



HAL
open science

**Null controllability and finite-time stabilization in
minimal time of one-dimensional first-order 2×2 linear
hyperbolic systems**

Long Hu, Guillaume Olive

► **To cite this version:**

Long Hu, Guillaume Olive. Null controllability and finite-time stabilization in minimal time of one-dimensional first-order 2×2 linear hyperbolic systems. 2020. hal-02983321

HAL Id: hal-02983321

<https://hal.science/hal-02983321>

Preprint submitted on 29 Oct 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 Null controllability and finite-time stabilization in minimal
2 time of one-dimensional first-order 2×2 linear hyperbolic
3 systems

4 Long Hu* Guillaume Olive†

5 October 29, 2020

6 **Abstract**

7 The goal of this article is to present the minimal time needed for the null controllability
8 and finite-time stabilization of one-dimensional first-order 2×2 linear hyperbolic systems.
9 The main technical point is to show that we cannot obtain a better time. The proof combines
10 the backstepping method with the Titchmarsh convolution theorem.

11 **Keywords.** Hyperbolic systems, Boundary controllability, Minimal control time, Backstep-
12 ping method, Titchmarsh convolution theorem

13 **AMS Subject Classifications.** 35L40, 93B05, 93D15, 45D05

14 **1 Introduction and main result**

15 **1.1 Problem description**

16 In this paper we are interested in the characterization of the minimal time needed for the con-
17 trollability of the following class of one-dimensional first-order 2×2 linear hyperbolic systems:

18
$$\begin{cases} \frac{\partial y_1}{\partial t}(t, x) + \lambda_1(x) \frac{\partial y_1}{\partial x}(t, x) = a(x)y_1(t, x) + b(x)y_2(t, x), \\ \frac{\partial y_2}{\partial t}(t, x) + \lambda_2(x) \frac{\partial y_2}{\partial x}(t, x) = c(x)y_1(t, x) + d(x)y_2(t, x), & t \in (0, +\infty), x \in (0, 1). \\ y_1(t, 1) = u(t), & y_2(t, 0) = 0, \\ y_1(0, x) = y_1^0(x), & y_2(0, x) = y_2^0(x), \end{cases} \quad (1)$$

19 Such systems appear in linearized versions of various physical models of balance laws, see e.g.
20 [BC16, Chapter 1]. For instance, the telegrapher equations of Heaviside form a linear system of
21 the form (1) for some parameters (see e.g. [BC16, Section 1.2 and (1.20)] with $-1 + \lambda R_0 C_\ell = 0$).

In (1), $(y_1(t, \cdot), y_2(t, \cdot))$ is the state at time t , (y_1^0, y_2^0) is the initial data and $u(t)$ is the control at time t . We assume that the speeds $\lambda_1, \lambda_2 \in C^{0,1}([0, 1])$ are such that

$$\lambda_1(x) < 0 < \lambda_2(x), \quad \forall x \in [0, 1].$$

*School of Mathematics, Shandong University, Jinan, Shandong 250100, China. E-mail: hul@sdu.edu.cn

†Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30-348 Krakow, Poland. E-mail: math.golive@gmail.com or guillaume.olive@uj.edu.pl

22 Finally, $a, b, c, d \in L^\infty(0, 1)$ couple the equations of the system inside the domain (the matrix
 23 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will also be referred in the sequel to as the internal coupling matrix).

24 We recall that the system (1) is well-posed: for every $u \in L^2_{\text{loc}}(0, +\infty)$ and $(y_1^0, y_2^0) \in L^2(0, 1)^2$,
 25 there exists a unique solution $(y_1, y_2) \in C^0([0, +\infty); L^2(0, 1)^2)$ to the system (1). By solution we
 26 mean “solution along the characteristics” or “broad solution” (see e.g. [CHOS21, Appendix A]).
 27 The same statement remains true if, in the boundary condition at $x = 1$, u is replaced by

$$28 \quad u(t) = \int_0^1 (f_1(\xi)y_1(t, \xi) + f_2(\xi)y_2(t, \xi)) d\xi, \quad (2)$$

29 for any $f_1, f_2 \in L^\infty(0, 1)$. The relation (2) is called the “feedback law”.

30 Let us now introduce the notions of controllability that we are interested in:

31 **Definition 1.1.** Let $T > 0$. We say that the system (1) is:

32 • **finite-time stable with settling time T** if, for every $y_1^0, y_2^0 \in L^2(0, 1)$, the corresponding
 33 solution to the system (1) with $u = 0$ satisfies

$$34 \quad y_1(T, \cdot) = y_2(T, \cdot) = 0. \quad (3)$$

35 • **finite-time stabilizable with settling time T** if there exist $f_1, f_2 \in L^\infty(0, 1)$ such that,
 36 for every $y_1^0, y_2^0 \in L^2(0, 1)$, the corresponding solution to the system (1) with u given by
 37 (2) satisfies (3).

38 • **null controllable in time T** if, for every $y_1^0, y_2^0 \in L^2(0, 1)$, there exists $u \in L^2_{\text{loc}}(0, +\infty)$
 39 such that the corresponding solution to the system (1) satisfies (3).

40 Obviously, finite-time stability implies finite-time stabilization, which in turn implies null
 41 controllability.

42 **Remark 1.2.** As we are trying to bring the solution of the system (1) to the state zero, let
 43 us first mention that, in general, $u = 0$ does not work. Not only this, but in fact any static
 44 boundary output feedback laws, that is of the form $u(t) = ky_2(t, 1)$ with $k \in \mathbb{R}$, does not work
 45 either in general. A simple example is provided by the following 2×2 system with constant
 46 coefficients (see also [BC16, Section 5.6] when $y_2(t, 0) = y_1(t, 0)$):

$$47 \quad \begin{cases} \frac{\partial y_1}{\partial t}(t, x) - \frac{\partial y_1}{\partial x}(t, x) = \pi y_2(t, x), \\ \frac{\partial y_2}{\partial t}(t, x) + \frac{\partial y_2}{\partial x}(t, x) = \pi y_1(t, x), & t \in (0, +\infty), x \in (0, 1). \\ y_1(t, 1) = ky_2(t, 1), \quad y_2(t, 0) = 0, \\ y_1(0, x) = y_1^0(x), \quad y_2(0, x) = y_2^0(x), \end{cases} \quad (4)$$

Indeed, for this system we can always construct a smooth initial data (y_1^0, y_2^0) which is an eigen-
 function of the operator associated with (4) and whose corresponding eigenvalue σ is a positive
 real number, which makes the system (4) exponentially unstable. This can be done as follows.
 We take

$$y_1^0(x) = \frac{1}{\pi} \left(\sigma y_2^0(x) + \frac{\partial y_2^0}{\partial x}(x) \right),$$

48 (so that the second equation in (4) will always be satisfied) and

- 49 • If $k < 1 + 1/\pi$, then we take $\sigma = \pi\sqrt{1 - \theta^2}$ and $y_2^0(x) = \sin(\theta\pi x)$, where $\theta \in (0, 1)$ is any
50 solution to the equation $\sqrt{1 - \theta^2} + \theta \cot(\theta\pi) = k$.
- 51 • If $k = 1 + 1/\pi$, then we take $\sigma = \pi$ and $y_2^0(x) = \pi x$.
- 52 • If $k > 1 + 1/\pi$, then we take $\sigma = \pi\sqrt{1 + \theta^2}$ and $y_2^0(x) = 2 \sinh(\theta\pi x)$, where $\theta > 0$ is any
53 solution to the equation $\sqrt{1 + \theta^2} + \theta \coth(\theta\pi) = k$.

54 The goal of this work is to establish a necessary and sufficient condition on the time T for
55 the system (1) to be null controllable in time T (resp. finite-time stabilizable with settling time
56 T).

57 Let us now introduce some notations that will be used all along the rest of this article. Let
58 $\phi_1, \phi_2 \in C^{1,1}([0, 1])$ be the increasing functions defined for every $x \in [0, 1]$ by

$$59 \quad \phi_1(x) = \int_0^x \frac{1}{-\lambda_1(\xi)} d\xi, \quad \phi_2(x) = \int_0^x \frac{1}{\lambda_2(\xi)} d\xi. \quad (5)$$

We then denote by

$$T_1(\Lambda) = \phi_1(1) = \int_0^1 \frac{1}{-\lambda_1(\xi)} d\xi, \quad T_2(\Lambda) = \phi_2(1) = \int_0^1 \frac{1}{\lambda_2(\xi)} d\xi.$$

60 Finally, we set

$$61 \quad T_{\text{opt}}(\Lambda) = \max\{T_1(\Lambda), T_2(\Lambda)\}, \quad T_{\text{unif}}(\Lambda) = T_1(\Lambda) + T_2(\Lambda). \quad (6)$$

62 The naming of the notations in (6) will be explained in Remark 1.8 below.

63 1.2 Literature

64 Boundary null controllability and stabilization of hyperbolic systems of balance laws have at-
65 tracted numerous attention of both mathematicians and engineers during the last decades. In
66 the pioneering work [Rus78b], the author established the null controllability of general $n \times n$
67 coupled linear hyperbolic systems of the form (1) in a control time that is given by the sum of
68 the two largest times from the states convecting in opposite directions ([Rus78b, Theorem 3.2]).
69 It was also observed that this time can be shorten in some cases ([Rus78b, Proposition 3.4]), and
70 the problem to find the minimal control time for hyperbolic partial differential equations (PDEs)
71 was then raised ([Rus78b, Remark p. 656]).

72 For systems of linear conservation laws (i.e. when no internal coupling matrix is present in
73 the system), this problem was completely solved few years later in [Wec82], where the minimal
74 control time has been characterized in terms of the boundary coupling matrix, that is the matrix
75 coupling the equations at the boundary on the uncontrolled side. For systems of balance laws,
76 the story is far from over. A first improvement of the control time of [Rus78b] was recently
77 obtained in [CN19b] thanks to the introduction of some rank condition on the boundary coupling
78 matrix. However, this was first done for some generic internal coupling matrices or under rather
79 stringent conditions ([CN19b, Theorem 1.1 and 1.5]). The same authors were then able to
80 remove some of these restrictions in [CN19a]. For the present paper it is especially important
81 to emphasize that the new time introduced in [CN19b, CN19a] is only shown to be sufficient
82 for the null controllability in these works. On the other hand, the minimal control time needed
83 to achieve the exact controllability property (that is when we want to reach any final data and
84 not only zero), was completely characterized in [HO19, Theorem 1.9] by a simple and calculable
85 formula. It is also pointed out that null and exact controllability are equivalent properties if

86 the boundary coupling matrix has a full row rank. For quasilinear systems, it has been shown
 87 in [Li10, Theorem 3.2] that the time of [Rus78b] yields the (local) exact controllability of such
 88 systems if the boundary coupling matrix has a full row rank in a neighborhood of the state
 89 zero. For homogeneous quasilinear systems, a smaller control time was then obtained in [Hu15,
 90 Theorem 1.1].

91 Concerning now the stabilization property, the first works seem [GL84, Qin85] for the expo-
 92 nential stabilization of homogeneous quasilinear hyperbolic systems in a C^1 framework by using
 93 the method of characteristics. To the best of our knowledge, the weakest sufficient condition
 94 using this technique can be found in [Li94, Theorem 1.3, p. 173]. This condition was then im-
 95 proved in [CBdN08, Theorem 2.3] in a H^2 framework thanks to the construction of an explicit
 96 strict Lyapunov function. In all the previous references, the feedback laws were static boundary
 97 output feedback laws (that is, depending only on the state values at the boundaries). However,
 98 due to the locality of such kind of feedback laws, these two strategies may not be effective to deal
 99 with general systems of balance laws ([BC16, Section 5.6] and Remark 1.2). Another method
 100 was then used to address this problem, the backstepping method. For PDEs, this method now
 101 consists in transforming our initial system into another system - called target system - for which
 102 the stabilization properties are simpler to study. The transformation used is usually a Volterra
 103 transformation of the second kind. One can refer to the tutorial book [KS08] to design boundary
 104 feedback laws stabilizing systems modeled by various PDEs and to the introduction of [CHOS21]
 105 for a complementary state of the art on this method. This technique turned out to be a powerful
 106 tool to stabilize general coupled hyperbolic systems, moreover in finite time. In [CVKB13] the
 107 authors adapted this technique to obtain the first finite-time stabilization result for 2×2 linear
 108 hyperbolic system. This method was then developed, notably with a more careful choice of the
 109 target system, to treat 3×3 systems in [HDM15] and then to treat general $n \times n$ systems in
 110 [HDMVK16, HVDMK19]. However, the control time obtained in these works was larger than
 111 the one in [Rus78b] and it was only shown in [ADM16, CHO17] that we can stabilize with the
 112 same time as the one of [Rus78b]. These works have recently been generalized to time-dependent
 113 systems in [CHOS21]. Finally, let us also mention the two recent works [CN20a, CN20b] con-
 114 cerning the finite-time stabilization of homogeneous quasilinear systems, with the same control
 115 time as in [CN19b, CN19a].

116 In spite of quite a number of contributions dealing with these two problems (controllability
 117 and stabilization), we see that there are no references concerning the optimality of the control
 118 time for systems of linear balance laws with spatial-varying internal coupling matrix, especially
 119 when null and exact controllability are not equivalent, so that the results in [CVKB13, HO19]
 120 cannot be considered. This is of course a nontrivial task and it requires the addition of new
 121 techniques as we shall see below. The goal of this article is to fill this gap, at least for 2×2
 122 systems. We will provide an explicit formula of the minimal control time for any 2×2 system
 123 of linear balance laws with spacial-varying internal coupling matrix. We will see that one of the
 124 main differences between null and exact controllability is that such a critical time is sensitive to
 125 the behavior of the internal coupling matrix for the null controllability, whereas it is known to
 126 never be the case for the exact controllability ([CVKB13, HO19]).

127 1.3 Main result and comments

128 The important quantity in the present work is the following:

Definition 1.3. For $\varepsilon > 0$ and a function $f : (0, \varepsilon) \rightarrow \mathbb{R}$, we denote by

$$\ell_\varepsilon(f) = \begin{cases} \sup I_\varepsilon(f) & \text{if } I_\varepsilon(f) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

129 where $I_\varepsilon(f) = \{\ell \in (0, \varepsilon) \mid f = 0 \text{ a.e. in } (0, \ell)\}$.

130 The quantity $\ell_\varepsilon(f)$ is the length of the largest interval of the form $(0, \ell)$ where the function
 131 f vanishes.

132 **Example 1.4.**

(E1) The simplest example of function f with $\ell_\varepsilon(f) = \ell$ ($\ell \in [0, \varepsilon]$) is obviously the step function

$$f(x) = \begin{cases} 0 & \text{if } x \leq \ell, \\ 1 & \text{if } x > \ell. \end{cases}$$

133 (E2) If $f \in C^k([0, \varepsilon])$ ($k \in \mathbb{N}$) and satisfies $f^{(k)}(0) \neq 0$, then $\ell_\varepsilon(f) = 0$. In particular, if f has
 134 an analytic extension in a neighborhood of $x = 0$, then $\ell_\varepsilon(f) = 0$.

135 (E3) An example of smooth function f with $\ell_\varepsilon(f) = 0$ but that does not satisfy the previous
 136 conditions is

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \exp\left(-\frac{1}{x}\right) & \text{if } x > 0. \end{cases} \quad (7)$$

138 The main result of this article is the following complete characterization of the controllability
 139 properties of the system (1):

140 **Theorem 1.5.** Let $T > 0$.

141 (i) If the system (1) is null controllable in time T , then necessarily

$$142 \quad T \geq \max \left\{ T_{\text{opt}}(\Lambda), \int_{\ell_{x_\Lambda}(c)}^1 \left(\frac{1}{-\lambda_1(\xi)} + \frac{1}{\lambda_2(\xi)} \right) d\xi \right\}, \quad (8)$$

143 where $x_\Lambda \in (0, 1)$ is the unique solution to $\phi_1(x_\Lambda) + \phi_2(x_\Lambda) = T_2(\Lambda)$ ($= \phi_2(1)$).

144 (ii) If the time T satisfies (8), then the system (1) is finite-time stabilizable with settling time
 145 T .

146 Note in particular that the system (1) is then null controllable in time T if, and only if, it is
 147 finite-time stabilizable with settling time T .

148 **Example 1.6.** For c satisfying the properties in (E2) or given by the function in (E3), this
 149 result shows that the time $T_{\text{unif}}(\Lambda)$ cannot be improved. This is not trivial, especially when c is
 150 given by the function in (E3).

Remark 1.7. When λ_1, λ_2 do not depend on space, the condition (8), in the situation $T_{\text{opt}}(\Lambda) \leq T < T_{\text{unif}}(\Lambda)$, simply becomes

$$c = 0 \quad \text{in} \quad \left(0, 1 - \frac{T}{T_{\text{unif}}(\Lambda)} \right).$$

151 In particular, we see that we can possibly obtain any intermediate time between $T_{\text{opt}}(\Lambda)$ and
 152 $T_{\text{unif}}(\Lambda)$. Moreover, note that the value $T_{\text{opt}}(\Lambda)$ is reachable even when c is not identically equal
 153 to zero.

154 **Remark 1.8.** As we shall see in the proof below, the most difficult part of this result is the
 155 necessary condition, that is the item (i). It is also thanks to this part that we can call the time
 156 on the right-hand side of (8) the minimal control time. It is sometimes find in the literature
 157 that the time $T_{\text{unif}}(\Lambda)$ is “the theoretical lower bound for control time” or “the optimal time”.
 158 However, the failure of the controllability before this time is never proved in these works (and,
 159 in fact, it cannot be in general), which brings some confusion to our point of view. This is why
 160 we carefully introduced a different naming and use the notations

- 161 • $T_{\text{unif}}(\Lambda)$ as “uniform time”, for the smallest time after which all the systems of the form
 162 (1) are null controllable,
- 163 • $T_{\text{opt}}(\Lambda)$ as “optimal time”, for the smallest control time that can be obtained among all
 164 the possible control times for the systems of the form (1).

165 **Remark 1.9.** Let us comment other possibilities for the boundary conditions at $x = 0$:

- 166 (i) When the boundary condition $y_2(t, 0) = 0$ is replaced by $y_2(t, 0) = qy_1(t, 0)$ with boundary
 167 coupling “matrix” $q \neq 0$, the result [CVKB13, Theorem 3.2] shows that the time $T_{\text{unif}}(\Lambda)$
 168 is the minimal control time (more precisely, it is shown that the system (1) with such a
 169 boundary condition is equivalent to the same system with no internal coupling matrix, for
 170 which $T_{\text{unif}}(\Lambda)$ is clearly minimal). However, when $q = 0$, we see that our time is smaller
 171 than the one obtained in this reference.
- 172 (ii) When a second control is applied at the boundary $x = 0$, i.e. the boundary condition
 173 $y_2(t, 0) = 0$ is replaced by $y_2(t, 0) = v(t)$ with $v \in L^2_{\text{loc}}(0, +\infty)$ a second control at our
 174 disposal, then the time $T_{\text{opt}}(\Lambda)$ is the minimal control time. The null controllability for $T \geq$
 175 $T_{\text{opt}}(\Lambda)$ can be shown using for instance the well-known constructive method developed
 176 in [Li10, Theorem 3.1]. On the other hand, the failure of the null controllability for $T <$
 177 $T_{\text{opt}}(\Lambda)$ follows from the backstepping method (by means of Volterra transformation of the
 178 second kind) and a simple adaptation of Lemma 3.3 below.

Therefore, combining the previous results of the literature with the new results of the present
 paper, we see that all the following possibilities for the boundary conditions have been handled:

$$y_1(t, 1) = py_2(t, 1) + ru(t), \quad y_2(t, 0) = qy_1(t, 0) + sv(t),$$

$$p, q, r, s \in \mathbb{R} \text{ with } (r, s) \neq (0, 0).$$

179 The rest of this article is organized as follows. In Section 2, we use the backstepping method
 180 to show that our initial system (1) is equivalent to a canonical system from a controllability point
 181 of view. In Section 3 we use the Titchmarsh convolution theorem to completely characterize the
 182 minimal control time for this canonical system. In Section 4 we characterize this time in terms
 183 of the parameters of the initial system. Finally, in Section 5 we discuss possible extensions to
 184 systems with more than two equations.

185 2 Reduction to a canonical form

186 In this section, we perform some changes of unknown to transform our initial system (1) into
 187 a new system whose controllability properties will be simpler to study, this is the so-called
 188 backstepping method for PDEs. The content of section is quite standard by now, we refer for
 189 instance to [CVKB13, Section 3.2] for more details on the computations below.

190 First of all, we remove the diagonal terms in the system (1). Using the invertible spatial
 191 transformation (seen as an operator from $L^2(0, 1)^2$ onto itself)

$$192 \quad \begin{cases} \tilde{y}_1(t, x) = e_1(x)y_1(t, x), \\ \tilde{y}_2(t, x) = e_2(x)y_2(t, x), \end{cases} \quad (9)$$

193 with

$$194 \quad e_1(x) = \exp\left(-\int_0^x \frac{a(\xi)}{\lambda_1(\xi)} d\xi\right), \quad e_2(x) = \exp\left(-\int_0^x \frac{d(\xi)}{\lambda_2(\xi)} d\xi\right), \quad (10)$$

195 we easily see that the system (1) is null controllable in time T (resp. finite-time stabilizable with
 196 settling time T) if, and only if, so is the system

$$197 \quad \begin{cases} \frac{\partial \tilde{y}_1}{\partial t}(t, x) + \lambda_1(x) \frac{\partial \tilde{y}_1}{\partial x}(t, x) = \tilde{b}(x)\tilde{y}_2(t, x), \\ \frac{\partial \tilde{y}_2}{\partial t}(t, x) + \lambda_2(x) \frac{\partial \tilde{y}_2}{\partial x}(t, x) = \tilde{c}(x)\tilde{y}_1(t, x), & t \in (0, +\infty), x \in (0, 1), \\ \tilde{y}_1(t, 1) = \tilde{u}(t), \quad \tilde{y}_2(t, 0) = 0, \\ \tilde{y}_1(0, x) = \tilde{y}_1^0(x), \quad \tilde{y}_2(0, x) = \tilde{y}_2^0(x), \end{cases} \quad (11)$$

198 where

$$199 \quad \tilde{b}(x) = b(x) \frac{e_1(x)}{e_2(x)}, \quad \tilde{c}(x) = c(x) \frac{e_2(x)}{e_1(x)}. \quad (12)$$

Let us now remove the coupling term on the first equation of (11) thanks to a second trans-
 formation. Set

$$\mathcal{T} = \{(x, \xi) \in (0, 1) \times (0, 1) \mid x > \xi\}.$$

200 Let $k_{11}, k_{12}, k_{21}, k_{22} \in L^\infty(\mathcal{T})$. Using the spatial transformation

$$201 \quad \begin{cases} \hat{y}_1(t, x) = \tilde{y}_1(t, x) - \int_0^x (k_{11}(x, \xi)\tilde{y}_1(t, \xi) + k_{12}(x, \xi)\tilde{y}_2(t, \xi)) d\xi, \\ \hat{y}_2(t, x) = \tilde{y}_2(t, x) - \int_0^x (k_{21}(x, \xi)\tilde{y}_1(t, \xi) + k_{22}(x, \xi)\tilde{y}_2(t, \xi)) d\xi, \end{cases} \quad (13)$$

202 which is invertible since it is a Volterra transformation of the second kind (see e.g. [Hoc73,
 203 Chapter 2, Theorem 5]), we see that the system (11) is null controllable in time T (resp. finite-
 204 time stabilizable with settling time T) if, and only if, so is the system

$$205 \quad \begin{cases} \frac{\partial \hat{y}_1}{\partial t}(t, x) + \lambda_1(x) \frac{\partial \hat{y}_1}{\partial x}(t, x) = 0, \\ \frac{\partial \hat{y}_2}{\partial t}(t, x) + \lambda_2(x) \frac{\partial \hat{y}_2}{\partial x}(t, x) = g(x)\hat{y}_1(t, 0), & t \in (0, +\infty), x \in (0, 1), \\ \hat{y}_1(t, 1) = \hat{u}(t), \quad \hat{y}_2(t, 0) = 0, \\ \hat{y}_1(0, x) = \hat{y}_1^0(x), \quad \hat{y}_2(0, x) = \hat{y}_2^0(x), \end{cases} \quad (14)$$

206 with g given by

$$207 \quad g(x) = -k_{21}(x, 0)\lambda_1(0), \quad (15)$$

208 provided that the kernels $k_{11}, k_{12}, k_{21}, k_{22}$ satisfy the so-called kernel equations:

$$209 \quad \begin{cases} \lambda_1(x) \frac{\partial k_{11}}{\partial x}(x, \xi) + \frac{\partial k_{11}}{\partial \xi}(x, \xi) \lambda_1(\xi) + k_{11}(x, \xi) \frac{\partial \lambda_1}{\partial \xi}(\xi) + k_{12}(x, \xi) \tilde{c}(\xi) = 0, \\ \lambda_1(x) \frac{\partial k_{12}}{\partial x}(x, \xi) + \frac{\partial k_{12}}{\partial \xi}(x, \xi) \lambda_2(\xi) + k_{11}(x, \xi) \tilde{b}(\xi) + k_{12}(x, \xi) \frac{\partial \lambda_2}{\partial \xi}(\xi) = 0, \\ k_{11}(x, 0) = 0, \\ k_{12}(x, x) = \frac{\tilde{b}(x)}{\lambda_1(x) - \lambda_2(x)}, \end{cases} \quad (x, \xi) \in \mathcal{T}, \quad (16)$$

210 and

$$211 \quad \begin{cases} \lambda_2(x) \frac{\partial k_{21}}{\partial x}(x, \xi) + \frac{\partial k_{21}}{\partial \xi}(x, \xi) \lambda_1(\xi) + k_{21}(x, \xi) \frac{\partial \lambda_1}{\partial \xi}(\xi) + k_{22}(x, \xi) \tilde{c}(\xi) = 0, \\ \lambda_2(x) \frac{\partial k_{22}}{\partial x}(x, \xi) + \frac{\partial k_{22}}{\partial \xi}(x, \xi) \lambda_2(\xi) + k_{21}(x, \xi) \tilde{b}(\xi) + k_{22}(x, \xi) \frac{\partial \lambda_2}{\partial \xi}(\xi) = 0, \\ k_{21}(x, x) = \frac{\tilde{c}(x)}{\lambda_2(x) - \lambda_1(x)}, \end{cases} \quad (x, \xi) \in \mathcal{T}. \quad (17)$$

212 Note that (16) and (17) are not coupled.

213 From [CVKB13, Theorem A.1], we know that the kernel equations (16)-(17) have a solution.

214 More precisely, we have the following result:

Theorem 2.1. *For every $k^0 \in L^\infty(0, 1)$, there exists a unique solution $(k_{11}, k_{12}, k_{21}, k_{22}) \in L^\infty(\mathcal{T})^4$ to the kernel equations (16)-(17) with*

$$k_{22}(x, 0) = k^0(x), \quad x \in (0, 1).$$

215 In the aforementioned reference this result is stated in a C^0 framework (assuming that
216 $a, b, c, d \in C^0([0, 1])$) but its proof readily shows that it is valid in L^∞ as well. As before,
217 the notion of solution is to be understood in the sense of solution along the characteristics. The
218 boundary terms such as $k_{21}(x, 0)$, which defines g (see (15)), or $k_{11}(1, \xi), k_{12}(1, \xi)$, that will ap-
219 pear shortly below in our feedback law (see (19)), etc. are also understood in this sense. We
220 refer for instance to the formula (42) below for the precise meaning of $k_{21}(x, 0)$.

221 3 Study of the canonical system

222 We call the system (14) the “control canonical form of the system (1)” or “canonical system” in
223 short, by analogy with [Bru70, Rus78a] and since we will see in this section that we are able
224 to directly read its controllability properties (a task that seems impossible on the initial system
225 (1)).

226 The goal of this section is to establish the following result:

227 **Theorem 3.1.** *Let $T > 0$ and $g \in L^\infty(0, 1)$.*

228 (i) *If the system (14) is null controllable in time T , then necessarily*

$$229 \quad T \geq \max \left\{ T_1(\Lambda) + \int_{\ell_1(g)}^1 \frac{1}{\lambda_2(\xi)} d\xi, \quad T_2(\Lambda) \right\}. \quad (18)$$

230 (ii) If the time T satisfies (18), then the system (14) is finite-time stable with settling time T .

231 Let us emphasize once again that the difficult point is the first item.

232 **Remark 3.2.** Since $\hat{u} = 0$ stabilizes the canonical system (14) by (ii) of Theorem 3.1, we see
 233 from the formula (9) and (13) that our feedback for the system (1) is then

$$234 \quad u(t) = \int_0^1 \frac{k_{11}(1, \xi)e_1(\xi)}{e_1(1)} y_1(t, \xi) d\xi + \int_0^1 \frac{k_{12}(1, \xi)e_2(\xi)}{e_1(1)} y_2(t, \xi) d\xi. \quad (19)$$

235 Note that $u \in C^0([0, +\infty))$.

236 3.1 The characteristics

237 Before proving Theorem 3.1 we need to introduce the characteristic curves associated with the
 238 system (14) and recall some useful properties.

239 First of all, it is convenient to extend λ_1, λ_2 to functions of \mathbb{R} (still denoted by the same)
 240 such that $\lambda_1, \lambda_2 \in C^{0,1}(\mathbb{R})$ and

$$241 \quad \lambda_1(x) \leq -\varepsilon < 0 < \varepsilon < \lambda_2(x), \quad \forall x \in \mathbb{R}, \quad (20)$$

242 for some $\varepsilon > 0$ small enough. Since all the results of the present paper depend only on the values
 243 of λ_1, λ_2 in $[0, 1]$, they do not depend on such an extension.

244 In what follows, $i \in \{1, 2\}$. Let χ_i be the flow associated with λ_i , i.e. for every $(t, x) \in \mathbb{R} \times \mathbb{R}$,
 245 the function $s \mapsto \chi_i(s; t, x)$ is the solution to the ODE

$$246 \quad \begin{cases} \frac{\partial \chi_i}{\partial s}(s; t, x) = \lambda_i(\chi_i(s; t, x)), & \forall s \in \mathbb{R}, \\ \chi_i(t; t, x) = x. \end{cases} \quad (21)$$

247 The existence and uniqueness of a (global) solution to the ODE (21) follows from the (global)
 248 Cauchy-Lipschitz theorem (see e.g. [Har02, Theorem II.1.1]). The uniqueness also yields the
 249 important group property

$$250 \quad \chi_i(\sigma; s, \chi_i(s; t, x)) = \chi_i(\sigma; t, x), \quad \forall \sigma, s \in \mathbb{R}. \quad (22)$$

251 By classical regularity results on ODEs (see e.g. [Har02, Theorem V.3.1]), we have $\chi_i \in C^1(\mathbb{R}^3)$
 252 and

$$253 \quad \frac{\partial \chi_i}{\partial t}(s; t, x) = -\lambda_i(\chi_i(s; t, x)), \quad \frac{\partial \chi_i}{\partial x}(s; t, x) = \frac{\lambda_i(\chi_i(s; t, x))}{\lambda_i(x)}. \quad (23)$$

Let us now introduce the entry and exit times $s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x) \in \mathbb{R}$ of the flow $\chi_i(\cdot; t, x)$
 inside the domain $[0, 1]$, i.e. the respective unique solutions to

$$\begin{cases} \chi_1(s_1^{\text{in}}(t, x); t, x) = 1, & \chi_1(s_1^{\text{out}}(t, x); t, x) = 0, \\ \chi_2(s_2^{\text{in}}(t, x); t, x) = 0, & \chi_2(s_2^{\text{out}}(t, x); t, x) = 1. \end{cases}$$

254 Their existence and uniqueness are guaranteed by the condition (20). It readily follows from (22)
 255 and the uniqueness of s_i^{in} that

$$256 \quad s_i^{\text{in}}(s, \chi_i(s; t, x)) = s_i^{\text{in}}(t, x), \quad \forall s \in \mathbb{R}. \quad (24)$$

257 By the implicit function theorem we have $s_i^{\text{in}} \in C^1(\mathbb{R}^2)$ with (using (23))

$$258 \quad \begin{cases} \frac{\partial s_1^{\text{in}}}{\partial t}(t, x) > 0, & \frac{\partial s_1^{\text{in}}}{\partial x}(t, x) > 0, \\ \frac{\partial s_2^{\text{in}}}{\partial t}(t, x) > 0, & \frac{\partial s_2^{\text{in}}}{\partial x}(t, x) < 0. \end{cases} \quad (25)$$

259 Combined with the group property (24), this yields the following inverse formula for every $s, t \in$
260 \mathbb{R} :

$$261 \quad \begin{cases} s < s_1^{\text{out}}(t, 1) & \iff & s_1^{\text{in}}(s, 0) < t, \\ s < s_2^{\text{out}}(t, 0) & \iff & s_2^{\text{in}}(s, 1) < t. \end{cases} \quad (26)$$

Finally, since λ_i does not depend on time, we have an explicit formula for the inverse function $\theta \mapsto \chi_i^{-1}(\theta; t, x)$. Indeed, it solves

$$\begin{cases} \frac{\partial(\chi_i^{-1})}{\partial \theta}(\theta; t, x) = \frac{1}{\frac{\partial \chi_i}{\partial s}(\chi_i^{-1}(\theta; t, x); t, x)} = \frac{1}{\lambda_i(\theta)}, & \forall \theta \in \mathbb{R}, \\ \chi_i^{-1}(x; t, x) = t, \end{cases}$$

262 which gives

$$263 \quad \chi_i^{-1}(\theta; t, x) = t + \int_x^\theta \frac{1}{\lambda_i(\xi)} d\xi. \quad (27)$$

This also yields an explicit formula for $s_1^{\text{in}}, s_2^{\text{in}}$ and $s_1^{\text{out}}, s_2^{\text{out}}$ and, in particular,

$$T_1(\Lambda) = s_1^{\text{out}}(0, 1), \quad T_2(\Lambda) = s_2^{\text{out}}(0, 0).$$

264 3.2 Proof of Theorem 3.1

265 First of all, the solution of the canonical system (14) is explicitly given by:

$$266 \quad \hat{y}_1(t, x) = \begin{cases} \hat{y}_1^0(\chi_1(0; t, x)) & \text{if } s_1^{\text{in}}(t, x) < 0, \\ \hat{u}(s_1^{\text{in}}(t, x)) & \text{if } s_1^{\text{in}}(t, x) > 0, \end{cases} \quad (28)$$

267 and

$$268 \quad \hat{y}_2(t, x) = \begin{cases} \hat{y}_2^0(\chi_2(0; t, x)) + \int_0^t g(\chi_2(s; t, x)) \hat{y}_1(s, 0) ds & \text{if } s_2^{\text{in}}(t, x) < 0, \\ \int_{s_2^{\text{in}}(t, x)}^t g(\chi_2(s; t, x)) \hat{y}_1(s, 0) ds & \text{if } s_2^{\text{in}}(t, x) > 0. \end{cases} \quad (29)$$

269 Next, we show a uniform lower bound for the control time:

Lemma 3.3. *Let $T > 0$. If the system (14) is null controllable in time T , then necessarily*

$$T \geq T_{\text{opt}}(\Lambda).$$

270 This result states that the control time cannot be better than the one of the case $g = 0$.

Proof. For $i \in \{1, 2\}$, let ω_i be the open subset defined by

$$\omega_i = \{x \in (0, 1) \mid s_i^{\text{in}}(T, x) < 0\}.$$

271 From (26) and (25), we see that

$$272 \quad T \geq T_i(\Lambda) \iff \omega_i = \emptyset. \quad (30)$$

Therefore, if $T < T_1(\Lambda)$, then we see from (28) that \hat{y}_1^0 can be chosen so that $\hat{y}_1(T, x) \neq 0$ for $x \in \omega_1$, whatever \hat{u} is. On the other hand, if $T < T_2(\Lambda)$ and if the system (14) is null controllable in time T , then for every $\hat{y}_2^0 \in L^2(0, 1)$, there exists $\hat{u} \in L^2(0, T)$ such that, for a.e. $x \in \omega_2$, we have

$$0 = \hat{y}_2^0(\chi_2(0; T, x)) + \int_0^T g(\chi_2(s; T, x)) \hat{y}_1(s, 0) ds.$$

Since $x \in \omega_2 \mapsto \chi_2(0; T, x)$ is bijective (it is increasing by (23) and ω_2 is an interval by (25)), this implies that the bounded linear operator $K : L^2(0, T) \rightarrow L^2(\omega_2)$ defined by

$$(Kh)(x) = - \int_0^T g(\chi_2(s; T, x)) h(s) ds,$$

273 is surjective. This is impossible since its range is clearly a subset of $L^\infty(\omega_2)$, which is a proper
 274 subset of $L^2(\omega_2)$ (alternatively, one could note that K is compact and therefore it cannot be
 275 surjective over an infinite dimensional space, see e.g. [Rud91, Theorem 4.18 (b)]). □

277 The proof of the item (i) of Theorem 3.1 crucially relies on the Titchmarsh convolution
 278 theorem [Tit26, Theorem VII] (see also [Mik78, Chapter XV]):

279 **Theorem 3.4.** *Let $\alpha, \beta \in L^1(0, \bar{\tau})$ ($\bar{\tau} > 0$). We have*

$$280 \quad \int_0^{\bar{\tau}} \alpha(\tau - \sigma)\beta(\sigma) d\sigma = 0, \quad \text{a.e. } 0 < \tau < \bar{\tau}, \quad (31)$$

if, and only if,

$$\ell_{\bar{\tau}}(\alpha) + \ell_{\bar{\tau}}(\beta) \geq \bar{\tau}.$$

281 **Remark 3.5.** The difficulty in the proof of this result is the necessary condition, i.e. the
 282 implication “ \implies ”, just like it is the case for our main result. Let us however mention that its
 283 proof is easy in case α satisfies the condition in (E2) of Example 1.4 (by taking derivatives of
 284 (31) and using the injectivity of Volterra transformations of the second kind). It does not seem
 285 trivial for functions of the form (7) though.

286 We are now ready to prove the main result of Section 3:

287 *Proof of Theorem 3.1.*

288 1) Thanks to Lemma 3.3, we can assume that $T \geq T_1(\Lambda)$ and $T \geq T_2(\Lambda)$. This means that
 289 $s_1^{\text{in}}(T, x) > 0$ and $s_2^{\text{in}}(T, x) > 0$ for every $x \in (0, 1)$ (see (30) and (25)). It then follows from
 290 the explicit formula (28) and (29) that $\hat{y}_1(T, \cdot) = 0$ if, and only if,

$$291 \quad \hat{u}(s_1^{\text{in}}(T, x)) = 0, \quad 0 < x < 1, \quad (32)$$

292 and $\hat{y}_2(T, \cdot) = 0$ if, and only if,

$$293 \quad \int_{s_2^{\text{in}}(T, x)}^T g(\chi_2(s; T, x)) \hat{y}_1(s, 0) ds = 0, \quad 0 < x < 1. \quad (33)$$

294 2) Let us focus on the second condition (33). Writing $x = \chi_2(T; t, 0)$, which belongs to $(0, 1)$
 295 for $t \in (s_2^{\text{in}}(T, 1), T)$ (recall in particular (26)), and using the group properties (22) and
 296 (24) with the identity $s_2^{\text{in}}(t, 0) = t$, we obtain that $\hat{y}_2(T, \cdot) = 0$ if, and only if,

$$297 \int_t^T g(\chi_2(s; t, 0)) \hat{y}_1(s, 0) ds = 0, \quad s_2^{\text{in}}(T, 1) < t < T. \quad (34)$$

Now we use the fact that $g(\chi_2(s; t, 0))$ is actually a function of $s - t$. Indeed, by uniqueness to the solution to the ODE (21), we see that the characteristics take the form

$$\chi_i(s; t, x) = \tilde{\chi}_i(s - t; x),$$

where $s \mapsto \tilde{\chi}_i(s; x)$ is the unique solution to

$$\begin{cases} \frac{\partial \tilde{\chi}_i}{\partial s}(s; x) = \lambda_i(\tilde{\chi}_i(s; x)), & \forall s \in \mathbb{R}, \\ \tilde{\chi}_i(0; x) = x. \end{cases}$$

Using the change of variables $\sigma = s - t$ and introducing

$$\alpha(\theta) = \hat{y}_1(-\theta + T, 0), \quad \beta(\theta) = g(\tilde{\chi}_2(\theta; 0)), \quad 0 < \theta < T - s_2^{\text{in}}(T, 1),$$

298 we see that (34) is equivalent to (setting $\tau = T - t$)

$$299 \int_0^\tau \alpha(\tau - \sigma) \beta(\sigma) d\sigma = 0, \quad 0 < \tau < \bar{\tau}, \quad (35)$$

where

$$\bar{\tau} = T - s_2^{\text{in}}(T, 1).$$

3) Applying the Titchmarsh convolution theorem (Theorem 3.4) we deduce that (35) is equivalent to

$$\ell_{\bar{\tau}}(\alpha) + \ell_{\bar{\tau}}(\beta) \geq \bar{\tau}.$$

From the explicit expression (28) and the inverse formula (26), we see that

$$\alpha(\theta) = \begin{cases} \hat{y}_1^0(\chi_1(0; -\theta + T, 0)) & \text{if } \theta > T - s_1^{\text{out}}(0, 1), \\ \hat{u}(s_1^{\text{in}}(-\theta + T, 0)) & \text{if } \theta < T - s_1^{\text{out}}(0, 1). \end{cases}$$

Therefore, we can choose \hat{y}_1^0 so that

$$\alpha(\theta) \neq 0, \quad \forall \theta \in (T - s_1^{\text{out}}(0, 1), T - s_1^{\text{out}}(0, 1) + \varepsilon),$$

for some $0 < \varepsilon < s_1^{\text{out}}(0, 1)$. This yields the bound

$$\ell_{\bar{\tau}}(\alpha) \leq T - s_1^{\text{out}}(0, 1).$$

300 Consequently, we necessarily have

$$301 \ell_{\bar{\tau}}(\beta) \geq s_1^{\text{out}}(0, 1) - s_2^{\text{in}}(T, 1). \quad (36)$$

Since $s \mapsto \tilde{\chi}_2(s; 0)$ is increasing with $\tilde{\chi}_2(0; 0) = 0$, this is equivalent to

$$\begin{aligned} \ell_1(g) &\geq \tilde{\chi}_2(s_1^{\text{out}}(0, 1) - s_2^{\text{in}}(T, 1); 0) = \chi_2(s_1^{\text{out}}(0, 1); s_2^{\text{in}}(T, 1), 0) \\ &= \chi_2(s_1^{\text{out}}(0, 1); T, 1) \quad (\text{by (22) with } s = s_2^{\text{in}}(T, 1)). \end{aligned}$$

Since $s \mapsto \chi_2(s; T, 1)$ is increasing, this is also equivalent to

$$\chi_2^{-1}(\ell_1(g); T, 1) \geq s_1^{\text{out}}(0, 1) = T_1(\Lambda).$$

302 Using the explicit expression (27), we then obtain the desired condition $T \geq T_1(\Lambda) +$
 303 $\int_{\ell_1(g)}^1 \frac{1}{\lambda_2(\xi)} d\xi.$

4) Conversely, assume that T satisfies this condition and $T \geq T_2(\Lambda)$. Then, (36) holds by the previous equivalences. Taking $\hat{u} = 0$, we see that $\alpha = 0$ in $(0, T - T_1(\Lambda))$, which yields

$$\ell_{\bar{\tau}}(\alpha) + \ell_{\bar{\tau}}(\beta) \geq T - T_1(\Lambda) + s_1^{\text{out}}(0, 1) - s_2^{\text{in}}(T, 1) = T - s_2^{\text{in}}(T, 1) = \bar{\tau}.$$

304 This implies (35) (here we only use the “easy part” of the Titchmarsh convolution theorem)
 305 and thus $\hat{y}_2(T, \cdot) = 0$. Finally, note that $\hat{u} = 0$ also obviously satisfies (32) and thus
 306 $\hat{y}_1(T, \cdot) = 0$ as well.

307

□

308 **Remark 3.6.** Let us point out that the space dependence of the speeds brings up more technical
 309 difficulties than the case of constant speeds (especially the step 2)).

310 4 Proof of the main result

311 In this section we show how to deduce our main result from Theorem 3.1.

312 *Proof of Theorem 1.5.*

1) First of all, let us recall that the initial system (1) is null controllable in time T (resp. finite-time stabilizable with settling time T) if, and only if, so is the canonical system (14) (with g given by (15)). Therefore, thanks to Theorem 3.1 it suffices to show that

$$T \geq T_1(\Lambda) + \int_{\ell_1(g)}^1 \frac{1}{\lambda_2(\xi)} d\xi \iff T \geq \int_{\ell_{x_\Lambda}(c)}^1 \left(\frac{1}{-\lambda_1(\xi)} + \frac{1}{\lambda_2(\xi)} \right) d\xi,$$

313 which amounts to characterize $\ell_1(g)$ in terms of $\ell_{x_\Lambda}(c)$ (we recall that x_Λ is defined in the
 314 statement of Theorem 1.5). To this end, we are going to prove the identity

$$315 \phi_2(\ell_1(g)) = \phi_1(\ell_{x_\Lambda}(c)) + \phi_2(\ell_{x_\Lambda}(c)), \quad (37)$$

316 where we recall that $\phi_1, \phi_2 \in C^{1,1}([0, 1])$ are defined in (5).

317 2) We recall that $g(x) = -k_{21}(x, 0)\lambda_1(0)$, where k_{21} is the solution in \mathcal{T} to

$$318 \begin{cases} \lambda_2(x) \frac{\partial k_{21}}{\partial x}(x, \xi) + \frac{\partial k_{21}}{\partial \xi}(x, \xi)\lambda_1(\xi) + k_{21}(x, \xi) \frac{\partial \lambda_1}{\partial \xi}(\xi) + k_{22}(x, \xi)\tilde{c}(\xi) = 0, \\ k_{21}(x, x) = \frac{\tilde{c}(x)}{\lambda_2(x) - \lambda_1(x)}, \end{cases} \quad (38)$$

319 and where \tilde{c} is defined in (12) and (10) (note that $\ell_\varepsilon(\tilde{c}) = \ell_\varepsilon(c)$ for any $\varepsilon \in (0, 1)$). Let
 320 $s \mapsto \chi(s; x)$ be the associated characteristic passing through $(x, \xi) = (x, 0)$, i.e. the
 321 solution to the ODE

$$322 \quad \begin{cases} \frac{\partial \chi}{\partial s}(s; x) = \frac{\lambda_1(\chi(s; x))}{\lambda_2(s)}, & \forall s \in \mathbb{R}, \\ \chi(x; x) = 0, \end{cases} \quad (39)$$

(we recall that λ_1, λ_2 have been extended to \mathbb{R} in Section 3.1). We have $\chi \in C^1(\mathbb{R}^2)$ by classical regularity results on ODEs with

$$\frac{\partial \chi}{\partial x}(s; x) = \frac{-\lambda_1(\chi(s; x))}{\lambda_2(x)} > 0.$$

Since $f : s \mapsto s - \chi(s; x)$ is continuous and increasing with $\lim_{s \rightarrow \mp\infty} f(s) = \mp\infty$, there exists a unique solution $s^{\text{in}}(x) \in \mathbb{R}$ to

$$\chi(s^{\text{in}}(x); x) = s^{\text{in}}(x).$$

Besides, for every $x \in (0, 1)$, we have $0 < s^{\text{in}}(x) < x$ and

$$(s, \chi(s; x)) \in \mathcal{T}, \quad \forall s \in (s^{\text{in}}(x), x).$$

323 By the implicit function theorem we have $s^{\text{in}} \in C^1(\mathbb{R})$ with, for every $x \in \mathbb{R}$,

$$324 \quad (s^{\text{in}})'(x) = \frac{\frac{\partial \chi}{\partial x}(s^{\text{in}}(x); x)}{1 - \frac{\partial \chi}{\partial s}(s^{\text{in}}(x); x)} > 0. \quad (40)$$

325 In particular, the inverse function $(s^{\text{in}})^{-1} : [0, s^{\text{in}}(1)] \rightarrow [0, 1]$ exists. We are going to
 326 show that

$$327 \quad \ell_{s^{\text{in}}(1)}(\tilde{c}) = s^{\text{in}}(\ell_1(g)). \quad (41)$$

Along the characteristics, the solution to (38) satisfies, for $s \in (s^{\text{in}}(x), x)$,

$$\begin{cases} \frac{d}{ds} k_{21}(s, \chi(s; x)) = \frac{-\frac{\partial \lambda_1}{\partial \xi}(\chi(s; x))}{\lambda_2(s)} k_{21}(s, \chi(s; x)) + \frac{-k_{22}(s, \chi(s; x))}{\lambda_2(s)} \tilde{c}(\chi(s; x)), \\ k_{21}(s^{\text{in}}(x), s^{\text{in}}(x)) = \frac{\tilde{c}(s^{\text{in}}(x))}{\lambda_2(s^{\text{in}}(x)) - \lambda_1(s^{\text{in}}(x))}. \end{cases}$$

328 Consequently,

$$329 \quad k_{21}(x, 0) = r(x) \tilde{c}(s^{\text{in}}(x)) + \int_{s^{\text{in}}(x)}^x h(x, \sigma) \tilde{c}(\chi(\sigma; x)) d\sigma, \quad (42)$$

with

$$r(x) = \exp\left(\int_{s^{\text{in}}(x)}^x \frac{-\frac{\partial \lambda_1}{\partial \xi}(\chi(s; x))}{\lambda_2(s)} ds\right) \frac{1}{\lambda_2(s^{\text{in}}(x)) - \lambda_1(s^{\text{in}}(x))},$$

and

$$h(x, \sigma) = \exp\left(\int_{\sigma}^x \frac{-\frac{\partial \lambda_1}{\partial \xi}(\chi(s; x))}{\lambda_2(s)} ds\right) \frac{-k_{22}(\sigma, \chi(\sigma; x))}{\lambda_2(\sigma)}.$$

Using the change of variable $\theta = (s^{\text{in}})^{-1}(\chi(\sigma; x))$, we obtain

$$\frac{1}{r(x)}k_{21}(x, 0) = \tilde{c}(s^{\text{in}}(x)) + \int_0^x \tilde{h}(x, \theta)\tilde{c}(s^{\text{in}}(\theta)) d\theta,$$

with kernel

$$\tilde{h}(x, \theta) = \frac{1}{r(x)}h(x, \chi^{-1}(s^{\text{in}}(\theta); x)) \frac{(s^{\text{in}})'(\theta)}{\frac{\partial \chi}{\partial s}(\chi^{-1}(s^{\text{in}}(\theta); x); x)}.$$

We can check that $\tilde{h} \in L^\infty(\mathcal{T})$ (recall (40)). It follows from the injectivity of Volterra transformations of the second kind that

$$\ell_1(g) = \ell_1(\tilde{c} \circ s^{\text{in}}),$$

330 which is equivalent to (41) since s^{in} is increasing with $s^{\text{in}}(0) = 0$.

3) To conclude the proof, it remains to observe that the solution to the ODE (39) satisfies

$$\phi_1(\chi(s; x)) = \phi_2(x) - \phi_2(s),$$

331 for every $x \in [0, 1]$ and $s \in [s^{\text{in}}(x), x]$. Taking $x = 1$ and $s = s^{\text{in}}(1)$, we see that $s^{\text{in}}(1) = x_\Lambda$
 332 (by uniqueness of the solution to the equation $\phi_1(x_\Lambda) + \phi_2(x_\Lambda) = \phi_2(1)$). Taking then
 333 $x = \ell_1(g)$ and $s = s^{\text{in}}(\ell_1(g)) = \ell_{x_\Lambda}(\tilde{c})$ (recall (41)), we obtain the desired identity (37).

334

□

335 **Remark 4.1.** In the proof of Theorem 1.5, we have not used the apparent freedom for the
 336 boundary data of k_{22} provided by Theorem 2.1.

337 5 Extensions and open problems

338 The results of this paper can be partially extended to systems of more than 2 equations. More
 339 precisely, we can consider the following $n \times n$ systems ($n \geq 2$):

$$340 \left\{ \begin{array}{l} \frac{\partial y_1}{\partial t}(t, x) + \lambda_1(x) \frac{\partial y_1}{\partial x}(t, x) = a(x)y_1(t, x) + B(x)y_+(t, x), \\ \frac{\partial y_+}{\partial t}(t, x) + \Lambda_+(x) \frac{\partial y_+}{\partial x}(t, x) = C(x)y_1(t, x) + D(x)y_+(t, x), \quad t \in (0, +\infty), x \in (0, 1). \\ y_1(t, 1) = u(t), \quad y_+(t, 0) = Qy_1(t, 0), \\ y_1(0, x) = y_1^0(x), \quad y_+(0, x) = y_+^0(x), \end{array} \right. \quad (43)$$

341 In (43), $(y_1(t, \cdot), y_+(t, \cdot)) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is the state at time t , (y_1^0, y_+^0) is the initial data and $u(t) \in \mathbb{R}$
 342 is the control at time t . We assume that we have one negative speed $\lambda_1 \in C^{0,1}([0, 1])$ and $n - 1$
 343 positive speeds $\lambda_2, \dots, \lambda_n \in C^{0,1}([0, 1])$ such that:

$$344 \lambda_1(x) < 0 < \lambda_2(x) < \dots < \lambda_n(x), \quad \forall x \in [0, 1], \quad (44)$$

345 and we use the notation $\Lambda_+ = \text{diag}(\lambda_2, \dots, \lambda_n)$. Finally, $a \in L^\infty(0, 1)$, $B \in L^\infty(0, 1)^{1 \times (n-1)}$,
 346 $C \in L^\infty(0, 1)^{(n-1) \times (n-1)}$, $D \in L^\infty(0, 1)^{(n-1) \times (n-1)}$ couple the equations of the system inside the domain
 347 and the constant matrix $Q \in \mathbb{R}^{n-1}$ couples the equations of the system on the boundary $x = 0$.

Let us now introduce the times defined by

$$T_1(\Lambda) = \int_0^1 \frac{1}{-\lambda_1(\xi)} d\xi, \quad T_i(\Lambda) = \int_0^1 \frac{1}{\lambda_i(\xi)} d\xi, \quad \forall i \in \{2, \dots, n\}.$$

348 Note that $T_n(\Lambda) < \dots < T_2(\Lambda)$ by (44).

349 It was established in [DMVK13] and [HDMVK16, Lemma 3.1] that the system (43) is finite-
350 time stabilizable with settling time T if $T \geq T_{\text{unif}}(\Lambda)$, where $T_{\text{unif}}(\Lambda)$ is still given by (6).

351 Using the backstepping method (see e.g. [HVDKM19, Section 2.2]), it can be shown as before
352 that the system (43) is null controllable in time T (resp. finite-time stabilizable with settling
353 time T) if, and only if, so is the system

$$354 \quad \begin{cases} \frac{\partial \hat{y}_1}{\partial t}(t, x) + \lambda_1(x) \frac{\partial \hat{y}_1}{\partial x}(t, x) = 0, \\ \frac{\partial \hat{y}_+}{\partial t}(t, x) + \Lambda_+(x) \frac{\partial \hat{y}_+}{\partial x}(t, x) = G(x) \hat{y}_1(t, 0), & t \in (0, +\infty), x \in (0, 1), \\ \hat{y}_1(t, 1) = \hat{u}(t), & \hat{y}_+(t, 0) = Q \hat{y}_1(t, 0), \\ \hat{y}_1(0, x) = \hat{y}_1^0(x), & \hat{y}_+(0, x) = \hat{y}_+^0(x), \end{cases} \quad (45)$$

355 for some $G \in L^\infty(0, 1)^{n-1}$ depending on all the parameters $\lambda_1, \Lambda_+, a, B, C, D$ and Q .

356 By mimicking the proof of Theorem 3.1, we can obtain the following result:

357 **Theorem 5.1.** *Let $T > 0$.*

358 (i) *If the system (45) is null controllable in time T , then necessarily*

$$359 \quad T \geq \max \left\{ T_1(\Lambda) + \max_{i \in \{2, \dots, n\}} T(\lambda_i, g_{i-1}, q_{i-1}), \quad T_2(\Lambda) \right\}, \quad (46)$$

where

$$T(\lambda_i, g_{i-1}, q_{i-1}) = \begin{cases} \int_{\ell_1(g_{i-1})}^1 \frac{1}{\lambda_i(\xi)} d\xi & \text{if } q_{i-1} = 0, \\ T_i(\Lambda) & \text{if } q_{i-1} \neq 0. \end{cases}$$

360 (ii) *If the time T satisfies (46), then the system (45) is finite-time stable with settling time T .*

361 However, we are unable so far to deduce from this result some explicit condition for the initial
362 system (43). The main technical problem is that G is heavily coupled on the parameters λ_1, Λ_+ ,
363 a, B, C, D and Q (see e.g. [HVDKM19, Section 2.2]). We leave it as an open problem that could
364 be investigated in future works.

365 Acknowledgements

366 The first author would like to thank Institute of Mathematics in Jagiellonian University for its
367 hospitality. This work was initiated while he was visiting there. This project was supported by
368 National Natural Science Foundation of China (No. 12071258) and the Young Scholars Program
369 of Shandong University (No. 2016WLJH52).

References

- 370 [ADM16] Jean Auriol and Florent Di Meglio, *Minimum time control of heterodirectional*
371 *linear coupled hyperbolic PDEs*, Automatica J. IFAC **71** (2016), 300–307.
- 372 [BC16] Georges Bastin and Jean-Michel Coron, *Stability and boundary stabilization of 1-D*
373 *hyperbolic systems*, Progress in Nonlinear Differential Equations and their Appli-
374 *cations*, vol. 88, Birkhäuser/Springer, [Cham], 2016, Subseries in Control.
- 375 [Bru70] Pavol Brunovský, *A classification of linear controllable systems*, Kybernetika
376 (Prague) **6** (1970), 173–188.
- 377 [CBdN08] Jean-Michel Coron, Georges Bastin, and Brigitte d’Andréa Novel, *Dissipative*
378 *boundary conditions for one-dimensional nonlinear hyperbolic systems*, SIAM J.
379 *Control Optim.* **47** (2008), no. 3, 1460–1498.
- 380 [CHO17] Jean-Michel Coron, Long Hu, and Guillaume Olive, *Finite-time boundary stabi-*
381 *lization of general linear hyperbolic balance laws via Fredholm backstepping trans-*
382 *formation*, Automatica J. IFAC **84** (2017), 95–100.
- 383 [CHOS21] Jean-Michel Coron, Long Hu, Guillaume Olive, and Peipei Shang, *Boundary stabi-*
384 *lization in finite time of one-dimensional linear hyperbolic balance laws with coef-*
385 *ficients depending on time and space*, J. Differential Equations **271** (2021), 1109–
386 1170.
- 387 [CN19a] Jean-Michel Coron and Hoai-Minh Nguyen, *Null-controllability of linear hyperbolic*
388 *systems in one dimensional space*, preprint: [https://hal.archives-ouvertes.](https://hal.archives-ouvertes.fr/hal-02334761)
389 [fr/hal-02334761](https://hal.archives-ouvertes.fr/hal-02334761) (2019).
- 390 [CN19b] ———, *Optimal time for the controllability of linear hyperbolic systems in one-*
391 *dimensional space*, SIAM J. Control Optim. **57** (2019), no. 2, 1127–1156.
- 392 [CN20a] ———, *Finite-time stabilization in optimal time of homogeneous quasilinear hy-*
393 *perbolic systems in one dimensional space*, preprint: [https://arxiv.org/abs/](https://arxiv.org/abs/2005.13269)
394 [2005.13269](https://arxiv.org/abs/2005.13269) (2020).
- 395 [CN20b] ———, *Lyapunov functions and finite time stabilization in optimal time for homo-*
396 *geneous linear and quasilinear hyperbolic systems*, preprint: [https://arxiv.org/](https://arxiv.org/abs/2007.04104)
397 [abs/2007.04104](https://arxiv.org/abs/2007.04104) (2020).
- 398 [CVKB13] Jean-Michel Coron, Rafael Vazquez, Miroslav Krstic, and Georges Bastin, *Local*
399 *exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using back-*
400 *stepping*, SIAM J. Control Optim. **51** (2013), no. 3, 2005–2035.
- 401 [DMVK13] Florent Di Meglio, Rafael Vazquez, and Miroslav Krstic, *Stabilization of a system*
402 *of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input*,
403 *IEEE Trans. Automat. Control* **58** (2013), no. 12, 3097–3111.
- 404 [GL84] J. M. Greenberg and Ta Tsien Li, *The effect of boundary damping for the quasilinear*
405 *wave equation*, J. Differential Equations **52** (1984), no. 1, 66–75.
- 406 [Har02] Philip Hartman, *Ordinary differential equations*, Classics in Applied Mathematics,
407 vol. 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia,
408 PA, 2002, Corrected reprint of the second (1982) edition [Birkhäuser, Boston, MA;
409 MR0658490 (83e:34002)], With a foreword by Peter Bates.
- 410

- 411 [HDM15] Long Hu and Florent Di Meglio, *Finite-time backstepping boundary stabilization of*
412 *3×3 hyperbolic systems*, Proceedings of the European Control Conference (ECC),
413 July 2015, pp. 67–72.
- 414 [HDMVK16] Long Hu, Florent Di Meglio, Rafael Vazquez, and Miroslav Krstic, *Control of*
415 *homodirectional and general heterodirectional linear coupled hyperbolic PDEs*, IEEE
416 Trans. Automat. Control **61** (2016), no. 11, 3301–3314.
- 417 [HO19] Long Hu and Guillaume Olive, *Minimal time for the exact controllability of one-*
418 *dimensional first-order linear hyperbolic systems by one-sided boundary controls*,
419 preprint: <https://arxiv.org/abs/1901.06005v2> (2019).
- 420 [Hoc73] Harry Hochstadt, *Integral equations*, John Wiley & Sons, New York-London-
421 Sydney, 1973, Pure and Applied Mathematics.
- 422 [Hul15] Long Hu, *Sharp time estimates for exact boundary controllability of quasilinear*
423 *hyperbolic systems*, SIAM J. Control Optim. **53** (2015), no. 6, 3383–3410.
- 424 [HVDMK19] Long Hu, Rafael Vazquez, Florent Di Meglio, and Miroslav Krstic, *Boundary ex-*
425 *ponential stabilization of 1-dimensional inhomogeneous quasi-linear hyperbolic sys-*
426 *tems*, SIAM J. Control Optim. **57** (2019), no. 2, 963–998.
- 427 [KS08] Miroslav Krstic and Andrey Smyshlyaev, *Boundary control of PDEs*, Advances
428 in Design and Control, vol. 16, Society for Industrial and Applied Mathematics
429 (SIAM), Philadelphia, PA, 2008, A course on backstepping designs.
- 430 [Li94] Ta Tsien Li, *Global classical solutions for quasilinear hyperbolic systems*, RAM:
431 Research in Applied Mathematics, vol. 32, Masson, Paris; John Wiley & Sons,
432 Ltd., Chichester, 1994.
- 433 [Li10] Tatsien Li, *Controllability and observability for quasilinear hyperbolic systems*,
434 AIMS Series on Applied Mathematics, vol. 3, American Institute of Mathemat-
435 ical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.
- 436 [Mik78] Jan Mikusiński, *The Bochner integral*, Birkhäuser Verlag, Basel-Stuttgart, 1978,
437 Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften,
438 Mathematische Reihe, Band 55.
- 439 [Qin85] Tie Hu Qin, *Global smooth solutions of dissipative boundary value problems for first*
440 *order quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B **6** (1985), no. 3,
441 289–298, A Chinese summary appears in Chinese Ann. Math. Ser. A **6** (1985), no.
442 4, 514.
- 443 [Rud91] Walter Rudin, *Functional analysis*, second ed., International Series in Pure and
444 Applied Mathematics, McGraw-Hill, Inc., New York, 1991.
- 445 [Rus78a] David L. Russell, *Canonical forms and spectral determination for a class of hyper-*
446 *bolic distributed parameter control systems*, J. Math. Anal. Appl. **62** (1978), no. 1,
447 186–225.
- 448 [Rus78b] ———, *Controllability and stabilizability theory for linear partial differential equa-*
449 *tions: recent progress and open questions*, SIAM Rev. **20** (1978), no. 4, 639–739.
- 450 [Tit26] E. C. Titchmarsh, *The Zeros of Certain Integral Functions*, Proc. London Math.
451 Soc. (2) **25** (1926), 283–302.

452 [Wec82] N. Weck, *A remark on controllability for symmetric hyperbolic systems in one space*
453 *dimension*, SIAM J. Control Optim. **20** (1982), no. 1, 1–8.