\psi_x^n(L) \rightarrow 0.$$

Step 3. The claim here is the showing that

$$\|\varphi_x^n\|^2 \rightarrow 0. \quad (2.49)$$

First, eliminating v^n in (3.10) by (3.7), taking the inner product of the resulting equation with $\overline{\varphi_x^n}$ in $L^2(0, L)$, and then integrating by parts, we get

$$\begin{aligned} & \operatorname{Re} \left\{ -\langle \lambda_n^2 \psi^n, \overline{\varphi_x^n} \rangle + \frac{b}{\rho_2} \langle \overline{\varphi_{xx}^n}, \psi_x^n \rangle - \frac{b}{\rho_2} [\overline{\varphi_x^n} \psi_x^n]_0^L + \frac{k}{\rho_2} \|\varphi_x^n\|^2 \right. \\ & \quad \left. + \frac{k}{\rho_2} \langle \overline{\varphi_x^n}, \psi^n \rangle + \frac{lk}{\rho_2} \langle \overline{\varphi_x^n}, \omega^n \rangle + \frac{\gamma_{11}}{\rho_2} \langle \theta_{1x}^n, \overline{\varphi_x^n} \rangle + \frac{\gamma_{12}}{\rho_2} \langle P_{1x}^n, \overline{\varphi_x^n} \rangle \right\} \\ & = \operatorname{Re} \left\{ \langle f_n^5, \overline{\varphi_x^n} \rangle + i \langle f_n^2, \lambda_n \overline{\varphi_x^n} \rangle \right\}. \end{aligned} \quad (2.50)$$

The term in the right side from the expression (2.50) can be written as

$$\operatorname{Re} \left\{ \langle f_n^5, \overline{\varphi_x^n} \rangle - i \langle f_{n_x}^2, \lambda_n \overline{\varphi^n} \rangle + i [f_n^2 \lambda_n \overline{\varphi^n}]_0^L \right\}.$$

Since $\lambda_n \rightarrow +\infty$, f_n^5 , f_n^2 , $f_{n_x}^2 \rightarrow 0$, $(\lambda_n \varphi^n)$ and (φ_x^n) are bounded, and $\varphi^n|_{0,L} = 0$, we deduce that

$$\operatorname{Re} \left\{ \langle f_n^5, \overline{\varphi_x^n} \rangle + i \langle f_n^2, \lambda_n \overline{\varphi_x^n} \rangle \right\} \rightarrow 0. \quad (2.51)$$

Next, using (2.51) in (2.50), we get

$$\begin{aligned} & \operatorname{Re} \left\{ -\langle \lambda_n^2 \psi^n, \overline{\varphi_x^n} \rangle + \frac{b}{\rho_2} \langle \overline{\varphi_{xx}^n}, \psi_x^n \rangle - \frac{b}{\rho_2} [\overline{\varphi_x^n} \psi_x^n]_0^L + \frac{k}{\rho_2} \|\varphi_x^n\|^2 \right. \\ & \quad \left. + \frac{k}{\rho_2} \langle \overline{\varphi_x^n}, \psi^n \rangle + \frac{lk}{\rho_2} \langle \overline{\varphi_x^n}, \omega^n \rangle + \frac{\gamma_{11}}{\rho_2} \langle \theta_{1x}^n, \overline{\varphi_x^n} \rangle + \frac{\gamma_{12}}{\rho_2} \langle P_{1x}^n, \overline{\varphi_x^n} \rangle \right\} \rightarrow 0. \end{aligned} \quad (2.52)$$

On the other hand, the sequence $(\varphi_x^n(L))$ is bounded. Indeed, eliminating u^n in (2.28) by (2.25), taking the inner product of the resulting equation with $\rho_1 x \overline{\varphi_x^n}$ in $L^2(0, L)$, and then integrating by parts, we obtain

$$\begin{aligned} & \frac{\rho_1}{2} \|\lambda_n \varphi^n\|^2 + \frac{k}{2} \|\varphi_x^n\|^2 - \frac{k}{2} L |\varphi_x^n(L)|^2 \\ & + \operatorname{Re} \left\{ k \langle x \psi_x^n, \overline{\varphi_x^n} \rangle - lk \langle x \omega_x^n, \overline{\varphi_x^n} \rangle - lk_0 \langle x \omega_x^n, \overline{\varphi_x^n} \rangle + \frac{l^2 k_0}{2} \|\varphi^n\|^2 + l \gamma_{21} \langle \theta_2^n, x \overline{\varphi_x^n} \rangle \right. \\ & \quad \left. + l \gamma_{22} \langle P_2^n, x \overline{\varphi_x^n} \rangle \right\} \rightarrow 0. \end{aligned} \quad (2.53)$$

Hence, $(\varphi_x^n(L))$ is bounded because $(\lambda_n \varphi^n)$, (φ_x^n) , (ψ_x^n) , (ω_x^n) , (θ_2^n) , and (P_2^n) are bounded. By the same way, we prove that $(\varphi_x^n(0))$ is bounded by multiplying (2.28) by $\rho_1 (L-x) \overline{\varphi_x^n}$. Therefore, due to (2.41), the boundary terms in (2.53) converge to zero. Moreover, the convergences (2.36) and (2.37), and the boundedness of (φ_x^n) made the last fourth terms in (2.53) converge to zero. Consequently, (2.53) gives

$$\operatorname{Re} \left\{ -\langle \lambda_n^2 \psi^n, \overline{\varphi_x^n} \rangle + \frac{b}{\rho_2} \langle \overline{\varphi_{xx}^n}, \psi_x^n \rangle \right\} + \frac{k}{\rho_2} \|\varphi_x^n\|^2 \rightarrow 0. \quad (2.54)$$

Now, eliminating u^n in (3.9) by (3.6), taking the inner product of the resulting equation with $\overline{\psi_x^n}$ in $L^2(0, L)$, and then integrating by parts, we get

$$\begin{aligned} & \operatorname{Re} \left\{ \langle \lambda_n^2 \varphi_x^n, \overline{\psi^n} \rangle + \frac{k}{\rho_1} + \frac{k}{\rho_1} \langle \varphi_{xx}^n, \overline{\psi_x^n} \rangle - \frac{k}{\rho_1} \|\psi_x^n\|^2 \right. \\ & \quad \left. - \frac{kl}{\rho_1} \langle \omega_x^n, \overline{\psi_x^n} \rangle - \frac{lk_0}{\rho_1} \langle \omega_x^n, \overline{\psi_x^n} \rangle + \frac{k_0 l^2}{\rho_1} \langle \varphi^n, \overline{\psi_x^n} \rangle \right\} \\ & = \operatorname{Re} \left\{ \langle f_n^4, \overline{\psi_x^n} \rangle + i \langle \lambda_n f_n^1, \overline{\psi_x^n} \rangle \right\}. \end{aligned} \quad (2.55)$$

The second term in the right side of (2.55) can be expressed as

$$\langle \lambda_n f_n^1, \overline{\psi_x^n} \rangle = -\langle f_{n_x}^1, \lambda_n \overline{\psi^n} \rangle + \left[\lambda_n f_n^1 \overline{\psi^n} \right]_0^L.$$

Hence,

$$\begin{aligned} & \operatorname{Re} \left\{ \langle f_n^4, \overline{\psi_x^n} \rangle + i \langle \lambda_n f_n^1, \overline{\psi_x^n} \rangle \right\} \\ & = \operatorname{Re} \left\{ \langle f_n^4, \overline{\psi_x^n} \rangle - i \langle f_{n_x}^1, \lambda_n \overline{\psi^n} \rangle \right\} + \left[\lambda_n f_n^1 \overline{\psi^n} \right]_0^L. \end{aligned}$$

Since $f_n^1, f_{n_x}^1 \rightarrow 0$, $(\lambda_n \psi^n)$, (ψ_x^n) are bounded and $f_n^1|_{0,L} = 0$, we have

$$\operatorname{Re} \left\{ \langle \lambda_n f_n^1, \overline{\psi_x^n} \rangle \right\} \rightarrow 0. \quad (2.56)$$

In addition, the fact that $f_n^4 \rightarrow 0$ and (ψ_x^n) is bounded imply

$$\operatorname{Re} \left\{ \langle f_n^4, \overline{\psi_x^n} \rangle \right\} \rightarrow 0. \quad (2.57)$$

Therefore, using (2.56) and (2.57) in (2.55), we get

$$\begin{aligned} & \operatorname{Re} \left\{ \langle \lambda_n^2 \varphi_x^n, \overline{\psi^n} \rangle + \frac{k}{\rho_1} \langle \varphi_{xx}^n, \overline{\psi_x^n} \rangle - \frac{k}{\rho_1} \|\psi_x^n\|^2 \right. \\ & \quad \left. - \frac{kl}{\rho_1} \langle \omega_x^n, \overline{\psi_x^n} \rangle - \frac{lk_0}{\rho_1} \langle \omega_x^n, \overline{\psi_x^n} \rangle + \frac{k_0 l^2}{\rho_1} \langle \varphi^n, \overline{\psi_x^n} \rangle \right\} \rightarrow 0. \end{aligned} \quad (2.58)$$

By (2.39), the last four terms in (2.58) converge to zero. Thereby, (2.58) gives

$$\operatorname{Re} \left\{ \langle \lambda_n^2 \varphi_x^n, \overline{\psi^n} \rangle + \frac{k}{\rho_1} \langle \varphi_{xx}^n, \overline{\psi_x^n} \rangle \right\} \rightarrow 0. \quad (2.59)$$

In view of $\frac{k}{\rho_1} = \frac{b}{\rho_2}$ and combining (2.54) and (2.59), we get (2.49).

To finish and complete the proof and get a contradiction with (2.23), it remains to show that

$$u^n, z^n \rightarrow 0 \quad \text{in } L^2(0, L).$$

Indeed, multiplying (2.27) by $\frac{\overline{u^n}}{\lambda_n}$. Then taking the inner product of the resulting equation with in $L^2(0, L)$, integrating by parts, and using the fact that $u^n \in H_0^1(0, L)$, we obtain

$$\begin{aligned} & i \|u^n\|^2 + \frac{k}{\rho_1} \langle \varphi_x^n + \psi^n + l\omega^n, \frac{\overline{u_x^n}}{\lambda_n} \rangle - \frac{k}{\rho_1} \left[\frac{1}{\lambda_n} (\varphi_x^n + \psi^n + l\omega^n) \overline{u^n} \right]_0^L \\ & - \frac{lk_0}{\rho_1} \langle \omega_x^n - l\varphi^n, \frac{\overline{u^n}}{\lambda_n} \rangle + l\gamma_{21} \langle \theta_2, \frac{\overline{u^n}}{\lambda_n} \rangle + l\gamma_{22} \langle P_2, \frac{\overline{u^n}}{\lambda_n} \rangle \rightarrow 0. \end{aligned} \quad (2.60)$$

Next, $\frac{\overline{u_x^n}}{\lambda_n}$ due to (2.26) and (2.49). In addition, $\frac{\overline{u^n}}{\lambda_n} \rightarrow 0$ since u^n is bounded and $\lambda_n \rightarrow +\infty$. Thus, due to the boundedness of the terms $\varphi_x^n + \psi^n + l\omega^n$ and $\omega_x^n - l\varphi^n$, and due to (2.36), the second and the last three terms in (2.60) converges to zero. The boundary terms converge to zero in the case when $j = 1$. However, in the case when $j = 2$, $x = 0, L$, we have

$$\begin{aligned} & \left| \frac{1}{\lambda_n} (\varphi_x^n + \psi^n + l\omega^n)(x) u^n(x) \right| \\ & \lesssim \|\varphi_x^n + \psi^n + l\omega^n\|^{\frac{1}{2}} \left\| \frac{(\varphi_x^n + \psi^n + l\omega^n)_x}{\lambda_n} \right\|^{\frac{1}{2}} \|u^n\|^{\frac{1}{2}} \left\| \frac{u_x^n}{\lambda_n} \right\|^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

due to (2.37), (2.49), and the boundedness of $\frac{(\varphi_x^n + \psi^n + l\omega^n)_x}{\lambda_n}$, u^n , and $\frac{u_x^n}{\lambda_n}$. Thereby, (2.60) implies that

$$u^n \rightarrow 0 \quad \text{in } L^2(0, L). \quad (2.61)$$

Repeating the same arguments by multiplying (2.29) by $\frac{u^n}{\lambda_n}$, we conclude that

$$z^n \rightarrow 0 \quad \text{in } L^2(0, L). \quad (2.62)$$

Finally, (2.39), (2.40), (2.49), (2.43), and (2.61) lead to

$$\|W^n\|_{\mathcal{H}_j} \rightarrow 0 \quad j = 1, 2$$

which contradicts (2.23). So, the proof is achieved.

3. POLYNOMIAL STABILITY

The polynomial stabilisation is summarized in the following theorem.

Theorem 3.1. *If $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$ then there exists a positive constant $C > 0$ such that*

$$\|\mathcal{A}_j(t)W^0\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}} \|W^0\|_{D(\mathcal{A}_j)}, \quad (3.1)$$

for all $W^0 \in D(\mathcal{A}_j)$. $j = 1, 2$

Proof. To apply the theorem 0.2, it will be sufficient to show only the second member in (0.7), because the first member is proved in the previous section, that is we must prove that

$$\lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda^{16}} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty. \quad (3.2)$$

Let us argue by contradiction. Then, we suppose that the condition (3.2) is false.

There exists a real sequence (λ_n) and a sequence $W^n = (\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta_1^n, P_1^n, \theta_2^n, P_2^n) \in D(\mathcal{A}_j)$ such that $|\lambda_n| \rightarrow +\infty$,

$$\|W^n\|_{\mathcal{H}_j} = 1 \quad j = 1, 2, \quad (3.3)$$

and

$$\lambda_n^{16} (i\lambda_n - \mathcal{A}_j) W^n = (f_n^1, f_n^2, \dots, f_n^{10}) \rightarrow 0 \quad \text{dans } \mathcal{H}_j \quad j = 1, 2. \quad (3.4)$$

Then, thanks to (3.4), we have

$$\lambda_n^{16} [i\lambda_n \varphi^n - u^n] = f_n^1 \longrightarrow 0 \quad \text{in} \quad H_0^1(0, L), \quad (3.5)$$

$$\lambda_n^{16} [i\lambda_n \psi^n - v^n] = f_n^2 \longrightarrow 0 \quad \text{in} \quad \begin{cases} H_0^1(0, L) & \text{pour } j = 1 \\ H_*^1(0, L) & \text{pour } j = 2 \end{cases}, \quad (3.6)$$

$$\lambda_n^{16} [i\lambda_n \omega^n - z^n] = f_n^3 \longrightarrow 0 \quad \text{in} \quad \begin{cases} H_0^1(0, L) & \text{pour } j = 1 \\ H_*^1(0, L) & \text{pour } j = 2 \end{cases}. \quad (3.7)$$

We also have the following convergence in $L^2(0, L)$:

$$\begin{cases} \lambda_n^{16} [i\lambda_n u^n - \frac{k}{\rho_1}(\varphi_{xx}^n + \psi_x^n + l\omega_x^n) - \frac{lEh}{\rho_1}(\omega_x^n - l\varphi_x^n) + l\gamma_{21}\theta_2 + l\gamma_{22}P_2] \\ = f_n^4 \longrightarrow 0, \end{cases} \quad (3.8)$$

$$\lambda_n^{16} [i\lambda_n v^n - \frac{b}{\rho_2}\psi_{xx}^n + \frac{k}{\rho_2}(\varphi_x^n + \psi^n + l\omega^n) + \frac{\gamma_{11}}{\rho_2}\theta_{1x}^n + \frac{\gamma_{12}}{\rho_2}P_{1x}^n] = f_n^5 \longrightarrow 0, \quad (3.9)$$

$$\lambda_n^{16} [i\lambda_n z^n - \frac{k_0}{\rho_1}(\omega_{xx}^n - l\varphi_x^n) + \frac{lk}{\rho_1}(\varphi_x^n + \psi^n + l\omega^n) + \frac{\gamma_{21}}{\rho_1}\theta_{2x}^n + \frac{\gamma_{22}}{\rho_1}P_{2x}^n] = f_n^6 \longrightarrow 0, \quad (3.10)$$

$$\lambda_n^{16} [i\lambda_n \theta_1^n - \delta^{-1}(f_1 m_1 \theta_{1xx}^n - f_2 h_1 P_{1xx}^n + l_1 v_x^n)] = f_n^7 \longrightarrow 0, \quad (3.11)$$

$$\lambda_n^{16} [i\lambda_n P_1^n - \delta^{-1}(-f_2 m_1 \theta_{1xx} + f_1 h_1 P_{1xx} + l_2 v_x^n)] = f_n^8 \longrightarrow 0, \quad (3.12)$$

$$\lambda_n^{16} [i\lambda_n \theta_2^n - \delta^{-1}(g_1 m_2 \theta_{2xx} - g_2 h_2 P_{2xx} + h_1(z_x^n - l u^n))] = f_n^9 \longrightarrow 0, \quad (3.13)$$

and

$$\lambda_n^{16} [i\lambda_n P_2^n - \delta^{-1}(-g_2 m_2 \theta_{2xx} + g_1 h_2 P_{2xx} + h_2(z_x^n - l u^n))] = f_n^{10} \longrightarrow 0. \quad (3.14)$$

Due to (3.4), we have

$$\begin{aligned} \text{Re}(\langle \lambda_n^{16}(i\lambda_n - \mathcal{A}_j) \rangle W^n, W^n)_{\mathcal{H}_j} &= -\text{Re}(\langle \mathcal{A}_j U^n, U^n \rangle_{\mathcal{H}_j}) \\ &= \langle (|\lambda_n^8 \theta_{1x}^n|^2 + |\lambda_n^8 P_{1x}^n|^2 + |\lambda_n^8 \theta_{2x}^n|^2 + |\lambda_n^8 P_{2x}^n|^2) dx \rangle_{\mathcal{H}_j} \quad j = 1, 2. \end{aligned} \quad (3.15)$$

Therefore, due to (3.4) et (3.15), on a

$$\lambda_n^8 \theta_{1x}^n, \lambda_n^8 \theta_{2x}^n, \lambda_n^8 P_{1x}^n, \lambda_n^8 P_{2x}^n \longrightarrow 0, \quad \text{in} \quad L^2(0, L). \quad (3.16)$$

Now, eliminating v^n in (2.29) by (2.26), taking the inner product of the resulting equation with $\frac{\psi_x^n}{\lambda_{13}^n}$ in $L^2(0, L)$, and then integrating by parts, we obtain

$$\begin{aligned} i\langle \lambda_n^4 \theta_1^n, \psi_x^n \rangle + \delta^{-1} f_1 m_1 \langle \lambda_n^3 \theta_{1x}^n, \frac{\psi_{xx}^n}{\lambda_n} \rangle - \delta^{-1} f_1 m_1 \left[\lambda_n^3 \theta_{1x}^n(x) \psi_x^n(x) \right]_0^L \\ - \delta^{-1} f_2 h_1 \langle \lambda_n^3 P_{1x}^n, \frac{\psi_{xx}^n}{\lambda_n} \rangle + \delta^{-1} f_2 h_1 \left[\lambda_n^3 P_{1x}^n(x) \psi_x^n(x) \right]_0^L + i l_1 \|\lambda_n^2 \psi_x^n\|^2 \longrightarrow 0. \end{aligned} \quad (3.17)$$

Using (2.36), the first term in (2.38) converges to zero since ψ_x^n is bounded by (2.23).

On the other hand, dividing (2.29) by λ_n^{17} , we conclude that $\frac{\psi_{xx}^n}{\lambda_n}$ is bounded because all the other terms in the resulting equation are bounded due to (2.23). Analogously, the fourth term tends to zero due to (2.36). The boundary terms vanish in the case

when $j = 2$. However, in the case when $j = 1$, using the Gagliardo-Nirenberg inequality, for $x = 0, L$ we have

$$|\lambda_n^3 \theta_{1x}^n(x) \psi_x^n(x)| \leq K \|\lambda_n^8 \theta_{1x}^n\|^{\frac{1}{2}} \left\| \frac{\theta_{1xx}^n}{\lambda_n} \right\|^{\frac{1}{2}} \|P_{1x}^n\|^{\frac{1}{2}} \left\| \frac{P_{1xx}^n}{\lambda_n} \right\|^{\frac{1}{2}}$$

for some $K > 0$. Dividing (2.31) and (2.32) by λ_n^{17} , we obtain the boundedness of $\frac{\theta_{1xx}^n}{\lambda_n}$ and $\frac{P_{1xx}^n}{\lambda_n}$. So, (3.8) implies that $\frac{1}{\lambda_n} \theta_{1x}^n(x) \psi_x^n(x)$ converges to zero for $x = 0, L$. Repeating same arguments leads to $\frac{1}{\lambda_n} P_{1x}^n(x) \psi_x^n(x)$ also converges to zero for $x = 0, L$. Consequently, in the two cases ($j = 1$ or $j = 2$), the expression (2.38) implies that

$$\lambda_n^2 \psi_x^n \longrightarrow 0 \quad \text{in } L^2(0, L). \quad (3.18)$$

Repeating the same procedures by eliminating z^n in (2.30) by (3.8), we obtain

$$\lambda_n^2 (\omega_x^n - l \varphi^n) \longrightarrow 0 \quad \text{in } L^2(0, L). \quad (3.19)$$

Next, repeating the same arguments in the step 3. in the previous section by multiplying (3.9) by $\frac{\varphi_x^n}{\lambda_n^{16}}$, we get

$$\begin{aligned} \text{Re} \left\{ -\langle \lambda_n^2 \psi^n, \overline{\varphi_x^n} \rangle + \frac{b}{\rho_2} \langle \overline{\varphi_{xx}^n}, \psi_x^n \rangle - \frac{b}{\rho_2} [\overline{\varphi_x^n} \psi_x^n]_0^L + \frac{k}{\rho_2} \|\varphi_x^n\|^2 \right. \\ \left. + \frac{k}{\rho_2} \langle \overline{\varphi_x^n}, \psi^n \rangle + \frac{lk}{\rho_2} \langle \overline{\varphi_x^n}, \omega^n \rangle + \frac{\gamma_{11}}{\rho_2} \langle \theta_{1x}^n, \overline{\varphi_x^n} \rangle + \frac{\gamma_{12}}{\rho_2} \langle P_{1x}^n, \overline{\varphi_x^n} \rangle \right\} \longrightarrow 0. \end{aligned} \quad (3.20)$$

The first term converges to zero by (3.18). The second term can be written as

$$\langle \overline{\varphi_{xx}^n}, \psi_x^n \rangle = \langle \frac{\overline{\varphi_{xx}^n}}{\lambda_n}, \lambda_n \psi_x^n \rangle \longrightarrow 0$$

due to (3.18). The boundary terms converges to zero since

$$\lambda_n \psi_x^n(0), \lambda_n \psi_x^n(L) \longrightarrow 0.$$

Consequently,

$$\|\varphi_x^n\|^2 \longrightarrow 0. \quad (3.21)$$

Therefore, (3.18), (3.19), and (3.21) imply that $\|W^n\|_j \longrightarrow 0$. This is a contradiction. \square

Remark 3.1. *If the Bresse system is damped by only a thermodiffusion effects on the bending moment i.e. the system is in the form*

$$\left\{ \begin{array}{l} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) + \mu \varphi_t = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - \alpha_1 \psi_{xx} + k(\varphi_x + \psi + l\omega) - \gamma_1 \theta_x - \gamma_2 P_x = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_1 \omega_{tt} - \alpha_2 (\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) = 0 \quad \text{in } (0, L) \times (0, \infty), \\ c \theta_t + d P_t - m \theta_{xx} - \gamma_1 \psi_{tx} = 0 \quad \text{in } (0, L) \times (0, \infty), \\ d \theta_t + r P_t - h_1 P_{xx} - \gamma_2 \psi_{tx} = 0 \quad \text{in } (0, L) \times (0, \infty), \end{array} \right. \quad (3.22)$$

we prove the exponential and the polynomial stability in the same ways as the previous section.

Similarly, If the Bresse system is damped by only a thermodiffusion effects on the axial force i.e.

$$\left\{ \begin{array}{l} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) + \mu \varphi_t = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - \alpha_1 \psi_{xx} + k(\varphi_x + \psi + l\omega) = 0 \quad \text{in } (0, L) \times (0, \infty), \\ \rho_1 \omega_{tt} - \alpha_2(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) - \gamma_1 \theta_x - \gamma_2 P_x = 0 \quad \text{in } (0, L) \times (0, \infty), \\ c\theta_t + dP_t - m\theta_{xx} - \gamma_1(\omega_x - l\varphi)_t = 0 \quad \text{in } (0, L) \times (0, \infty), \\ d\theta_t + rP_t - hP_{xx} - \gamma_2(\omega_x - l\varphi)_t = 0 \quad \text{in } (0, L) \times (0, \infty), \end{array} \right. \quad (3.23)$$

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