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Integral Eisenstein cocycles on $\mathbf{GL}_n$, I: Sczech's cocycle and p-adic L-functions of Totally Real Fields

Pierre Charollois
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September 3, 2013

Abstract

We define an integral version of Sczech's Eisenstein cocycle on $\mathbf{GL}_n$ by smoothing at a prime ℓ. As a result we obtain a new proof of the integrality of the values at nonpositive integers of the smoothed partial zeta functions associated to ray class extensions of totally real fields. We also obtain a new construction of the p-adic L-functions associated to these extensions. Our cohomological construction allows for a study of the leading term of these p-adic L-functions at $s = 0$. We apply Spiess's formalism to prove that the order of vanishing at $s = 0$ is at least equal to the expected one, as conjectured by Gross. This result was already known from Wiles' proof of the Iwasawa Main Conjecture.

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Introduction

Let F be a totally real field of degree n, and let $\mathfrak{f}$ be an integral ideal of F. For a fractional ideal $\mathfrak{a}$ of F relatively prime to the conductor $\mathfrak{f}$, consider the partial zeta function

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, s) = \sum_{\mathfrak{b} \sim \mathfrak{a}} \frac{1}{\mathrm{N}\mathfrak{b}^s}, \qquad \mathrm{Re}(s) > 1. \tag{1}$$

Here the sum ranges over integral ideals $\mathfrak{b} \subset F$ equivalent to $\mathfrak{a}$ in the narrow ray class group modulo $\mathfrak{f}$, which we denote $G_{\mathfrak{f}}$. A classical result of Siegel and Klingen states that the partial zeta functions $\zeta_{\mathfrak{f}}(\mathfrak{a}, s)$, which may be extended to meromorphic functions on the complex plane, assume rational values at nonpositive integers s. Siegel proved this fact by realizing these special values as the constant terms of certain Eisenstein series on the Hilbert modular group associated to F. The rationality of the constant terms follows from the rationality of the other Fourier coefficients, which have a simple form.

Shintani gave an alternate proof of the Siegel–Klingen result using a "geometry of numbers" approach. Shintani fixed an isomorphism $F \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathbf{R}^n$, and considered a fundamental domain D for the action of the group of totally positive units in F congruent to 1 modulo $\mathfrak{f}$ on the totally positive orthant of $\mathbf{R}^n$. The partial zeta functions of F could then be expressed as a sum indexed by the points of D contained in various lattices in $\mathbf{R}^n$. Shintani evaluated these sums using standard techniques from complex analysis and expressed them explicitly in terms of sums of products of Bernoulli polynomials.

In 1993, Sczech gave yet another proof of the Siegel–Klingen rationality theorem. He defined an "Eisenstein" cocycle Ψ on $\mathbf{GL}_n(\mathbf{Q})$ valued in a space of $\mathbf{Q}$-valued distributions denoted $M_{\mathbf{Q}}$. He then showed that the cohomology class $[\Psi] \in H^{n-1}(\mathbf{GL}_n(\mathbf{Q}), M_{\mathbf{Q}})$ could be paired with certain classes in a dual homology group to yield the special values of all totally real fields F of degree n at nonpositive integers, thereby demonstrating their rationality.

Each of these proofs of the Siegel–Klingen rationality theorem bears an integral refinement. Deligne and Ribet gave such a refinement of Siegel's method following an idea initiated by Serre. They constructed a model over $\mathbf{Z}$ of the relevant Hilbert modular scheme, and proved that its fibers in characteristic p are geometrically irreducible. Meanwhile, Barsky and Pi. Cassou-Noguès proved an integral refinement of Shintani's formulas and interpreted these results in terms of p-adic measures.

The first goal of the present paper is to provide an integral refinement of Sczech's cocycle Ψ. We introduce a "smoothing" operation with respect to a prime ℓ, and use it to define a cocycle Ψ_ℓ that satisfies an important integrality property (see Theorem 4 below for a precise statement). As an application of our results, we give new proofs of the following two celebrated theorems of Deligne–Ribet and Cassou-Noguès.

Theorem 1. *Let $\mathfrak{c}$ be an integral ideal of F relatively prime to $\mathfrak{f}$ with prime norm ℓ. The smoothed zeta function*

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s) = \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, s) - \mathrm{N}\mathfrak{c}^{1-s}\zeta_{\mathfrak{f}}(\mathfrak{a}, s) \tag{2}$$

assumes values in $\mathbf{Z}[1/\ell]$ at nonpositive integers s.[†]

Our integrality results further allow for a new construction of the Deligne–Ribet–Cassou-Noguès p-adic zeta functions $\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s)$ interpolating the classical zeta values $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s)$. Define $\zeta^*_{\mathfrak{f}}(\mathfrak{a}, s)$ as in (1), but with the sum restricted to ideals $\mathfrak{b}$ relatively prime to p; define $\zeta^*_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s)$ from $\zeta^*_{\mathfrak{f}}(\mathfrak{a}, s)$ as in (2). Let $\mathcal{W}$ denote the *weight space* of continuous homomorphisms from $\mathbf{Z}_p^*$ to $\mathbf{C}_p^*$, with $k \in \mathbf{Z}$ embedded as $x \mapsto x^k$.

Theorem 2. *Let $\mathfrak{c}$ be an integral ideal of F relatively prime to $\mathfrak{f}p$ with prime norm ℓ. There exists a unique $\mathbf{Z}_p$-valued analytic function $\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s)$ of the variable $s \in \mathcal{W}$ such that*

$$\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, -k) = \zeta^*_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k)$$

for all nonnegative integers k.

Our construction of the p-adic zeta functions of totally real fields allows us to embark on a new study of the behavior of the leading terms of these functions at $s = 0$. In order to state our main result, it is convenient to work with the p-adic L-functions associated to characters rather than the p-adic zeta functions associated to ideal classes. To this end, let $\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^*$ be a totally odd finite order character with conductor $\mathfrak{f}$. We fix

[†]See Remark 3.2 for a discussion of the condition that $\mathfrak{c}$ has prime norm. Also, we note that Theorem 1 has the corollary that the "twice smoothed" zeta function

$$\begin{aligned}\zeta_{\mathfrak{f},\mathfrak{c},\mathfrak{b}}(\mathfrak{a}, s) &= \zeta_{\mathfrak{f},\mathfrak{b}}(\mathfrak{a}\mathfrak{c}, s) - \mathrm{N}\mathfrak{c}^{1-s}\zeta_{\mathfrak{f},\mathfrak{b}}(\mathfrak{a}, s)\\ &= \zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}\mathfrak{b}, s) - \mathrm{N}\mathfrak{b}^{1-s}\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s)\end{aligned}$$

assumes *integer* values at nonpositive integers s when $(\mathrm{N}\mathfrak{b}, \mathrm{N}\mathfrak{c}) = 1$. Gross has recently provided an interpretation of these integers in terms of dimensions of certain spaces of automorphic forms [G2].

embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, so that χ can be viewed as taking values in $\mathbf{C}$ or $\overline{\mathbf{Q}}_p$. Let[‡]

$$\omega : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mu_{p-1} \subset \overline{\mathbf{Q}}^*$$

denote the Teichmüller character. There is a p-adic L-function $L_{\mathfrak{c},p}(\chi\omega, s)\colon \mathbf{Z}_p \longrightarrow \mathbf{C}_p^*$ associated to the totally even character $\chi\omega$, given by

$$L_{\mathfrak{c},p}(\chi\omega, s) := \sum_{\mathfrak{a}\in G_{\mathfrak{f}}} \chi(\mathfrak{a}\mathfrak{c})\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, \langle\cdot\rangle^s),$$

where $\langle x\rangle = x/\omega(x)$ for $x \in \mathbf{Z}_p^*$. The function $L_{\mathfrak{c},p}(\chi\omega, s)$ satisfies the interpolation property

$$\begin{aligned} L_{\mathfrak{c},p}(\chi\omega, -k) &= L^*_{\mathfrak{c}}(\chi\omega^{-k}, -k) \\ &:= L^*(\chi\omega^{-k}, -k)(1 - \chi\omega^{-k}(\mathfrak{c})\mathrm{N}\mathfrak{c}^{1+k}) \end{aligned}$$

for integers $k \geq 0$, where $L^*(\chi, s)$ denotes the classical L-function with Euler factors at the primes dividing p removed.

Let r_χ denote the number of primes $\mathfrak{p}$ of F above p such that $\chi(\mathfrak{p}) = 1$. It is well-known that

$$\mathrm{ord}_{s=0}\, L^*_{\mathfrak{c}}(\chi, s) = \mathrm{ord}_{s=0}\, L^*(\chi, s) = r_\chi$$

(see [T, 2.6]). In [G], Gross proposed the following:

Conjecture 1 (Gross)**.** *We have*

$$\mathrm{ord}_{s=0}\, L_{\mathfrak{c},p}(\chi\omega, s) = r_\chi.$$

Combining our cohomological construction of the p-adic L-function with Spiess's formalism (see §5), we prove the following partial result towards Gross's conjecture:

Theorem 3. *We have*

$$\mathrm{ord}_{s=0}\, L_{\mathfrak{c},p}(\chi\omega, s) \geq r_\chi.$$

The result of Theorem 3 was already known from Wiles' proof of the Iwasawa Main Conjecture under the auxiliary assumption

$$F_\infty \cap H = F,$$

where H denotes the fixed field of χ and F_∞ denotes the cyclotomic $\mathbf{Z}_p$-extension of F. This assumption (namely, that χ has "type S" in Greenberg's terminology) was removed by Snaith in [Sn, Theorem 6.2.5] using Brauer induction (see also [BG, pp. 165-166] for further discussion).

Our method contrasts with that of Wiles in that it is purely analytic; we calculate the first $r_\chi - 1$ derivatives of $L_{\mathfrak{c},p}(\chi\omega, s)$ at $s = 0$ directly and show that they vanish. Spiess proved Theorem 3 as well using his formalism [Sp2]. His cohomology classes are defined

[‡]As usual, replace μ_{p-1} by $\{\pm 1\}$ when $p = 2$.

using Shintani's method. In [CDG] we provide a direct comparison between the cocycles defined using the methods of Sczech and Shintani.

We conclude the introduction by recalling Sczech's method, which is central to this article, and describing our integral refinement. Let $\mathcal{P} = \mathbf{Q}[X_1, \ldots, X_n]$. We denote by $\mathcal{Q}$ a certain space of tuples of linear forms on $\mathbf{R}^n$, defined precisely in Section 1.1. The spaces $\mathcal{P}$ and $\mathcal{Q}$ are endowed with a left action of $\Gamma = \mathbf{GL}_n(\mathbf{Q})$ given by $(AP)(X) = P(XA)$. Let $M_{\mathbf{Q}}$ denote the $\mathbf{Q}$-vector space of functions

$$\phi : \mathcal{P} \times \mathcal{Q} \times (\mathbf{Q}/\mathbf{Z})^n \longrightarrow \mathbf{Q}$$

that are $\mathbf{Q}$-linear in the first variable and satisfy the distribution relation

$$\phi(P, Q, v) = \operatorname{sgn}(\lambda)^n \sum_{\substack{w \in (\mathbf{Q}/\mathbf{Z})^n \\ \lambda w = v}} \phi(\lambda^{\deg P} P, \lambda^{-1} Q, w) \tag{3}$$

for all nonzero integers λ, when $P \in \mathcal{P}$ is homogeneous.

We view the elements of $(\mathbf{Q}/\mathbf{Z})^n$ as column vectors and define a left Γ-action on $M_{\mathbf{Q}}$ as follows. Given $\gamma \in \Gamma$, choose a nonzero scalar multiple $A = \lambda\gamma$ with $\lambda \in \mathbf{Z}$ such that $A \in M_n(\mathbf{Z})$. For $f \in M_{\mathbf{Q}}$, define

$$(\gamma f)(P, Q, v) = \operatorname{sgn}(\det(A)) \sum_{r \in \mathbf{Z}^n / A\mathbf{Z}^n} f(A^t P, A^{-1} Q, A^{-1}(r + v)). \tag{4}$$

The distribution relation (3) implies that (4) does not depend on the auxiliary choice of λ.

Sczech defined a homogeneous cocycle

$$\Psi \in Z^{n-1}(\Gamma, M_{\mathbf{Q}}) \subset C^{n-1}(\Gamma, M_{\mathbf{Q}}) = \operatorname{Hom}_\Gamma(\mathbf{Z}[\Gamma^n], M_{\mathbf{Q}})$$

called the Eisenstein cocycle, representing a class

$$[\Psi] \in H^{n-1}(\Gamma, M_{\mathbf{Q}}).$$

We recall this definition precisely in Section 1. The values at nonpositive integers of the zeta functions of all totally real fields F of degree n can be obtained from certain specializations of $[\Psi]$ as follows.

Fix a conductor $\mathfrak{f}$ and an integral ideal $\mathfrak{a}$ as above. Associated to $F, \mathfrak{f}, \mathfrak{a}$, and a nonnegative integer k, we define a certain homology class

$$[\mathfrak{Z}_{\mathfrak{a},\mathfrak{f},k}] \in H_{n-1}(\Gamma, M_{\mathbf{Q}}^\vee),$$

where $M_{\mathbf{Q}}$ denotes the $\mathbf{Q}$-linear dual of $M_{\mathbf{Q}}$. Fix a $\mathbf{Z}$-basis $\{w_1, \ldots, w_n\}$ for $\mathfrak{a}^{-1}\mathfrak{f}$. Let $P \in \mathbf{Z}[X_1, \ldots, X_n]$ denote the homogeneous polynomial of degree n given up to scalar by the norm:

$$P(X_1, \ldots, X_n) = \mathrm{N}(\mathfrak{a})\mathrm{N}(w_1 X_1 + \cdots + w_n X_n). \tag{5}$$

Let $Q = (Q_1, \ldots, Q_n)$ be the n-tuple of linear forms given by

$$Q_i = \tau_i(w_1^*)X_1 + \cdots + \tau_i(w_n^*)X_n, \tag{6}$$

where $\{w_1^*, \ldots, w_n^*\}$ denotes the dual basis with respect to the trace form on F, and the τ_i for $i = 1, \ldots, n$ denote the embeddings $F \hookrightarrow \mathbf{R}$. Let

$$v = (\mathrm{Tr}(w_1^*), \ldots, \mathrm{Tr}(w_n^*)). \tag{7}$$

Denote by $f_k = f_{P,Q,v}$ the element of $M_{\mathbf{Q}}^\vee$ defined by evaluation at (P^k, Q, v):

$$f_k(\phi) := \phi(P^k, Q, v).$$

Finally, we define elements $A_1, \ldots, A_{n-1} \in \Gamma$ by considering the action of a basis of totally positive units of F congruent to 1 modulo $\mathfrak{f}$ via multiplication on $\mathfrak{a}^{-1}\mathfrak{f}$, in terms of the basis $\{w_i\}$ for $\mathfrak{a}^{-1}\mathfrak{f}$. Homogenizing and symmetrizing the tuple $(A_1, \ldots, A_{n-1})$ yields a certain homogeneous $(n-1)$-chain $A \in \mathbf{Z}[\Gamma^n]$ (see (56) for the precise formula).

We then define

$$\mathfrak{Z}_{\mathfrak{a},\mathfrak{f},k} = A \otimes f_k \in C_{n-1}(\Gamma, M_{\mathbf{Q}}^\vee) = \mathbf{Z}[\Gamma^n] \otimes_\Gamma M_{\mathbf{Q}}^\vee.$$

Using the definition of A in (56), it follows from the fact that the A_i commute and that f_k is invariant under the A_i that $\mathfrak{Z}_{\mathfrak{a},\mathfrak{f},k}$ is in fact an $(n-1)$-cycle. The homology class that it represents depends only on $\mathfrak{f}$, $\mathfrak{a}$, and k, and not on any other choices made.

The cap product yields a canonical pairing

$$\langle \cdot, \cdot \rangle : H^{n-1}(\Gamma, M_{\mathbf{Q}}) \times H_{n-1}(\Gamma, M_{\mathbf{Q}}^\vee) \longrightarrow \mathbf{Q}$$

given by $\langle [\Psi], [A \otimes f] \rangle = f(\Psi(A))$ and extended by linearity. Sczech proved the formula

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, -k) = \langle [\Psi], [\mathfrak{Z}_{\mathfrak{a},\mathfrak{f},k}] \rangle \in \mathbf{Q} \tag{8}$$

for integers $k \geq 0$, thereby completing his proof of the Siegel–Klingen rationality theorem.

In this paper, we fix a prime ℓ and consider the congruence subgroup $\Gamma_\ell \subset \Gamma \cap \mathbf{GL}_n(\mathbf{Z}_\ell)$ consisting of matrices whose first column has all elements but the first divisible by ℓ. We define a cocycle $\Psi_\ell^+ \in Z^{n-1}(\Gamma_\ell, M_{\mathbf{Q}})$ derived from Sczech's Ψ by smoothing at the prime ℓ (see (24) for a precise formula). We also define a cocycle Ψ_ℓ refining Ψ_ℓ^+ from which Ψ_ℓ^+ can be recovered by projecting on to the subspace invariant under the action of multiplication by -1 on $\mathcal{Q}$. Our key result is the following.

Theorem 4. *The cocycle Ψ_ℓ takes values in the $\mathbf{Z}[\frac{1}{\ell}][\Gamma_\ell]$-submodule $M_\ell \subset M_{\mathbf{Q}}$ consisting of distributions ϕ such that $\phi(P, Q, v) \in \frac{1}{m}\mathbf{Z}[\frac{1}{\ell}]$ when $P \in \mathbf{Z}[\frac{1}{\ell}][X_1, \ldots, X_n]$ is homogeneous and has the property*

$$P(v + \tfrac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1}) \subset \mathbf{Z}[\tfrac{1}{\ell}], \tag{9}$$

and $Q \in \mathcal{Q}$ is an m-tuple of linear forms. The cocycle Ψ_ℓ^+ takes values in $\frac{1}{2}M_\ell$.

In particular, taking $P = 1$, we find that $\Psi_\ell(A, 1, Q, v) \in \frac{1}{m}\mathbf{Z}[1/\ell]$ for all A, Q, and v. This integrality property of our cocycle Ψ_ℓ lies in sharp contrast to Sczech's Ψ, which assumes fractional values with p-adically unbounded denominator for each p as v varies (with $P = 1$ and fixed A, Q).

Now let F be as above, and let $\mathfrak{c}$ be a prime ideal of F with norm ℓ. Using Sczech's formula (8), we prove that certain specializations of the class Ψ_ℓ^+ yield the smoothed partial zeta functions of F at nonpositive integers:

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) = \langle [\Psi_\ell^+], [\mathfrak{Z}_{\mathfrak{a},\mathfrak{f},k,\ell}] \rangle \tag{10}$$

where $[\mathfrak{Z}_{\mathfrak{a},\mathfrak{f},k,\ell}] \in H_{n-1}(\Gamma_\ell, M_\ell^\vee)$ is defined similarly to $\mathfrak{Z}_{\mathfrak{a},\mathfrak{f},k}$, but slightly modified to account for the ℓ-smoothing (see (50)–(56) in Section 3 for the precise definition). Here $M_\ell^\vee$ denotes the $\mathbf{Z}[\frac{1}{\ell}]$-dual of M_ℓ. Since $H^{n-1}(\Gamma_\ell, M_\ell)$ and $H_{n-1}(\Gamma_\ell, M_\ell^\vee)$ pair to $\mathbf{Z}[\frac{1}{\ell}]$, we arrive at our proof of Theorem 1.[1]

We prove Theorem 2 by interpreting the cocycle Ψ_ℓ in terms of p-adic measures and thereby defining a certain measure-valued cocycle μ_ℓ. Using the cohomology class $[\mu_\ell]$ and applying equation (10), we construct the p-adic zeta functions $\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s)$ with the desired interpolation property. Theorem 3 is proven by using our cohomological construction of the p-adic L-function to recognize the values $L_{\mathfrak{c},p}^{(k)}(\chi\omega, 0)$ for nonnegative integers k as the pairing of $[\mu_\ell]$ with certain homology classes denoted $[\mathfrak{L}_k]$, and applying results of Spiess (Theorems 5.1 and 5.2) that implies that these homology classes vanish for $k < r_\chi$.

The fact that the p-adic zeta functions of all totally real fields of degree n that contain a prime of norm ℓ arise as the specializations of a single cohomology class $[\Psi_\ell]$ is striking. It seems promising to study these p-adic zeta functions by taking certain *other* specializations of the same class. As an example of this phenomenon, we hope to study Gross's conjectural formula for the leading term $L_{\mathfrak{c},p}^{(r_\chi)}(\chi\omega, 0)$ in future work, building on our prior investigations ([DD], [Das]). Also in future work ([CDG]), we will demonstrate how to provide an alternate construction of the Eisenstein class $[\Psi_\ell]$ using Shintani's method, following prior works by Solomon, Hill, Colmez, and the second author.

This paper is organized as follows. In Section 1, we give the precise definition of Sczech's cocycle Ψ and recall the formula for Ψ in terms of Dedekind sums derived in [GS]. In Section 2 we define our smoothed cocycle Ψ_ℓ^+ and its refinement Ψ_ℓ. We prove Theorem 4, the key integrality result concerning Ψ_ℓ and the technical heart of the paper. In Section 3 we combine Theorem 4 with a suitable generalization of Sczech's formula (8) to prove Theorem 1. In Section 4, we interpret our construction in terms of p-adic measures, and thereby prove Theorem 2. We conclude in Section 5 by proving Theorem 3 using Spiess' results.

[1] In fact, since the cocycle Ψ_ℓ^+ defined from Sczech's method takes values in $\frac{1}{2}M_\ell$, and since the Q defined in (6) and used by Sczech is an n-tuple, the results of this paper prove that the value $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s)$ lies in $\frac{1}{2n}\mathbf{Z}[1/\ell]$ for nonpositive integers s. The factor $1/(2n)$ can be eliminated by considering the refined cocycle Ψ_ℓ and proving that one need only consider an individual linear form Q_i rather than the entire tuple Q. These aspects are studied in [CDG]; see Remark 2.6 and Section 3 below for further details.

1 The Eisenstein cocycle

The goal of this section is to recall the definition of Sczech's cocycle Ψ and some of its salient properties.

1.1 Sczech's Eisenstein cocycle for $\mathbf{GL}_n(\mathbf{Q})$

We begin by recalling Sczech's Eisenstein cocycle for $\Gamma = \mathbf{GL}_n(\mathbf{Q})$. See [Sc], [GS, Section 6], or [CGS] Section 2 for a more detailed exposition.

Let $A = (A_1, \ldots, A_n) \in \Gamma^n$ be an n-tuple of matrices. Fix $x \in \mathbf{R}^n - \{0\}$. For each matrix A_i, let σ_i denote the first (i.e. leftmost) column of A_i such that $\langle x, \sigma_i \rangle \neq 0$. Denote by $\sigma = (\sigma_{ij})$ the square matrix with columns σ_i, for $1 \leq i \leq n$, and define

$$\psi_A(x) = \frac{\det(\sigma)}{\langle x, \sigma_1 \rangle \cdots \langle x, \sigma_n \rangle}.$$

More generally, for any homogeneous polynomial $P(X_1, \ldots, X_n)$, we consider the partial differential operator $P(-\partial_{x_1}, \ldots, -\partial_{x_n})$ and define the function

$$\begin{aligned} \psi_A(P, x) &= P(-\partial_{x_1}, \ldots, -\partial_{x_n})\psi_A(x) \\ &= \det(\sigma) \sum_r P_r(\sigma) \prod_{j=1}^n \frac{1}{\langle x, \sigma_j \rangle^{1+r_j}}, \end{aligned} \tag{11}$$

where $r = (r_1, \ldots, r_n)$ runs over all partitions of $\deg(P)$ into nonnegative integers r_j, and $P_r(\sigma)$ is the homogeneous polynomial in the σ_{ij} satisfying the relation

$$P(X\sigma^t) = \sum_r P_r(\sigma) \frac{X_1^{r_1} \cdots X_n^{r_n}}{r_1! \cdots r_n!}. \tag{12}$$

The Eisenstein cocycle is essentially given by summing the value of $\psi_A(P, x)$ over all $x \in \mathbf{Z}^n - \{0\}$:

$$\text{“ } \Psi(A, P, v) = (2\pi i)^{-n-\deg(P)} \sum_{x \in \mathbf{Z}^n - \{0\}} \mathbf{e}(\langle x, v \rangle) \psi_A(P, x). \text{ ”} \tag{13}$$

The sum in (13) converges only conditionally, so to make sense of it one uses Sczech's Q-summation trick. To this end we fix a family of m linear forms $Q_1, \ldots, Q_m$ on $\mathbf{R}^n$ such that each form Q_i is nonvanishing on $\mathbf{Q}^n - \{0\}$. Let $\mathcal{Q}$ denote the set of such m-tuples $Q = (Q_1, \ldots, Q_m)$ of linear forms. We view each Q_i as a row vector, and for any row vector $x \in \mathbf{R}^n$ we adopt the notation

$$Q_i(x) = Q_i x^t = \sum_{j=1}^n Q_{ij} x_j, \qquad Q(x) = \prod_{i=1}^m Q_i(x). \tag{14}$$

We can identify Q with an $m \times n$ matrix with real rows Q_i. The set $\mathcal{Q}$ is endowed with a left action of Γ described in terms of matrices by $AQ \leftrightarrow (Q_{ij})A^t$. The action with respect to the corresponding functions on row vectors is given by $AQ(x) = Q(xA)$.

Given $Q \in \mathcal{Q}$, and a sequence $a(x)$ indexed by a subset L of a lattice in $\mathbf{R}^n$, the Q-summation of $a(x)$ over L is defined by

$$\sum_{x \in L} a(x)|_Q = \lim_{t \to \infty} \sum_{\substack{x \in L \\ |Q(x)| < t}} a(x), \tag{15}$$

under the assumption that the sum of $a(x)$ over the $x \in L$ such that $|Q(x)| < t$ converges absolutely for all t, and that the limit in (15) exists. Sczech proved that the Q-summation

$$\Psi(A, P, Q, v) = (2\pi i)^{-n-\deg(P)} \sum_{x \in \mathbf{Z}^n - \{0\}} \mathbf{e}(\langle x, v \rangle) \psi_A(P, x)\,|_Q \tag{16}$$

exists.

Let us recall some notation from the Introduction. Define the $\mathbf{Q}[\Gamma]$-module $M_{\mathbf{Q}}$ to be the space of functions

$$\phi : \mathcal{P} \times \mathcal{Q} \times \mathbf{Q}^n/\mathbf{Z}^n \longrightarrow \mathbf{Q}$$

that are $\mathbf{Q}$-linear in the first variable and satisfy the distribution relation

$$\phi(P, Q, v) = \operatorname{sgn}(\lambda)^n \sum_{\substack{w \in (\mathbf{Q}/\mathbf{Z})^n \\ \lambda w = v}} \phi(\lambda^{\deg P} P, \lambda^{-1} Q, w)$$

for all nonzero integers λ, when $P \in \mathcal{P}$ is homogeneous. The Γ-action on $M_{\mathbf{Q}}$ is given in equation (4). Sczech proved:

Theorem 1.1. *The map $A \mapsto \Psi(A, \cdot, \cdot, \cdot)$ defines a $M_{\mathbf{Q}}$-valued homogenous $(n-1)$-cocycle on Γ. Moreover, it represents a non-trivial cohomology class in $H^{n-1}(\Gamma, M_{\mathbf{Q}})$.*

This is [Sc, Theorem 4], restricted to $u = 0$.

1.2 Decomposition

For future calculations, it is convenient to decompose the sum in (16) according to the various matrices σ that may occur. To this end, for each $d = (d_1, \ldots, d_n) \in \mathbf{Z}^n$ such that $1 \le d_i \le n$, let $\sigma(d)$ denote the $n \times n$ matrix whose ith column is the d_ith column of A_i. Let $X(d) \subset \mathbf{R}^n - \{0\}$ denote the set of x whose associated matrix σ is equal to $\sigma(d)$, i.e. such that the first column of A_i not orthogonal to x is the d_ith, for each $i = 1, \ldots, n$.

Write $\mathbf{1} = (1, 1, \ldots, 1)$, and for an n-tuple $e = (e_1, \ldots, e_n)$, write $\underline{e} = \sum_{i=1}^n e_i$. Gathering the terms together according to the finite partition $\{X(d)\}_d$ of $\mathbf{R}^n - \{0\}$, we obtain

$$\Psi(A, P, Q, v) = (-1)^n \sum_d \operatorname{sgn}(\det(\sigma(d)) \sum_r \frac{P_r(\sigma)}{(\mathbf{1} + r)!} \mathbf{D}^+(X(d) \cap \mathbf{Z}^n, \sigma(d), \mathbf{1} + r, Q, v), \tag{17}$$

where

$$\mathbf{D}^+(L, \sigma, e, Q, v) = \frac{(-1)^n |\det(\sigma)| e!}{(2\pi i)^{\underline{e}}} \sum_{x \in L} \frac{\mathbf{e}(\langle x, v \rangle)}{\langle x, \sigma_1 \rangle^{e_1} \cdots \langle x, \sigma_n \rangle^{e_n}} |_Q. \tag{18}$$

We have introduced the factor $(-1)^n |\det(\sigma)| e!$ in (18) to simplify future calculations.

1.3 Bernoulli distributions

The sums in (18) are Dedekind sums that can be evaluated in terms of Bernoulli distributions, whose definition we now recall. For each integer $k \geq 0$, the Bernoulli polynomial $b_k(x)$ is defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} b_k(x) \frac{t^k}{k!}.$$

For $k \neq 1$, we define the periodic Bernoulli function $B_k(x) = b_k(\{x\})$, where $\{x\} \in [0, 1)$ denotes the fractional part of x. For $k = 1$, we reconcile the discrepancy between $b_1(0)$ and $b_1(1)$ by defining

$$B_1(x) = \begin{cases} b_1(\{x\}) = \{x\} - 1/2 & x \notin \mathbf{Z} \\ 0 & x \in \mathbf{Z}. \end{cases}$$

Given an n-tuple of positive integers $e = (e_1, \ldots, e_n)$ and an element $x \in \mathbf{R}^n$, define

$$\mathbf{B}_e(x) = \prod_{j=1}^{n} B_{e_j}(x_j).$$

Writing $\underline{e} = \sum e_j$, the function $\mathbf{B}_e$ provides a $\mathbf{Q}$-valued distribution on $(\mathbf{Q}/\mathbf{Z})^n$ of weight $\underline{e} - n$, in the sense that for each integer N, we have

$$\mathbf{B}_e(x) = N^{\underline{e}-n} \sum_{y \in (\mathbf{Z}/N\mathbf{Z})^n} \mathbf{B}_e\left(\frac{x+y}{N}\right). \tag{19}$$

To relate the distributions $\mathbf{B}_e$ to Sczech's cocycle Ψ, they must be altered by a defect arising from the Q-limit summation process discussed in Section 1.1.

Definition 1.2. Let $e = (e_1, \ldots, e_n)$ be a vector of positive integers, and $v \in (\mathbf{Q}/\mathbf{Z})^n$. Let

$$J = \{1 \leq j \leq n \mid e_j = 1 \text{ and } v_j \in \mathbf{Z}\}. \tag{20}$$

Define

$$\mathbf{B}_e(v, Q) = \frac{1}{m} \sum_{i=1}^{m} \left(\prod_{j \in J} \frac{\operatorname{sign} Q_{ij}}{2} \right) \prod_{j \notin J} B_{e_j}(v_j).$$

(In particular, if J is empty, then $\mathbf{B}_e(v, Q) = \mathbf{B}_e(v)$ does not depend on Q.) Define

$$\mathbf{B}_e^+(v, Q) = \frac{1}{2}(\mathbf{B}_e(v, Q) + \mathbf{B}_e(v, -Q)) = \begin{cases} \mathbf{B}_e(v, Q) & \text{if } \#J \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The functions $\mathbf{B}_e(\,.\,, Q)$ and $\mathbf{B}_e^+(\,.\,, Q)$ are distributions on $(\mathbf{Q}/\mathbf{Z})^n$ of weight $\underline{e} - n$.

Proposition 1.3 ([GS], Proposition 2.7)**.** *Let e be an n-tuple of nonnegative integers, $Q \in \mathcal{Q}$, $v \in (\mathbf{Q}/\mathbf{Z})^n$, and $\sigma \in M_n(\mathbf{Z})$. Let*

$$L = \{x \in \mathbf{Z}^n : \langle x, \sigma_i \rangle \neq 0 \text{ for } 1 \leq i \leq n\}. \tag{21}$$

We have

$$\mathbf{D}^+(L, \sigma, e, Q, v) = \sum_{x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n} \mathbf{B}_e^+(\sigma^{-1}(x+v), \sigma^{-1}Q), \tag{22}$$

where the right side is understood to have the value 0 when $\det(\sigma) = 0$.

Proposition 1.3 gives the value of the Dedekind sum appearing in (17) for $d = \mathbf{1}$; this will be sufficient for our applications. Whenever L is given in terms of σ as in (21), we drop it from our notation and write simply $\mathbf{D}^+(\sigma, e, v, Q)$.

2 The smoothed cocycle

Let ℓ be a prime number, and let $\mathbf{Z}_{(\ell)} = \mathbf{Z}[1/p, p \neq \ell]$ denote the localization of $\mathbf{Z}$ at the prime ideal (ℓ). Our aim in this section is to smooth the cocycle Ψ at the prime ℓ, yielding a cocycle Ψ_ℓ^+ defined on the congruence subgroup

$$\Gamma_\ell := \Gamma_0(\ell\mathbf{Z}_{(\ell)}) = \{A \in \mathbf{GL}_n(\mathbf{Z}_{(\ell)}) : A \equiv \begin{pmatrix} * & * & * \\ 0 & * & * \\ \vdots & \vdots & \vdots \\ 0 & * & * \end{pmatrix} \mod \ell\}.$$

We will then prove Theorem 4, an integrality result for the smoothed cocycle Ψ_ℓ^+ and a refinement Ψ_ℓ.

2.1 Definition of the smoothed cocycle

Consider the diagonal matrix whose first entry is ℓ and other diagonal entries are equal to 1:

$$\pi_\ell = \begin{pmatrix} \ell & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

For $A = (A_1, \ldots, A_n) \in \Gamma_\ell^n$, let

$$A' = \pi_\ell A \pi_\ell^{-1} = (\pi_\ell A_1 \pi_\ell^{-1}, \ldots, \pi_\ell A_n \pi_\ell^{-1}) \in \mathbf{GL}_n(\mathbf{Z}_{(\ell)})^n.$$

Fix a polynomial $P \in \mathcal{P}$ and define $P' = \pi_\ell^{-1} P$. It is a straightforward computation to check that the coefficients of P and P' defined in (12) satisfy

$$P'_r(\sigma'(d)) = P_r(\sigma(d)) \cdot \ell^{-\Sigma(r,d)} \qquad \text{where} \qquad \Sigma(r,d) = \sum_{i:\, d_i = 1} r_i. \tag{23}$$

Here as in Section 1.2, $d = (d_1, \ldots, d_n)$ is a tuple of integers with $1 \leq d_i \leq n$, and $\sigma(d)$ denotes the square matrix whose ith column is the d_ith column of A_i. The matrix $\sigma'(d)$ denotes the similarly constructed matrix with A_i replaced by A'_i.

Next fix $Q \in \mathcal{Q}$ as in (14), and let $Q' = \pi_\ell Q$. For $v \in (\mathbf{Q}/\mathbf{Z})^n$, let $v' = \pi_\ell v$. We define

$$\Psi_\ell^+(A, P, Q, v) = \Psi(A', P', Q', v') - \ell\Psi(A, P, Q, v). \tag{24}$$

The following is a straightforward computation using the fact that Ψ is a Γ-cocycle.

Proposition 2.1. *The function Ψ_ℓ^+ is a homogeneous $(n-1)$-cocycle on Γ_ℓ valued in $M_{\mathbf{Q}}$:*

$$\Psi_\ell^+ \in Z^{n-1}(\Gamma_\ell, M_{\mathbf{Q}}).$$

2.2 A Dedekind sum formula

In this section we prove that in the analogue of the Dedekind sum formula (17) for Ψ_ℓ, all terms other than those arising from $d = \mathbf{1}$ cancel.

In what follows, all objects associated to A' instead of A will be denoted with a "prime", such as $X'(d) = X(A', d)$.

Lemma 2.2. *For $d \neq \mathbf{1}$, the map*

$$\pi_\ell : (x_1, x_2, \ldots, x_n) \mapsto (\ell x_1, x_2, \ldots, x_n)$$

induces a bijection between $X'(d) \cap \mathbf{Z}^n$ and $X(d) \cap \mathbf{Z}^n$.

Proof. Denoting the jth column of the matrix A_i by A_{ij}, note that for $x \in \mathbf{R}^n$, we have $\langle x, A'_{ij}\rangle = 0 \Leftrightarrow \langle \pi_\ell x, A_{ij}\rangle = 0$. In particular, π_ℓ gives a bijection between $X'(d)$ and $X(d)$, and hence induces an injection from $X'(d) \cap \mathbf{Z}^n$ to $X(d) \cap \mathbf{Z}^n$.

To show that map is surjective, we make use of the assumption $d \neq \mathbf{1}$ to obtain an index i, say $i = 1$, for which $d_i > 1$. For any given $z \in X(d) \cap \mathbf{Z}^n$, the condition $\langle z, A_{11}\rangle = 0$ ensures that its coordinates satisfy

$$a_{11}z_1 + \ldots + a_{n1}z_n = 0.$$

This equation implies $a_{11}z_1 \equiv 0 \pmod{\ell}$ since $A_1 \in \Gamma_\ell$. Moreover, a_{11} is coprime to ℓ since $\det(A_1) \in \mathbf{Z}^*_{(\ell)}$. We conclude that ℓ divides z_1, hence z is of the form $\pi_\ell x$ for some $x \in \mathbf{Z}^n$ as desired. □

Next we use Lemma 2.2 to compare $\psi_{A'}$ and ψ_A on the sets $X'(d) \cap \mathbf{Z}^n$ and $X(d) \cap \mathbf{Z}^n$, respectively. A direct computation shows that for $x \in X'(d)$, one has

$$\langle x, \sigma'_i(d)\rangle = \begin{cases} \langle \pi_\ell x, \sigma_i(d)\rangle & \text{if } d_i > 1, \\ \ell^{-1}\langle \pi_\ell x, \sigma_i(d)\rangle & \text{if } d_i = 1. \end{cases} \tag{25}$$

On the other hand,

$$\det(\sigma'(d)) = \ell^{1-\#\{i:\ d_i=1\}} \det(\sigma(d)). \tag{26}$$

Equations (25) and (26) yield

$$\frac{\det(\sigma'(d))}{\prod_{i=1}^n \langle x, \sigma_i'(d)\rangle^{1+r_i}} = \frac{\ell^{1+\Sigma(r,d)}\det(\sigma(d))}{\prod_{i=1}^n \langle \pi_\ell x, \sigma_i(d)\rangle} \tag{27}$$

for $x \in X'(d)$, where $\Sigma(r,d)$ is defined as in (23). Multiplying (27) by (23) and recalling the definition of ψ given in (11), we obtain

$$\psi_{A'}(P', x) = \ell\psi_A(P, \pi_\ell x).$$

Multiplying by

$$\mathbf{e}(\langle x, \pi_\ell v\rangle) = \mathbf{e}(\langle \pi_\ell x, v\rangle)$$

and taking the Q'-summation over all $x \in X'(d) \cap \mathbf{Z}^n$, Lemma 2.2 implies that in the evaluation of

$$\Psi_\ell^+(A, P, Q, v) = \Psi(A', P', Q', \pi_\ell v) - \ell\Psi(A, P, Q, v),$$

the terms in (17) for $d \neq \mathbf{1}$ cancel.

We have therefore proven the following explicit formula for Ψ_ℓ. Given $A \in \Gamma_\ell^n$, the matrix $\sigma = \sigma(\mathbf{1})$ has all rows but the first divisible by ℓ (and all entries in the first row relatively prime to ℓ). Therefore, the matrix $\sigma_\ell = \pi_\ell \ell^{-1}\sigma$ has entries in $\mathbf{Z}_{(\ell)}$. Define the ℓ-smoothed Dedekind sum:

$$\begin{aligned}\mathbf{D}_\ell^+(\sigma, e, Q, v) &= \mathbf{D}^+(\sigma_\ell, e, \pi_\ell Q, \pi_\ell v) - \ell^{1-n+\underline{e}}\mathbf{D}^+(\sigma, e, Q, v).\\ &= \sum_{x' \in \mathbf{Z}^n/\sigma_\ell\mathbf{Z}^n} \mathbf{B}_e^+(\sigma_\ell^{-1}(x' + \pi_\ell v), \sigma^{-1}Q)\\ &\qquad - \ell^{1-n+\underline{e}} \sum_{x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n} \mathbf{B}_e^+(\sigma^{-1}(x+v), \sigma^{-1}Q).\end{aligned} \tag{28}$$

Proposition 2.3. *We have*

$$\Psi_\ell^+(A, P, Q, v) = (-1)^n \mathrm{sgn}(\det\sigma) \sum_r \frac{P_r(\sigma)}{(r+\mathbf{1})!\ell^{\underline{r}}} \mathbf{D}_\ell^+(\sigma, \mathbf{1}+r, v, Q).$$

2.3 A refined cocycle

Define $\mathbf{D}_\ell$ as in (28) with $\mathbf{B}^+$ replaced with $\mathbf{B}$. Define Ψ_ℓ as in Proposition 2.3, with $\mathbf{D}_\ell^+$ replaced by $\mathbf{D}_\ell$:

$$\Psi_\ell(A, P, Q, v) := (-1)^n \mathrm{sgn}(\det\sigma) \sum_r \frac{P_r(\sigma)}{(r+\mathbf{1})!\ell^{\underline{r}}} \mathbf{D}_\ell(\sigma, \mathbf{1}+r, Q, v). \tag{29}$$

Then Ψ_ℓ^+ is recovered from Ψ_ℓ by projection onto the $+1$ eigenspace of the action of multiplication by -1 on $\mathcal{Q}$:

$$\Psi_\ell^+(A, P, Q, v) = \frac{1}{2}(\Psi_\ell(A, P, Q, v) + \Psi_\ell(A, P, -Q, v)).$$

In [CDG], we show that the result of Proposition 2.1 holds still for Ψ_ℓ:

Proposition 2.4. *The function Ψ_ℓ is a homogeneous $(n-1)$-cocycle on Γ_ℓ valued in $M_{\mathbf{Q}}$:*

$$\Psi_\ell \in Z^{n-1}(\Gamma_\ell, M_{\mathbf{Q}}).$$

Remark 2.5. The Γ_ℓ-invariance of Ψ_ℓ (i.e. that Ψ_ℓ is a homogeneous cochain) is easy to check directly from the definition. However, the alternating property of Ψ_ℓ (i.e. that Ψ_ℓ is a cocycle) is mysterious using our explicit definition. In [CDG], we show that for $P = 1$, the function Ψ_ℓ is equal to a cocycle defined using Shintani's method; the case for general P then follows from Theorem 4.2 below.

2.4 Decomposition of the Dedekind Sum

Consider the following $\mathbf{Z}[\frac{1}{\ell}][\Gamma]$-submodule of $M_{\mathbf{Q}}$:

$$\begin{aligned} M_\ell = \{\phi \in M_{\mathbf{Q}} : \phi(P,Q,v) \in \tfrac{1}{m}\mathbf{Z}[\tfrac{1}{\ell}] \text{ when } P \in \mathbf{Z}[\tfrac{1}{\ell}][X_1,\ldots,X_n] \\ \text{is homogeneous and } P(v + (\tfrac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1})) \subset \mathbf{Z}[\tfrac{1}{\ell}]\}. \end{aligned} \tag{30}$$

In (30), the constant m denotes the number of linear forms defining the element $Q \in \mathcal{Q}$, as in Section 1.1. In the remainder of this section, we prove Theorem 4, which states that the smoothed cocycle Ψ_ℓ takes values in M_ℓ.

Remark 2.6. The cocycle Ψ_ℓ was introduced in Section 2.3 because it takes values in M_ℓ. The smoothed Sczech Ψ_ℓ^+ takes values in $\frac{1}{2}M_\ell$. Furthermore, we prove in [CDG] that the cocycle Ψ_ℓ may be paired with an appropriate cycle involving $Q \in \mathcal{Q}$ with $m = 1$ to yield the special values of zeta functions, thereby leading to a proof of Theorem 1 (see Section 3). Sczech's formulas relating Ψ to zeta values involves a Q with $m = n = [F : \mathbf{Q}]$. This leads to a slightly weaker version of Theorem 1, where the zeta values are shown to lie in $\frac{1}{2n}\mathbf{Z}[1/\ell]$.

In view of the definition (29), our first step in proving the integrality of Ψ_ℓ is to decompose $\mathbf{D}_\ell(\sigma, e, v, Q)$ into a sum of terms that individually share an analogous integrality property. To this end, fix a linear map $L \in \mathrm{Hom}(\mathbf{Z}^n, \mathbf{F}_\ell)$, and assume that its values on the standard basis of $\mathbf{Z}^n$ are all nonzero. For a tuple $e = (e_1, \ldots, e_n)$ of positive integers, $x \in \mathbf{R}^n$, and $z \in \mathbf{F}_\ell$ define

$$\mathbf{B}_e^{L,z}(x,Q) = \mathbf{B}_e(x,Q) - \ell^{1-n+\underline{e}} \sum_{\substack{y \in \mathbf{F}_\ell^n \\ L(y)=z}} \mathbf{B}_e\left(\frac{x+y}{\ell}, Q\right), \tag{31}$$

where the summation runs over all $y \in \mathbf{F}_\ell^n$ such that $L(y) = z$. Note that $\mathbf{B}_e^{L,z}$ depends on $x \bmod \ell\mathbf{Z}^n$ rather than mod $\mathbf{Z}^n$, since the summation over y is restricted. It satisfies the following distribution relation for integers N relatively prime to ℓ:

$$\mathbf{B}_e^{L,Nz}(x,Q) = N^{\underline{e}-n} \sum_{k \in (\ell\mathbf{Z}/\ell N\mathbf{Z})^n} \mathbf{B}_e^{L,z}\left(\frac{x+k}{N}, Q\right). \tag{32}$$

As in the previous section, consider $A \in \Gamma_\ell^n$ and $\sigma = \sigma(\mathbf{1})$ the square matrix consisting of the first columns of the matrices in the tuple A. Recall $\sigma_\ell = \pi_\ell \ell^{-1}\sigma$. For $y \in \mathbf{Z}^n/\ell\mathbf{Z}^n$, define $L(y) = \langle R, y\rangle \pmod{\ell}$, where R denotes the first row of σ. Our desired decomposition is:

Lemma 2.7. *Let $\{x = (x_1, \ldots, x_n)\} \subset \mathbf{Z}^n$ denote a set of representatives for $\mathbf{Z}^n/\sigma_\ell \mathbf{Z}^n$. We have*

$$\mathbf{D}_\ell(\sigma, e, v, Q) = \sum_x \mathbf{B}_e^{L,-x_1}(\sigma_\ell^{-1}(x + \pi_\ell v), \sigma^{-1}Q). \tag{33}$$

Remark 2.8. One easily checks that the summand in (33) is independent of the choice of representative $x \in \mathbf{Z}^n$ for each class in $\mathbf{Z}^n/\sigma_\ell \mathbf{Z}^n$.

Proof. The map $(x, y) \mapsto z = \pi_\ell^{-1}x + \ell^{-1}\sigma y$ induces a bijection between $\mathbf{Z}^n/\sigma_\ell \mathbf{Z}^n \times \mathbf{Z}^n/\ell \mathbf{Z}^n$ and $\pi_\ell^{-1}\mathbf{Z}^n/\sigma \mathbf{Z}^n$. Furthermore, under this bijection

$$L(y) \equiv -x_1 \pmod{\ell} \iff z \in \mathbf{Z}^n.$$

The result follows immediately from the definitions (28) and (31). □

From (29) and Lemma 2.7, we obtain:

Proposition 2.9. *We have*

$$\Psi_\ell(A, P, Q, v) = \pm \sum_r \frac{P_r(\sigma)}{(r + \mathbf{1})! \ell^{\underline{r}}} \sum_{x \in \mathbf{Z}^n/\sigma_\ell \mathbf{Z}^n} \mathbf{B}_{1+r}^{L,-x_1}(\sigma_\ell^{-1}(x + \pi_\ell v), \sigma^{-1}Q),$$

where the $\pm$ sign is given by $(-1)^n \mathrm{sgn}(\det \sigma)$.

In the next two subsections, we show that the individual terms in (33) are integral when $e = \mathbf{1}$, thereby proving the integrality property for Ψ_ℓ when $P = 1$. The integrality of Ψ_ℓ in general will follow by bootstrapping from this base case.

2.5 A cyclotomic Dedekind sum

The following "cyclotomic Dedekind sum" attached to a real number x will play an important role in our computations:

$$B_1^{\exp}(x, r) = \sum_{m=1}^{\ell} \mathbf{e}\left(\frac{rm}{\ell}\right) B_1\left(\frac{x + m}{\ell}\right) \tag{34}$$

for any $x \in \mathbf{R}$ and $r \in \mathbf{F}_\ell^\times$.

Lemma 2.10. *The value of the cyclotomic Dedekind sum is given by*

$$B_1^{\exp}(x, r) = \frac{\mathbf{e}(\frac{-r[x]}{\ell})}{\mathbf{e}(\frac{r}{\ell}) - 1} + \frac{\delta_x}{2} \mathbf{e}\left(-\frac{rx}{\ell}\right), \tag{35}$$

where $\delta_x = 1$ if $x \in \mathbf{Z}$ and is 0 otherwise.

Proof. Translating m by the integer $[x]$ gives $m' = m + [x]$, which runs over the same set of classes mod ℓ. This yields

$$B_1^{\exp}(x,r) = \sum_{m' \bmod \ell} \mathbf{e}\left(\frac{r(m'-[x])}{\ell}\right) B_1\left(\frac{m'+\{x\}}{\ell}\right),$$

where $0 \le \{x\} < 1$ is the fractional part of x. Choosing representatives m' between 0 and $\ell - 1$ gives

$$B_1\left(\frac{m'+\{x\}}{\ell}\right) = \frac{m'+\{x\}}{\ell} - \frac{1}{2}$$

unless $m' = \{x\} = 0$, in which case the $-1/2$ term does not occur. Therefore

$$B_1^{\exp}(x,r) = \mathbf{e}\left(\frac{-r[x]}{\ell}\right) \sum_{m'=0}^{\ell-1} \frac{m'}{\ell} \mathbf{e}\left(\frac{rm'}{\ell}\right) + \frac{\delta_x}{2} \mathbf{e}\left(\frac{-rx}{\ell}\right), \tag{36}$$

since the terms coming from $\{x\}$ and $\frac{1}{2}$ each vanish in the sum over m'. The value of the sum in (36) can be obtained by evaluating at $z = r$ the derivative of

$$\frac{1}{2\pi i} \sum_{m'=0}^{\ell-1} \mathbf{e}\left(\frac{m'z}{\ell}\right) = \frac{1}{2\pi i} \cdot \frac{\mathbf{e}(z) - 1}{\mathbf{e}(z/\ell) - 1}.$$

The lemma follows. □

2.6 The case $e = \mathbf{1}$

The goal of this section is to bound the denominators of the individual terms in (33) when $e = \mathbf{1}$.

Proposition 2.11. *Let $x \in \mathbf{Q}^n$. The quantity $\mathbf{B}_{\mathbf{1}}^{L,z}(x,Q)$ lies in $\frac{1}{m}\mathbf{Z}[1/\ell]$, and lies in $\frac{1}{m}\mathbf{Z}$ if $\ell > n+1$.*

Proof. This proof follows the argument of [Das, §6.1]. We begin by relaxing the restricted summation. Since the map

$$y \mapsto \frac{1}{\ell} \sum_{k=0}^{\ell-1} \mathbf{e}\left(\frac{kL(y)}{\ell}\right)$$

is the characteristic function of the kernel of L, we obtain

$$\mathbf{B}_e^{L,z}(x,Q) = -\ell^{\underline{e}-n} \sum_{k=1}^{\ell-1} \sum_{y \in \mathbf{F}_\ell^n} \mathbf{e}\left(\frac{k(L(y)-z)}{\ell}\right) \mathbf{B}_e\left(\frac{x+y}{\ell}, Q\right). \tag{37}$$

Note that the term $k = 0$ cancels the leading term of $\mathbf{B}_e^{L,z}$ using the distribution relation for $\mathbf{B}_e$.

Let a_j be the value of the linear form L on the j-th term of the standard basis of $\mathbf{Z}^n$. We specialize to $e = \mathbf{1}$ and consider the case where no component of x is an integer (so there is no dependence on Q). The sum (37) then decomposes as

$$-\sum_{k=1}^{\ell-1} \mathbf{e}\left(-\frac{kz}{\ell}\right) \sum_{y_1=1}^{\ell} \cdots \sum_{y_n=1}^{\ell} \prod_{j=1}^{n} \mathbf{e}\left(\frac{ka_j y_j}{\ell}\right) B_1\left(\frac{x_j + y_j}{\ell}\right).$$

Since each a_j is non-zero by assumption, we can repeatedly use Lemma 2.10 to obtain

$$\begin{aligned} \mathbf{B}_{\mathbf{1}}^{L,z}(x, Q) &= -\sum_{k=1}^{\ell-1} \mathbf{e}(-\frac{kz}{\ell}) \left(\prod_{j=1}^{n} \frac{\mathbf{e}(-\frac{ka_j[x_j]}{\ell})}{\mathbf{e}(\frac{ka_j}{\ell}) - 1} \right) \\ &= -\operatorname{Tr}_{\mathbf{Q}(\zeta_\ell)/\mathbf{Q}} \left(\frac{\mathbf{e}(\frac{-z-L([x])}{\ell})}{\prod_{j=1}^{n}(\mathbf{e}(\frac{a_j}{\ell}) - 1)} \right). \end{aligned} \tag{38}$$

For any primitive ℓ-th root of unity ζ_ℓ, the element $\zeta_\ell - 1$ is supported at ℓ and has ℓ-adic valuation $1/(\ell - 1)$. Therefore, the expression (38) lies in $\mathbf{Z}[\frac{1}{\ell}]$ and has denominator at most $\ell^{\frac{n}{\ell-1}}$. The lemma is proven in the case where no component x_j is integral.

For the general case, let $J_0 = \{j : x_j \in \mathbf{Z}\}$. An argument as above yields the following value for $\mathbf{B}_{\mathbf{1}}^{L,z}(x, Q)$:

$$-\frac{1}{m2^{\#J_0}} \sum_{i=1}^{m} \sum_{J \subset J_0} \operatorname{Tr}_{\mathbf{Q}(\zeta_\ell)/\mathbf{Q}} \left(\frac{\mathbf{e}(\frac{-z-L([x])}{\ell})}{\prod_{j \notin J_0}(\mathbf{e}(\frac{a_j}{\ell}) - 1)} \prod_{j \in J_0 - J} \frac{\mathbf{e}(\frac{a_j}{\ell}) + 1}{\mathbf{e}(\frac{a_j}{\ell}) - 1} \prod_{j \in J} \operatorname{sign} Q_{ij} \right). \tag{39}$$

As in (38), the sum in (39) lies in $\mathbf{Z}[1/\ell]$ and in fact lies in $\mathbf{Z}$ if $\ell > n + 1$. It only remains to eliminate the factor $1/2^{\#J_0}$. When summed over all $J \subset J_0$, the argument of the trace in (39) is equal to a constant depending on J_0 (but not J) times

$$\sum_{J \subset J_0} \left(\prod_{j \in J_0 - J} \frac{\mathbf{e}(\frac{a_j}{\ell}) + 1}{\mathbf{e}(\frac{a_j}{\ell}) - 1} \prod_{j \in J} \operatorname{sign} Q_{ij} \right) = \prod_{j \in J_0} \left(\frac{\mathbf{e}(\frac{a_j}{\ell}) + 1}{\mathbf{e}(\frac{a_j}{\ell}) - 1} + \operatorname{sign} Q_{ij} \right) \tag{40}$$

$$= 2^{\#J_0} \prod_{j \in J_0} \left(\frac{\alpha_{ij}}{\mathbf{e}(\frac{a_j}{\ell}) - 1} \right), \tag{41}$$

where $\alpha_{ij} = \mathbf{e}(a_j/\ell)$ or $\alpha_{ij} = 1$ in the cases $\operatorname{sign} Q_{ij} = 1$ or $\operatorname{sign} Q_{ij} = -1$, respectively. The factor $2^{\#J_0}$ in (41) cancels that in (39). □

2.7 Proof of Theorem 4

We are now ready to complete the proof of Theorem 4 from the introduction, which states that the rational number $\Psi_\ell(A, P, Q, v)$ lies in $\mathbf{Z}[1/\ell]$ when (9) is satisfied. We do this by showing that it lies in $\mathbf{Z}_p$ for each prime $p \neq \ell$.

Proposition 2.12. *Let $x \in \mathbf{Q}^n$ and let $p \neq \ell$ be a prime number. Let r be an n-tuple of nonnegative integers. There exists an integer ϵ depending only on r, ℓ and the denominator of x, such that for all positive integers M we have the following congruence between rational numbers:*

$$p^{M\cdot \underline{r}}\mathrm{N}(r+\mathbf{1})^{-1}\mathbf{B}_{\mathbf{1}+r}^{L,z}\left(\frac{x}{p^M}, Q\right) \equiv \mathbf{B}_{\mathbf{1}}^{L,z}\left(\frac{x}{p^M}, Q\right)\mathrm{N}x^r \mod p^{M-\epsilon}\mathbf{Z}_p. \tag{42}$$

Remark 2.13. Here $\mathrm{N}x^r$ is shorthand for $\prod_{j=1}^n x_j^{r_j}$. If it happens that both r_j and x_j are zero, the corresponding term $x_j^{r_j}$ is understood to equal 1.

Before proving Proposition 2.12, we show how it enables the proof of Theorem 4.

Proof of Theorem 4. We recall Proposition 2.9:

$$\Psi_\ell(A,P,Q,v) = \pm\sum_r \frac{P_r(\sigma)}{r!\ell^{\underline{r}}}\mathrm{N}(r+1)^{-1}\sum_{x\in\mathbf{Z}^n/\sigma_\ell\mathbf{Z}^n}\mathbf{B}_{\mathbf{1}+r}^{L,-x_1}(\sigma_\ell^{-1}(x+\pi_\ell v), \sigma^{-1}Q). \tag{43}$$

For each x in the sum above we let $y = \sigma_\ell^{-1}(x+\pi_\ell v)$ and note that y has the property

$$\tfrac{1}{\ell}\sigma(y) \in v + \tfrac{1}{\ell}\mathbf{Z}\oplus\mathbf{Z}^{n-1}. \tag{44}$$

Fix a prime $p \neq \ell$. For each y we let ϵ be as in Proposition 2.12 and fix a positive integer $M \geq \epsilon$. Applying the distribution relation (32) we replace the term $\mathbf{B}_{1+r}^{L,-x_1}(y,\sigma^{-1}Q)$ in (43) with

$$p^{M\cdot\underline{r}}\sum_{k\in(\ell\mathbf{Z}/\ell p^M\mathbf{Z})^n}\mathbf{B}_{\mathbf{1}+r}^{L,z}\left(\frac{y+k}{p^M}, \sigma^{-1}Q\right), \tag{45}$$

where $z \equiv -p^{-M}x_1 \pmod{\ell}$. By Proposition 2.12 and the choice of M, the quantity in (45) multiplied by $\mathrm{N}(r+1)^{-1}$ is congruent modulo $\mathbf{Z}_p$ to

$$\sum_{k\in(\ell\mathbf{Z}/\ell p^M\mathbf{Z})^n}\mathbf{B}_{\mathbf{1}}^{L,z}\left(\frac{y+k}{p^M}, \sigma^{-1}Q\right)\mathrm{N}(y+k)^r$$

Plugging this expression into (43), we note that each coefficient $\frac{P_r(\sigma)}{r!}$ lies in $\mathbf{Z}_p$, and hence $\Psi_\ell(A,P,Q,v)$ is congruent modulo $\mathbf{Z}_p$ to

$$\pm\sum_x\sum_{k\in(\ell\mathbf{Z}/\ell p^M\mathbf{Z})^n}\mathbf{B}_{\mathbf{1}}^{L,z}\left(\frac{y+k}{p^M}, \sigma^{-1}Q\right)\sum_r\frac{P_r(\sigma)}{r!}\mathrm{N}\left(\frac{y+k}{\ell}\right)^r. \tag{46}$$

By the definition (12), the sum over r in (46) is equal to $P(\sigma(y+k)/\ell)$, which by (44) and the given property

$$P(v + (\tfrac{1}{\ell}\mathbf{Z}\oplus\mathbf{Z}^{n-1})) \subset \mathbf{Z}[\tfrac{1}{\ell}]$$

lies in $\mathbf{Z}[\frac{1}{\ell}]$. Therefore, by Proposition 2.11, each term in (46) lies in $\mathbf{Z}_p$, and the theorem is proven. □

Proof of Proposition 2.12. As in the classical Kubota–Leopoldt construction of p-adic L-functions over $\mathbf{Q}$, the proof relies on the fact that the Bernoulli polynomial $b_k(x)$ begins

$$b_k(x) = x^k - \frac{k}{2}x^{k-1} + \cdots . \tag{47}$$

We recall equation (37) for $\mathbf{B}^{L,z}_{\mathfrak{1}+r}$:

$$\mathbf{B}^{L,z}_{\mathfrak{1}+r}(x,Q) = -\ell^{\underline{r}} \sum_{k=1}^{\ell-1} \sum_{y\in\mathbf{F}_\ell^n} \mathbf{e}\left(\frac{k(L(y)-z)}{\ell}\right) \mathbf{B}_{\mathfrak{1}+r}\left(\frac{x+y}{\ell}, Q\right). \tag{48}$$

At the expense of altering z, we may translate x by an element of $p^M\mathbf{Z}^n$ and assume that x/p^M belongs to $[0,1)^n$. Furthermore, for each class in $\mathbf{F}_\ell^n$ we choose the representative $y \in \mathbf{Z}^n$ with $0 \le y_j \le \ell-1$. Let $J_0 = \{j \mid x_j = 0 \text{ and } r_j = 0\}$. For $j \notin J_0$, (47) yields

$$p^{Mr_j} B_{1+r_j}\left(\frac{\frac{x_j}{p^M}+y_j}{\ell}\right) \equiv p^{-M}\left(\frac{x_j}{\ell}\right)^{1+r_j} + (1+r_j)\left(\frac{x_j}{\ell}\right)^{r_j}\left(\frac{y_j}{\ell}-\frac{1}{2}\right) \mod p^{M-\epsilon_j}\mathbf{Z}_p,$$

where ϵ_j depends only on r_j, ℓ and the power of p in the denominator of x_j. Let $a_j \in \mathbf{F}_\ell$ denote the value of L on the jth standard basis vector of $\mathbf{Z}^n$, and multiply the previous congruence by $\mathbf{e}(\frac{ka_jy_j}{\ell})$. Summing over all $0 \le y_j \le \ell-1$, the leading term of the right side vanishes and we obtain

$$\begin{aligned} p^{Mr_j} \sum_{y_j\in\mathbf{F}_\ell} \mathbf{e}\left(\frac{ka_jy_j}{\ell}\right) B_{1+r_j}\left(\frac{\frac{x_j}{p^M}+y_j}{\ell}\right) \equiv \\ (1+r_j)\left(\frac{x_j}{\ell}\right)^{r_j} \sum_{y_j\in\mathbf{F}_\ell} \mathbf{e}\left(\frac{ka_jy_j}{\ell}\right) B_1\left(\frac{\frac{x_j}{p^M}+y_j}{\ell}\right) \mod p^{M-\epsilon_j}\mathbf{Z}_p[\zeta_\ell]. \end{aligned} \tag{49}$$

Take the product of these congruences over all $j \notin J_0$ and multiply by

$$-\frac{\mathbf{e}(-\frac{kz}{\ell})}{m} \sum_{i=1}^{m} \left(\prod_{j\in J_0} \frac{\operatorname{sign} Q_{ij}}{2}\right).$$

In view of (48), summing over $k = 1, \ldots, \ell-1$ gives the desired result (after dividing by $\mathrm{N}(\mathfrak{1}+r)$ and increasing ϵ accordingly). □

3 Integrality of smoothed zeta functions

Let F be a totally real field of degree n. In this section we combine the $\mathbf{Z}[1/\ell]$-integrality property of Ψ_ℓ proved in Theorem 4 with a suitable generalization of Sczech's formula (8) to prove Theorem 1, recalled below. Let $\mathfrak{a}$ and $\mathfrak{f}$ be coprime integral ideals of F. Let $\mathfrak{c}$ be an integral ideal of F with norm ℓ such that $(\mathfrak{c},\mathfrak{f}) = 1$.

Theorem 1. *The smoothed zeta function*

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s) = \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, s) - \mathrm{N}\mathfrak{c}^{1-s}\zeta_{\mathfrak{f}}(\mathfrak{a}, s)$$

assumes values in $\mathbf{Z}[1/\ell]$ *at nonpositive integers* s.

Proof. Fix a basis $\{w_1, \ldots, w_n\}$ for $\mathfrak{a}^{-1}\mathfrak{f}$ such that $\{\frac{1}{\ell}w_1, w_2, \ldots, w_n\}$ is a basis for $\mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{f}$. Let $\{\epsilon_1, \ldots, \epsilon_{n-1}\}$ denote a basis of the group of totally positive units of F congruent to 1 modulo $\mathfrak{f}$. Let $A_1, \ldots, A_{n-1}$ be the matrices representing multiplication by the ϵ_i on $\mathfrak{a}^{-1}\mathfrak{f}$ in terms of the basis $w = (w_1, \ldots, w_n)$, i.e. such that

$$w\epsilon_i = wA_i \qquad i = 1, \ldots, n.$$

The matrices A_i lie in Γ_ℓ, since left multiplication by these matrices preserves the lattice of column vectors $\frac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1} \cong \mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{f}$ (where the isomorphism is given by dot product with w).

Let $P \in \mathbf{Z}[\frac{1}{\ell}][X_1, \ldots, X_n]$ denote the homogeneous polynomial of degree n given up to a scalar by the norm:

$$P(X_1, \ldots, X_n) = \mathrm{N}(\mathfrak{a}\mathfrak{c})\mathrm{N}(w_1X_1 + \cdots + w_nX_n). \tag{50}$$

Let $\tilde{Q} = (Q_1, \ldots, Q_n)$ be the n-tuple of linear forms given by

$$Q_i = \tau_i(w_1^*)X_1 + \cdots + \tau_i(w_n^*)X_n, \tag{51}$$

where $\{w_1^*, \ldots, w_n^*\}$ is the dual basis with respect to the trace form on F (i.e. $\mathrm{Tr}(w_iw_j^*) = \delta_{ij}$), and the τ_i denote the embeddings $F \hookrightarrow \mathbf{R}$. Define the column vector

$$v = (\mathrm{Tr}(w_1^*), \ldots, \mathrm{Tr}(w_n^*)), \tag{52}$$

so that

$$1 = v_1w_1 + v_2w_2 + \cdots + v_nw_n.$$

Dot product with $(w_1, \ldots, w_n)$ provides a bijection

$$v + \tfrac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1} \leftrightarrow 1 + \mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{f}, \tag{53}$$

so P and v satisfy the key property

$$P(v + \tfrac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1}) \subset \mathbf{Z}[\tfrac{1}{\ell}] \tag{54}$$

of Theorem 4.

The matrices $\{A_i\}$ give rise to a homogeneous $(n-1)$-chain in the standard way, which we write using the bar notation:

$$[A_1 \mid \ldots \mid A_{n-1}] = (1, A_1, A_1A_2, \ldots, A_1A_2 \cdots A_{n-1}) \in \Gamma_\ell^n.$$

We symmetrize this chain by defining

$$\mathcal{A}(A_1, \ldots, A_{n-1}) := \sum_{\pi \in S_{n-1}} \operatorname{sgn}(\pi)[A_{\pi(1)} \mid \cdots \mid A_{\pi(n-1)}] \in \mathbf{Z}[\Gamma_\ell^n]. \tag{55}$$

Let

$$A = \rho \cdot \mathcal{A}(A_1, \ldots, A_{n-1}). \tag{56}$$

Here $\rho = \pm 1$ is a sign defined as follows: let $\tau_1, \ldots, \tau_n$ denote the real embeddings of F, and consider the square matrices

$$W = (\tau_i(w_j))_{i,j=1}^n \quad \text{and} \quad R = (\log \tau_i(\epsilon_j))_{i,j=1}^{n-1}.$$

Then $\rho = (-1)^{n-1} \operatorname{sign}(\det W) \operatorname{sign}(\det R)$.

For integers $k \geq 0$, Sczech's formula ([Sc2, Corollary p. 595]) reads

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, -k) = \ell^{-k} \Psi(A, P^k, \tilde{Q}, v), \tag{57}$$

where the ℓ-power term arises from the extra factor of $\mathrm{N}\mathfrak{c} = \ell$ in the definition of P. Applying this formula again with $\mathfrak{a}$ replaced by $\mathfrak{a}\mathfrak{c}$ gives

$$\zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, -k) = \Psi(A', (P')^k, \tilde{Q}', v'), \tag{58}$$

where

$$A' = \pi_\ell A \pi_\ell^{-1}, \quad P' = \pi_\ell^{-1} P, \quad \tilde{Q}' = \pi_\ell \tilde{Q}, \quad \text{and} \quad v' = \pi_\ell v$$

as in Section 2.

Combining (57) and (58), we find

$$\begin{aligned} \zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) &= \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, -k) - \ell^{1+k} \zeta_{\mathfrak{f}}(\mathfrak{a}, -k). \\ &= \Psi(A', (P')^k, \tilde{Q}', v') - \ell \Psi(A, P^k, \tilde{Q}, v) \\ &= \Psi_\ell^+(A, P^k, \tilde{Q}, v). \end{aligned} \tag{59}$$

The result $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) \in \frac{1}{2n}\mathbf{Z}[1/\ell]$ now follows from Theorem 4.

The denominator $2n$ may be removed by invoking the following result proven in [CDG], which states that (59) still holds with Ψ_ℓ^+ replaced by Ψ_ℓ, and with $\tilde{Q}$ replaced by any one of the n individual linear forms defining it:

Theorem 3.1. *Let the notation be as above, and let $Q(X_1, \ldots, X_n) = \sum_{i=1}^n \tau(w_i^*) X_i$ for a real embedding $\tau : F \hookrightarrow \mathbf{R}$. Then*

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) = \Psi_\ell(A, P^k, Q, v).$$

Now $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) \in \mathbf{Z}[1/\ell]$ follows from Theorem 4. □

Remark 3.2. Note that

$$\zeta_{\mathfrak{f},\mathfrak{b}\mathfrak{c}}(\mathfrak{a}, s) = \zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}\mathfrak{b}, s) + \mathrm{N}\mathfrak{c}^{1-s} \zeta_{\mathfrak{f},\mathfrak{b}}(\mathfrak{a}, s),$$

so we obtain more generally that $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) \in \mathbf{Z}[1/\mathrm{N}\mathfrak{c}]$ for nonnegative integers k when $\mathfrak{c}$ is a product of ideals with prime norm. Deligne–Ribet show that this result holds for arbitrary integral ideals $\mathfrak{c}$. Extending our methods to obtain this general result seems difficult, since the fact that the level ℓ of our modular group Γ_ℓ is squarefree appears at face value to be essential in the argument of Proposition 2.11.

4 p-adic measures and p-adic zeta functions

In this section we interpret the cocycle Ψ_ℓ in terms of p-adic measures and use this perspective to prove Theorem 2 of the Introduction on the existence of p-adic zeta functions.

4.1 p-adic measures associated to Ψ_ℓ

Let $\mathbf{X} = \mathbf{Z}_p^n$, and let

$$\Gamma_{\ell,p} := \Gamma_0(\ell\mathbf{Z}[1/p]) = \Gamma_\ell \cap \mathbf{GL}_2(\mathbf{Z}[1/p]).$$

Given $A \in \Gamma_{\ell,p}^n$, $Q \in \mathcal{Q}$, and $v \in \mathbf{Q}^n$, we define a $\mathbf{Z}[1/\ell]$-valued measure $\mu_\ell = \mu_\ell(A, Q, v)$ on

$$\mathbf{X}_v := v + \mathbf{X} \subset \mathbf{Q}_p^n$$

as follows. Let σ denote the matrix whose columns are the first columns of the matrices in the tuple A. If $\det(\sigma) = 0$, then μ_ℓ is the 0 measure. Suppose now that $\det(\sigma) \neq 0$.

A vector $a \in \mathbf{Z}^n$ and a nonnegative integer r give rise to the compact open subset

$$a + p^r\mathbf{X} \subset \mathbf{X}.$$

These sets form a basis of compact open subsets of $\mathbf{X}$, and hence their translates by v form a basis of compact open subsets of $\mathbf{X}_v$. We define μ_ℓ by applying Ψ_ℓ with the constant polynomial $P = 1$:

$$\mu_\ell(A, Q, v)(v + a + p^r\mathbf{X}) = \Psi_\ell\left(A, 1, Q, \frac{v+a}{p^r}\right) \in \tfrac{1}{m}\mathbf{Z}[\tfrac{1}{\ell}] \subset \tfrac{1}{m}\mathbf{Z}_p. \tag{60}$$

It is easily checked that this assignment is well-defined, and that the distribution relation for Ψ_ℓ yields a corresponding distribution relation for μ_ℓ.

Let $\mathcal{M}_p$ denote the space of functions that assigns to each $(Q, v) \in \mathcal{Q} \times \mathcal{V}$ a $\mathbf{C}_p$-valued measure $\alpha(Q, v)$ on $\mathbf{X}_v$ such that $\alpha(Q, pv)(pU) = \alpha(Q, v)(U)$ for all $U \subset \mathbf{X}_v$. The space $\mathcal{M}_p$ naturally has the structure of a Γ-module given by

$$(\gamma\alpha)(Q, v)(U) := \alpha(AQ, Av)(AU),$$

where $A = \lambda\gamma$ is chosen such that λ is a power of p and $A \in M_n(\mathbf{Z})$.

Proposition 4.1. *The function $\mu_\ell : \Gamma_{\ell,p}^n \longrightarrow \mathcal{M}_p$ is a homogenous $(n-1)$-cocycle.*

Proposition 4.1 follows directly from the fact that Ψ_ℓ is a cocycle. The following theorem shows that the cocycle Ψ_ℓ can be recovered from the cocycle of measures μ_ℓ; in other words, the cocycle Ψ_ℓ specialized to $P = 1$ determines its value on all $P \in \mathcal{P}$.

Theorem 4.2. *For any $P(x) \in \mathcal{P}$ we have*

$$\Psi_\ell(A, P, Q, v) = \int_{\mathbf{X}_v} P(x)\ d\mu_\ell(A, Q, v)(x). \tag{61}$$

Proof. It suffices to prove the result when P is homogeneous of degree d. We follow closely the proof of Theorem 4 given in Section 2.7. It was shown there (see (46)) that there exists an integer ϵ such that for each positive integer $M \geq \epsilon$, the quantity $\Psi_\ell(A, P, Q, v)$ is congruent to

$$\pm \sum_{x} \sum_{k \in (\ell \mathbf{Z}/\ell p^M \mathbf{Z})^n} \mathbf{B}_{\mathbf{1}}^{L, -p^{-M} x_1} \left(\frac{y+k}{p^M}, \sigma^{-1} Q \right) P \left(\frac{\sigma(y+k)}{\ell} \right) \tag{62}$$

modulo $p^{M-\epsilon}\mathbf{Z}_p$. Here σ again denotes the matrix of first columns of A, and x sums over representatives in $\mathbf{Z}^n$ for $\mathbf{Z}^n/\sigma_\ell \mathbf{Z}^n$. Meanwhile, $y = \sigma_\ell^{-1} x + \sigma^{-1} \ell v$. The $\pm$ sign is $(-1)^n \operatorname{sign}(\det \sigma)$. The expression (62) is simplified with a change of variables. First replace the variable k by arbitrary representatives j for $\mathbf{Z}^n/p^M\mathbf{Z}^n$ (not necessarily divisible by ℓ) such that $j \equiv k \pmod{p^M}$; the expression $\mathbf{B}_{\mathbf{1}}^{L, -p^{-M} x_1} \left(\frac{y+k}{p^M}, \sigma^{-1} Q \right)$ is seen from the definitions to equal $\mathbf{B}_{\mathbf{1}}^{L, -p^{-M}(x_1 + L(j))} \left(\frac{y+j}{p^M}, \sigma^{-1} Q \right)$. Then let $u = x + \sigma_\ell(j)$; the expression (62) is congruent modulo p^M to:

$$\pm \sum_{u \in \mathbf{Z}^n/p^M \sigma_\ell \mathbf{Z}^n} \mathbf{B}_{\mathbf{1}}^{L, -p^{-M} u_1} \left(\frac{\sigma_\ell^{-1}(u) + \sigma^{-1} \ell v}{p^M}, \sigma^{-1} Q \right) P(\pi_\ell^{-1} u + v). \tag{63}$$

Let us meanwhile evaluate the Riemann sums approximating the integral on the right side of (61). There is a δ depending on the powers of p in the denominator of $P(v)$ such that for M large we have

$$\begin{aligned} \int_{\mathbf{X}_v} P(x)\, d\mu_\ell(x) &\equiv \sum_{j \in \mathbf{Z}^n/p^M\mathbf{Z}^n} P(v+j) \mu_\ell(j + p^M \mathbf{X}) \pmod{p^{M-\delta}\mathbf{Z}_p} \\ &= \sum_{j \in \mathbf{Z}^n/p^M\mathbf{Z}^n} P(v+j) \mathbf{D}_\ell \left(\sigma, \mathbf{1}, Q, \frac{v+j}{p^M} \right) \end{aligned} \tag{64}$$

As j runs over $(\mathbf{Z}/p^M\mathbf{Z})^n$, let k run over representatives for $(\mathbf{Z}/p^M\mathbf{Z})^n$ such that $k \equiv \pi_\ell j \pmod{p^M}$. An argument similar to the proof of Lemma 2.7 shows that

$$\mathbf{D}_\ell \left(\sigma, \mathbf{1}, Q, \frac{v+j}{p^M} \right) = \sum_{z \in \mathbf{Z}^n/\sigma_\ell \mathbf{Z}^n} \mathbf{B}_{\mathbf{1}}^{L, -z_1 - p^{-M} k} \left(\sigma_\ell^{-1}(z) + \frac{\sigma_\ell^{-1}(k) + \sigma^{-1} \ell v}{p^M}, \sigma^{-1} Q \right).$$

Substituting this equation into (64) and using the change of variables $u = p^M z + k$ gives exactly the expression in (63). The values are therefore congruent modulo $p^{M - \max\{\epsilon, \delta\}}$, so taking the limit as $M \longrightarrow \infty$ gives the desired result. □

The following more general result may be proved using the same technique.

Theorem 4.3. *For any $P \in \mathcal{P}$, $a \in \mathbf{Z}^n$, and $M \in M_n(\mathbf{Z}) \cap \Gamma_{\ell,p}$, we have*

$$\int_{v + a + M(\mathbf{X})} P(x) d\mu_\ell(A, Q, v) = \operatorname{sgn}(\det(M)) \cdot \Psi_\ell(M^{-1}A, M^t P, M^{-1}Q, M^{-1}(v+a)).$$

4.2 p-adic zeta functions

We now return to the setting of a totally real field F. Let p be a rational prime. Let $\mathfrak{a}$ be an integral ideal of F and let $\mathfrak{c}$ be an ideal of norm ℓ such that $(\mathfrak{a}\mathfrak{c}, \mathfrak{f}) = 1$. We will use the p-adic measures defined above to construct the p-adic zeta function $\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s)$ of Theorem 2, which we recall below.

Theorem 2. *There exists a unique $\mathbf{Z}_p$-valued analytic function $\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s)$ of the p-adic variable $s \in \mathcal{W}$ such that*

$$\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, -k) = \zeta^*_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) \tag{65}$$

for all nonnegative integers k.

Proof. First we note that it suffices to consider the case where $\mathfrak{f}$ is divisible by all primes of F above p. Indeed, if we let $\mathfrak{g}$ denote the least common multiple of $\mathfrak{f}$ and the primes above p, then we can define the p-adic zeta functions attached to $\mathfrak{f}$ from the ones attached to $\mathfrak{g}$ as follows:

$$\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s) = \sum_{\mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{a}} \zeta_{\mathfrak{g},\mathfrak{c},p}(\mathfrak{b}, s).$$

Here the sum ranges over representatives $\mathfrak{b}$ for the narrow ideal class group $G_{\mathfrak{g}}$ whose images in $G_{\mathfrak{f}}$ are equivalent to $\mathfrak{a}$. The analogous equation for the classical partial zeta functions ζ^* follows from the fact that the sum defining ζ^* ranges over ideals relatively prime to p.

Therefore, suppose that $\mathfrak{f}$ is divisible by all primes of F above p. Then $\zeta^*_{\mathfrak{f},\mathfrak{c}} = \zeta_{\mathfrak{f},\mathfrak{c}}$. Let P, Q, v, and A be as in Section 3, and let $\mu_\ell = \mu_\ell(A, Q, v)$ as above. As we saw in (53)–(54), for any $x \in \mathbf{Z}^n$ the quantity $P(v + x)$ is the norm of an integral ideal of F relatively prime to $\mathfrak{f}$. Since $\mathfrak{f}$ is divisible by all the primes above p, this quantity is an integer relatively prime to p. By the continuity of P, we find that

$$P(v + x) \in \mathbf{Z}_p^\times \text{ for all } x \in \mathbf{X},$$

i.e. $P(\mathbf{X}_v) \subset \mathbf{Z}_p^\times$. We may therefore define a p-adic analytic function on $\mathcal{W}$:

$$\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s) := \int_{\mathbf{X}_v} P(x)^{-s} \, d\mu_\ell(A, Q, v). \tag{66}$$

Here we have followed the usual convention of writing x^{-s} for $s(x)^{-1}$ when $x \in \mathbf{Z}_p^\times, s \in \mathcal{W}$.

For a nonnegative integer k, equation (59) and Theorem 4.2 yield

$$\begin{aligned} \zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) &= \Psi_\ell(A, P^k, Q, v) \\ &= \int_{\mathbf{X}_v} P(x)^k \, d\mu_\ell(A, Q, v). \end{aligned} \tag{67}$$

Equation (67) gives the desired interpolation property (65). □

With our applications to Gross's Conjecture 1 in mind, it is useful to have a formula such as (66) for the p-adic zeta-function when $\mathfrak{f}$ is not necessarily divisible by the primes above p.

Write $\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_1$, where $\mathfrak{f}_0$ is the prime-to-p part of $\mathfrak{f}$ and $\mathfrak{f}_1$ is divisible only by primes above p. Let $\mathfrak{f}_p$ denote the product of the primes dividing p that do not divide $\mathfrak{f}$. Fix an integral ideal $\mathfrak{a}$ relatively prime to p. An elementary calculation following directly from the definitions shows that for $s \in \mathbf{C}$,

$$\zeta^*_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s) = \sum_{\mathfrak{b}|\mathfrak{f}_p} \mu(\mathfrak{b})\mathrm{N}\mathfrak{b}^{-s}\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}\mathfrak{b}^{-1}, s), \tag{68}$$

where $\mu(\mathfrak{b}) = \pm 1$ is determined by the parity of the number of prime factors of $\mathfrak{b}$. For integers $s \leq 0$, we will express each term of (68) as an integral with respect to the measure μ_ℓ in such a way that the sum can be interpolated p-adically.

Define the variables A, P, Q, v as in Section 3 using the ideals $\mathfrak{a}$ and $\mathfrak{f}_0$. In particular, $\{w_i\}$ is a $\mathbf{Z}$-basis of $\mathfrak{a}^{-1}\mathfrak{f}_0$. Dot product with $w = (w_1, \ldots, w_n)$ gives a bijection between the spaces $\mathbf{X}_v = \mathbf{X}$ and $\mathcal{O}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}$. Using this bijection we view $\mu_\ell(A, Q, v)$ as a measure on $\mathcal{O}_p$. For each prime ideal $\mathfrak{p} \mid p$, define $\mathcal{O}_{\mathfrak{p},\mathfrak{f}} := 1 + \mathfrak{f}\mathcal{O}_\mathfrak{p}$, and write

$$\mathcal{O}_{p,\mathfrak{f}} := 1 + \mathfrak{f}\mathcal{O}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p},\mathfrak{f}}.$$

Let $\mathfrak{b}$ denote an integral ideal of F with p-power norm such that $(\mathfrak{b}, \mathfrak{f}) = 1$ (so $\mathfrak{b}$ is a product of prime ideals dividing $\mathfrak{f}_p$). Define

$$\mathcal{O}_{p,\mathfrak{b},\mathfrak{f}} := \mathfrak{b}\mathcal{O}_p \cap \mathcal{O}_{p,\mathfrak{f}} = \prod_{\mathfrak{p}|\mathfrak{b}} \mathfrak{b}\mathcal{O}_\mathfrak{p} \times \prod_{\mathfrak{p}|p,\ \mathfrak{p}\nmid\mathfrak{b}} \mathcal{O}_{\mathfrak{p},\mathfrak{f}}.$$

Write

$$\mathcal{O}^*_{\mathfrak{p},\mathfrak{f}} := \mathcal{O}^*_\mathfrak{p} \cap \mathcal{O}_{p,\mathfrak{f}}, \qquad \mathcal{O}^*_{p,\mathfrak{f}} := \prod_{\mathfrak{p}|p} \mathcal{O}^*_{\mathfrak{p},\mathfrak{f}}, \qquad \mathcal{O}^*_{p,\mathfrak{b},\mathfrak{f}} := \prod_{\mathfrak{p}|p} \mathfrak{b}\mathcal{O}^*_\mathfrak{p} \cap \mathcal{O}_{p,\mathfrak{f}}.$$

The following formula generalizes (66) to the current setting, where $\mathfrak{f}$ is not necessarily divisible by all primes above p.

Proposition 4.4. *We have*

$$\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s) = (\mathrm{N}\mathfrak{a}\mathfrak{c})^{-s} \int_{\mathcal{O}^*_{p,\mathfrak{f}}} (\mathrm{N}x)^{-s}\ d\mu_\ell(A, Q, v). \tag{69}$$

More generally, with $\mathfrak{b}$ as above we have

$$\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}\mathfrak{b}^{-1}, s) = (\mathrm{N}\mathfrak{a}\mathfrak{c})^{-s} \int_{\mathcal{O}^*_{p,\mathfrak{b},\mathfrak{f}}} (\mathrm{N}x)_p^{-s}\ d\mu_\ell(A, Q, v), \tag{70}$$

where $x_p := x/p^{\mathrm{ord}_p(x)}$ is the unit part of $x \in \mathbf{Q}_p^\times$.

Proof. We first express the zeta value $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}\mathfrak{b}^{-1}, -k)$ for a nonnegative integer k as an integral over the space $\mathcal{O}_{p,\mathfrak{b},\mathfrak{f}}$. Fix an integral ideal $\mathfrak{q}$ relatively prime to $\mathfrak{f}p$ whose image in $G_\mathfrak{f}$ is the inverse class of $\mathfrak{b}$. Therefore $\mathfrak{b}\mathfrak{q} = (\pi)$ for a totally positive $\pi \equiv 1 \pmod{\mathfrak{f}}$.

Fix a $\mathbf{Z}$-basis $u = (u_1, \ldots, u_n)$ for $\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{f}$ such that $(\frac{1}{\ell}u_1, u_2, \ldots, u_n)$ is a basis for $\mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{b}\mathfrak{f}$. Let M be the matrix such that $wM = u$. It is clear that $M \in M_n(\mathbf{Z})$, and that $|\det M| = \mathrm{N}\mathfrak{b}\mathfrak{f}_1$ is a power of p. Furthermore, it is clear that $M \in \Gamma_{\ell,p}$, so M satisfies the conditions of Theorem 4.3. For simplicity let us choose the basis u such that $\det M > 0$.

Next, let R note the matrix representing multiplication by π with respect to the basis w, i.e. such that $w\pi = wR$. Note that R commutes with the matrices A_i in the definition of the chain A. Now $\mathfrak{a}^{-1}\mathfrak{q}^{-1}\mathfrak{f} = \pi^{-1}(\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{f})$ has basis $\pi^{-1}u = wR^{-1}M$. If we use this basis to define the variables $A_{\mathfrak{a}\mathfrak{q}}, P_{\mathfrak{a}\mathfrak{q}}, Q_{\mathfrak{a}\mathfrak{q}}, v_{\mathfrak{a}\mathfrak{q}}$ as in Section 3 using the ideals $\mathfrak{a}\mathfrak{q}$ and $\mathfrak{f}$, then we find

$$\begin{aligned} A_{\mathfrak{a}\mathfrak{q}} &= M^{-1}AM, \\ P_{\mathfrak{a}\mathfrak{q}} &= \mathrm{N}\mathfrak{b}^{-1} \cdot (M^t P), \\ Q_{\mathfrak{a}\mathfrak{q}} &\sim M^{-1}Q, \\ v_{\mathfrak{a}\mathfrak{q}} &\equiv M^{-1}(v + a) \pmod{\mathbf{Z}^n}, \end{aligned}$$

where $a \in \mathbf{Z}^n$ is chosen so that $w \cdot a = \pi - 1 \in \mathfrak{a}^{-1}\mathfrak{f}_0$. Here $Q_{\mathfrak{a}\mathfrak{q}} \sim M^{-1}Q$ means that the corresponding tuples of linear forms are equal up to up to scaling the linear forms by positive reals (which does not affect the value of the cocycle Ψ_ℓ).

For any nonnegative integer k, we therefore find

$$\begin{aligned} \zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}\mathfrak{q}, -k) &= \Psi_\ell(A_{\mathfrak{a}\mathfrak{q}}, P_{\mathfrak{a}\mathfrak{q}}^k, Q_{\mathfrak{a}\mathfrak{q}}, v_{\mathfrak{a}\mathfrak{q}}) \\ &= \mathrm{N}\mathfrak{b}^{-k}\Psi_\ell(M^{-1}AM, (M^tP)^k, M^{-1}Q, M^{-1}(v+a)) \\ &= \mathrm{N}\mathfrak{b}^{-k}\Psi_\ell(M^{-1}A, (M^tP)^k, M^{-1}Q, M^{-1}(v+a)) \qquad (71) \\ &= \mathrm{N}\mathfrak{b}^{-k}\int_{v+a+M(\mathbf{X})} P(x)^k \, d\mu_\ell(A, Q, v) \qquad (72) \\ &= (\mathrm{N}\mathfrak{a}\mathfrak{c}\mathfrak{b}^{-1})^k \int_{\mathcal{O}_{p,\mathfrak{b},\mathfrak{f}}} (\mathrm{N}x)^k \, d\mu_\ell(A, Q, v). \qquad (73) \end{aligned}$$

Equation (71) follows from [Sc, Lemma 4]. Equation (72) follows from Theorem 4.3. In equation (73) we have identified $v + a + M(\mathbf{X})$ with $\mathcal{O}_{p,\mathfrak{b},\mathfrak{f}}$ via dot product with w. From (68) and (73) and an inclusion-exclusion argument, it follows that

$$\zeta^*_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) = (\mathrm{N}\mathfrak{a}\mathfrak{c})^k \int_{\mathcal{O}^*_{p,\mathfrak{f}}} (\mathrm{N}x)^k \, d\mu_\ell(A, Q, v).$$

This proves (69) by interpolation, and (70) holds similarly. □

5 Order of Vanishing at $s = 0$

5.1 p-adic L-functions

As in the introduction, let $\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^*$ be a totally odd finite order character with conductor $\mathfrak{f}$. The p-adic L-function $L_{\mathfrak{c},p}(\chi\omega, s) \colon \mathbf{Z}_p \longrightarrow \mathbf{C}_p^*$ is given by

$$L_{\mathfrak{c},p}(\chi\omega, s) = \sum_{\mathfrak{a} \in G_{\mathfrak{f}}} \chi(\mathfrak{a}\mathfrak{c})\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, \langle\cdot\rangle^s).$$

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{r_\chi}$ denote the primes above p such that $\chi(\mathfrak{p}_i) = 1$. (In particular each $\mathfrak{p}_i \nmid \mathfrak{f}$.) The goal of the rest of the paper is to prove Theorem 3, which states that

$$L^{(k)}_{\mathfrak{c},p}(\chi\omega, 0) = 0 \text{ for } k < r_\chi.$$

In the sequel, we write simply r for r_χ. Let $G_{p,\chi} \subset G_{\mathfrak{f}}$ denote the subgroup generated by the images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Let e_i denote the order of $\mathfrak{p}_i$ in $G_{\mathfrak{f}}$, and write $\mathfrak{p}_i^{e_i} = (\pi_i)$ for a totally positive $\pi_i \equiv 1 \pmod{\mathfrak{f}}$. Let

$$e = \left(\prod_i^r e_i\right) / \# G_{p,\chi} \in \mathbf{Z}.$$

We then have

$$\begin{aligned} L_{\mathfrak{c},p}(\chi\omega, s) &= \sum_{\mathfrak{a} \in G_{\mathfrak{f}}/G_{p,\chi}} \chi(\mathfrak{a}\mathfrak{c}) \sum_{\mathfrak{b} \in G_{p,\chi}} \zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}\mathfrak{b}^{-1}, s) \\ &= \frac{1}{e} \sum_{\mathfrak{a} \in G_{\mathfrak{f}}/G_{p,\chi}} \chi(\mathfrak{a}\mathfrak{c}) \sum_{\mathfrak{b} \mid \prod_{i=1}^r \mathfrak{p}_i^{e_i - 1}} \zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}\mathfrak{b}^{-1}, s) \\ &= \frac{1}{e} \sum_{\mathfrak{a} \in G_{\mathfrak{f}}/G_{p,\chi}} \chi(\mathfrak{a}\mathfrak{c}) \langle \mathrm{N}\mathfrak{a}\mathfrak{c} \rangle^{-s} \int_{\mathbf{O}} \langle \mathrm{N}x \rangle^{-s} d\mu_\ell(A_{\mathfrak{a}}, Q_{\mathfrak{a}}, v_{\mathfrak{a}}), \end{aligned} \tag{74}$$

where

$$\mathbf{O} = \prod_{i=1}^r (\mathcal{O}_{\mathfrak{p}_i} - \pi_i \mathcal{O}_{\mathfrak{p}_i}) \times \prod_{\mathfrak{p} \mid p,\ \mathfrak{p} \neq \mathfrak{p}_i} \mathcal{O}^*_{\mathfrak{p},\mathfrak{f}}.$$

Equation (74) follows from Proposition 4.4. (As usual, the representative ideals $\mathfrak{a}$ are chosen relatively prime to $\mathfrak{f}p$.)

In order to prove Theorem 3, it therefore suffices to show that the integral in (74) has order of vanishing at least r, i.e. that

$$\int_{\mathbf{O}} (\log_p \mathrm{N}x)^k d\mu_\ell(A, Q, v) = 0 \text{ for } 0 \leq k < r. \tag{75}$$

5.2 Spiess' Theorems and the proof of Theorem 3

In this section, we explain how the cocycle of measures μ_ℓ can be combined with Spiess's cohomological formalism for p-adic L-functions to deduce (75), and thereby prove Theorem 3. All of the definitions, results, and proofs in this section are due to Spiess [Sp1].

Denote by $E \subset \mathcal{O}_F^*$ the group of totally positive units of F congruent to 1 modulo $\mathfrak{f}$. Denote by T the subgroup of F^* generated by the π_i, for $i = 1, \ldots, r$. Denote by $U \cong E \times T$ the subgroup of F^* generated by E and T.

Write $F_p = \prod_{\mathfrak{p} \mid p} F_{\mathfrak{p}}$ for the completion of F at p. Let $C_c(F_p)$ denote the $\mathbf{C}_p$-algebra of $\mathbf{C}_p$-valued continuous functions on F_p with compact support, and similarly for $C_c(F_{\mathfrak{p}})$, for

$\mathfrak{p} \mid p$. Write $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, and for each $\mathfrak{p} \in S$ define $C_c^\flat(F_\mathfrak{p})$ to be the subspace of $C_c(F_\mathfrak{p})$ consisting of those functions that are constant in a neighborhood of 0. Finally, define

$$C_c^\flat(F_p) := \bigotimes_{\mathfrak{p} \in S} C_c^\flat(F_\mathfrak{p}) \otimes \bigotimes_{\mathfrak{p} \mid p,\ \mathfrak{p} \notin S} C(\mathcal{O}_\mathfrak{p}^*) \subset C_c(F_p).$$

In other words, $C_c^\flat(F_p)$ consists of functions on F_p that can be written as finite $\mathbf{C}_p$-linear combinations of products $\prod_{\mathfrak{p}|p} f_\mathfrak{p}$, with $f_\mathfrak{p} \in C_c^\flat(F_\mathfrak{p})$ for $\mathfrak{p} \in S$ and $f_\mathfrak{p} \in C(\mathcal{O}_\mathfrak{p}^*)$ for $\mathfrak{p} \mid p$, $\mathfrak{p} \notin S$. Note that, for instance, $\mathbf{1_O} \cdot (\log_p \mathrm{N}x)^k \in C_c^\flat(F_p)$, since for a tuple $x = (x_\mathfrak{p}) \in F_p$ with $x_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$, $x_\mathfrak{p} \neq 0$, we have

$$\log_p \mathrm{N}x = \sum_\mathfrak{p} \ell_\mathfrak{p}(x), \text{ where } \ell_\mathfrak{p}(x) := \log_p \mathrm{N}_{F_\mathfrak{p}/\mathbf{Q}_p} x_\mathfrak{p}. \tag{76}$$

Note that U acts on $C_c^\flat(F_p)$ by $(u \cdot f)(x) := f(x/u)$. The cocycle μ_ℓ along with the ideal $\mathfrak{a}$ allow for the definition of a homogeneous cocycle

$$\kappa_\mathfrak{a} \in Z^{n-1}(U, C_c^\flat(F_p)^\vee)$$

as follows. Given $\epsilon_1, \ldots, \epsilon_n \in U$, let $A_i \in \Gamma_{\ell,p}$ denote the matrix for multiplication by ϵ_i with respect to the basis $\{w_i\}$ of $\mathfrak{a}^{-1}\mathfrak{f}_0$, and define

$$\kappa_\mathfrak{a}(\epsilon_1, \ldots, \epsilon_n)(f) = \int_{F_p} f(x) d\mu_\ell(A_1, \ldots, A_n)(Q_\mathfrak{a}, v_\mathfrak{a}).$$

As in the previous section, we have identified $\mathbf{Q}_p^n$ with F_p via dot product with $(w_1, \ldots, w_n)$, and thereby view μ_ℓ as a compactly supported measure on F_p. The cocycle $\kappa_\mathfrak{a}$ yields a class $[\kappa_\mathfrak{a}] \in H^{n-1}(U, C_c^\flat(F_p)^\vee)$.

Meanwhile, for each nonnegative integer k we define a class $[\mathfrak{L}_k] \in H_{n-1}(U, C_c^\flat(F_p))$ as follows. Let $\epsilon_1, \ldots, \epsilon_{n-1}$ denote a basis of E, and define

$$\mathfrak{L}_k := \mathcal{A}(\epsilon_1, \ldots, \epsilon_{n-1}) \otimes \mathbf{1_O} \cdot (\log_p \mathrm{N}x)^k \in Z_{n-1}(U, C_c^\flat(F_p)),$$

where $\mathcal{A}$ is defined as in (55). Tracing through these notations, it is clear that the left side of (75) is given by

$$\int_\mathbf{O} (\log_p \mathrm{N}x)^k d\mu_\ell(A, Q, v) = \pm\langle [\kappa_\mathfrak{a}], [\mathfrak{L}_k] \rangle, \tag{77}$$

where the pairing on the right is the usual cap product

$$H^{n-1}(U, C_c^\flat(F_p)^\vee) \times H_{n-1}(U, C_c^\flat(F_p)) \longrightarrow \mathbf{C}_p$$

(and the $\pm$ is given by the sign ρ appearing in the definition (56) of A). In view of the discussion of Section 5.1 (in particular (75)) and (77), Theorem 3 is reduced to proving that

$$[\mathfrak{L}_k] = 0 \text{ in } H_{n-1}(U, C_c^\flat(F_p)) \text{ for } k < r. \tag{78}$$

The proof of (78) is broken into two steps. Let I denote the augmentation ideal of the group ring $\mathbf{C}_p[T]$.

Theorem 5.1 (Spiess, [Sp1]). *The natural map $H_{n-1}(U, C_c^\flat(F_p)) \longrightarrow H_{n-1}(E, C_c^\flat(F_p)/I)$ is an isomorphism.*

Theorem 5.2 (Spiess, [Sp1]). *For $k < r$, we have*

$$\mathbf{1_O} \cdot (\log_p \mathrm{N}x)^k \in I \cdot C_c^\flat(F_p),$$

and in particular the image of $[\mathfrak{L}_k]$ in $H_{n-1}(E, C_c^\flat(F_p)/I)$ vanishes.

Theorems 5.1 and 5.2 combine to yield (78), which in turn combined with (77) and (74) yields the proof of Theorem 3. For completeness, we recall Spiess's proofs of these results.

Lemma 5.3. *The space $C_c^\flat(F_p)$ is a free $\mathbf{C}_p[T]$-module.*

Proof. We show by induction on r that

$$C_c^\flat(F_S) := \bigotimes_{\mathfrak{p} \in S} C_c^\flat(F_\mathfrak{p})$$

is a free T-module. To this end, fix $\mathfrak{p} = \mathfrak{p}_r \in S$ and write $\pi = \pi_r$, $S' = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{r-1}\}$, and $T' = \langle \pi_i \rangle_{i=1}^{r-1}$, so $T = T' \times \langle \pi \rangle$. Our inductive hypothesis is that $C_c^\flat(F_{S'})$ is free as a $\mathbf{C}_p[T']$-module.

The space $C_c(F_\mathfrak{p}^*)$ of compactly supported continuous functions on $F_\mathfrak{p}^*$ can be identified with the subspace of $C_c^\flat(F_\mathfrak{p})$ consisting of those functions that vanish on a neighborhood of 0. Write $C_c^0(F_\mathfrak{p})$ for the space of compactly supported locally constant functions on $F_\mathfrak{p}$, so in particular $C_c^\flat(F_\mathfrak{p})$ is generated by its subspaces $C_c(F_\mathfrak{p}^*)$ and $C_c^0(F_\mathfrak{p})$, and we have

$$C_c^\flat(F_\mathfrak{p})/C_c^0(F_\mathfrak{p}) \cong C_c(F_\mathfrak{p}^*)/C_c^0(F_\mathfrak{p}^*).$$

We therefore obtain an exact sequence

$$0 \longrightarrow C_c^\flat(F_{S'}) \otimes C_c^0(F_\mathfrak{p}) \longrightarrow C_c^\flat(F_S) \longrightarrow C_c^\flat(F_{S'}) \otimes C_c(F_\mathfrak{p}^*)/C_c^0(F_\mathfrak{p}^*) \longrightarrow 0. \tag{79}$$

It suffices to prove that the first and third terms of the sequence are free $\mathbf{C}_p[T]$-modules.

Since $\mathcal{F}_\mathfrak{p} := \mathcal{O}_\mathfrak{p} - \pi\mathcal{O}_\mathfrak{p}$ is a fundamental domain for the action of π on $F_\mathfrak{p}^*$, we have

$$C_c^0(F_\mathfrak{p}) = C_c^0(F_\mathfrak{p}^*) \oplus \mathbf{C}_p 1_{\mathcal{O}_\mathfrak{p}} = (\mathrm{Ind}^{\langle \pi \rangle} C^0(\mathcal{F}_\mathfrak{p})) \oplus \mathbf{C}_p 1_{\mathcal{O}_\mathfrak{p}} \tag{80}$$

as $\mathbf{C}_p[T']$-modules. Choose a T'-stable decomposition $C^0(\mathcal{F}_\mathfrak{p}) = V \oplus \mathbf{C}_p 1_{\mathcal{F}_\mathfrak{p}}$ (for instance, we may take $V \subset C^0(\mathcal{F}_\mathfrak{p})$ to be the subspace of functions that have integral against Haar measure on $\mathcal{O}_\mathfrak{p}$ equal to 0). Using $1_{\mathcal{F}_\mathfrak{p}} = (1 - \pi) 1_{\mathcal{O}_\mathfrak{p}}$, one sees from (80) that

$$C_c^0(F_\mathfrak{p}) = \mathrm{Ind}^{\langle \pi \rangle}(V \oplus 1_{\mathcal{O}_\mathfrak{p}})$$

as $\mathbf{C}_p[T]$-modules. Lemma 5.4 below then implies that $C_c^\flat(F_{S'}) \otimes C_c^0(F_\mathfrak{p})$ is a free $\mathbf{C}_p[T]$-module.

Similarly,

$$C_c(F_\mathfrak{p}^*)/C_c^0(F_\mathfrak{p}^*) \cong \mathrm{Ind}^{\langle \pi \rangle}(C(\mathcal{F}_\mathfrak{p})/C^0(\mathcal{F}_\mathfrak{p}))$$

as a $\mathbf{C}_p[T]$-module, and hence the third term of (79) is a free $\mathbf{C}_p[T]$-module by Lemma 5.4. We conclude that $C_c^\flat(F_S)$ is a free $\mathbf{C}_p[T]$-module, and hence $C_c^\flat(F_p)$ is as well. $\square$

Lemma 5.4. *Let G_1 and G_2 be groups, $G = G_1 \times G_2$, and let K be a field. Let M_1, M_2 two $K[G]$-modules such that $M_2 \cong \mathrm{Ind}^{G_2} N_2$ as a $K[G]$-module for some G_1-module N_2, and $M_1 \cong \mathrm{Ind}^{G_1} N_1$ as $K[G_1]$-modules for some K-vector space N_1. Then $M_1 \otimes_K M_2$ is a free $K[G]$-module.*

Proof. One verifies that $M_1 \otimes_K M_2 \cong \mathrm{Ind}^G(N_1 \otimes_K \mathrm{Res}_1^{G_1} N_2)$. □

Lemma 5.3 immediately yields:

Proof of Theorem 5.1. Since $C_c^\flat(F_p)$ is a free $\mathbf{C}_p[T]$-module, it has homological dimension 1, and the Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(E, H_q(T, C_c^\flat(F_p))) \Longrightarrow H_{p+q}(U, C_c^\flat(F_p))$$

degenerates at E^2. Therefore the maps $E_n \longrightarrow E^2_{n,0}$ are isomorphisms as desired. □

We now move on to:

Proof of Theorem 5.2. As above we write $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. For any subset $R \subset S$, let

$$\mathbf{O}_R := \prod_{\mathfrak{p}_i \in R} (\mathcal{O}_{\mathfrak{p}_i} - \pi_i \mathcal{O}_{\mathfrak{p}_i}) \times \prod_{\mathfrak{p}_i \in S - R} \mathcal{O}_{\mathfrak{p}_i} \times \prod_{\mathfrak{p}|p,\ \mathfrak{p} \notin S} \mathcal{O}^*_{\mathfrak{p},\mathfrak{f}},$$

so in particular $\mathbf{O}_S = \mathbf{O}$.

From (76) it follows that $(\log_p \mathrm{N}x)^k$ can be expanded as a sum of monomials of the form

$$\ell(x)^n := \prod_{\mathfrak{p}|p} \ell_{\mathfrak{p}}(x)^{n_{\mathfrak{p}}},$$

where $n = (n_{\mathfrak{p}})_{\mathfrak{p}|p}$ is a tuple of nonnegative integers such that $|n| := \sum_{\mathfrak{p}} n_{\mathfrak{p}} = k$. We will prove by induction on $|n|$ that if $R \subset S$ is a subset such that $n_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in S - R$ and $|R| > |n|$, then

$$1_{\mathbf{O}_R} \cdot \ell(x)^n \in I \cdot C_c^\flat(F_p).$$

The desired result will then follow from the case $R = S$, $|n| = k$.

For the base case, $|n| = 0$, we choose $\mathfrak{p}_i \in R$ (which is possible since $|R| > 0$), let $R' = R - \{\mathfrak{p}_i\}$, and note

$$1_{\mathbf{O}_R} = (1 - \pi_i) \cdot 1_{\mathbf{O}_{R'}}. \tag{81}$$

The inductive step is similar, using the linearity of the functions $\ell_{\mathfrak{p}}$. Given $R \subset S$ such that $n_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in S - R$ and $|R| > |n|$, we may choose a $\mathfrak{p}_i \in R$ such that $n_{\mathfrak{p}_i} = 0$. Let $R' = R - \{\mathfrak{p}_i\}$. Then:

$$\begin{aligned}(1 - \pi_i)\ell(x)^n &= \prod_{\mathfrak{p}|p} \ell_{\mathfrak{p}}(x)^{n_{\mathfrak{p}}} - \prod_{\mathfrak{p}|p} (\ell_{\mathfrak{p}}(x) - \ell_{\mathfrak{p}}(\pi_i))^{n_{\mathfrak{p}}} \\ &= \sum_{n', |n'| < |n|} a_{n'} \ell(x)^{n'} \end{aligned} \tag{82}$$

for some coefficients $a_{n'}$ (where $n'_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in S - R'$ if $a_{n'} \neq 0$). Using the formula

$$((1-\pi)f) \cdot g = (1-\pi)(f \cdot g) - f \cdot ((1-\pi)g) + ((1-\pi)f) \cdot ((1-\pi)g)$$

applied with $f = 1_{\mathbf{O}_{R'}}$, $g = \ell(x)^n$, and $\pi = \pi_i$, equations (81) and (82) combine to give

$$\begin{aligned} 1_{\mathbf{O}_R} \cdot \ell(x)^n =& ((1-\pi_i)1_{\mathbf{O}_{R'}}) \cdot \ell(x)^n \\ \equiv& -1_{\mathbf{O}_{R'}} \cdot \sum_{n'} a_{n'} \ell(x)^{n'} + 1_{\mathbf{O}_R} \cdot \sum_{n'} a_{n'} \ell(x)^{n'} \pmod{I \cdot C_c^\flat(F_p)}. \end{aligned}$$

Each of the terms $1_{\mathbf{O}_{R'}} \cdot \ell(x)^{n'}$ and $1_{\mathbf{O}_R} \cdot \ell(x)^{n'}$ lies in $I \cdot C_c^\flat(F_p)$ by the inductive hypothesis, yielding $1_{\mathbf{O}_R} \cdot \ell(x)^n \in I \cdot C_c^\flat(F_p)$ as desired. □

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