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Multipoint formulas for inverse scattering at high energies

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We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 1, \quad E > 0, \quad (1)$$

where v is complex-valued,

$$v \in \mathcal{C}_c^\infty(\mathbb{R}^d), \quad (2)$$

where \mathcal{C}_c^∞ denotes infinitely smooth compactly supported functions.

For equation (1) we consider the scattering solutions $\psi^+(x, k) = e^{ikx} + \psi^{sc}(x, k)$, $k \in \mathbb{R}^d$, $k^2 = E$, where ψ^{sc} satisfies the Sommerfeld radiation condition:

$$|x|^{(d-1)/2} \left(\frac{\partial}{\partial|x|} - i|k| \right) \psi^{sc}(x, k) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty \quad (3)$$

uniformly in $x/|x|$. This implies that

$$\psi^{sc}(x, k) = \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f_1(k, |k| \frac{x}{|x|}) + O\left(\frac{1}{|x|^{(d+1)/2}}\right), \quad |x| \rightarrow +\infty, \quad (4)$$

where f_1 is the scattering amplitude for equation (1). For more details about definitions of ψ^+ and f_1 , see, e.g., [BSh], [N2] and references therein.

It is convenient to represent f_1 as follows

$$f_1 = c(d, |k|) f(\theta, \omega, E), \quad (\theta, \omega) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}, \quad \text{where} \quad (5)$$

$$c(d, |k|) = -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \quad \theta = k/|k|, \quad \omega = x/|x|.$$

In order to formulate our results we also use the following notations:

$$\hat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d, \quad (6)$$

$$\omega^\perp = \{p \in \mathbb{R}^d : p\omega = 0\}, \quad \omega \in \mathbb{S}^{d-1}, \quad (7)$$

$$\theta(p, \omega, E) = E^{-1/2} (p + (E - p^2)^{1/2} \omega), \quad p \in \omega^\perp, \quad \omega \in \mathbb{S}^{d-1}, \quad E^{1/2} > 0, \quad (8)$$

$$\begin{aligned}\alpha_j(\vec{\xi}) &= \prod_{i=1}^{j-1} (\xi_j - \xi_i) \quad \text{for } 1 < j \leq n, \quad \alpha_1(\vec{\xi}) = 1, \\ \beta_{n,j}(\vec{\xi}) &= \prod_{i=j+1}^n (\xi_i - \xi_j) \quad \text{for } 1 \leq j < n, \quad \beta_{n,n}(\vec{\xi}) = 1,\end{aligned}\tag{9}$$

where $\vec{\xi} = (\xi_1, \dots, \xi_n)$.

Theorem 1. *Let v satisfy (2). Then the following formulas hold:*

$$\hat{v}(p) = \sum_{j=1}^n \frac{(-1)^{n-j} (s + \tau_j)^{n-1} f(\theta_j(s), \omega, E_j(s))}{\alpha_j(\vec{\tau}) \beta_{n,j}(\vec{\tau})} + O(s^{-n}) \quad \text{as } s \rightarrow +\infty,\tag{10a}$$

$$\begin{aligned}\theta_j(s) &= \theta(p, \omega, E_j(s)), \quad E_j(s) = (s + \tau_j)^2, \quad s > 0, \\ \vec{\tau} &= (\tau_1, \dots, \tau_n), \quad \tau_1 = 0, \quad \tau_{j_1} < \tau_{j_2} \quad \text{for } j_1 < j_2,\end{aligned}$$

$$\hat{v}(p) = \sum_{j=1}^n \frac{(-1)^{n-j} \lambda_j^{n-1} f(\theta_j(s), \omega, E_j(s))}{\alpha_j(\vec{\lambda}) \beta_{n,j}(\vec{\lambda})} + O(s^{-n}) \quad \text{as } s \rightarrow +\infty,\tag{10b}$$

$$\begin{aligned}\theta_j(s) &= \theta(p, \omega, E_j(s)), \quad E_j(s) = (\lambda_j s)^2, \quad s > 0, \\ \vec{\lambda} &= (\lambda_1, \dots, \lambda_n), \quad \lambda_1 = 1, \quad \lambda_{j_1} < \lambda_{j_2} \quad \text{for } j_1 < j_2,\end{aligned}$$

where $p \in \omega^\perp$, $\omega \in \mathbb{S}^{d-1}$ (and ω , p are fixed).

Formulas (10a), (10b) are explicit asymptotic formulas for finding the Fourier transform $\hat{v}(p)$ at fixed $p \in \mathbb{R}^d$, $d \geq 2$, from the scattering amplitude f at n points at high energies E_1, \dots, E_n . The precision of these formulas is $O(s^{-n})$ as $s \rightarrow +\infty$ and in this sense is proportional to n . To our knowledge these formulas are new for $n \geq 2$. For $n = 1$, formulas (10a), (10b) are a known variation of the Born formula at high energies for smooth v ; see, e.g., Proposition 3.4 of [M] and formula (5.1) of [N1]. (For $n = 1$, formulas (10a) and (10b) coincide.)

Formulas (10a) and (10b) follow from Proposition 3.4 of [M] about the asymptotic expansion of $f(\theta, \omega, s^2)$ as $s \rightarrow +\infty$, for $s(\theta - (\theta\omega)\omega) = p$, and from applying to this expansion Theorems 3.1 and 3.2 of [N2].

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Multipoint formulas for scattered far field

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