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MAXIMAL ESTIMATES FOR THE KRAMERS-FOKKER-PLANCK OPERATOR WITH ELECTROMAGNETIC FIELD

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ABSTRACT. In continuation of a former work by the first author with F. Nier (2009) and of a more recent work by the second author on the torus (2019), we consider the Kramers-Fokker-Planck operator (KFP) with an external electromagnetic field on \mathbb{R}^d . We show a maximal type estimate on this operator using a nilpotent approach for vector field polynomial operators and induced representations of a nilpotent graded Lie algebra. This estimate leads to an optimal characterization of the domain of the closure of the KFP operator and a criterion for the compactness of the resolvent.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. The Fokker-Planck equation was introduced by Fokker and Planck at the beginning of the twentieth century, to describe the evolution of the density of particles under the Brownian motion. In recent years, global hypoelliptic estimates have led to new results motivated by applications to the kinetic theory of gases. In this direction many authors have shown maximal estimates to deduce the compactness of the resolvent of the Fokker-Planck operator and to have resolvent estimates in order to address the issue of return to the equilibrium. F. Hérau and F. Nier in [8] have highlighted the links between the Fokker-Planck operator with a confining potential and the associated Witten Laplacian. Later this work has been extended in the book of B. Helffer and F. Nier [4], and we refer more specifically to their Chapter 9 for a proof of the maximal estimate.

In this article, we continue the study of the model case of the operator of Fokker-Planck with an external magnetic field B_e , which was initiated in the case of the torus \mathbb{T}^d ($d = 2, 3$) in [12, 13], by considering \mathbb{R}^d and reintroducing an electric potential as in [4]. In this context we establish a maximal-type estimate for this model, giving a characterization of the domain of its closed extension and giving sufficient conditions for the compactness of the resolvent.

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1.2. Statement of the result. For $d = 2$ or 3 , we consider, for a given external electromagnetic field B_e defined on \mathbb{R}^d with value in $\mathbb{R}^{d(d-1)/2}$ and a real valued electric potential V defined on \mathbb{R}^d , the associated Kramers-Fokker-Planck operator K (in short KFP) defined by:

$$(1) \quad K = v \cdot \nabla_x - \nabla_x V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v - \Delta_v + v^2/4 - d/2,$$

where $v \in \mathbb{R}^d$ represents the velocity, $x \in \mathbb{R}^d$ represents the space variable, and the notation $(v \wedge B_e) \cdot \nabla_v$ means:

$$(v \wedge B_e) \cdot \nabla_v = \begin{cases} b(x)(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 2 \\ b_1(x)(v_2 \partial_{v_3} - v_3 \partial_{v_2}) + b_2(x)(v_3 \partial_{v_1} - v_1 \partial_{v_3}) \\ + b_3(x)(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 3. \end{cases}$$

The operator K is initially considered as an unbounded operator on the Hilbert space $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ whose domain is $D(K) = C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

We then denote by:

- K_{\min} the minimal extension of K where $D(K_{\min})$ is the closure of $D(K)$ with respect to the graph norm;
- K_{\max} the maximal extension of K where $D(K_{\max})$ is given by:

$$D(K_{\max}) = \{u \in L^2(\mathbb{R}^d \times \mathbb{R}^d) / Ku \in L^2(\mathbb{T}^d \times \mathbb{R}^d)\}.$$

We will use the notation \mathbf{K} for the operator K_{\min} or $\mathbf{K}_{B_e, V}$ if we want to mention the reference to B_e and V .

The existence of a strongly continuous semi-group associated to operator \mathbf{K} is shown in [12] when the magnetic field is regular and $V = 0$. We will improve this result by considering a much lower regularity. In order to obtain the maximal accretivity, we are led to substitute the hypoellipticity argument by a regularity argument for the operators with coefficients in L_{loc}^∞ , which will be combined with more classical results of Rothschild-Stein in [15] for Hörmander operators of type 2 (see [9] for more details of this subject). Our first result is:

Theorem 1.1. *If $B_e \in L_{loc}^\infty(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$ and $V \in W_{loc}^{1, \infty}(\mathbb{R}^d)$, then $\mathbf{K}_{B_e, V}$ is maximally accretive.*

The theorem implies that the domain of the operator $\mathbf{K} = K_{\min}$ has the following property:

$$(2) \quad D(\mathbf{K}) = D(K_{\max}).$$

We are next interested in specifying the domain of the operator \mathbf{K} introduced in (2). For this goal, we will establish a maximal estimate for \mathbf{K} , using techniques which were developed initially for the study of hypoellipticity of invariant operators on nilpotent groups and the proof of the Rockland conjecture. Before we state our main result, we introduce the following functional spaces:

- $B^2(\mathbb{R}^d)$ (or B_v^2 to indicate the name of the variables) denotes the space:

$$B^2(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) / \forall (\alpha, \beta) \in \mathbb{N}^{2d}, |\alpha| + |\beta| \leq 2, v^\alpha \partial_v^\beta u \in L^2(\mathbb{R}^d)\},$$

which is equipped with its natural Hilbertian norm.

- $\tilde{B}^2(\mathbb{R}^d \times \mathbb{R}^d)$ is the space $L_x^2 \hat{\otimes} B_v^2$ (in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ identified with $L_x^2 \hat{\otimes} L_v^2$) with its natural Hilbert norm.

We can now state the second theorem of this article:

Theorem 1.2. *Let $d = 2$ or 3 . We assume that $B_e \in C^1(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2}) \cap L^\infty$ and that there exist positive constants C , $\rho_0 > \frac{1}{3}$ and $\gamma_0 < \frac{1}{3}$ such that*

$$(3) \quad |\nabla_x B_e(x)| \leq C \langle \nabla V(x) \rangle^{\gamma_0},$$

$$(4) \quad |D_x^\alpha V(x)| \leq C \langle \nabla V(x) \rangle^{1-\rho_0}, \forall \alpha \text{ s.t. } |\alpha| = 2,$$

where

$$\langle \nabla V(x) \rangle = \sqrt{|\nabla V(x)|^2 + 1}.$$

Then there exists $C_1 > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, the operator K satisfies the following maximal estimate:

$$(5) \quad \|\langle \nabla V(x) \rangle^{\frac{2}{3}} u\| + \|(v \cdot \nabla_x - \nabla_x V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v)u\| + \|u\|_{\tilde{B}^2} \leq C_1 (\|Ku\| + \|u\|).$$

The proofs will combine the previous works of [4] (in the case $B_e = 0$) and [13] (in the case $V = 0$) with in addition two differences:

- \mathbb{T}^d is replaced by \mathbb{R}^d .
- The reference operator in the enveloping algebra of the nilpotent algebra is different.

Notice also that, when $B_e = 0$, our assumptions are weaker than in the book [4] where the property that $|\nabla V(x)|$ tends to $+\infty$ as $|x| \rightarrow +\infty$ was used to construct the partition of unity.

Using the density of $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ in the domain of \mathbf{K} , we obtain the following characterization of this domain:

Corollary 1.3.

$$(6) \quad D(\mathbf{K}) = \{u \in \tilde{B}^2(\mathbb{R}^d \times \mathbb{R}^d) / (v \cdot \nabla_x - \nabla V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v)u \text{ and } |\nabla V(x)|^{\frac{2}{3}} u \in L^2(\mathbb{R}^d \times \mathbb{R}^d)\}.$$

In particular this implies that under the assumptions of Theorem 1.2 the operator $\mathbf{K}_{B_e, V}$ has compact resolvent if and only if $\mathbf{K}_{B_e=0, V}$ has the same property. This is in particular the case (see [4]) when

$$|\nabla V(x)| \rightarrow +\infty \text{ when } |x| \rightarrow +\infty,$$

as can also be seen directly from (6).

Remark 1.4. *For more results in the case without magnetic field we refer to [4] and recent results obtained in 2018 by Wei-Xi Li [14] and in 2019 by M. Ben Said [1] in connexion with a conjecture of Helffer-Nier relating the compact resolvent property for the (KFP)-operator with the same property for the Witten Laplacian on (0)-forms: $-\Delta_{x,v} + \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$, with $\Phi(x, v) = V(x) + \frac{v^2}{2}$. Its proof involves also nilpotent techniques. One can then naturally ask on results when $B_e(x)$ is unbounded. In particular, the existence of (KFP)-magnetic bottles (i.e. the compact resolvent property for the (KFP)-operator) when $V = 0$ is natural. Here we simply observe that Proposition 5.19 in [4] holds (with exactly the same proof) for $\mathbf{K}_{B_e, V}$ when B_e and V are C^∞ . Hence there are no (KFP)-magnetic bottles.*

2. MAXIMAL ACCRETIVITY FOR THE KRAMERS-FOKKER-PLANCK OPERATOR
WITH A WEAKLY REGULAR ELECTROMAGNETIC FIELD

To prove Theorem 1.1, we will show the Sobolev regularity associated to the following problem

$$K^* f = g \text{ with } f, g \in L^2_{loc}(\mathbb{R}^{2d}),$$

where K^* is the formal adjoint of K :

$$(7) \quad K^* = -v \cdot \nabla_x - \Delta_v + (v \wedge B_e + \nabla_x V) \cdot \nabla_v + v^2/4 - d/2.$$

The result of Sobolev regularity is the following:

Theorem 2.1. *Let $d = 2$ or 3 . We suppose that $B_e \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$ and $V \in W^{1,\infty}_{loc}(\mathbb{R}^d \times \mathbb{R}^d)$. Then for all $f \in L^2_{loc}(\mathbb{R}^{2d})$, such that $K^* f = g$ with $g \in L^2_{loc}(\mathbb{R}^{2d})$, then $f \in \mathcal{H}^2_{loc}(\mathbb{R}^{2d})$.*

Before proving Theorem 2.1, we recall the following result :

Proposition 2.2 (Proposition A.3 in [13]). *Let $c_j \in L^{\infty,2}_{loc}(\mathbb{R}^d \times \mathbb{R}^d)$, $\forall j = 1, \dots, d$, where $L^{\infty,2}_{loc}(\mathbb{R}^d \times \mathbb{R}^d) = \{u \in L^2_{loc}, \forall \varphi \in C^\infty_0(\mathbb{R}^d \times \mathbb{R}^d) \text{ such that } \varphi u \in L^\infty_x(L^2_v)\}$, such that*

$$(8) \quad \partial_{v_j}(c_j(x, v)) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^{2d}), \forall j = 1, \dots, d.$$

Let P_0 be the Kolmogorov operator

$$(9) \quad P_0 := -v \cdot \nabla_x - \Delta_v.$$

If $h \in L^2_{loc}(\mathbb{R}^{2d})$ satisfies

$$(10) \quad \begin{cases} P_0 h = \sum_{j=1}^d c_j(x, v) \partial_{v_j} h_j + \tilde{g} \\ h_j, \tilde{g} \in L^2_{loc}(\mathbb{R}^{2d}), \forall j = 1, \dots, d, \end{cases}$$

then $\nabla_v h \in L^2_{loc}(\mathbb{R}^{2d}, \mathbb{R}^d)$.

We can now give the proof of Theorem 2.1.

Proof of Theorem 2.1. The proof is similar to that of Theorem A.2 in [13]. In the following, we will only focus on the differences appearing in our case. To show the Sobolev regularity of the problem $K^* f = g$ with f and $g \in L^2_{loc}(\mathbb{R}^{2d})$, we can reformulate the problem as follows :

$$\begin{cases} P_0 f = \sum_{j=1}^d c_j(x, v) \partial_{v_j} h_j + \tilde{g} \\ h_j = f \in L^2_{loc}(\mathbb{R}^{2d}), \\ \tilde{g} = g - \frac{v^2}{4} f + \frac{d}{2} f \in L^2_{loc}(\mathbb{R}^{2d}), \forall j = 1, \dots, d. \end{cases}$$

Here the coefficients c_j are defined by :

$$c_j(x, v) = -(v \wedge B_e)_j - \partial_{x_j} V \in L^\infty(\mathbb{R}^d, L^2_{loc}(\mathbb{R}^d)), \forall j = 1, \dots, d.$$

We note that the coefficients c_j verify the condition (8) of the Proposition 2.2 because

$$\partial_{v_j}(v \wedge B_e)_j = 0 \text{ and } \partial_{v_j} \partial_{x_j} V = 0, \forall j = 1, \dots, d.$$

□

2.1. Proof of Theorem 1.1.

The accretivity of the operator K is clear. To show that the operator \mathbf{K} is maximally accretive, it suffices to show that there exists $\lambda_0 > 0$ such that the operator $T = K + \lambda_0 Id$ is of dense range in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. As in [4], we take $\lambda_0 = \frac{d}{2} + 1$.

We have to prove that if $u \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies

$$(11) \quad \langle u, (K + \lambda_0 Id) w \rangle = 0, \quad \forall w \in D(K),$$

then $u = 0$.

For this we observe that equation (11) implies that

$$K^* u = -\left(\frac{d}{2} + 1\right)u \text{ in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d),$$

where K^* is the operator defined in (7).

Under the assumption that $B_e \in L_{loc}^\infty(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$, $V \in W_{loc}^{1,\infty}(\mathbb{R}^d)$ and $u \in D(K^*) \subset L_{loc}^2(\mathbb{R}^d \times \mathbb{R}^d)$, Theorem 2.1 shows that $u \in \mathcal{H}_{loc}^2(\mathbb{R}^d \times \mathbb{R}^d)$. So we have $\chi(x, v)u \in \mathcal{H}^2(\mathbb{R}^d \times \mathbb{R}^d)$ for any $\chi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. The rest of the proof is standard. The regularity obtained for u allows us to justify the integrations by parts and the cut-off argument given in [4, Proposition 5.5].

Remark 2.3. *One can also prove that the operator $K = K_{V, B_e}$ is maximally accretive by using the Kato perturbation theory by an unbounded operator but this can only be done under the stronger assumption that $B_e \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$ and $V \in W^{1,\infty}(\mathbb{R}^d)$. Under this assumption, one can prove that $\mathcal{B} = -\nabla_x V \cdot \nabla_v$ is K_{B_e} -bounded operator with a K_{B_e} -bound which is strictly smaller than 1. (cf. [11, Section A.4]), where K_{B_e} is the same operator with $V = 0$ already studied in [11].*

3. PROOF OF THEOREM 1.2

3.1. General strategy. For technical reasons, it is easier to work with

$$\tilde{K} := K + \frac{d}{2}.$$

It is clear that it is equivalent to prove the maximal estimate for \tilde{K} .

The proof consists in constructing \mathcal{G} , a graded and stratified algebra of type 2, and, at any point $x \in \mathbb{R}^d$, an homogeneous element \mathcal{F}_x in the enveloping algebra $\mathcal{U}_2(\mathcal{G})$ which satisfies the Rockland condition. We recall that Helffer-Nourrigat's proof is based on maximal estimates which not only hold for the operator but also (and uniformly) for $\pi(\mathcal{F}_x)$ where π is any induced representation of the Lie Algebra. It remains to find π_x such that $\pi_x(\mathcal{F}_x) = \mathcal{K}_x + \frac{d}{2}$ is a good approximation of \mathbf{K} in a suitable ball centered at x and to patch together the estimates through a partition of unity. This strategy was used in [4] in the case $B = 0$ and in [11, 12] in the case of the torus \mathbb{T}^d with $V = 0$.

Actually, we first define \mathcal{K}_x and then look for the Lie Algebra, the operator and the induced representation.

3.2. Maximal estimate. Let us present for simplification the approach when $d = 2$ (but this restriction is not important). The dependence on x will actually appear through the two parameters $(b, w) \in \mathbb{R} \times \mathbb{R}^2$ where

$$b = B_e(x) \text{ and } w = \nabla V(x).$$

We then consider the following model:

(12)

$$\mathcal{K}_x + 1 := K_{w,b} = v \cdot \nabla_x - w \cdot \nabla_v + b(v_1 \partial_{v_2} - v_2 \partial_{v_1}) - \Delta_v + v^2/4, \text{ in } \mathbb{R}^2 \times \mathbb{R}^2.$$

We would like to have uniform estimates with respect to the parameters b, w . This is the object of the following proposition.

Proposition 3.1. *For any compact interval I , there exists a constant C such that for any $b \in I$, any $w \in \mathbb{R}^2$, any $f \in \mathcal{S}(\mathbb{R}^4)$ we have the maximal estimate*

(13)

$$|w|^{4/3} \|f\|^2 + |w|^{2/3} \|\nabla_v f\|^2 + |w|^{2/3} \|v f\|^2 + \sum_{|\alpha|+|\beta| \leq 2} \|v^\alpha \partial_v^\beta f\|^2 \leq C (\|f\|^2 + \|K_{w,b} f\|^2).$$

3.3. Proof of Proposition 3.1. Step 1.

Following [4], we consider, after application of the $x \rightarrow \xi$ partial Fourier transform, the family indexed by w, b, ξ :

$$(14) \quad \hat{K}_{w,b,\xi} = i v \cdot \xi - w \cdot \nabla_v + b(v_1 \partial_{v_2} - v_2 \partial_{v_1}) - \Delta_v + v^2/4, \text{ in } \mathbb{R}^2 \times \mathbb{R}^2,$$

We denote by σ the symbol of the operator $\hat{K}_{w,b,\xi}$ considered as acting in $L^2(\mathbb{R}_v^2)$:

$$\sigma(v, \eta) = i \xi \cdot v - i w \cdot \eta - i b(v_1 \eta_2 - v_2 \eta_1) + \eta^2 + v^2/4.$$

We introduce the symplectic map on \mathbb{R}^4 associated with the matrix

$$A = \begin{pmatrix} \cos t_1 & -\sin t_1 & 0 & 0 \\ \sin t_1 & \cos t_1 & 0 & 0 \\ 0 & 0 & \cos t_2 & -\sin t_2 \\ 0 & 0 & \sin t_2 & \cos t_2 \end{pmatrix}.$$

Then there exists (see [4] for an explicit definition) an associate unitary metaplectic operator \tilde{T} acting in $L^2(\mathbb{R}^4)$ such that

(15)

$$\begin{aligned} & \tilde{T}^{-1} \hat{K}_{w,b,\xi} \tilde{T} \\ &= i v \cdot \xi'(t) - w'(t) \cdot \nabla_v + b'_1(t)(v_1 \partial_{v_2} - v_2 \partial_{v_1}) + i b'_2(t)(v_1 v_2 - \partial_{v_1} \partial_{v_2}) - \Delta_v + v^2/4, \end{aligned}$$

with $t = (t_1, t_2) \in \mathbb{R}^2$

$$(16) \quad \begin{cases} w'_k(t) = w_k \cos t_k - \xi_k \sin t_k \\ \xi'_k(t) = w_k \sin t_k + \xi_k \cos t_k \\ b'_1(t) = b \cos(t_2 - t_1) \\ b'_2(t) = b \sin(t_2 - t_1) \end{cases}$$

for $k = 1, 2$.

We now choose t_k so that

$$w'_k(t) = 0 \text{ and } \xi'_k(t) = \sqrt{w_k^2 + \xi_k^2} := \rho_k.$$

With this choice of t_k , this leads by (15) to the analysis of a maximal estimate for

$$(17) \quad \check{K}_{\rho,b'} = i v \cdot \rho + b'_1(v_1 \partial_{v_2} - v_2 \partial_{v_1}) + i b'_2(v_1 v_2 - \partial_{v_1} \partial_{v_2}) - \Delta_v + v^2/4,$$

which is considered, for fixed (ρ, b') , as an operator on $\mathcal{S}(\mathbb{R}^2)$.

We notice that $|b'| = |b|$. Hence, if b belongs to a compact interval, then b' belongs to a compact set in \mathbb{R}^2 .

It remains to show that for a suitable Lie Algebra \mathcal{G} , this is the image by an induced representation π_ρ of an element $\mathcal{F}_{b'}$ satisfying the Rockland condition.

3.4. Nilpotent techniques for the analysis of $\check{K}_{\rho,b'}$.

We construct a graded Lie algebra \mathcal{G} of type 2, a subalgebra \mathcal{H} , for $b' \in \mathbb{R}^2$ an element $\mathcal{F}_{b'}$ in $\mathcal{U}_2(\mathcal{G})$, and for $\rho \in \mathbb{R}^2$ a linear form $\ell_\rho \in \mathcal{G}^*$ such that $\ell_\rho([\mathcal{H}, \mathcal{H}]) = 0$ and

$$\pi_{\ell_\rho, \mathcal{H}}(\mathcal{F}_{b'}) = \check{K}_{\rho,b'}.$$

$\check{K}_{\rho,b'}$ can be written as a ρ -independent polynomial of five differential operators

$$(18) \quad \check{K}_{\rho,b'} = X_{1,2} - \sum_{k=1}^2 \left((X'_{k,1})^2 + \frac{1}{4}(X''_{k,1})^2 \right) - ib'_1 \left(X'_{1,1}X''_{2,1} - X'_{2,1}X''_{1,1} \right) - ib'_2 \left(X''_{1,1}X''_{2,1} + X'_{1,1}X'_{2,1} \right),$$

which are given by

$$(19) \quad X'_{1,1} = \partial_{v_1}, \quad X''_{1,1} = iv_1, \quad X'_{2,1} = \partial_{v_2}, \quad X''_{2,1} = iv_2, \quad X_{1,2} = iv \cdot \rho.$$

We now look at the Lie algebra generated by these five operators and their brackets. This leads us to introduce three new elements that verify the following relations :

$$\begin{aligned} X_{2,2} &:= [X'_{1,1}, X''_{1,1}] = [X'_{2,1}, X''_{2,1}] = i, \\ X_{1,3} &:= [X_{1,2}, X'_{1,1}] = -i\rho_1, \quad X_{2,3} := [X_{1,2}, X'_{2,1}] = -i\rho_2. \end{aligned}$$

We also observe that we have the following properties:

$$\begin{aligned} [X'_{1,1}, X'_{2,1}] &= [X''_{1,1}, X''_{2,1}] = 0, \\ [X'_{j,1}, X_{k,3}] &= [X''_{j,1}, X_{k,3}] = [X_{k,3}, X_{2,2}] = \dots = 0 \quad , \forall j, k = 1, 2. \end{aligned}$$

We then construct a graded Lie algebra \mathcal{G} verifying the same commutator relations. More precisely, \mathcal{G} is stratified of type 2, nilpotent of rank 3, its underlying vector space is \mathbb{R}^8 , and \mathcal{G}_1 is generated by four elements $Y'_{1,1}, Y'_{2,1}, Y''_{1,1}$ and $Y''_{2,1}$, \mathcal{G}_2 is generated by $Y_{1,2}$ and $Y_{2,2}$ and \mathcal{G}_3 is generated by $Y_{1,3}$ and $Y_{2,3}$.

$$\begin{aligned} Y_{2,2} &:= [Y'_{1,1}, Y''_{1,1}] = [Y'_{2,1}, Y''_{2,1}], \\ Y_{1,3} &:= [Y_{1,2}, Y'_{1,1}], \quad Y_{2,3} := [Y_{1,2}, Y'_{2,1}], \end{aligned}$$

and

$$\begin{aligned} [Y'_{1,1}, Y'_{2,1}] &= [Y''_{1,1}, Y''_{2,1}] = 0, \\ [Y'_{j,1}, Y_{k,3}] &= [Y''_{j,1}, Y_{k,3}] = [Y_{k,3}, Y_{2,2}] = \dots = 0 \quad , \forall j, k = 1, 2. \end{aligned}$$

We note that \mathcal{G} is the same graded Lie algebra given in [13] and is indeed independent of the parameters (ρ, b') .

We have to check that for a given $\rho = (\rho_1, \rho_2)$ the representation π (with the convention that if $\diamond = \emptyset$ there is no exponent) defined on its basis by

$$(20) \quad \pi(Y_{i,j}^\diamond) = X_{i,j}^\diamond \quad \text{with } i = 1, 2, j = 1, 2, 3 \text{ and } \diamond \in \{\emptyset, \prime, \prime\prime\}.$$

defines an induced representation of the Lie algebra \mathcal{G} .

By applying the same steps given in [13, page 11] (following the techniques of [7]), we actually obtain $\pi = \pi_{\ell, \mathcal{H}}$ with

$$(21) \quad \mathcal{H} = \text{Vect}(Y''_{1,1}, Y''_{2,1}, Y_{1,2}, Y_{2,2}, Y_{1,3}, Y_{2,3}),$$

and $\ell_\rho \in \mathcal{G}^*$ defined by 0 for the elements of the basis of \mathcal{G} except

$$(22) \quad \ell_\rho(Y_{1,3}) = -\rho_1, \quad \ell_\rho(Y_{2,3}) = -\rho_2 \text{ and } \ell_\rho(Y_{2,2}) = 1.$$

With (18) in mind, we introduce

$$(23) \quad \mathcal{F}_{b'} = Y_{1,2} - \sum_{k=1}^2 \left((Y'_{k,1})^2 + \frac{1}{4}(Y''_{k,1})^2 \right) \\ - ib'_1 \left(Y'_{1,1} Y''_{2,1} - Y'_{2,1} Y''_{1,1} \right) - ib'_2 \left(Y''_{1,1} Y''_{2,1} + Y'_{1,1} Y'_{2,1} \right)$$

and get

$$(24) \quad \pi_{\ell, \rho, \mathcal{H}}(\mathcal{F}_{b'}) = \check{K}_{\rho, b'}.$$

The verification of the Rockland condition is the same as in [13]. For any non trivial irreducible representation π , we consider a C^∞ -vector of the representation such that $\pi(\mathcal{F}_{b'})u = 0$ and write $\Re\langle \pi(\mathcal{F}_{b'})u, u \rangle = 0$. By using that the operator $\pi(Y)$ is a formally skew-adjoint operator for all $Y \in \mathcal{G}$ (see [13, Proposition 2.7]), we first get that $\pi(Y)u = 0$ for any $Y \in \mathcal{G}_1$ and by difference $\pi(Y_{1,2})u = 0$.

Observing that \mathcal{G} is the algebra generated by \mathcal{G}_1 and Y_{12} , we get $\pi(Y)u = 0$ for any $Y \in \mathcal{G}$, which implies, π being non trivial, that $u = 0$.

Therefore, according to Helffer-Nourrigat theorem [5] the operator $\mathcal{F}_{b'}$ is maximal hypoelliptic and this implies also that $\pi(\mathcal{F}_{b'})$ satisfies a maximal estimate for any induced representation π with a constant which is independent of π . By applying this argument to $\check{K}_{\rho, b'} = \pi_{\ell, \mathcal{H}}(\mathcal{F}_{b'})$, we obtain for any compact $K_0 \subset \mathbb{R}^2$ the existence of $C > 0$ such that, $\forall b' \in K_0, \forall \rho \in \mathbb{R}^2$ and $\forall u \in \mathcal{S}(\mathbb{R}^2)$,

$$(25) \quad \|X_{1,2}u\|^2 + \sum_{k=1}^2 \left(\|(X'_{k,1})^2 u\|^2 + \|(X''_{k,1})^2 u\|^2 \right) + \sum_{k,\ell=1}^2 \|X'_{k,1} X''_{\ell,1} u\|^2 \\ \leq C \left(\|\check{K}_{\rho, b'} u\|^2 + \|u\|^2 \right).$$

Notice here that we first prove the inequality for fixed b' and then show the local uniformity with respect to b' .

In particular, we have

$$(26) \quad \|(v \cdot \rho) u\|^2 + \sum_{|\alpha|+|\beta| \leq 2} \|v^\alpha \partial_v^\beta u\|^2 \leq C \left(\|\check{K}_{\rho, b'} u\|^2 + \|u\|^2 \right).$$

Using the treatment of the operator introduced in (V5.52) in [4], there exists \check{C} , such that, $\forall \rho \in \mathbb{R}^2$ and $\forall u \in \mathcal{S}(\mathbb{R}^2)$, we have

$$(27) \quad |\rho|^{\frac{4}{3}} \|u\|^2 \leq \check{C} \left(\|(-\Delta + v \cdot \rho)u\|^2 \right).$$

Combining (26) and (27), we finally obtain, for any compact $K_0 \subset \mathbb{R}^2$ the existence of $\hat{C} > 0$ such that, $\forall b' \in K_0, \forall \rho \in \mathbb{R}^2$ and $\forall u \in \mathcal{S}(\mathbb{R}^2)$,

$$(28) \quad \|(v \cdot \rho) u\|^2 + |\rho|^{\frac{4}{3}} \|u\|^2 + \sum_{|\alpha|+|\beta| \leq 2} \|v^\alpha \partial_v^\beta u\|^2 \leq \hat{C} \left(\|\check{K}_{\rho, b'} u\|^2 + \|u\|^2 \right).$$

3.5. End of the proof of Proposition 3.1. Coming back to the initial coordinates, we get for a new constant $C > 0$,

$$(29) \quad \|f\|^2 + \|\hat{K}_{w, b, \xi} f\|^2 \geq \frac{1}{C} \left(|w|^{\frac{4}{3}} \|f\|^2 + \sum_{|\alpha|+|\beta| \leq 2} \|v^\alpha \partial_v^\beta f\|^2 \right), \forall f \in \mathcal{S}(\mathbb{R}^2),$$

Note that by complex interpolation, this implies also for $f \in \mathcal{S}(\mathbb{R}^2)$:

$$(30) \quad \|f\|^2 + \|\hat{K}_{w,b,\xi} f\|^2 \geq \frac{1}{C} \left(|w|^{\frac{2}{3}} \|vf\|^2 + |w|^{\frac{2}{3}} \|\nabla_v f\|^2 + \sum_{|\alpha|+|\beta| \leq 2} \|v^\alpha \partial_v^\beta f\|^2 \right).$$

This achieves the proof of the proposition.

3.6. Proof of Theorem 1.2.

3.6.1. Step 1: construction of the partition of unity.

We start with the following lemma

Lemma 3.2. *Let V satisfies (4) for some $\rho_0 \in (\frac{1}{3}, 1)$ and, for $s > 0$, let us define*

$$(31) \quad r(x) := \langle \nabla_x V(x) \rangle^{-s}.$$

Then there exists $\delta_0 > 0$ such that if $|x - y| \leq \delta_0 r(x)$ then

$$\delta_0 \leq \frac{r(x)}{r(y)} \leq \frac{1}{\delta_0}.$$

Proof. For $c > 0$, let $\hat{v}(x, c) := \sup_{|x-y| \leq cr(x)} \langle \nabla V(y) \rangle$. Using our assumptions, there exists C_0 , such that, for any $c > 0$, we have

$$\hat{v}(x, c) \leq \langle \nabla V(x) \rangle + C_0 c r(x) \hat{v}(x, c)^{1-\rho_0}.$$

Hence

$$\begin{aligned} \frac{\hat{v}(x, c)}{\langle \nabla V(x) \rangle} &\leq 1 + C_0 c \langle \nabla V(x) \rangle^{-1-s} \hat{v}(x, c)^{1-\rho_0} \\ &\leq 1 + C_0 c \left(\frac{\hat{v}(x, c)}{\langle \nabla V(x) \rangle} \right)^{1-\rho_0}. \end{aligned}$$

This inequality implies for all $c > 0$ the existence of $C_1(c)$ such that, $\forall x \in \mathbb{R}^2$,

$$1 \leq \frac{\hat{v}(x, c)}{\langle \nabla V(x) \rangle} \leq C_1(c),$$

and in particular:

$$\frac{\langle \nabla V(y) \rangle}{\langle \nabla V(x) \rangle} \leq C_1(c), \text{ for } |x - y| \leq cr(x).$$

To get the reverse inequality, we fix some c_0 and consider $c \in (0, c_0)$. We then get for $|x - y| \leq cr(x)$,

$$\begin{aligned} \langle \nabla V(x) \rangle &\leq \langle \nabla V(y) \rangle + C_0 c |\nabla V(x)|^{-s} C_1(c_0) |\nabla V(x)|^{1-\rho_0} \\ &\leq \langle \nabla V(y) \rangle + C_0 c C_1(c_0) |\nabla V(x)|. \end{aligned}$$

Choosing c small enough, we obtain

$$\langle \nabla V(x) \rangle \leq (1 - C_0 c C_1(c_0))^{-1} |\nabla V(y)|.$$

Hence we have found $c_1 \in (0, c_0)$ and $C_2 > 1$ such that $|x - y| \leq c_1 r(x)$,

$$(32) \quad \frac{1}{C_2} \leq \frac{\langle \nabla V(y) \rangle}{\langle \nabla V(x) \rangle} \leq C_2.$$

It is then easy to get the lemma for $\delta_0 \in (0, c_1)$ small enough. \square

The parameter δ_0 is now fixed by the lemma and we now consider $\delta \in (0, \delta_0)$ and $r(x, \delta) = \delta r(x)$. According to Lemma 18.4.4 (and around this lemma in Section 8.4) in [10], one can now introduce, for any $\delta \in (0, \delta_0]$, a δ -dependent partition of unity ϕ_j in the x variable corresponding to a covering by balls $B(x_j, r(x_j, \delta))$ (with the property of uniform finite intersection N_δ) where, for $r > 0$ and $\hat{x} \in \mathbb{R}^2$, the

ball $B(\hat{x}, r)$ is defined by $B(\hat{x}, r) := \{x \in \mathbb{R}^2 / |x - \hat{x}| < r\}$.
So the support of each ϕ_j is contained in $B(x_j, \delta r(x_j))$ and we have

$$(33) \quad \sum_j \phi_j^2(x) = 1,$$

and

$$(34) \quad |\nabla \phi_j(x)| \leq C_\delta \langle \nabla V(x_j) \rangle^s \leq \hat{C}_\delta \langle \nabla V(x) \rangle^s.$$

This implies (using the finite intersection property)

$$(35) \quad \sum_j |\nabla \phi_j(x)|^2 \leq \check{C}_\delta \langle \nabla V(x) \rangle^{2s}.$$

3.6.2. Step 2.

The proof is inspired by the Chapter 9 of [4] but different due to the presence of the magnetic field and the absence of the assumption $|\nabla V(x)| \rightarrow +\infty$ at infinity. We start, for $u \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, from

$$\begin{aligned} \|\check{K}u\|^2 &= \sum_j \|\phi_j \check{K}u\|^2 \\ &= \sum_j \|\check{K} \phi_j u\|^2 - \sum_j \|[\check{K}, \phi_j]u\|^2 \\ &= \sum_j \|\check{K} \phi_j u\|^2 - \sum_j \|(X_0 \phi_j)u\|^2 \\ &= \sum_j \|\check{K} \phi_j u\|^2 - \sum_j \|(\nabla \phi_j) \cdot v u\|^2 \\ &\geq \sum_j \|\check{K} \phi_j u\|^2 - \check{C}_\delta \|\langle \nabla V \rangle^s v u\|^2. \end{aligned}$$

For the analysis of $\|\check{K} \phi_j u\|^2$ let us now write, with $w_j = \nabla V(x_j)$ and $b_j = B_e(x_j)$,

$$\check{K} = \check{K} - K_{w_j, b_j} + K_{w_j, b_j}.$$

We verify that, by using the construction of the partition of unity and the assumptions of Theorem 1.2

$$\begin{aligned} \|(\check{K} - K_{w_j, b_j})\phi_j u\|^2 &\leq 2\|\phi_j(x)(\nabla V(x) - w_j) \cdot \nabla_v u\|^2 + 2\|\phi_j(x)(B_e(x) - b_j)(v_1 \partial_{v_2} - v_2 \partial_{v_1})u\|^2 \\ &\leq C\delta^2 (\|\nabla V\|^{1-\rho_0-s} \|\phi_j \nabla_v u\|^2 + \|\nabla V\|^{-s+\gamma_0} \|\phi_j (v_1 \partial_{v_2} - v_2 \partial_{v_1})u\|^2). \end{aligned}$$

These errors have to be controlled by the main term.

We note that by using the inequality (30), we have

$$\begin{aligned} \|\phi_j u\|^2 + \|K_{w_j, b_j} \phi_j u\|^2 &\geq \frac{1}{C} \|\nabla V\|^{\frac{2}{3}} \|\phi_j u\|^2 + \frac{1}{C} \|\nabla V\|^{\frac{1}{3}} \|\phi_j \nabla_v u\|^2 \\ &\quad + \frac{1}{C} \|\nabla V\|^{\frac{1}{3}} \|\phi_j v u\|^2 + \frac{1}{C} \sum_{|\alpha|+|\beta| \leq 2} \|v^\alpha \partial_v^\beta \phi_j u\|^2. \end{aligned}$$

Finally, we observe that

$$\|\check{K} \phi_j u\|^2 \geq \frac{1}{2} \|K_{w_j, b_j} \phi_j u\|^2 - \|(\check{K} - K_{w_j, b_j})\phi_j u\|^2.$$

We now choose

$$(36) \quad \max\left(\frac{2}{3} - \rho_0, \gamma_0\right) < s < \frac{1}{3}.$$

Summing up over j , we have obtained the existence of a constant C such that for all $u \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$

$$\begin{aligned}
 \|u\|^2 + \|\check{K}u\|^2 &\geq \frac{1}{C} \|\ |\nabla V|^{\frac{2}{3}} u \|^2 + \frac{1}{C} \|\ |\nabla V|^{\frac{1}{3}} \nabla_v u \|^2 + \frac{1}{C} \|\ |\nabla V|^{\frac{1}{3}} vu \|^2 \\
 &\quad + \frac{1}{C} \sum_{|\alpha|+|\beta|\leq 2} \|v^\alpha \partial_v^\beta u\|^2 \\
 (37) \quad &\quad - C\delta^2 \|\ |\nabla V|^{1-s-\rho_0} \nabla_v u \|^2 - C\delta^2 \|\ |\nabla V|^{-s+\gamma_0} (v_1 \partial_{v_2} - v_2 \partial_{v_1}) u \|^2 \\
 &\quad - C \left(\int_{\mathbb{R}^d} \int_{\Omega(R)} |\nabla V|^{2s} |\nabla_v u|^2 dx dv + \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{2s} |\nabla_v u|^2 dx dv \right) \\
 &\quad - \hat{C} \left(\int_{\mathbb{R}^d} \int_{\Omega(R)} |\nabla V|^{2s} |vu|^2 dx dv + \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{2s} |vu|^2 dx dv \right) \\
 &\quad - C\delta^2 (\|vu\|^2 + \|u\|^2 + \|\nabla_v u\|^2)
 \end{aligned}$$

where $\Omega(R)$ is defined by $\Omega(R) = \{x \in \mathbb{R}^d / |\nabla V(x)| < R\}$ and $\Omega(R)^c := \mathbb{R}^d \setminus \Omega(R)$ is the complement of $\Omega(R)$ in \mathbb{R}^d .

Using the definition of $\Omega(R)$ and the assumption that $s < \frac{1}{3}$, we obtain

$$\begin{aligned}
 \int_{\mathbb{R}^d} \int_{\Omega(R)} |\nabla V|^{2s} |\nabla_v u|^2 dx dv &\leq R^{2s} \int_{\mathbb{R}^d} \int_{\Omega(R)} |\nabla_v u|^2 dx dv, \\
 \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{2s} |\nabla_v u|^2 dx dv &\leq R^{2s-\frac{2}{3}} \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{\frac{2}{3}} |\nabla_v u|^2 dx dv,
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{\mathbb{R}^d} \int_{\Omega(R)} |\nabla V|^{2s} |vu|^2 dx dv &\leq R^{2s} \int_{\mathbb{R}^d} \int_{\Omega(R)} |vu|^2 dx dv, \\
 \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{2s} |vu|^2 dx dv &\leq R^{2s-\frac{2}{3}} \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{\frac{2}{3}} |vu|^2 dx dv.
 \end{aligned}$$

Similarly for $\|\ |\nabla V|^{1-s-\rho_0} \nabla_v u \|^2$, we have, for $1 > s + \rho_0 > \frac{2}{3}$

$$\begin{aligned}
 \int_{\mathbb{R}^d} \int_{\Omega(R)} |\nabla V|^{2-2s-2\rho_0} |\nabla_v u|^2 dx dv &\leq R^{2-2s-2\rho_0} \int_{\mathbb{R}^d} \int_{\Omega(R)} |\nabla_v u|^2 dx dv, \\
 \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{2-2s-2\rho_0} |\nabla_v u|^2 dx dv &\leq R^{\frac{4}{3}-2s-2\rho_0} \int_{\mathbb{R}^d} \int_{\Omega(R)^c} |\nabla V|^{\frac{2}{3}} |\nabla_v u|^2 dx dv,
 \end{aligned}$$

By combining the previous inequalities in (37), we obtain for a possibly new larger $C > 0$ and a (δ, R) dependent constant

$$\begin{aligned}
 \|u\|^2 + \|\check{K}u\|^2 &\geq \frac{1}{C} \|\ |\nabla V|^{\frac{2}{3}} u \|^2 \\
 &\quad + \left(\frac{1}{C} - \check{C}_\delta (R^{2s-2/3} + R^{\frac{4}{3}-2s-2\rho_0}) \right) \left(\|\ |\nabla V|^{\frac{1}{3}} \nabla_v u \|^2 + \|\ |\nabla V|^{\frac{1}{3}} vu \|^2 \right) \\
 &\quad + \left(\frac{1}{C} - C\delta^2 \right) \|(v_1 \partial_{v_2} - v_2 \partial_{v_1}) u\|^2 \\
 &\quad - C_{\delta,R} (\|\nabla_v u\|^2 + \|vu\|^2 + \|u\|^2).
 \end{aligned}$$

We can achieve the proof by observing our conditions on s, ρ_0, γ_0 and by choosing first δ small enough and then R large enough. With this choice of δ and R , we obtain

the existence of a constant $C > 0$ such that

$$\begin{aligned} & \|u\|^2 + \|\check{K}u\|^2 \\ & \geq \frac{1}{C} \left(\|\nabla V|^{\frac{2}{3}} u\|^2 + \|\nabla V|^{\frac{1}{3}} \nabla_v u\|^2 + \|\nabla V|^{\frac{1}{3}} vu\|^2 + \|(v_1 \partial_{v_2} - v_2 \partial_{v_1})u\|^2 \right) \\ & \quad - C(\|\nabla_v u\|^2 + \|vu\|^2 + \|u\|^2). \end{aligned}$$

Combining this inequality with the standard inequality

$$\Re \langle \check{K}u, u \rangle \geq \|\nabla_v u\|^2 + \|vu\|^2, \quad \forall u \in C_0^\infty(\mathbb{R}^{2d}),$$

we can achieve the proof of the theorem, keeping in mind that the maximal estimate for \check{K} is equivalent to the maximal estimate for K .

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