Cumulated Effects in Learning

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Positioning of this talk

- Starting point: a game theory result, formally verified in Coq
- Methodology: probabilistic analysis of entire classes of games
- Objective: discuss implications of this result, possible generalizations and connection with learning

Outline

1. A Coq theory of bool. games with random formulas as payoff functions
2. Discussion on modeling cumulative effects in learning

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$\rightsquigarrow$ Game represented by a Boolean function $F : \underbrace{\mathbf{2}^k \times \mathbf{2}^{n-k}}_{\simeq\, \mathbf{2}^n} \to \mathbf{2}$

- Alice wins (with strat. a) against Bob (with strat. b) iff $F(a,b) = 1$

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- The size of the considered class (number of n-var. Boolean functions) grows very fast (2^{2^n}) $\rightsquigarrow$ exhaustive computation does not scale
- Derive symbolic results (that can then be numerically evaluated)

Overview of the library

- Coq library "`RandBoolGames`"
- Based on SSReflect/MathComp (`fintype`, `finfun`, `finset`, `bigop`) as well as on the `infotheo` library [Affeldt et al.]
- 3.1k lines of Coq code – automation: introduce a new tactic "`under`"
- https://sourcesup.renater.fr/coq-bool-games/

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- First step: consider any $\mathbb{P} : \mathcal{S} \to [0, 1]$

First result

Theorem (Pr_ex_winA)

For any finite probability space $(\Omega, \mathcal{S}, \mathbb{P})$, the probability that there exists some strategy $a = (a_1, \ldots, a_k)$ of A that is winning satisfies:

$$\mathbb{P}(\exists a.\, \mathrm{win}_A(a)) = \sum_{m=1}^{2^k} (-1)^{m-1} \sum_{\substack{J \subseteq \mathbf{2}^k \\ \mathrm{Card}\, J = m}} \mathbb{P}\left(\bigcap_{a \in J} W_a \right),$$

denoting for any $a \in \mathbf{2}^k$,

$$\mathrm{win}_A(a) := \forall b \in \mathbf{2}^{n-k}.\ F(a,b) = 1$$
$$\omega_a := \{v \in \mathbf{2}^n \mid v_1 = a_1 \wedge \cdots \wedge v_k = a_k\} \in \Omega$$
$$W_a := \{\omega \in \Omega \mid \omega_a \subseteq \omega\} \in \mathcal{S}.$$

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Remark

This setting subsumes the simpler case where all functions have the same elementary probability (just take $p = \frac{1}{2}$)

Results I

Theorem (Pr_ex_winA_Bern)

For all $p \in [0,1]$ and for all integers n, k, the probability that player A has a winning strategy is:

$$\mathbb{P}_{n;\,p}(\exists a \in \mathbf{2}^k.\ \mathrm{win}_A(a)) = 1 - \left(1 - p^{2^{n-k}}\right)^{2^k}.$$

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Proba. that a winning strategy exists neither for A nor for B ($n = 10$)

p\k	1	2	3	4	5	6	7	8	9
0.25	1.52e-184	5.11e-43	1.37e-6	**0.525**	**0.997**	**≈1**	**0.998**	0.367	4.46e-15
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The order of moves matters: no winning strat $\wedge$ B plays first $\Rightarrow$ A can always win!
But what happens if player A just knows a few bits from player B's strategy?

Results II

Theorem (Pr_ex_winA_knowing_Bern)

For all $p \in [0,1]$ and for all integers n, k, s satisfying $0 \leq s \leq n-k \leq n$, the probability of guaranteed win for player A knowing s choices of player B among his $n-k$ variables is:

$$\mathbb{P}_{n;p}\left(\forall b_{1:s} \in \mathbf{2}^s.\ \exists a \in \mathbf{2}^k.\ \mathrm{win}_A(a \mid b_{1:s})\right) = \underbrace{\left(1-\left(1-p^{2^{n-k-s}}\right)^{2^k}\right)^{2^s}}_{g(s)}.$$

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Theorem (phi_ineq)

For any $p \in (0,1)$, $n,k \in \mathbb{N}^*$ *such that* $0 \leq s \leq n-k$, *if*

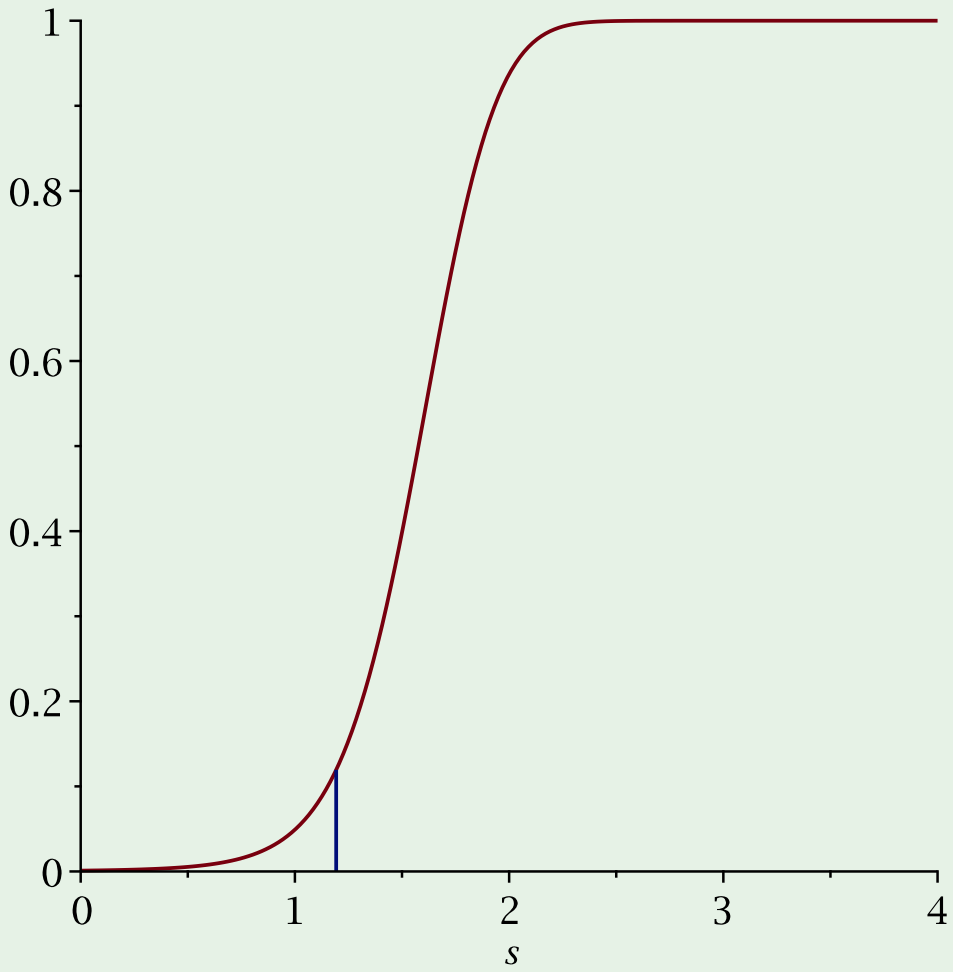
$$s \leq (n-k) - \log_2(k+1) + \log_2(|\log_2 p|) \tag{1}$$

then

$$\phi(s) := g(s) - g(0) > \left(2^{(k-1)2^s} - 2^k\right) p^{2^{n-k}}.$$

Growth rate of guaranteed win w.r.t knowledge on player B

Example (Graph of $g(s)$ for parameters $p = \frac{1}{2}$, $n = 10$, and $k = 6$)



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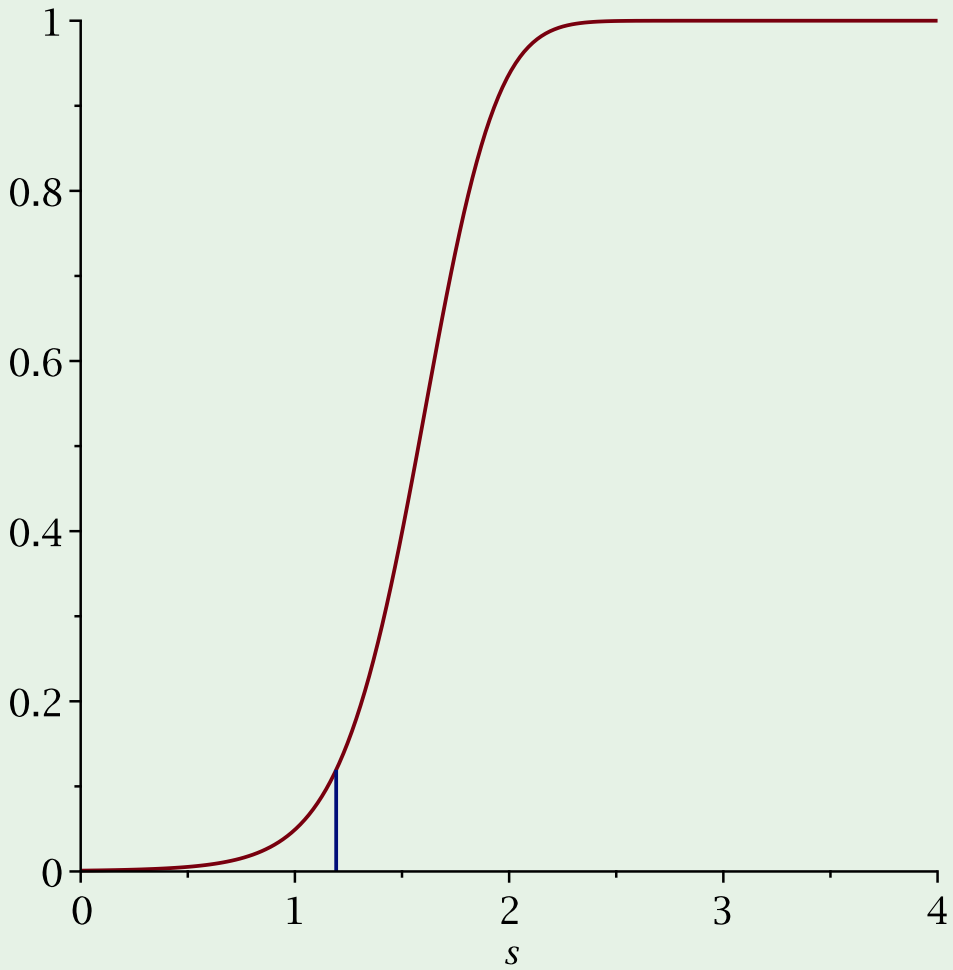

The blue vertical line indicates the largest $s \in \mathbb{R}$ satisfying (1)

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$\rightsquigarrow$ can this be related to other phenomena as well as to learning?

Can we learn Boolean functions?

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PAC learning [Valiant]

- $X = \mathbf{2}^n$, $\mathcal{C} = (X \to \mathbf{2})$, $\mathcal{D}$ distribution over X
- unknown $c \in \mathcal{C}$ (target concept)
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Active learning – membership queries – ACRE learning [Lowd et al.]

- $X = \mathbf{2}^n$, $\mathcal{C} = (X \to \mathbf{2})$
- access to black-box fun. $c \in \mathcal{C}$, $a : X \to \mathbb{R}_+$ (adversarial cost fun.) and $x_+, x_- \in X$ s.t. $c(x_+) = 1$ and $c(x_-) = 0$
- (efficiently) find $x_0 \in X$ s.t. $c(x_0) = 0$ and $a(x_0) = \min_{x|c(x)=0} a(x)$

Possible extensions of our approach

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  - Player A plays repeatedly against player B with random $F \in \mathrm{BG}(n;k)$
    1. Players A and B choose a strategy (a, b) and evaluate $F(a, b)$
    2. Player A memorizes (part of) the profile strategy and the outcome
    3. Repeat at step 1 with (another) function F and try to get $F(a, b) = 1$

    $\rightsquigarrow$ what are the conditions on info./choices to make the scenario realistic?

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- This will increase their probability to win. The win will then let them pass to the next level, and so on.

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- If one aims to get a distributed ITP platform that scales as much as S.O., the "social" interactions (discussions/reward/reputation/etc.) as well as the user-friendliness of the interface are maybe as important as the "logical" aspects.

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Perspectives

- Refine the model sketched in this talk $\rightsquigarrow$ amenable to formal methods?
- Consider other distributions/methods for constructing Boolean games
- Consider more general settings such as n-player Boolean games
- Strengthen the "`under`" tactic and propose to integrate it in MathComp