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# Linear turnpike theorem

Emmanuel Trélat\*

## Abstract

The turnpike phenomenon stipulates that the solution of an optimal control problem in large time remains essentially close to a steady-state of the dynamics, itself being the optimal solution of an associated static optimal control problem. Under general assumptions, it is known that not only the optimal state and the optimal control, but also the adjoint state coming from the application of the Pontryagin maximum principle, are exponentially close to that optimal steady-state, except at the beginning and at the end of the time frame. In such a result, the turnpike set is a singleton, which is a steady-state.

In this paper, we establish a turnpike result for finite-dimensional optimal control problems in which some of the coordinates evolve in a monotone way, and some others are partial steady-states of the dynamics. We prove that the discrepancy between the optimal trajectory and the turnpike set is linear, but not exponential: we thus speak of a linear turnpike theorem.

## 1 Introduction and main results

### 1.1 Reminders on the exponential turnpike phenomenon

Let  $n, m \in \mathbb{N}^*$  and let  $T > 0$  be arbitrary. We consider a general nonlinear optimal control problem in  $\mathbb{R}^n$ , in fixed final time  $T$ :

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) \in M_0, \quad x(T) \in M_1 \quad (1)$$

$$(2)$$

$$\min \int_0^T f^0(x(t), u(t)) dt \quad (3)$$

where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $f^0 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  are mappings of class  $C^2$ ,  $M_0$  and  $M_1$  are subsets of  $\mathbb{R}^n$ , and  $u \in L^\infty([0, T], \Omega)$  is the control. For simplicity of the exposition, we assume that there exists  $T_0 > 0$  such that, for every  $T \geq T_0$ , there exists a unique optimal solution of (1)-(2)-(3), denoted by  $(x^*, u^*)$  (sufficient conditions ensuring existence are standard, see, e.g., [33]; uniqueness is ensured under differentiability properties of the value function and thus is generic in some sense). By the Pontryagin maximum principle (see [26]), there exist  $p^0 \leq 0$  and an absolutely continuous mapping (called adjoint vector), satisfying  $\dot{p} = -H_p(x, p, u)$ , such that for almost every  $t \in [0, T]$ , where  $H(x, p_x, p^0, u) = \langle p_x, f(x, u) \rangle + p^0 f^0(x, u)$  is the Hamiltonian of the optimal control problem (1)-(2)-(3) (here and throughout,  $\langle \cdot, \cdot \rangle$  is the Euclidean scalar product in  $\mathbb{R}^n$ , and  $\| \cdot \|$  is the corresponding norm). We have moreover the transversality conditions on the adjoint vector at initial and final time:

$$p_x^T(0) \perp T_{x^T(0)} M_0, \quad p_x^T(T) \perp T_{x^T(T)} M_1, \quad (4)$$

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provided that the tangent space  $T_{x^T(0)}M_0$  to  $M_0$  at  $x^T(0)$  and the tangent space  $T_{x^T(T)}M_1$  to  $M_1$  at  $x^T(T)$  exist (these conditions are empty when the initial and final points are fixed in the optimal control problem). We assume that the extremal lift  $(x^T(\cdot), p_x^T(\cdot), p^0, u^T(\cdot))$  is not abnormal and thus that one can take  $p^0 = -1$ . We also assume, for the simplicity of the exposition, that the extremal lift is unique (these two conditions are generic for large classes of optimal control problems, see [9, 10]).

The turnpike phenomenon stipulates that, when the final time  $T$  is large, the optimal solution  $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$  remains, along the time interval  $[0, T]$  and except around the initial time  $t = 0$  and around the final time  $t = T$ , “essentially close” to some static point  $(\bar{x}, \bar{p}_x, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^n \times \Omega$ , i.e., roughly speaking,

$$x^T(t) \simeq \bar{x}, \quad p_x^T(t) \simeq \bar{p}_x, \quad u^T(t) \simeq \bar{u} \quad \forall t \in [\eta, T - \eta] \quad (5)$$

(for some  $\eta > 0$ ). Moreover,  $(\bar{x}, \bar{p}_x, \bar{u})$  is the solution of a minimization problem called *static optimal control problem*, which is:

$$\min \{ f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, f(x, u) = 0 \} \quad (6)$$

i.e., in other words, the problem of minimizing the instantaneous cost  $f^0(x, u)$  over all possible equilibrium points of the dynamics. Note that, by the Lagrange multiplier rule, expressed in a Hamiltonian form, applied to the static optimal control problem, we have

$$\frac{\partial H}{\partial p_x}(\bar{x}, \bar{p}_x, p^0, \bar{u}) = 0, \quad \frac{\partial H}{\partial x}(\bar{x}, \bar{p}_x, p^0, \bar{u}) = 0, \quad \frac{\partial H}{\partial u}(\bar{x}, \bar{p}_x, p^0, \bar{u}) = 0. \quad (7)$$

Here we assume as well, for the simplicity of the exposition, that the minimizer  $(\bar{x}, \bar{u}) \in \mathring{\mathbb{R}}^n \times \mathring{\Omega}$  is unique, and also that the quadruple  $(\bar{x}, \bar{p}_x, p^0, \bar{u})$  is not abnormal, so that one can take  $p^0 = -1$  (usual qualification condition in optimization theory) and that the Lagrange multiplier is unique (the latter two conditions are generic).

We note then that  $(\bar{x}, \bar{p}_x, -1, \bar{u})$  is an equilibrium point of (??). Hence, the informal property (5) stipulates that the optimal triple  $(x(\cdot), p_x(\cdot), u(\cdot))$ , solution of (1)-(2)-(3)-(??) remains essentially close to an *equilibrium point* of the dynamics (??), this equilibrium being itself an optimal solution of the static problem (6). This is the turnpike phenomenon.

The intuition explaining why such a property is to be expected is the following. Since  $T$  is assumed to be large, we set  $\varepsilon = \frac{1}{T}$ , which is considered as a small parameter. Now, we make a time reparametrization, by setting  $s = \varepsilon t$ , so that, when  $t$  ranges over  $[0, T]$ ,  $s$  ranges over  $[0, 1]$ . Setting  $x_\varepsilon(s) = x^T(t)$ ,  $p_\varepsilon(s) = p_x^T(t)$  and  $u_\varepsilon(s) = u^T(t)$ , we have, still in an informal way,

$$\begin{aligned} \frac{\partial H}{\partial p_x}(x_\varepsilon(s), p_\varepsilon(s), -1, u_\varepsilon(s)) &= \varepsilon x'_\varepsilon(s) \simeq 0 \\ \frac{\partial H}{\partial x}(x_\varepsilon(s), p_\varepsilon(s), -1, u_\varepsilon(s)) &= -\varepsilon p'_\varepsilon(s) \simeq 0 \\ \frac{\partial H}{\partial u}(x_\varepsilon(s), p_\varepsilon(s), -1, u_\varepsilon(s)) &= 0 \end{aligned} \quad (8)$$

(the latter condition, assuming that the control is in the interior of  $\Omega$ ) which gives credibility to (5), at least when  $(\bar{x}, \bar{p}_x, \bar{u})$  is the unique solution of (7) (and under nondegenerate assumptions on the system of equations (7)). But this is not a proof and this remains informal: indeed, to justify (8) one would need that the derivatives  $x'_\varepsilon(s)$  and  $p'_\varepsilon(s)$  be bounded (except around  $t = 0$  and  $t = T$ ), uniformly with respect to  $\varepsilon$ , i.e., that the derivatives  $T\dot{x}^T(t)$  and  $T\dot{p}_x^T(t)$  be bounded

(except around  $t = 0$  and  $t = T$ ), uniformly with respect to  $T$ , which is precisely the hard part in establishing a turnpike result.

The main result of [37] states an *exponential turnpike property* which is of a local nature (see Remark 1). Hereafter, we combine this result with a general result of [35] in order to obtain a global exponential turnpike property. We use the notion of *strict dissipativity*, introduced in [39] and already used in [16, 19, 20, 21] to derive turnpike properties. We recall that the family of optimal control problems (1)-(2)-(3), indexed by  $T > 0$ , is said to be *strictly dissipative* at the optimal static point  $(\bar{x}, \bar{u})$  with respect to the *supply rate function*

$$w(x, u) = f^0(x, u) - f^0(\bar{x}, \bar{u}) \quad (9)$$

if there exists a locally bounded function  $S : \mathbb{R}^n \rightarrow \mathbb{R}$ , called *storage function*, such that

$$S(x(0)) + \int_0^T w(x(t), u(t)) dt \geq S(x(T)) + \int_0^T \alpha(\|x(t) - \bar{x}\|, \|u(t) - \bar{u}\|) dt \quad (10)$$

(strict dissipation inequality) for every  $t \in [0, T]$ , for every  $T > 0$ , for any pair  $(x(\cdot), u(\cdot))$  solution of the control system (1), for some function  $\alpha$  of class  $\mathcal{K}$ , i.e.,  $\alpha : [0, +\infty) \rightarrow [0, +\infty)$  continuous, increasing, and  $\alpha(0) = 0$ . Comments on dissipativity can be found in [7, 35, 39].

We use hereafter the shorter notations

$$\bar{H}_{\star\#} = \frac{\partial^2 H}{\partial \star \partial \#}(\bar{x}, \bar{p}_x, -1, \bar{u})$$

where  $\star$  and  $\#$  will be replaced with such or such variable. We set

$$\bar{A}_1 = \bar{H}_{p_x x} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}), \quad \bar{B}_1 = \bar{H}_{p_x u} = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}), \quad \bar{U} = -\bar{H}_{uu}$$

and, since we assume hereafter that  $\bar{U}$  is invertible,

$$A_1 = \bar{A}_1 + \bar{B}_1 \bar{U}^{-1} \bar{H}_{ux}, \quad \bar{W} = -\bar{H}_{xx} - \bar{H}_{xu} \bar{U}^{-1} \bar{H}_{ux}.$$

**Theorem 1.** *We make the following assumptions:*

*Assumptions of global nature:*

- (i) *There exist  $K > 0$  and  $T_0 > 0$  such that, for every  $T \geq T_0$ , there exists a unique optimal triple  $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$  solution of (1)-(2)-(3)-(4) (assumed to be normal); moreover,  $\|x^T(t)\| + \|u^T(t)\| \leq K$  for every  $t \in [0, T]$ .*
- (ii) *There exists a unique optimal triple  $(\bar{x}, \bar{p}_x, \bar{u})$  solution of (6)-(7), assumed to be normal; .*
- (iii) *The family of optimal control problems (1)-(2)-(3), indexed by  $T > 0$ , is strictly dissipative at the optimal static point  $(\bar{x}, \bar{u})$  with respect to the supply rate function  $w$  defined by (9), with a storage function  $S$  (satisfying (10)) that is bounded on  $M_0$  and  $M_1$ .*
- (iv) *For  $i = 1, 2$ , there exist  $t_i \geq 0$ , a control  $u_i(\cdot) \in L^\infty([0, t_i], \Omega)$  and a trajectory  $x_i(\cdot)$  solution of  $\dot{x}_i(t) = f(x_i(t), u_i(t))$  on  $[0, t_i]$ , satisfying  $\|x_i(t)\| + \|u_i(t)\| \leq K$  for every  $t \in [0, t_i]$  and the terminal conditions  $x_1(0) \in M_0$ ,  $x_1(t_1) = \bar{x}$ , and  $x_2(0) = \bar{x}$ ,  $x_2(t_2) \in M_1$ .*

*Assumptions of local nature:*

- (v)  $\bar{U} = -\bar{H}_{uu} = -\frac{\partial^2 H}{\partial u^2}(\bar{x}, \bar{p}_x, -1, \bar{u})$  *is a positive definite symmetric matrix.*

(vi)  $\bar{W}$  is a positive definite symmetric matrix.

(vii) The pair  $(\bar{A}_1, \bar{B}_1)$  satisfies the Kalman condition<sup>1</sup>, i.e.,  $\text{rank}(\bar{B}_1, \bar{A}_1\bar{B}_1, \dots, \bar{A}_1^{n-1}\bar{B}_1) = n$ . In other words, the linearized system at  $(\bar{x}, \bar{u})$  is controllable.

Then, there exist  $C > 0$  and  $\nu > 0$  such that if  $T \geq T_0$  then

$$\|x^T(t) - \bar{x}\| + \|p_x^T(t) - \bar{p}_x\| + \|u^T(t) - \bar{u}\| \leq C \left( e^{-\nu t} + e^{-\nu(T-t)} \right) \quad \forall t \in [0, T] \quad (11)$$

In other words, except around  $t = 0$  and  $t = T$ , the optimal triple  $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$  solution of (1)-(2)-(3)-(4) is exponentially close to the optimal triple  $(\bar{x}, \bar{p}_x, \bar{u})$  solution of (6)-(7). The constants  $C$  and  $\nu$  (not depending on  $T$ ) can even be made explicit by solving a Riccati equation (see [37] for details).

The local assumptions (v)-(vi)-(vii) done in this theorem, which concern the linearization at the turnpike, have been chiefly discussed and commented in [37].

The assumption (iv) is a global controllability assumption: there exists a trajectory steering in finite time the control system (1) from  $M_0$  to the turnpike  $\bar{x}$ , and from the turnpike  $\bar{x}$  to the final set  $M_1$ . But, such comments are completely general in optimal control theory and we thus do not comment more on the generality of our assumptions.

**Remark 1.** The above statement does not appear as such in the literature. Compared with [37] where the main result is established under the assumptions of local nature only (i.e., (v)-(vi)-(vii)), we have added here assumptions of global nature (i.e., (i)-(ii)-(iii)-(iv)): compactness and uniqueness of minimizers, strict dissipativity, global controllability to and from the turnpike set, and then we obtain a *global* turnpike property.

When the minimizers are not unique or when the system is not dissipative, we obtain only a *local* turnpike property, in the sense that the turnpike estimate (11) can only be expected to be satisfied in a neighborhood of an extremal steady-state  $(\bar{x}, \bar{p}_x, \bar{u})$  (see [37]). This is so, because the dynamics is nonlinear. In dynamical systems theory, when an equilibrium is non degenerate, the solutions of the nonlinear system resemble the solutions of the linearized system at the equilibrium, only locally around the equilibrium. The above turnpike theorem is of this nature.

It is worth noting that the exponential turnpike estimate (11) is global for non-degenerate linear-quadratic optimal control problems (by strict convexity), without having to assume (i)-(ii)-(iii).

When the static problem has several local minimizers, then one has local turnpike properties. When the static problem has, for instance, two global minimizers, then there is a region of the state space in which globally optimal solutions of the optimal control problems have a turnpike property corresponding to the first global minimizer, and another region in which the turnpike property corresponds to the second global minimizer. We refer to Appendix A for a discussion on these topics and for some numerical simulations.

The turnpike result of [37] has been extended to the infinite-dimensional setting in [35, 36] and in [15, 22]. One can find many turnpike results in the literature (see [8, 16, 20, 21, 27, 28, 40, 41], and see historical references in [37]), but the specificity of the above result is that the turnpike property is established as well for the adjoint vector: as explained in [37], this is particularly important in view of deriving appropriate initializations in numerical computation methods (direct methods or shooting methods, see [34]).

**Remark 2.** It is proved in [35] that “strong duality implies dissipativity”. To be more precise, we recall that the static problem (6) has the *strong duality property* if  $(\bar{x}, \bar{u})$  minimizes the *Lagrangian*

<sup>1</sup>Equivalently, the pair  $(A_1, \bar{B}_1)$  satisfies the Kalman condition.

function  $L(\cdot, \cdot, \bar{p}_x) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $L(x, u, \bar{p}_x) = f^0(x, u) - \langle \bar{p}_x, f(x, u) \rangle$  (the sign minus is due to the fact that we took  $p^0 = -1$  for the Lagrange multiplier associated with the cost). This notion is well known in classical optimization theory, in relation with primal and dual problems. It is proved in [35, Theorem 3] that the strong duality property implies the dissipativity property, with the storage function  $S(x) = \langle \bar{p}_x, x \rangle$ .

Note that strong duality is in some sense an infinitesimal version of the dissipation inequality, at least when the storage function is continuously differentiable (see also [16, 18]).

Until now, we have discussed a turnpike phenomenon around an *equilibrium point* of the dynamics. In the next section, we state a turnpike theorem for systems having some coordinates that evolve in a monotone way and thus have no equilibrium point.

## 1.2 A new result: linear turnpike phenomenon

### 1.2.1 Framework

We keep the notations introduced in Section 1.1, and we add to the control system (1)-(2)-(3)  $p$  additional differential equations, for some  $p \in \mathbb{N}^*$ , as follows. We consider the general optimal control problem in  $\mathbb{R}^{n+p}$  in fixed final time  $T > 0$ , without control constraint:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) \in M_0, \quad x(T) \in M_1 \quad (12)$$

$$\dot{y}(t) = g(x(t), u(t)), \quad y(0) = y_0, \quad y(T) = y_1^T \quad (13)$$

$$u(t) \in \Omega \quad (14)$$

$$\min \int_0^T f^0(x(t), u(t)) dt \quad (15)$$

where  $y_0, y_1^T \in \mathbb{R}^p$  are fixed and  $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$  is of class  $C^2$  (the rest of the assumptions is unchanged, with respect to Section 1.1).

Compared with (1)-(2)-(3), we have added the  $p$  differential equations  $\dot{y}(t) = g(x(t), u(t))$ , where  $g$  does not depend on the new coordinate  $y \in \mathbb{R}^p$ . The instantaneous cost function  $f^0$  also, still does not depend on  $y$ . This is important to derive the result hereafter.

As in the previous section, the final time  $T$  is fixed in the optimal control problem (12)-(13)-(14)-(15) and the linear turnpike property will describe the behavior of optimal solutions when  $T$  is large.

In the optimal control problem (12)-(13)-(14)-(15), the data  $M_0, M_1$  and  $y_0$  do not depend on  $T$ , but the final condition  $y_1^T$  depends on  $T$ : we have thus put a superscript  $T$  in order to underline this important dependence. We have indeed in mind to consider cases where  $\|y_1^T\|$  is large as  $T$  is large: in such cases, we expect that the components of  $y(t)$  should evolve in a monotone way, i.e.,  $g(x(t), u(t)) \neq 0$  along the optimal trajectory: we have then no equilibrium in the  $y$  components.

We are going to derive a turnpike result in which the “turnpike set” consists of a *partial equilibrium*, i.e., the turnpike is a path  $t \mapsto (\bar{x}, \bar{y}(t), \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$  where  $f(\bar{x}, \bar{u}) = 0$ .

**Remark 3.** Following [13, 14], it is worth noting that such problems (12)-(13)-(14)-(15) frequently appear in applications, for instance in mechanical problems where, denoting by  $x(t)$  a position variable, by  $v(t) = \dot{x}(t)$  the speed variable, one has a system of the form

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = f(v(t), u(t)).$$

The result that we are going to derive allows one to obtain a turnpike phenomenon along a trajectory  $x(t) = \bar{v}t$ , where  $\bar{v} \in \mathbb{R}^n$  is such that  $f(\bar{v}, \bar{u}) = 0$ . Here,  $(\bar{v}, \bar{u})$  is an equilibrium of  $f$ , but

$(\bar{v}t, \bar{v}, \bar{u})$  is not an equilibrium of the complete dynamics. In [13, 14], this phenomenon is called a “velocity turnpike”. Because of the estimate that we are going to establish in Theorem 2 hereafter, we rather speak of a “linear turnpike” phenomenon.

Note that, in (12)-(13)-(14)-(15), we have fixed the initial and final conditions on  $y(t)$ . Indeed, otherwise, if either  $y(0)$  or  $y(T)$  is let free, then there is nothing new, because  $f$  and  $f^0$  do not depend on  $y$  and then (12)-(13)-(14)-(15) is then equivalent to (1)-(2)-(3), with the coordinate  $y(t)$  evolving without having any impact on the optimal control problem (1)-(2)-(3). But here, we impose  $y(0) = y_0$  and  $y(T) = y_1^T$ , which creates additional constraints with respect to (1)-(2)-(3), and then the turnpike result stated in Theorem 1 cannot be applied.

### 1.2.2 A motivating example

In order to shed light on what can be expected, let us consider the very simple example in  $\mathbb{R}^2$ :

$$\begin{aligned} \dot{x}(t) &= u(t), & x(0) &= 1, & x(T) &= 2 \\ \dot{y}(t) &= x(t), & y(0) &= 0, & y(T) &= y_1^T \\ \min \int_0^T (x(t)^2 + u(t)^2) dt \end{aligned} \tag{16}$$

for which the optimal trajectory can be computed explicitly. Here,  $y_1^T \in \mathbb{R}$  is a smooth function of  $T$  that is sublinear at infinity. For instance, one can take  $y_1^T = \alpha T$  or  $y_1^T = \alpha T \sin(T)$  for some  $\alpha \in \mathbb{R}$ . We also assume that  $\Omega = \mathbb{R}$ , i.e., there is no control constraint.

In a first step, if we ignore the dynamics in  $y$  in (16), then we are in the framework of Theorem 1. The static optimal control problem consists of minimizing  $x^2 + u^2$  under the constraint  $u = 0$ , which gives the optimal solution  $\bar{x} = 0$ ,  $\bar{u} = 0$  with the Lagrange multiplier  $\bar{p}_x = 0$  (which is normal; here, we take  $p^0 = -1/2$ ). By Theorem 1, we obtain  $|x^T(t)| + |p_x^T(t)| + |u^T(t)| \leq C(e^{-\nu t} + e^{-\nu(T-t)})$  for every  $t \in [0, T]$ . In other words, roughly,  $x(t) \simeq 0$  and  $u(t) \simeq 0$ , except around  $t = 0$  and  $t = T$ .

Now, let us take into account the additional dynamics in  $y$  with the terminal conditions on  $y$ . We are not anymore in the framework of Theorem 1, because there is no term in  $y(t)^2$  in the cost and therefore the matrix  $W$  is not positive definite. We define the as

$$\min\{x^2 + u^2 \mid (x, u) \in \mathbb{R} \times \mathbb{R}, \quad u = 0, \quad Tx = y_1^T\} \tag{17}$$

i.e., with respect to the previous static problem, we add the terminal constraints coming from the dynamics in  $y$ : the unique solution of the Cauchy problem  $\dot{y}(t) = x$  (with  $x$  constant),  $y(0) = 0$ , must satisfy  $y(T) = y_1^T$ .

It is easy to check that the (unique) optimal solution of (17) is  $\bar{x}^T = \frac{y_1^T}{T}$ ,  $\bar{u}^T = 0$ , with Lagrange multipliers  $\bar{p}_x^T = 0$  associated with the constraint  $u = 0$ , and  $\bar{p}_y^T = \frac{y_1^T}{T}$  associated with the constraint  $x = \frac{y_1^T}{T}$  (which are normal; here, we take  $p^0 = -1/2$ ). The corresponding trajectory in  $y$  is  $\bar{y}^T(t) = \frac{y_1^T}{T}t$  (and we have  $\bar{y}^T(T) = y_1^T$  as expected). For this example, we expect that, if  $T > 0$  is large, then  $x^T(t) \simeq 0$ ,  $u^T(t) \simeq 0$  and  $y^T(t) \simeq \bar{y}^T(t)$  for  $t \in [0, T]$ , where the order of this approximation is to be determined, what we do hereafter by explicit calculations.

Let us apply the Pontryagin maximum principle to (16). The Hamiltonian is  $H = p_x u + p_y x - \frac{1}{2}(x^2 + u^2)$  (there is no abnormal minimizer). Then  $u^T(t) = p_x^T(t)$  and the extremal system is

$$\dot{x}^T(t) = p_x^T(t), \quad \dot{y}^T(t) = x^T(t), \quad \dot{p}_x^T(t) = x^T(t) - p_y^T(t)$$

with  $p_y^T$  constant. We have then

$$\begin{aligned}\frac{d}{dt} \begin{pmatrix} x^T(t) - p_y^T \\ p_x^T(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^T(t) - p_y^T \\ p_x^T(t) \end{pmatrix} \\ \frac{d}{dt} y^T(t) &= x^T(t)\end{aligned}$$

In particular, the dynamics in  $(x^T - p_y^T, p_x^T)$  is hyperbolic: setting  $v^T(t) = x^T(t) - p_y^T + p_x^T(t)$  and  $w^T(t) = x^T(t) - p_y^T - p_x^T(t)$ , we have

$$\dot{v}^T(t) = v^T(t), \quad \dot{w}^T(t) = -w^T(t)$$

with terminal conditions  $v^T(0) + w^T(0) = 2 - 2p_y^T$ ,  $v^T(T) + w^T(T) = 4 - 2p_y^T$ . The solutions of this differential system are given by the familiar phase portrait around a saddle point, and we infer that

$$|v^T(t)| + |w^T(t)| = O(e^{-t} + e^{-(T-t)}) + O(|p_y^T|(e^{-t} + e^{-(T-t)}))$$

for every  $t \in [0, T]$ , where the  $O(\cdot)$  is uniform with respect to  $T$ , and thus

$$|x^T(t) - p_y^T| + |p_x^T(t)| = O(e^{-t} + e^{-(T-t)}) + O(|p_y^T|(e^{-t} + e^{-(T-t)})).$$

In particular,

$$x^T(t) = p_y^T + O(e^{-t} + e^{-(T-t)}) + O(|p_y^T|(e^{-t} + e^{-(T-t)}))$$

and, integrating  $\dot{y}^T(t) = x^T(t)$  we obtain  $y^T(t) = p_y^T t + O(1) + O(|p_y^T|)$  for every  $t \in [0, T]$ . Since  $y^T(T) = y_1^T$ , we obtain that

$$p_y^T = \frac{y_1^T}{T} + O\left(\frac{1}{T}\right) + O\left(\frac{|p_y^T|}{T}\right),$$

and since  $y_1^T$  is sublinear at infinity, we conclude that

$$p_y^T = \bar{p}_y^T + O\left(\frac{1}{T}\right).$$

Therefore

$$x^T(t) = \frac{y_1^T}{T} + O\left(\frac{1}{T} + e^{-t} + e^{-(T-t)}\right) = \bar{x}^T + O\left(\frac{1}{T} + e^{-t} + e^{-(T-t)}\right)$$

and

$$y^T(t) = \frac{y_1^T}{T} t + O(1) = \bar{y}^T(t) + O(1)$$

for every  $t \in [0, T]$ . In turn, we have also  $p_x^T(t) = O(e^{-t} + e^{-(T-t)})$ .

We conclude that there exists  $C > 0$  (independent of  $T$ ) such that

$$\boxed{\begin{aligned} |x^T(t) - \bar{x}^T| &\leq C \left( \frac{1}{T} + e^{-t} + e^{-(T-t)} \right) \\ |u^T(t) - \bar{u}^T| + |p_x^T(t) - \bar{p}_x^T| &\leq C \left( e^{-t} + e^{-(T-t)} \right) \\ |y^T(t) - \bar{y}^T(t)| &\leq C \\ |p_y^T - \bar{p}_y^T| &\leq \frac{C}{T} \end{aligned}} \quad (18)$$



for every  $t \in [0, T]$  (with  $\bar{p}_x^T = 0$  and  $\bar{p}_y^T = \frac{y_1^T}{T}$ ). Comparing with Theorem 1, we see on this example that we do not have an exponential turnpike phenomenon for the component  $x^T(t)$ : the above inequality is weaker; it says that, except around  $t = 0$  and  $t = T$ ,  $x^T(t)$  is bounded by  $\frac{1}{T}$  (which is small as  $T$  is large), while the component  $y^T(t)$  remains in a bounded neighborhood of  $\bar{y}^T(t) = \frac{y_1^T}{T}$ .

In what follows, we are going to generalize the above example.

### 1.2.3 Main result

As in Section 1.1, we assume that there exists a unique optimal solution  $(x^T(\cdot), y^T(\cdot), u^T(\cdot))$  of (12)-(13)-(14)-(15). With respect to (1)-(2)-(3), the Hamiltonian of (12)-(13)-(14)-(15) is now

$$H(x, p_x, p_y, p^0, u) = \langle p_x, f(x, u) \rangle + \langle p_y, g(x, u) \rangle + p^0 f^0(x, u).$$

Note that  $H$  does not depend on the variable  $y$ . Assuming as well that, for every  $T \geq T_0$ , the optimal solution has a unique extremal lift, which is not abnormal (so that we take  $p^0 = -1$ ), the application of the Pontryagin maximum principle leads to the existence of an absolutely continuous mapping  $p_x^T(\cdot) : [0, T] \rightarrow \mathbb{R}^n$  and of a constant  $p_y^T \in \mathbb{R}^p$  such that

$$\begin{aligned} \dot{x}^T(t) &= \frac{\partial H}{\partial p_x}(x^T(t), p_x^T(t), p_y^T, -1, u^T(t)) = f(x^T(t), u^T(t)) \\ \dot{y}^T(t) &= \frac{\partial H}{\partial p_y}(x^T(t), p_x^T(t), p_y^T, -1, u^T(t)) = g(x^T(t), u^T(t)) \\ \dot{p}_x^T(t) &= -\frac{\partial H}{\partial x}(x^T(t), p_x^T(t), p_y^T, -1, u^T(t)) \\ &= -\frac{\partial f}{\partial x}(x^T(t), u^T(t))^\top p_x^T(t) - \frac{\partial g}{\partial x}(x^T(t), u^T(t))^\top p_y^T + \frac{\partial f^0}{\partial x}(x^T(t), u^T(t))^\top \end{aligned} \tag{19}$$

( $p_y^T$  is constant because  $\frac{\partial H}{\partial y} = 0$ ) and

$$u^T(t) = \operatorname{argmax}_{v \in \Omega} H(x^T(t), p_x^T(t), p_y^T, -1, v) \tag{20}$$

for almost every  $t \in [0, T]$ .

With respect to (6), the is now the following one:

$$\min \left\{ f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, \quad f(x, u) = 0 \right. \\ \left. \dot{y}(t) = g(x, u), \quad y(0) = y_0, \quad y(T) = y_1^T \right\} \tag{21}$$

where we have added the constraint coming from the dynamics and terminal conditions in  $y$ .

We assume that, , there exists a unique optimal solution and that it has a unique extremal (Lagrange multiplier) lift, which is not abnormal. Integrating the differential equation in (21), noting that  $(x, u)$  is constant, the optimal control problem (21) is equivalent to

$$\min \left\{ f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, \quad f(x, u) = 0, \quad g(x, u) = \frac{y_1^T - y_0}{T} \right\} \tag{22}$$

which is a classical optimization problem under equality constraints. The Lagrange multiplier rule, applied to the solution  $(\bar{x}^T, \bar{y}^T(\cdot), \bar{u}^T)$  of this optimization problem, implies (since it is assumed not to be abnormal) that  $df^0(\bar{x}^T, \bar{u}^T)$  can be expressed linearly in function of  $df(\bar{x}^T, \bar{u}^T)$  and  $dg(\bar{x}^T, \bar{u}^T)$ . In the next lemma, we write these conditions in an equivalent Hamiltonian form, which seems more complicated but is exactly devised to be comparable to (19).

**Lemma 1.** *There exist  $\bar{p}_x^T \in \mathbb{R}^n$  and  $\bar{p}_y^T \in \mathbb{R}^p$  such that, for every  $t \in [0, T]$ ,*

$$\begin{aligned} 0 &= \frac{\partial H}{\partial p_x}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = f(\bar{x}^T, \bar{u}^T) \\ \dot{y}(t) &= \frac{\partial H}{\partial p_y}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = g(\bar{x}^T, \bar{u}^T), \quad \bar{y}^T(0) = y_0, \quad \bar{y}^T(T) = y_1^T \\ 0 &= -\frac{\partial H}{\partial x}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = -\frac{\partial f}{\partial x}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_x^T - \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_y^T + \frac{\partial f^0}{\partial x}(\bar{x}^T, \bar{u}^T) \\ 0 &= -\frac{\partial H}{\partial y}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) \end{aligned}$$

and

$$\frac{\partial H}{\partial u}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = \frac{\partial f}{\partial u}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_x^T + \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_y^T - \frac{\partial f^0}{\partial u}(\bar{x}^T, \bar{u}^T) = 0.$$

This lemma follows by applying the Lagrange multiplier rule to the optimization problem (22) (see also Lemma 2 in Section 2).

We observe that

$$\bar{y}^T(t) = y_0 + tg(\bar{x}^T, \bar{u}^T) \quad \forall t \in [0, T] \quad (23)$$

and therefore  $y_0 + Tg(\bar{x}^T, \bar{u}^T) = y_1^T$ . When  $g(\bar{x}^T, \bar{u}^T) \neq 0$ , this can only be if the norm of  $y_1^T - y_0$  is large, as  $T$  is large. In the proof of the turnpike theorem hereafter, we are going to consider  $T$  as a parameter, tending to  $+\infty$ . This is why we have underlined the dependence of  $y_1^T$  with respect to  $T$ : when  $T$  becomes larger, it may be required that  $y_1^T - y_0$  becomes larger as well (otherwise, the controllability problem may have no solution).

We use the same notations as before, and additionally, we set

$$\begin{aligned} \bar{A}_2^T &= \bar{H}_{p_y x} = \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T), \quad \bar{B}_2^T = \bar{H}_{p_y u} = \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T), \quad A_2^T = \bar{A}_2^T + \bar{B}_2^T(\bar{U}^T)^{-1} \bar{H}_{u x}, \\ \bar{A}^T &= \begin{pmatrix} \bar{A}_1^T \\ \bar{A}_2^T \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(\bar{x}^T, \bar{u}^T) \\ \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T) \end{pmatrix}, \quad \bar{B}^T = \begin{pmatrix} \bar{B}_1^T \\ \bar{B}_2^T \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u}(\bar{x}^T, \bar{u}^T) \\ \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T) \end{pmatrix}. \end{aligned}$$

In our main result hereafter, we make all assumptions done in Theorem 1 and we adapt them in an obvious way to the present context where the turnpike set is not anymore the singleton  $\{\bar{x}\}$ , but is now the path  $t \mapsto (\bar{x}^T, \bar{y}^T(t))$  that we call the *turnpike trajectory*. For instance, the controllability assumption (iv) now stipulates that there exists a trajectory steering in finite time some point of the initial set  $M_0$  to some point of the turnpike trajectory, and that any point of the turnpike trajectory can be steered in finite time to the final set  $M_1$ .

**Theorem 2.** *There exist  $K > 0$  and  $T_0 > 0$  such that, for every  $T \geq T_0$ :*

– *Assumptions of global nature:*

- (i) *There exists a unique optimal tuple  $(x^T(\cdot), y^T(\cdot), p_x^T(\cdot), p_y^T(\cdot), u^T(\cdot))$  solution of (12)-(13)-(14)-(15)-(19)-(20)-(4) (assumed to be normal); moreover,  $\|x^T(t)\| + \|u^T(t)\| \leq K$  for every  $t \in [0, T]$ .*
- (ii) *There exists a unique optimal tuple  $(\bar{x}^T, \bar{y}^T(\cdot), \bar{p}_x^T, \bar{p}_y^T, \bar{u}^T)$  solution of (21) (with  $\bar{y}^T(\cdot)$  given by (23)) and of the first-order optimality system stated in Lemma 1, assumed to be normal; moreover,  $\|\bar{x}^T\| + \|\bar{u}^T\| \leq K$  and*

(iii) The family of optimal control problems (12)-(13)-(14)-(15), indexed by  $T > 0$ , is with respect to the supply rate function  $w$  defined by (9), with a storage function  $S$  (satisfying (10)), with  $\alpha$  not depending on  $T$ ) that is bounded on  $M_0$  and  $M_1$ , uniformly with respect to  $T$ .

(iv) There exists a trajectory  $(\tilde{x}^T(\cdot), \tilde{y}^T(\cdot), \tilde{u}^T(\cdot))$  solution of (12)-(13) on  $[0, T]$ , such that  $\|\tilde{x}^T(t)\| + \|\tilde{u}^T(t)\| \leq K$  for every  $t \in [0, T]$  and  $\tilde{x}^T(t) = \bar{x}^T$  and  $\tilde{u}^T(t) = \bar{u}^T$  for every  $t \in [t_1^T, T - t_2^T]$ , for some  $t_1^T \geq 0$  and  $t_2^T \geq 0$  satisfying  $t_i^T/T \rightarrow 0$  as  $T \rightarrow +\infty$ .

– Assumptions of local nature:

(v)  $\bar{U}^T = -\bar{H}_{uu} = -\frac{\partial^2 H}{\partial u^2}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T)$  is a positive definite symmetric matrix.

(vi)  $\bar{W}^T$  is a positive definite symmetric matrix.

(vii) The pair  $(\bar{A}_1^T, \bar{B}_1^T)$  satisfies the Kalman condition<sup>2</sup>, i.e.,  $\text{rank}(\bar{B}_1^T, \bar{A}_1^T \bar{B}_1^T, \dots, (\bar{A}_1^T)^{n-1} \bar{B}_1^T) = n$ . In other words, the system (12) in  $x$  (not in  $y$ ), linearized at  $(\bar{x}^T, \bar{u}^T)$ , is controllable.

(viii)  $\ker((\bar{A}^T)^\top) \cap \ker((\bar{B}^T)^\top) = \{0\}$ .<sup>3</sup>

In addition, we assume that there exists  $C_1 > 0$  such that:

(ix)  $\|y_1^T\| \leq C_1 T$  for every  $T \geq T_0$ .

(x)

Then, there exist  $C > 0$  and  $\nu > 0$  such that if  $T \geq T_0$  then

$$\boxed{\begin{aligned} \|x^T(t) - \bar{x}^T\| + \|p_x^T(t) - \bar{p}_x^T\| + \|u^T(t) - \bar{u}^T\| &\leq C \left( \frac{1}{T} + e^{-\nu t} + e^{-\nu(T-t)} \right) \\ \|y^T(t) - \bar{y}^T(t)\| &\leq C, \quad \|p_y^T - \bar{p}_y^T\| \leq \frac{C}{T} \end{aligned}} \quad (24)$$

for every  $t \in [0, T]$ .

The first estimate in (24) states that, except around  $t = 0$  and  $t = T$ , the discrepancies  $x^T(t) - \bar{x}^T$ ,  $p_x^T(t) - \bar{p}_x^T$  and  $u^T(t) - \bar{u}^T$  are bounded above by  $\frac{1}{T}$ , which is small as  $T$  is large, but not exponentially small: it is weaker than (11), and we speak of a *linear turnpike* estimate. The second estimate in (24) gives a bound (uniform with respect to  $T$ ) on the discrepancy  $y^T(t) - \bar{y}^T(t)$  along the whole interval  $[0, T]$ : it says that  $y^T(t)$  remains at a uniform (wrt  $T$ ) distance of  $\bar{y}^T(t) = y_0 + tg(\bar{x}^T, \bar{u}^T)$  as  $t \in [0, T]$ . Finally, the third estimate in (24) says that  $p_y^T - \bar{p}_y^T$  is (linearly) small as  $T$  is large.

Of course, on specific examples it may happen that some of the components of the extremal triple  $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$  enjoy the exponential turnpike property, i.e., an estimate that is stronger than (24), without the term  $\frac{1}{T}$  at the right-hand side. This is the case in the example (16): one can see in (18) that the component  $x^T$  satisfies the linear turnpike estimate (24), while the components  $p_x^T$  and  $u^T$  satisfy the stronger exponential turnpike estimate (11).

**Remark 4.** As said earlier, to simplify the exposition we have assumed that, for every  $T \geq T_0$ , the optimal control problem (12)-(13)-(14)-(15) has a unique solution, whose extremal lift is as well unique and normal, and that the static optimal control problem (22) has a unique solution, whose

<sup>2</sup>Equivalently, the pair  $(A_1^T, B_1^T)$  satisfies the Kalman condition.

<sup>3</sup>This condition is satisfied if the pair  $(\bar{A}^T, \bar{B}^T)$  satisfies the Kalman condition, i.e.,  $\text{rank}(\bar{B}^T, \bar{A}^T \bar{B}^T, \dots, (\bar{A}^T)^{n+p-1} \bar{B}^T) = n + p$ , or equivalently, if the system (12)-(13) (i.e., the full system), linearized at  $(\bar{x}^T, \bar{u}^T)$ , is controllable. .

Lagrange multiplier is as well unique and normal. In case the optimal solution is not unique, or the extremal lift (still assumed to be normal) is not unique, the statement of Theorem (2) must be adapted as in [37]: we obtain in this case a result that is satisfied *locally* around the extremal triple.

When the optimal control problem (12)-(13)-(14)-(15) is linear-quadratic,

$$\begin{aligned} \dot{x}(t) &= \bar{A}_1^T x(t) + \bar{B}_1^T u(t), & x(0) &\in M_0, \quad x(T) \in M \\ \dot{y}(t) &= \bar{A}_2^T x(t) + \bar{B}_2^T u(t), & y(0) &= y_0, \quad y(T) = y_1^T \\ \min \int_0^T & (x(t)^\top \bar{W}^T x(t) + u(t)^\top \bar{U}^T u(t)) dt \end{aligned} \quad (25)$$

i.e., when  $f(x, u) = \bar{A}_1^T x + \bar{B}_1^T u$ ,  $g(x, u) = \bar{A}_2^T x + \bar{B}_2^T u$ ,  $f^0(x, u) = x^\top \bar{W}^T x + u^\top \bar{U}^T u$  and  $\Omega = \mathbb{R}^m$ , the result of Theorem 2 is automatically global, by strict convexity, without having to assume any dissipativity property.

Theorem 2 is proved in Section 2. In Section 3 we give several optimal control problems that illustrate our main result and we provide numerical simulations showing evidence of the linear turnpike phenomenon.

## 2 Proof of Theorem 2

### 2.1 Preliminaries on the turnpike-static optimal control problem

Let us consider the optimal control problem (21), written in the form of the optimization problem (22). Recall that, by Assumption (ii), there exists a unique minimizer  $(\bar{x}^T, \bar{u}^T)$  of (22) and that  $\bar{p}_x^T \in \mathbb{R}^n$  and  $\bar{p}_y^T \in \mathbb{R}^p$  are the (normal) Lagrange multipliers respectively associated to the two constraints.

**Lemma 2.**

$$(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T) = O(1) \quad (26)$$

as  $T \rightarrow +\infty$ .

*Proof.* The optimality system (??) is written as

$$F(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T) = \begin{pmatrix} df(\bar{x}^T, \bar{u}^T)^\top \bar{p}_x^T + dg(\bar{x}^T, \bar{u}^T)^\top \bar{p}_y^T - df^0(\bar{x}^T, \bar{u}^T)^\top \\ f(\bar{x}^T, \bar{u}^T) \\ g(\bar{x}^T, \bar{u}^T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{y_1^T - y_0}{T} \end{pmatrix}$$

which is a system of  $2n + m + p$  equations with the  $2n + m + p$  unknowns  $(x, u, p_x, p_y)$ , where  $F : \mathbb{R}^{2n+m+p} \rightarrow \mathbb{R}^{2n+m+p}$  is a  $C^1$  mapping. The Jacobian of  $F$  at the point  $(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)$ , which is the usual sensitivity matrix in optimization, is

$$dF(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T) = \left( \begin{array}{cc|cc} \bar{H}_{xx} & \bar{H}_{xu} & \bar{H}_{xp_x} & \bar{H}_{xp_y} \\ \bar{H}_{ux} & \bar{H}_{uu} & \bar{H}_{up_x} & \bar{H}_{up_y} \\ \hline \bar{H}_{p_x x} & \bar{H}_{p_x u} & 0 & 0 \\ \bar{H}_{p_y x} & \bar{H}_{p_y u} & 0 & 0 \end{array} \right) = \left( \begin{array}{c|c} E_1 & E_2^\top \\ \hline E_2 & 0 \end{array} \right)$$

Noting that

$$E_2 = \begin{pmatrix} \frac{\partial f}{\partial x}(\bar{x}^T, \bar{u}^T) & \frac{\partial f}{\partial u}(\bar{x}^T, \bar{u}^T) \\ \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T) & \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T) \end{pmatrix} = (\bar{A}^T \quad \bar{B}^T),$$

Assumption (viii) immediately implies that the matrix  $E_2$  is surjective.

Besides, let us prove that the matrix  $E_1$  is invertible. If  $E_1 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$  then  $\bar{H}_{xx}\xi_1 + \bar{H}_{xu}\xi_2 = 0$  and  $\bar{H}_{ux}\xi_1 + \bar{H}_{uu}\xi_2 = 0$ . From the latter equation, we get  $\xi_2 = -\bar{H}_{uu}^{-1}\bar{H}_{ux}\xi_1$ , and plugging into the first equation yields  $(\bar{H}_{xx} - \bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_{ux})\xi_1 = 0$ . Since, by Assumption (vi), the matrix  $\bar{W}^T = -\bar{H}_{xx} + \bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_{ux}$  is symmetric positive definite, we infer that  $\xi_1 = 0$ , and then  $\xi_2 = 0$ . Therefore  $E_1$  is invertible.

Now, since  $E_1$  is invertible and  $E_2$  is surjective, we conclude that  $dF(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)$  is invertible. By Assumption (ix), we have  $\frac{y_1^T - y_0^T}{T} = O(1)$  as  $T \rightarrow +\infty$ . Hence, we have to solve the nonlinear system of equations  $F(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T) = O(1)$ . The continuity property follows by applying the implicit function theorem. □

## 2.2 Convergence to the turnpike

Recall that, by Assumption (i), the unique solution  $(x^T(\cdot), y^T(\cdot), u^T(\cdot))$  of the optimal control problem (12)-(13)-(14)-(15), has a unique extremal lift  $(x^T(\cdot), y^T(\cdot), p_x^T(\cdot), p_y^T(\cdot), u^T(\cdot))$  which is moreover normal.

*Proof.* The partial  $x$ -strict dissipativity inequality (10) implies that

$$f^0(\bar{x}^T, \bar{u}^T) \leq \frac{1}{T} \int_0^T f^0(x^T(t), u^T(t)) dt + \frac{S(x^T(0)) - S(x^T(T))}{T} - \frac{1}{T} \int_0^T \alpha(\|x^T(t) - \bar{x}^T\|, \|u^T(t) - \bar{u}\|) dt \quad (27)$$

Let us prove that  $\frac{1}{T} \int_0^T \alpha(\|x^T(t) - \bar{x}^T\|, \|u^T(t) - \bar{u}^T\|) dt \rightarrow 0$  as  $T \rightarrow +\infty$ . By contradiction, assume that there exists  $\eta > 0$  and a sequence  $T_k \rightarrow +\infty$  such that

$$\frac{1}{T_k} \int_0^{T_k} \alpha(\|x^{T_k}(t) - \bar{x}^{T_k}\|, \|u^{T_k}(t) - \bar{u}^{T_k}\|) dt \geq \eta \quad (28)$$

for every  $k$  large enough. The cost of the trajectory  $(\tilde{x}^T(\cdot), \tilde{y}^T(\cdot))$  solution of (12)-(13) given by Assumption (iv) is

$$\begin{aligned} \frac{1}{T} \int_0^T f^0(\tilde{x}^T(t), \tilde{u}^T(t)) dt &= \frac{1}{T} \int_0^{t_1^T} f^0(\tilde{x}^T(t), \tilde{u}^T(t)) dt + \frac{1}{T} \int_{T-t_2^T}^T f^0(\tilde{x}^T(t), \tilde{u}^T(t)) dt \\ &\quad + \frac{T - t_1^T - t_2^T}{T} f^0(\bar{x}^T, \bar{u}^T) = f^0(\bar{x}^T, \bar{u}^T) + o(1) \end{aligned}$$

as  $T \rightarrow +\infty$ . Hence, for  $T$  large enough, using (27) and (28), we have obtained a trajectory solution of (12)-(13) whose cost is (strictly) less than the cost of the optimal trajectory  $(x^T(\cdot), y^T(\cdot), u^T(\cdot))$ : this is a contradiction. □

### 2.3 Linear turnpike estimates

Let us now introduce perturbation variables, by setting

$$\begin{aligned}\delta x^T(t) &= x^T(t) - \bar{x}^T, & \delta y^T(t) &= y^T(t) - \bar{y}^T(t), \\ \delta p_x^T(t) &= p_x^T(t) - \bar{p}_x^T, & \delta p_y^T &= p_y^T - \bar{p}_y^T, & \delta u^T(t) &= u^T(t) - \bar{u}^T\end{aligned}$$

Linearizing (??) and then (19) at  $(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, \bar{u}^T)$ , we get, at the first order (i.e., neglecting the terms in  $\delta x^T, \delta p_x^T, \delta p_y^T, \delta u^T$  of order  $\geq 2$ ),

$$\delta u^T(t) = (\bar{U}^T)^{-1} (\bar{H}_{ux} \delta x^T(t) + (\bar{B}_1^T)^\top \delta p_x^T(t) + (\bar{B}_2^T)^\top \delta p_y^T(t)) \quad (29)$$

(where we have used Assumption (v)) and then, still at the first order,

$$\begin{aligned}\delta \dot{x}^T(t) &= A_1^T \delta x^T(t) + \bar{B}_1^T (\bar{U}^T)^{-1} (\bar{B}_1^T)^\top \delta p_x^T(t) + \bar{B}_1^T (\bar{U}^T)^{-1} (\bar{B}_2^T)^\top \delta p_y^T(t) \\ \delta \dot{p}_x^T(t) &= \bar{W}^T \delta x^T(t) - (A_1^T)^\top \delta p_x^T(t) - (A_2^T)^\top \delta p_y^T(t)\end{aligned} \quad (30)$$

where we recall that  $\delta p_y^T$  is constant. Setting

$$z^T(t) = \begin{pmatrix} \delta x^T(t) \\ \delta p_x^T(t) \end{pmatrix}$$

and defining the matrices

$$M^T = \begin{pmatrix} A_1^T & \bar{B}_1^T (\bar{U}^T)^{-1} (\bar{B}_1^T)^\top \\ \bar{W}^T & -(A_1^T)^\top \end{pmatrix}, \quad V^T = \begin{pmatrix} -\bar{B}_1^T (\bar{U}^T)^{-1} (\bar{B}_2^T)^\top \\ (A_2^T)^\top \end{pmatrix}, \quad (31)$$

we write the system (30) in the form

$$\dot{z}^T(t) = M^T z^T(t) - V^T \delta p_y^T. \quad (32)$$

$$\delta \dot{y}^T(t) = L^T z^T(t) + \bar{B}_2^T (\bar{U}^T)^{-1} (\bar{B}_2^T)^\top \delta p_y^T \quad (33)$$

where

$$L^T = (A_2^T \quad \bar{B}_2^T (\bar{U}^T)^{-1} (\bar{B}_1^T)^\top).$$

Note that, when  $\bar{H}_{ux} = 0$ , the system (32)-(33) corresponds exactly to the extremal system associated with the linear-quadratic optimal control problem (25).

We have the following spectral property for the matrix  $M^T$ .

**Lemma 3.** *The matrix  $M^T$  is hyperbolic, i.e., all (complex) eigenvalues of  $M^T$  have a nonzero real part. Moreover, there exists  $\eta > 0$  such that, for every  $T \geq T_0$ , every eigenvalue  $\lambda$  of  $M_T$  satisfies  $|\operatorname{Re}(\lambda)| \geq \eta$ .*

*Proof.* Let  $\mu \in \mathbb{R}$  and let  $\xi \in \mathbb{C}^{2n}$  be such that  $(M^T - i\mu I_{2n})\xi = 0$ . Let us prove that  $\xi = 0$ . Denoting by  $(\xi_1, \xi_2)$  the coordinates of  $\xi$ , we have

$$\begin{aligned}(A_1^T - i\mu I_n)\xi_1 + \bar{B}_1^T (\bar{U}^T)^{-1} (\bar{B}_1^T)^\top \xi_2 &= 0 \\ \bar{W}^T \xi_1 - ((A_1^T)^\top + i\mu I_n)\xi_2 &= 0\end{aligned} \quad (34)$$

We stress that all matrices above are real, but the coordinates  $(\xi_1, \xi_2)$  are complex. Using Assumption (vi), the second equation in (34) gives  $\xi_1 = (\bar{W}^T)^{-1} ((A_1^T)^\top + i\mu I_n)\xi_2$ , and plugging this expression in the first equation of (34) yields

$$(A_1^T - i\mu I_n) (\bar{W}^T)^{-1} ((A_1^T)^\top + i\mu I_n)\xi_2 + \bar{B}_1^T (\bar{U}^T)^{-1} (\bar{B}_1^T)^\top \xi_2 = 0$$

and multiplying this equality to the left by the complex conjugate  $\bar{\xi}_1^\top$  gives

$$\|(\bar{W}^T)^{-1/2}((A_1^T)^\top + i\mu I_n)\xi_2\|^2 + \|(\bar{U}^T)^{-1/2}(\bar{B}_1^T)^\top \xi_2\|^2 = 0. \quad (35)$$

Here, we have used the fact that  $\bar{U}^T$  and  $\bar{W}^T$  are symmetric positive definite matrices (by Assumptions (v) and (vi)), and we have used that the complex conjugate of  $A_1^T - i\mu I_n$  is  $\overline{(A_1^T - i\mu I_n)}^\top = (A_1^T)^\top + i\mu I_n$ . The norms and scalar product used in (35) are Hermitian. Now, (35) implies that  $((A_1^T)^\top + i\mu I_n)\xi_2 = 0$  and  $(\bar{B}_1^T)^\top \xi_2 = 0$ , i.e.,

$$\xi_2 \in \ker((A_1^T)^\top + i\mu I_n) \cap \ker((\bar{B}_1^T)^\top). \quad (36)$$

Since the pair  $(A_1^T, \bar{B}_1^T)$  satisfies the Kalman condition (see Assumption (vii) and Footnote 2 in Theorem 1), we have  $\ker((A_1^T)^\top - \lambda I_n) \cap \ker((\bar{B}_1^T)^\top) = \{0\}$  for every  $\lambda \in \mathbb{C}$  (Hautus test: see, e.g., [30, 33]). Then, (36) gives  $\xi_2 = 0$ .

The second equation of (34) gives then  $\xi_1 = 0$ . We conclude that  $\xi = 0$ .  $\square$

Since  $M^T$  is hyperbolic, first of all, it is invertible and then (32) can be written as

$$\frac{d}{dt} (z^T(t) - (M^T)^{-1}V^T \delta p_y^T) = M^T (z^T(t) - (M^T)^{-1}V^T \delta p_y^T).$$

Second, there exists a real-valued invertible matrix  $P^T$  of size  $2n$  such that

$$(P^T)^{-1}M^T P^T = \begin{pmatrix} M_1^T & 0 \\ 0 & M_2^T \end{pmatrix}$$

where all (complex) eigenvalues of the real-valued matrix  $M_1^T$  have a negative real part and all (complex) eigenvalues of the real-valued matrix  $M_2^T$  have a positive real part. Setting<sup>4</sup>

$$z^T(t) - (M^T)^{-1}V^T \delta p_y^T = P^T \begin{pmatrix} v_1^T(t) \\ v_2^T(t) \end{pmatrix}$$

we have then  $\dot{v}_1^T(t) = M_1^T v_1^T(t)$  and  $\dot{v}_2^T(t) = M_2^T v_2^T(t)$ . Since all complex eigenvalues of  $M_1^T$  (resp., of  $M_2^T$ ) have a negative (resp., positive) real part, far from 0 uniformly with respect to  $T$ , there exist  $C > 0$  and  $\nu > 0$ , not depending on  $T$ , such that  $\|v_1^T(t)\| \leq C e^{-\nu t}$  and  $\|v_2^T(t)\| \leq C e^{-\nu(T-t)}$  for every  $t \in [0, T]$ .

It remains to estimate the norm of  $\delta p_y^T$ . Integrating (33), using that  $\int_0^T e^{-\nu t} dt \leq \frac{1}{\nu}$  for every  $T > 0$ , and defining the square matrix of size  $p$

$$R^T = L^T (M^T)^{-1} V^T + \bar{B}_2^T (U^T)^{-1} (\bar{B}_2^T)^\top \quad (37)$$

we infer from (??) that (taking a larger constant  $C$  if necessary)

$$\|\delta y^T(t) - t R^T \delta p_y^T\| \leq C \quad \forall t \in [\varepsilon T, (1 - \varepsilon)T]. \quad (38)$$

Since we must have  $\delta y(T) = 0$  (this comes from the final condition  $y(T) = y_1^T$  on the component  $y$ ), and since  $\varepsilon > 0$  is arbitrary, we infer that

$$\|R^T \delta p_y^T\| \leq \frac{C}{T}. \quad (39)$$

We have the following instrumental fact.

---

<sup>4</sup>Actually, since the matrix  $M^T$  is *Hamiltonian* (see [37] for some remarks on this issue), the size of  $v_1$  is equal to the size of  $v_2$  (equal to  $n$ ).

**Lemma 4.** *The matrix  $R^T$  defined by (37) is symmetric positive definite, and thus is invertible.*

Lemma 4 is proved in Section 2.4.

It follows from (39) and from Lemma 4 that (taking a larger constant  $C$  if necessary)  $\|\delta p_y^T\| \leq \frac{C}{T}$ . We infer from (??) and (38) that

$$\|z^T(t)\| \leq C \left( \frac{1}{T} + e^{-\nu t} + e^{-\nu(T-t)} \right), \quad \|\delta y^T(t)\| \leq C \quad \forall t \in [0, T],$$

which gives the estimates stated in Theorem 2.

**Remark 5.** In (12)-(13)-(14)-(15), the dynamics in  $y$  is written as  $\dot{y}(t) = h(x(t), y(t), u(t))$  with  $h(x, y, u) = g(x, u)$ : the mapping  $h$  does not depend on  $y$ . Our strategy of proof would not apply to a general mapping  $h$ , depending on  $y$ . Indeed, otherwise, when linearizing  $\dot{y}(t) = h(x(t), y(t), u(t))$  along the path  $(\bar{x}, \bar{y}(t), \bar{u})$ , we would get  $\delta \dot{y}(t) = A_2 \delta x(t) + \bar{B}_2 \delta u(t) + Q(t) \delta y(t)$  where  $Q(t) = \frac{\partial h}{\partial y}(\bar{x}, \bar{y}(t), \bar{u})$  depends on  $t$  and thus the matrix  $M$  defined by (31) would as well depend on  $t$  (anyway, only through the part in  $Q$ ). The argument of hyperbolicity, then, cannot be applied in general, in particular we think that it may fail whenever the eigenvectors of  $Q(t)$  are oscillating too fast. Nevertheless, we think that the technique of proof may be extended to the case where  $t \mapsto Q(t)$  is slowly-varying, as in the context of control or stabilization by quasi-static deformation (see [12] and references therein). We leave this issue open.

## 2.4 Proof of Lemma 4

In this section, we do not write the superscript  $T$  in all matrices, to facilitate the reading. The uniform property stated in the lemma is obtained by compactness.

Let us first express the matrix  $R$ , defined by (37), in a more explicit way. Recall that

$$R = (A_2 \bar{B}_2 U^{-1} \bar{B}_1^\top) M^{-1} \begin{pmatrix} -\bar{B}_1 U^{-1} \bar{B}_2^\top \\ A_2^\top \end{pmatrix} + \bar{B}_2 U^{-1} \bar{B}_2^\top \quad \text{with} \quad M = \begin{pmatrix} A_1 & \bar{B}_1 U^{-1} \bar{B}_1^\top \\ W & -A_1^\top \end{pmatrix}.$$

We set

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M^{-1} \begin{pmatrix} -\bar{B}_1 U^{-1} \bar{B}_2^\top \\ A_2^\top \end{pmatrix}$$

so that

$$\begin{aligned} A_1 C_1 + \bar{B}_1 U^{-1} \bar{B}_1^\top C_2 &= -\bar{B}_1 U^{-1} \bar{B}_2^\top \\ W C_1 - A_1^\top C_2 &= A_2^\top \end{aligned} \tag{40}$$

Since  $W$  is invertible, we infer from the second equation of (40) that  $C_1 = W^{-1} A_1^\top C_2 + W^{-1} A_2^\top$ , and plugging this expression into the first equation of (40) gives

$$(A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top) C_2 = -A_1 W^{-1} A_2^\top - \bar{B}_1 U^{-1} \bar{B}_2^\top \tag{41}$$

**Lemma 5.** *The matrix  $A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top$  is invertible.*

*Proof.* Let  $\xi \in \mathbb{R}^n$  be such that  $(A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top) \xi = 0$ . Multiplying to the left by  $\xi^\top$ , we obtain  $\|W^{-1/2} A_1^\top \xi\|^2 + \|U^{-1/2} \bar{B}_1^\top \xi\|^2 = 0$  and hence  $\xi \in \ker(A_1^\top) \cap \ker(\bar{B}_1^\top)$ . As in the proof of Lemma 3, since the pair  $(A_1, \bar{B}_1)$  satisfies the Kalman condition, the latter intersection is  $\{0\}$ . The claim follows.  $\square$



We infer from (41) and from Lemma 5 that

$$C_2 = -(A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top)^{-1} (A_1 W^{-1} A_2^\top + \bar{B}_1 U^{-1} \bar{B}_2^\top)$$

and thus

$$C_1 = -W^{-1} A_1^\top (A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top)^{-1} (A_1 W^{-1} A_2^\top + \bar{B}_1 U^{-1} \bar{B}_2^\top) + W^{-1} A_2^\top.$$

We conclude that

$$R = -(A_1 W^{-1} A_2^\top + \bar{B}_1 U^{-1} \bar{B}_2^\top)^\top (A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top)^{-1} (A_1 W^{-1} A_2^\top + \bar{B}_1 U^{-1} \bar{B}_2^\top) + A_2 W^{-1} A_2^\top + \bar{B}_2 U^{-1} \bar{B}_2^\top$$

(note that  $R$  is symmetric) that we can write as the sum of two symmetric matrices

$$\begin{aligned} R &= A_2 W^{-1/2} \left( -W^{-1/2} A_1^\top (A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top)^{-1} A_1 W^{-1/2} + I_n \right) W^{-1/2} A_2^\top \\ &\quad + \bar{B}_2 U^{-1/2} \left( -U^{-1/2} \bar{B}_1^\top (A_1 W^{-1} A_1^\top + \bar{B}_1 U^{-1} \bar{B}_1^\top)^{-1} \bar{B}_1 U^{-1/2} + I_m \right) U^{-1/2} \bar{B}_2^\top \\ &= \tilde{A}_2 \left( -\tilde{A}_1^\top (\tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top)^{-1} \tilde{A}_1 + I_n \right) \tilde{A}_2^\top + \tilde{B}_2 \left( -\tilde{B}_1^\top (\tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top)^{-1} \tilde{B}_1 + I_m \right) \tilde{B}_2^\top \end{aligned}$$

where we have set  $\tilde{A}_i = A_i W^{-1/2}$  and  $\tilde{B}_i = \bar{B}_i U^{-1/2}$  for  $i = 1, 2$ , and we are going to prove hereafter that the above two matrices are symmetric positive definite, so that  $R$  is itself symmetric positive definite.

We have the following general result.

**Lemma 6.** *Let  $A$  be an arbitrary real-valued square matrix of size  $n$ , and let  $B$  be an arbitrary real-valued matrix of size  $n \times m$ , where  $n$  and  $m$  are arbitrary nonzero integers. If  $\ker(A^\top) \cap \ker(B^\top) = \{0\}$  then*

$$B^\top (AA^\top + BB^\top)^{-1} B \preceq I_m \quad (42)$$

meaning that the symmetric matrix  $I_m - B^\top (AA^\top + BB^\top)^{-1} B$  is positive semi-definite.

*Proof.* Since  $\ker(A^\top) \cap \ker(B^\top) = \{0\}$ , the matrix  $AA^\top + BB^\top$  is invertible (same argument as in Lemma 5).

When  $B$  is invertible, the result is obvious: starting from

$$AA^\top + BB^\top \succeq BB^\top,$$

taking the inverse and multiplying to the left by  $B^\top$  and to the right by  $B$ , we obtain (42).

When  $B$  is not invertible, we follow anyway the above reasoning, adding  $\varepsilon I_m$  for  $\varepsilon > 0$  to recover an appropriate invertibility property: starting from

$$AA^\top + BB^\top + \varepsilon I_m \succeq BB^\top + \varepsilon I_m,$$

taking the inverse and multiplying to the left by  $B^\top$  and to the right by  $B$ , we obtain

$$B^\top (AA^\top + BB^\top + \varepsilon I_m)^{-1} B \preceq B^\top (BB^\top + \varepsilon I_m)^{-1} B.$$

Then, taking the limit  $\varepsilon \rightarrow 0$  gives (42).  $\square$

Noting that  $\ker(\tilde{A}_1^\top) \cap \ker(\tilde{B}_1^\top) = \{0\}$  (this comes again from the Kalman condition on the pair  $(A_1, \bar{B}_1)$ ), we infer from Lemma 6 that

$$\tilde{A}_1^\top \left( \tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top \right)^{-1} \tilde{A}_1 \preceq I_n \quad \text{and} \quad \tilde{B}_1^\top \left( \tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top \right)^{-1} \tilde{B}_1 \preceq I_m.$$

Therefore  $R \succeq 0$ , i.e.,  $R$  is positive semi-definite.

### 3 Examples

In this section, we illustrate the linear turnpike phenomenon on several practical examples, with numerical simulations. Hereafter, we compute numerically the optimal solutions either with a direct method, using `AMPL` (see [17]) combined with `IpOpt` (see [38]), following a full discretization of the problem, or with the variant of the shooting method introduced in [37]. In both cases, we initialize the numerical method at the turnpike, thus guaranteeing its convergence.

#### 3.1 The Zermelo problem

The Zermelo problem is a famous optimal control problem, often used as a simple model or exercise to illustrate theory or practice of optimal control (see [33]). In this problem, we consider a boat moving with constant speed along a river of constant width  $\ell$ , in which there is a current  $c(x)$ . The movement of the center of mass of the boat is governed by the control system

$$\begin{aligned} \dot{x}(t) &= v(t) \sin u(t), & x(0) &= 0, \\ \dot{y}(t) &= v(t) \cos u(t) + c(x(t)), & y(0) &= 0, \end{aligned}$$

where the controls are the modulus of the speed  $v(t) \in [0, v_{\max}]$  and the angle  $u(t) \in \mathbb{R}$  of the axis of the boat with respect to the axis  $(0y)$  (see Figure 2).

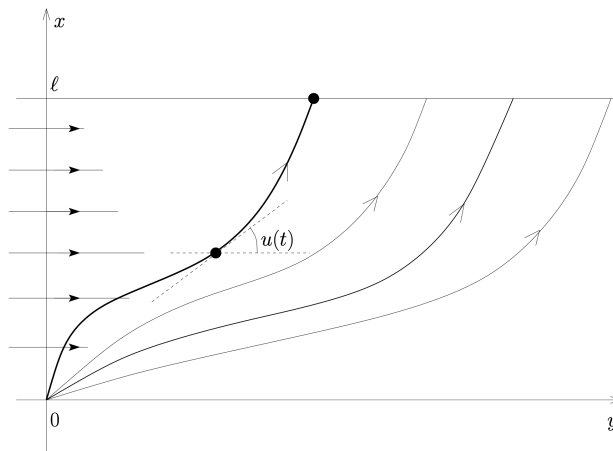


Figure 1: Zermelo problem.

Many variants of terminal conditions and of cost functionals can be considered. Here, in order to illustrate the linear turnpike phenomenon, given some  $T > 0$  fixed (to be chosen large enough), we consider the optimal control problem of steering in time  $T$  this control system from the initial point  $(0, 0)$  to the final point  $(\ell, 5T)$  (on the opposite river), by minimizing the cost functional  $\int_0^T v(t)^2 dt$ .

We take  $L = 2$ ,  $v_{\max} = 1.1$  and  $c(x) = 3 + x(L - x)$ . The current in the river is so strong that we always have  $\dot{y}(t) > 0$ , whatever the control may be; hence  $y(\cdot)$  is bound to be increasing.

The turnpike-static optimal control problem consists of minimizing  $v^2$  under the constraints  $v \sin u = 0$  and  $T(v \cos u + c(y)) = 5T$ . It has the unique solution  $\bar{x}^T = 1$ ,  $\bar{y}^T(t) = 5t$ ,  $\bar{x}^T = 1$ ,  $\bar{u}^T = 0$ ,  $\bar{v}^T = 1$ ,  $\bar{p}_x^T = 0$ ,  $\bar{p}_y^T = 2$ .

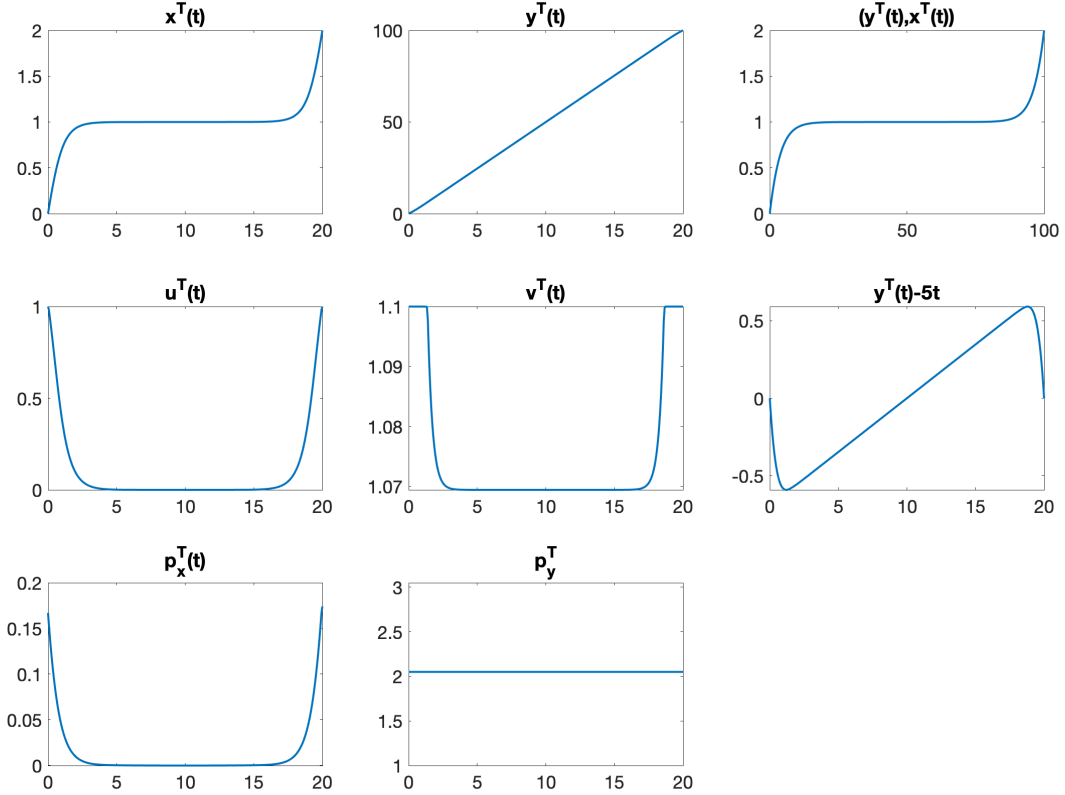


Figure 2: Numerical results for the Zermelo problem

The numerical results for  $T = 20$  are reported on Figure 2. We observe, as expected, that  $|y^T(t) - \bar{y}^T(t)|$  is bounded by a constant. Besides, we observe that, in the long middle part of the time interval,  $(x^T(t), u^T(t), p_x^T(\cdot))$  is exponentially close to  $(\bar{x}^T, \bar{u}^T, \bar{p}_x^T)$  while  $(v^T(t), p_y^T)$  is linearly close to  $(\bar{v}^T, \bar{p}_y^T)$ .

Note that the optimal control  $v^T(\cdot)$  has two bang arcs along which  $v^T(t) = v_{\max} = 1.1$ : a short one at the beginning, and another short one at the end.

### 3.2 Optimal control model of a runner

Consider the runner model developed in [2, 3, 4, 5], originating from [24]:

$$\begin{aligned}
 \dot{x}(t) &= v(t) & x(0) &= 0, & x(t_f) &= d \\
 \dot{v}(t) &= -\frac{v(t)}{\tau} + f(t) & v(0) &= v^0 \\
 \dot{e}(t) &= \sigma - f(t)v(t) & e(0) &= e^0, & e(t_f) &= 0, & e(t) &\geq 0 \\
 \dot{f}(t) &= \gamma(u(t)(F_{\max} - f(t)) - f(t)) & 0 &\leq f(t) \leq F_{\max}
 \end{aligned} \tag{43}$$

where  $d > 0$  is the prescribed distance to run,  $e^0 > 0$  is the initial energy,  $\tau > 0$  is the friction coefficient related to the runner's economy,  $\sigma > 0$  is a constant standing for the energetic equivalent of the oxygen uptake  $VO_2$ ,  $F_{\max} > 0$  is a threshold upper bound for the force  $f(t)$ ,  $\gamma > 0$  is the time constant of motor activation and  $u(t) \in [-M, M]$  is the neural drive which is the control, where  $M > 0$  is some control bound. Here,  $x(t)$  is the distance travelled at time  $t$  by the runner,  $v(t)$  the velocity,  $e(t)$  the anaerobic energy, and  $f(t)$  the propulsive force per unit mass. Actually, the typical values of the initial energy  $e_0$  are such that the energy variable  $e(t)$  is decreasing. The minimization criterion is

$$\min \left( t_f + \frac{\alpha}{2} \int_0^{t_f} u(t)^2 dt \right) \quad (44)$$

where  $\alpha > 0$  is a fixed constant and the final time  $t_f$  is free.

Although, in the optimal control problem (43)-(44),  $t_f$ ,  $v(t_f)$ ,  $f(0)$  and  $f(t_f)$  are let free, the problem can be reparametrized by the distance  $s$ , with the change of variable  $t'(s) = \frac{dt}{ds} = \frac{1}{v(s)}$ . In terms of  $s$ , the optimal control problem is

$$\begin{aligned} v'(s) &= -\frac{1}{\tau} + \frac{f(s)}{v(s)} & v(0) &= v^0 \\ e'(s) &= \frac{\sigma}{v(s)} - f(s) & e(0) &= e^0, \quad e(d) = 0, \quad e(s) \geq 0 \\ f'(s) &= \frac{\gamma}{v(s)} (u(s)(F_{\max} - f(s)) - f(s)) & 0 &\leq f(s) \leq F_{\max} \\ \min \int_0^d \frac{1}{v(s)} \left( 1 + \frac{\alpha}{2} u(s)^2 \right) ds \end{aligned}$$

and now fits in the general framework developed in this paper.

Solving the "static" problem leads to  $\bar{e}(t) = e_0 + (\sigma - \bar{f}\bar{v})t$  (we re-express it in function of  $t$ ) where

$$\bar{v} = \frac{\tau}{2d} \left( e_0 + \sqrt{e_0^2 + 4\frac{\sigma d^2}{\tau}} \right), \quad \bar{f} = \frac{\bar{v}}{\tau} = \frac{1}{2d} \left( e_0 + \sqrt{e_0^2 + 4\frac{\sigma d^2}{\tau}} \right), \quad \bar{u} = \frac{\bar{f}}{F_{\max} - \bar{f}}.$$

For the numerical simulations, we take  $d = 1500$ ,  $\tau = 0.932$ ,  $\sigma = 22$ ,  $\alpha = 10^{-5}$ ,  $F_{\max} = 8$ ,  $\gamma = 0.0025$ ,  $v_0 = 3$ ,  $e_0 = 4651$ . We obtain  $\bar{v} = 6.2$ ,  $\bar{f} = 6.65$ ,  $\bar{u} = 4.92$ . The distance run (1500 meters) is large enough so that we observe the linear turnpike phenomenon. The numerical results are reported on Figure 3.

This runner optimal control problem has actually been the initial point for the present paper, and the author warmly thanks Amandine Aftalion for having raised such an interesting problem. The linear turnpike phenomenon has been recently exploited in [5].

## A Appendix

This appendix is devoted to illustrating the comments done in Remark 1. We take a very simple example and we give numerical simulations showing the competition between two global turnpikes, or, at the level of the initialization of numerical methods, between local and global turnpikes.

We consider the one-dimensional optimal control problem

$$\begin{aligned} \dot{x} &= -3x + 3x^3 + u, & x(0) &= x_0, \quad x(T) = x_f \text{ or free} \\ \min \int_0^T & ((x(t) - x_d)^2 + (u(t) - u_d)^2) dt \end{aligned} \quad (45)$$

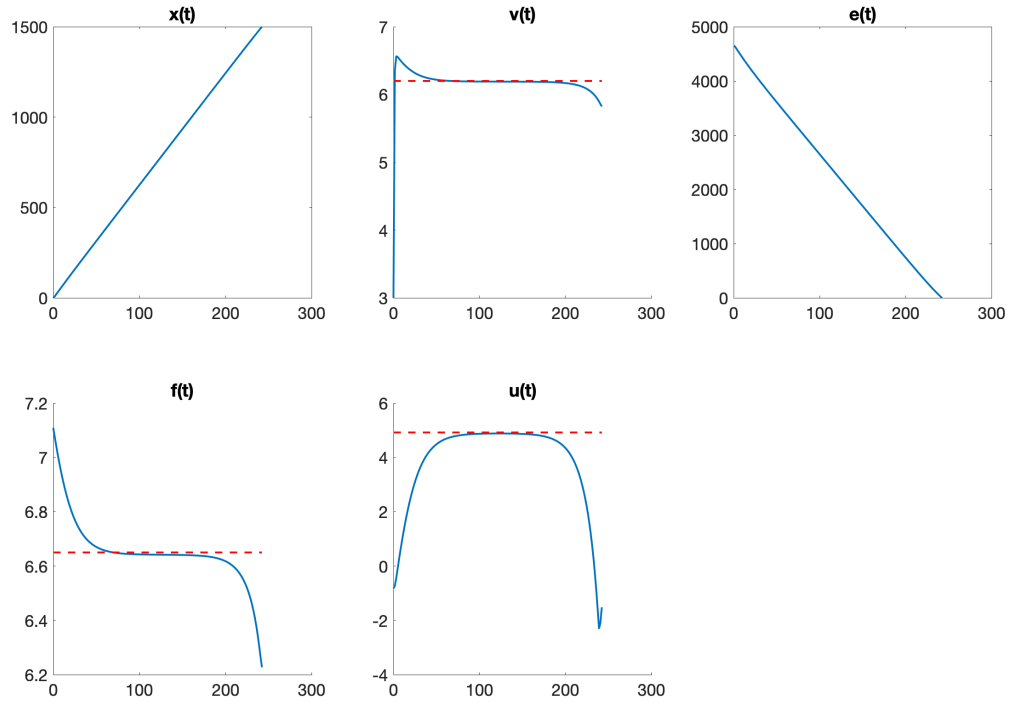


Figure 3: Numerical results for the runner problem

The corresponding static problem is

$$\min_{(x,u) \mid -3x+3x^3+u=0} ((x - x_d)^2 + (u - u_d)^2). \quad (46)$$

### A.1 Competition between two (global) turnpikes

We take  $x_d = 1$  and  $u_d = 3.47197$ . The choice of  $u_d$  is done so that the static problem (46) has two global minima, at  $\bar{x}_1 = -1.347372066$  and  $\bar{x}_2 = 0.5939615956$ , see Figure 4.

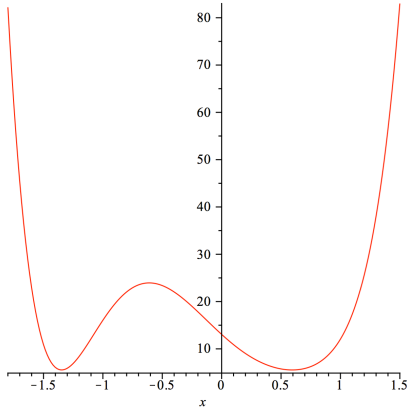


Figure 4: Plot of the function  $x \mapsto (x - 1)^2 + (3x - 3x^3 - 3.47197)^2$

On Figure 5, in dashed blue, we have computed the optimal trajectory with  $x_0 = -5$ ,  $x_f = -1$ ,  $T = 10$ : we observe a turnpike phenomenon around  $\bar{x}_1$ . In solid red, we have computed the optimal trajectory with  $x_0 = 2$ ,  $x_f = 1$ ,  $T = 10$ : we observe a turnpike phenomenon around  $\bar{x}_2$ .

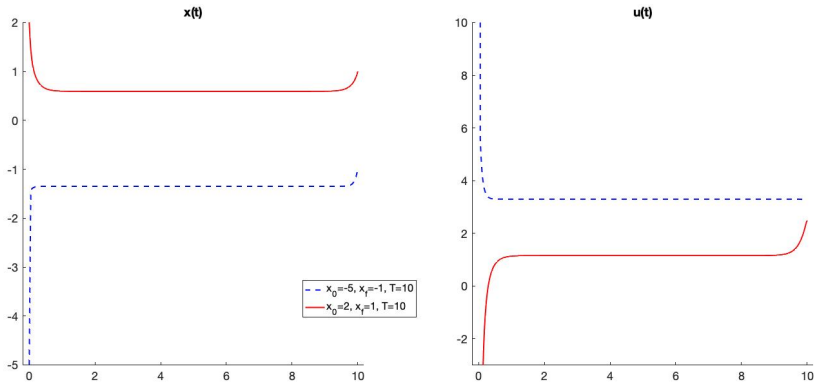


Figure 5: Global turnpikes around  $\bar{x}_1$  and  $\bar{x}_2$ .

The fact that the turnpike phenomenon is either around  $\bar{x}_1$  or around  $\bar{x}_2$  depends on the terminal conditions. For instance, if  $x_0$  and  $x_f$  are close to  $\bar{x}_1$  (resp.,  $\bar{x}_2$ ) then the optimal trajectory will make a turnpike around  $\bar{x}_1$  (resp.,  $\bar{x}_2$ ). But when the terminal conditions are farther, it is not clear to predict the behavior of the optimal trajectory.

Since the minima are global, we expect to observe a competition between both turnpikes, depending on the terminal conditions (see [29]). Let us provide some numerical simulations illustrating this competition. To facilitate the understanding, we consider the problem with  $x_f$  free. It is interesting to note that the numerical result strongly depends on the initialization of the numerical method. Here, to compute numerically the optimal solutions of (45), we use **AMPL** (see [17]) combined with **IpOpt** (see [38]): the trajectory and the controls are discretized (the control is piecewise constant and the trajectory is piecewise linear, on a given subdivision that is chosen fine enough), and we initialize the trajectory with the same constant value over all the subdivision.

Then according to the value of this initialization, we can make emerge such or such turnpike, and all solutions are anyway optimal.

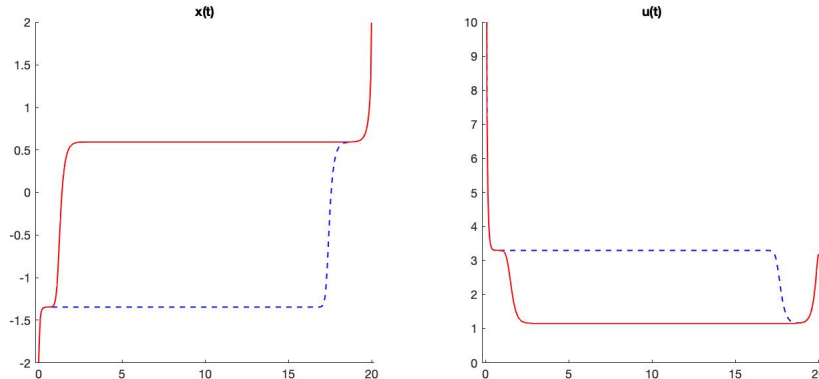


Figure 6:  $x_0 = -2$ ,  $x_f$  free,  $T = 20$ .

On Figure 6, we have taken  $T = 20$ ,  $x_0 = -2$  (and  $x_f$  is let free). In dashed blue, we have initialized the trajectory to the constant trajectory  $\bar{x}_1$ , and we then obtain an optimal trajectory which stays essentially near  $\bar{x}_1 = -1.347372066$ , with a kind of “hesitation” towards  $\bar{x}_2 = 0.5939615956$  near the end. In solid red, we have initialized the trajectory to the constant trajectory  $\bar{x}_2$ , and we obtain a trajectory staying essentially near  $\bar{x}_2$ , with a kind of “hesitation” towards  $\bar{x}_1$  near the beginning. We stress that the two solutions are optimal: both have a cost  $C \simeq 6.2822$ . We could make emerge other similar trajectories, which “hesitate” between the two turnpikes. All of them are optimal, or, at least, “quasi-optimal” (there is a small error due to switches from one turnpike to the other).

## A.2 Local versus global turnpike

We take  $x_d = 1$  and  $u_d = 1$ . The choice of  $u_d$  is now such that the static problem (46) has a unique global solution  $\bar{x} = 0.781538640850898$ , see Figure 7. But it has also a local solution  $\bar{x}_{loc} = -1.10551208794920$ .

The global minimum  $\bar{x}$  is a global turnpike, while the local minimum  $\bar{x}$  is a local turnpike. In the numerical simulations, when performing either an optimization or a Newton method (shooting), we compute local solutions. Hence, we must expect that the numerical results depend on the initialization. To check global optimality, we have to compare the costs.

On Figure 8, in dashed blue, we have initialized the code with the constant trajectory  $\bar{x}_{loc}$ . We observe a turnpike around  $\bar{x}_{loc} = -1.10551208794920$ . The cost is  $C \simeq 7.008$ . But this is a local turnpike only. This trajectory that we obtain is only locally optimal, and is not globally optimal. In solid red, we have initialized the code with the constant trajectory  $\bar{x}$ . We observe a turnpike around  $\bar{x} = 0.781538640850898$ . The cost is  $C \simeq 3.825$  and is lower than the one of the previous one. Here, we have actually computed the globally optimal trajectory. This is the global turnpike.

It is also interesting to see what happens if we take  $T$  much smaller. Let us take  $T = 2$ . On Figure 9, in dashed blue, we have initialized the code with the constant trajectory  $\bar{x}$ . We observe a trend to the turnpike around  $\bar{x} = 0.781538640850898$ . The cost is  $C \simeq 17.792$ . But this trajectory,

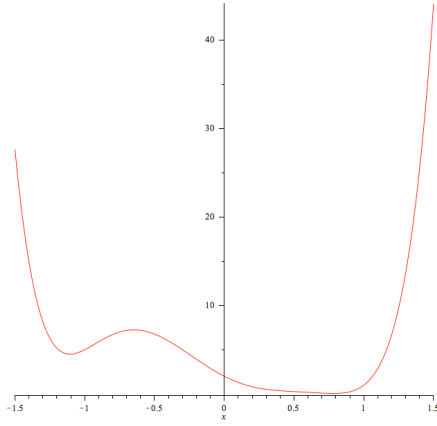


Figure 7: Plot of the function  $x \mapsto (x - 1)^2 + (3x - 3x^3 - 1)^2$

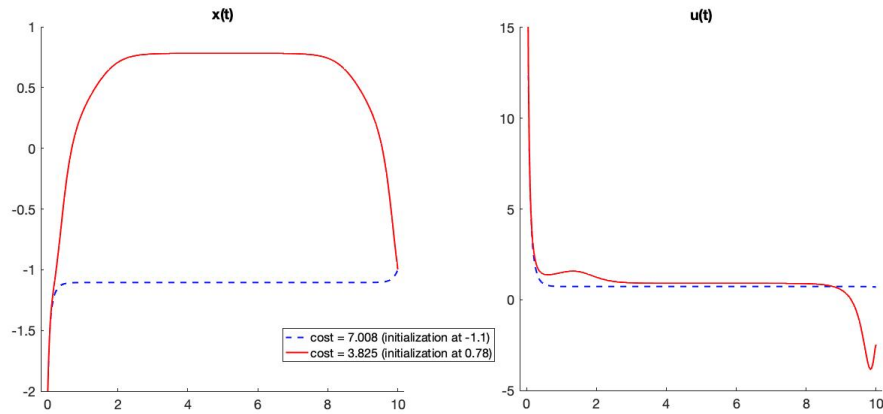


Figure 8:  $x_0 = -2$ ,  $x_f = -1$ ,  $T = 10$ .

now, is not globally optimal. In solid red, we have initialized the code with the constant trajectory  $\bar{x}_{loc}$ . We observe a turnpike around  $\bar{x}_{loc} = -1.347372066$ . The cost is  $C \simeq 16.322$ . It is the globally optimal trajectory.

We can search a time  $2 < T_c < 10$  for which both previous initializations give equivalent turnpikes around  $\bar{x}_{loc}$  and  $\bar{x}$  (with same cost). This is done hereafter. What is important is that, in large time  $T$ , we have indeed the global turnpike around  $\bar{x}$ .

On Figure 10, at the left, the code is initialized with  $x(t) \equiv -1.1$ , in order to promote the turnpike around  $\bar{x}_{loc} = -1.347372066$ . At the right, the code is initialized with  $x(t) \equiv 0.78$ , in order to promote the turnpike around  $\bar{x} = 0.781538640850898$ . We have computed trajectories for the following successive values of  $T$ : 2.5, 2.7, 2.9, 3.1, 3.3. We observe that the blue and cyan trajectories at the top are optimal (see the value of their cost); and that the red, green and black trajectories at the bottom are optimal. The bifurcation occurs around  $T = 2.9$ .

Finally, on Figure 11, we represent the global optimal trajectory. For  $T \lesssim 2.9$ , the global



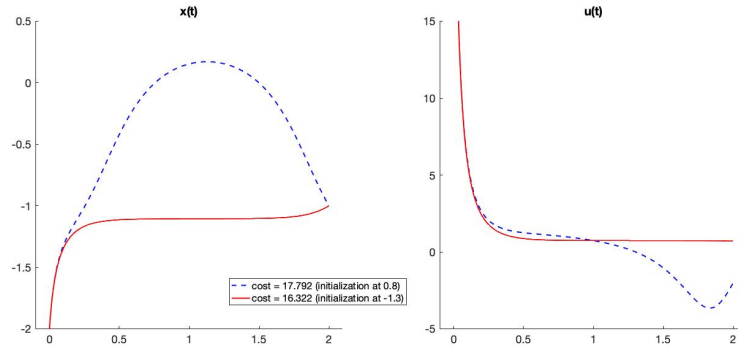


Figure 9:  $x_0 = -2$ ,  $x_f = -1$ ,  $T = 2$ .

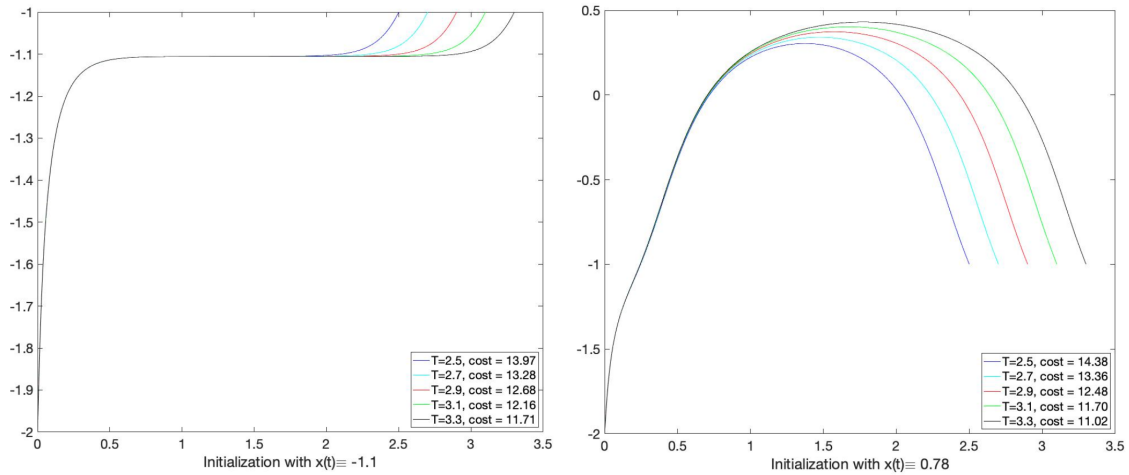


Figure 10:  $x_0 = -2$ ,  $x_f = -1$

optimal trajectory makes a turnpike around  $\bar{x}_{loc} = -1.347372066$ , which is a local minimizer of the optimal static problem. For  $T \geq 2.9$  we have a bifurcation and the global optimal trajectory makes a turnpike around  $\bar{x} = 0.781538640850898$ , which is the global minimizer of the optimal static problem.

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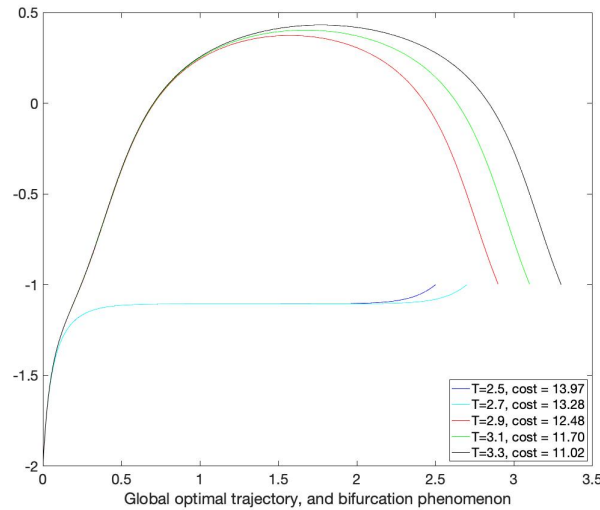


Figure 11:  $x_0 = -2$ ,  $x_f = -1$

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