



HAL
open science

Linear turnpike theorem

Emmanuel Trélat

► **To cite this version:**

Emmanuel Trélat. Linear turnpike theorem. *Mathematics of Control, Signals, and Systems*, 2023, 35 (3), pp.685–739. 10.1007/s00498-023-00354-5 . hal-02978505v4

HAL Id: hal-02978505

<https://hal.science/hal-02978505v4>

Submitted on 10 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Linear turnpike theorem

Emmanuel Trélat*

Abstract

The turnpike phenomenon stipulates that the solution of an optimal control problem in large time remains essentially close to a steady-state of the dynamics, itself being the optimal solution of an associated static optimal control problem. Under general assumptions, it is known that not only the optimal state and the optimal control, but also the adjoint state coming from the application of the Pontryagin maximum principle, are exponentially close to that optimal steady-state, except at the beginning and at the end of the time frame. In such a result, the turnpike set is a singleton, which is a steady-state.

In this paper, we establish a turnpike result for finite-dimensional optimal control problems in which some of the coordinates evolve in a monotone way, and some others are partial steady-states of the dynamics. We prove that the discrepancy between the optimal trajectory and the turnpike set is linear, but not exponential: we thus speak of a linear turnpike theorem.

1 Introduction and main results

1.1 Reminders on the exponential turnpike phenomenon

Let $n, m \in \mathbb{N}^*$ and let $T > 0$ be arbitrary. We consider a general nonlinear optimal control problem in \mathbb{R}^n , in fixed final time T :

$$\dot{x}(t) = f(x(t), u(t)) \quad (1)$$

$$x(0) \in M_0, \quad x(T) \in M_1 \quad (2)$$

$$u(t) \in \Omega \quad (3)$$

$$\min \int_0^T f^0(x(t), u(t)) dt \quad (4)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $f^0 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ are mappings of class C^2 , Ω is a measurable subset of \mathbb{R}^m , M_0 and M_1 are bounded subsets of \mathbb{R}^n , and $u \in L^\infty([0, T], \Omega)$ is the control. For simplicity of the exposition, we assume that there exists $T_0 > 0$ such that, for every $T \geq T_0$, there exists a unique optimal solution of (1)-(2)-(3)-(4), denoted by $(x^T(\cdot), u^T(\cdot))$ (sufficient conditions ensuring existence are standard, see, e.g., [41]; uniqueness is ensured under differentiability properties of the value function and of the Hamiltonian, see, e.g., [12, Theorem 7.3.9], and thus is generic in some sense). By the Pontryagin maximum principle (see [34]), there exist $p^0 \leq 0$ and an absolutely continuous mapping $p_x^T(\cdot) : [0, T] \rightarrow \mathbb{R}^n$ (called adjoint vector), satisfying $(p_x^T(\cdot), p^0) \neq (0, 0)$, such that

$$\begin{aligned} \dot{x}^T(t) &= \frac{\partial H}{\partial p_x}(x^T(t), p_x^T(t), p^0, u^T(t)), & \dot{p}_x^T(t) &= -\frac{\partial H}{\partial x}(x^T(t), p_x^T(t), p^0, u^T(t)) \\ u^T(t) &= \operatorname{argmax}_{v \in \Omega} H(x^T(t), p_x^T(t), p^0, v) \end{aligned} \quad (5)$$

*Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris, France (emmanuel.trelat@sorbonne-universite.fr).

for almost every $t \in [0, T]$, where $H(x, p_x, p^0, u) = \langle p_x, f(x, u) \rangle + p^0 f^0(x, u)$ is the Hamiltonian of the optimal control problem (1)-(2)-(3)-(4) (here and throughout, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in \mathbb{R}^n , and $\|\cdot\|$ is the corresponding norm). We have moreover the transversality conditions on the adjoint vector at initial and final time:

$$p_x^T(0) \perp T_{x^T(0)}M_0, \quad p_x^T(T) \perp T_{x^T(T)}M_1, \quad (6)$$

provided that the tangent space $T_{x^T(0)}M_0$ to M_0 at $x^T(0)$ and the tangent space $T_{x^T(T)}M_1$ to M_1 at $x^T(T)$ exist (these conditions are empty when the initial and final points are fixed in the optimal control problem). We assume that the extremal lift $(x^T(\cdot), p_x^T(\cdot), p^0, u^T(\cdot))$ is not abnormal and thus that we can take $p^0 = -1$. We also assume, for the simplicity of the exposition, that the extremal lift is unique (these two conditions are generic for large classes of optimal control problems, see [14, 15]).

Note that the last condition in (5) gives

$$\frac{\partial H}{\partial u}(x^T(t), p_x^T(t), p^0, u^T(t)) = 0$$

at every Lebesgue time $t \in [0, T]$ at which $u^T(\cdot)$ is approximately continuous and $u^T(t) \in \mathring{\Omega}$ (interior of Ω).

The turnpike phenomenon stipulates that, when the final time T is large, the optimal solution $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$ remains, along the time interval $[0, T]$ and except around the initial time $t = 0$ and around the final time $t = T$, “essentially close” to some static point $(\bar{x}, \bar{p}_x, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^n \times \Omega$, i.e., roughly speaking,

$$x^T(t) \simeq \bar{x}, \quad p_x^T(t) \simeq \bar{p}_x, \quad u^T(t) \simeq \bar{u} \quad \forall t \in [\eta, T - \eta] \quad (7)$$

(for some $\eta > 0$). Moreover, $(\bar{x}, \bar{p}_x, \bar{u})$ is the solution of a minimization problem called *static* optimal control problem (not depending on T), which is:

$$\min \{f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, f(x, u) = 0\} \quad (8)$$

i.e., in other words, the problem of minimizing the instantaneous cost $f^0(x, u)$ over all possible equilibrium points of the dynamics. Note that, assuming that $\bar{u} \in \mathring{\Omega}$ (interior of Ω), by the Lagrange multiplier rule, expressed in a Hamiltonian form, applied to the static optimal control problem, we have

$$\frac{\partial H}{\partial p_x}(\bar{x}, \bar{p}_x, p^0, \bar{u}) = 0, \quad \frac{\partial H}{\partial x}(\bar{x}, \bar{p}_x, p^0, \bar{u}) = 0, \quad \frac{\partial H}{\partial u}(\bar{x}, \bar{p}_x, p^0, \bar{u}) = 0. \quad (9)$$

Here we assume as well, for the simplicity of the exposition, that the minimizer $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathring{\Omega}$ is unique, and also that the quadruple $(\bar{x}, \bar{p}_x, p^0, \bar{u})$ is not abnormal, so that one can take $p^0 = -1$ (usual qualification condition in optimization theory) and that the Lagrange multiplier is unique (the latter two conditions are generic).

We note then that $(\bar{x}, \bar{p}_x, -1, \bar{u})$ is an equilibrium point of (5). Hence, the informal property (7) stipulates that the optimal triple $(x(\cdot), p_x(\cdot), u(\cdot))$, solution of (1)-(2)-(3)-(4)-(5)-(6) remains essentially close to an *equilibrium point* of the dynamics (5), this equilibrium being itself an optimal solution of the static problem (8). This is the turnpike phenomenon.

The intuition explaining why such a property is to be expected is the following. Since T is assumed to be large, we set $\varepsilon = \frac{1}{T}$, which is considered as a small parameter. Now, we make a

time reparametrization, by setting $s = \varepsilon t$, so that, when t ranges over $[0, T]$, s ranges over $[0, 1]$. Setting $x_\varepsilon(s) = x^T(t)$, $p_\varepsilon(s) = p_x^T(t)$ and $u_\varepsilon(s) = u^T(t)$, we have, still in an informal way,

$$\begin{aligned}\frac{\partial H}{\partial p_x}(x_\varepsilon(s), p_\varepsilon(s), -1, u_\varepsilon(s)) &= \varepsilon x'_\varepsilon(s) \simeq 0 \\ \frac{\partial H}{\partial x}(x_\varepsilon(s), p_\varepsilon(s), -1, u_\varepsilon(s)) &= -\varepsilon p'_\varepsilon(s) \simeq 0 \\ \frac{\partial H}{\partial u}(x_\varepsilon(s), p_\varepsilon(s), -1, u_\varepsilon(s)) &= 0\end{aligned}\tag{10}$$

(the latter condition, assuming that the control is in the interior of Ω) which gives credibility to (7), at least when $(\bar{x}, \bar{p}_x, \bar{u})$ is the unique solution of (9) (and under nondegenerate assumptions on the system of equations (9)). But this is not a proof and this remains informal: indeed, to justify (10) one would need that the derivatives $x'_\varepsilon(s)$ and $p'_\varepsilon(s)$ be bounded (except around $t = 0$ and $t = T$), uniformly with respect to ε , i.e., that the derivatives $T\dot{x}^T(t)$ and $T\dot{p}_x^T(t)$ be bounded (except around $t = 0$ and $t = T$), uniformly with respect to T , which is precisely the hard part in establishing a turnpike result.

The main result of [45] states an *exponential turnpike property* which is of a local nature (see Remark 2). Hereafter, we combine this result with a general result of [43] in order to obtain a global exponential turnpike property. We use the notion of *strict dissipativity*, introduced in [48] and already used in [24, 18, 27, 28] to derive turnpike properties. The family of optimal control problems (1)-(3)-(4), indexed by $T > 0$, is said to be *strictly dissipative* at the optimal static point (\bar{x}, \bar{u}) with respect to the *supply rate function*

$$w(x, u) = f^0(x, u) - f^0(\bar{x}, \bar{u})\tag{11}$$

if there exists a locally bounded function $S : \mathbb{R}^n \rightarrow \mathbb{R}$, called *storage function*, such that

$$S(x(t_0)) + \int_{t_0}^{t_1} w(x(t), u(t)) dt \geq S(x(t_1)) + \int_{t_0}^{t_1} \alpha(\|x(t) - \bar{x}\|, \|u(t) - \bar{u}\|) dt\tag{12}$$

(strict dissipativity inequality) for all $t_0 < t_1$, for any pair $(x(\cdot), u(\cdot))$ solution of (1) and (3), for some function α of class \mathcal{K} , i.e., $\alpha : [0, +\infty) \rightarrow [0, +\infty)$ continuous, increasing, and $\alpha(0) = 0$. Comments on dissipativity can be found in [11, 43, 48] (see also further).

We use hereafter the shorter notations

$$\bar{H}_{\star\#} = \frac{\partial^2 H}{\partial \star \partial \#}(\bar{x}, \bar{p}_x, -1, \bar{u})$$

where \star and $\#$ will be replaced with such or such variable. We set

$$\bar{A}_1 = \bar{H}_{p_x x} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}), \quad \bar{B}_1 = \bar{H}_{p_x u} = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}), \quad \bar{U} = -\bar{H}_{uu}$$

and, since we assume hereafter that \bar{U} is invertible,

$$A_1 = \bar{A}_1 + \bar{B}_1 \bar{U}^{-1} \bar{H}_{ux}, \quad \bar{W} = -\bar{H}_{xx} - \bar{H}_{xu} \bar{U}^{-1} \bar{H}_{ux}.$$

Theorem 1. *We make the following assumptions:*

Assumptions of global nature:

- (i) *There exist $K > 0$ and $T_0 > 0$ such that, for every $T \geq T_0$, there exists a unique optimal triple $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$ solution of (1)-(2)-(3)-(4)-(5)-(6) (assumed to be normal); moreover, $\|x^T(t)\| + \|u^T(t)\| \leq K$ for every $t \in [0, T]$.*

- (ii) There exists a unique optimal triple $(\bar{x}, \bar{p}_x, \bar{u})$ solution of (8)-(9), assumed to be normal; moreover, $\bar{u} \in \overset{\circ}{\Omega}$.
- (iii) The family of optimal control problems (1)-(3)-(4), indexed by $T > 0$, is strictly dissipative at the optimal static point (\bar{x}, \bar{u}) with respect to the supply rate function w defined by (11), with a storage function S (satisfying (12)) that is bounded on M_0 and M_1 .
- (iv) For $i = 1, 2$, there exist $t_i \geq 0$, a control $u_i(\cdot) \in L^\infty([0, t_i], \Omega)$ and a trajectory $x_i(\cdot)$ solution of $\dot{x}_i(t) = f(x_i(t), u_i(t))$ on $[0, t_i]$, satisfying $\|x_i(t)\| + \|u_i(t)\| \leq K$ for every $t \in [0, t_i]$ and the terminal conditions $x_1(0) \in M_0$, $x_1(t_1) = \bar{x}$, and $x_2(0) = \bar{x}$, $x_2(t_2) \in M_1$.

Assumptions of local nature:

- (v) $\bar{U} = -\bar{H}_{uu} = -\frac{\partial^2 H}{\partial u^2}(\bar{x}, \bar{p}_x, -1, \bar{u})$ is a positive definite symmetric matrix.
- (vi) \bar{W} is a positive definite symmetric matrix.
- (vii) The pair (\bar{A}_1, \bar{B}_1) satisfies the Kalman condition¹, i.e., $\text{rank}(\bar{B}_1, \bar{A}_1 \bar{B}_1, \dots, \bar{A}_1^{n-1} \bar{B}_1) = n$. In other words, the linearized system at (\bar{x}, \bar{u}) is controllable.

Then, there exist $C > 0$ and $\nu > 0$ such that if $T \geq T_0$ then

$$\boxed{\|x^T(t) - \bar{x}\| + \|p_x^T(t) - \bar{p}_x\| + \|u^T(t) - \bar{u}\| \leq C \left(e^{-\nu t} + e^{-\nu(T-t)} \right) \quad \forall t \in [0, T]} \quad (13)$$

In other words, except around $t = 0$ and $t = T$, the optimal triple $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$ solution of (1)-(2)-(3)-(4)-(5)-(6) is exponentially close to the optimal triple $(\bar{x}, \bar{p}_x, \bar{u})$ solution of (8)-(9). The constants C and ν (not depending on T) can even be made explicit by solving a Riccati equation (see [45] for details).

The local assumptions (v)-(vi)-(vii) done in this theorem, which concern the linearization at the turnpike, have been chiefly discussed and commented in [45].

The assumption (iv) is a global controllability assumption: there exists a trajectory steering in finite time the control system (1) from M_0 to the turnpike \bar{x} , and from the turnpike \bar{x} to the final set M_1 . Global controllability is usually established under appropriate Lie bracket assumptions (see, e.g., [30] for control-affine systems, and [6] for general nonlinear systems). Assumption (iv) can also follow from the local controllability at the turnpike (Assumption (vii)) combined with the (weak) assumption of stabilizability of the system to the turnpike. The latter assumption is a weak one and is related to the dissipativity assumption (iii).

In Assumption (i), the property of uniqueness of the solution of the optimal control problem and normality of the extremal triple is actually generic, in some sense: it is satisfied if the Hamiltonian system is sufficiently regular and if the value function corresponding to the optimal control problem is differentiable at the terminal points under consideration (see [12, Theorem 7.3.9]). The K -boundedness assumption (uniform with respect to T) precludes any trajectory blow-up. Note that the boundedness of the adjoint vector can be deduced from the normality assumption for a large class of optimal control problems (see [40]); here, we will deduce it from the dissipativity assumption and from the normality of the static problem. Assumption (ii) is generic as well, and is related to the qualification of a usual optimization problem under constraints. But, such comments are completely general in optimal control theory and we thus do not comment more on the generality of our assumptions.

Assumption (iii) on dissipativity is certainly the most difficult to check in practice. When a control system is dissipative, the question of determining a storage function is similar to the one

¹Equivalently, the pair (A_1, \bar{B}_1) satisfies the Kalman condition.

of finding an appropriate Lyapunov function in order to establish an asymptotic stability domain for a given equilibrium point of a dynamical system. Like in Lyapunov theory, storage functions can be determined by solving a nonlinear partial differential equation (see, e.g., [11]).

It is proved in [43] that “strong duality implies dissipativity”. To be more precise, we recall that the static problem (8) has the *strong duality property* if (\bar{x}, \bar{u}) minimizes the *Lagrangian function* $L(\cdot, \cdot, \bar{p}_x) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $L(x, u, \bar{p}_x) = f^0(x, u) - \langle \bar{p}_x, f(x, u) \rangle$ (the sign minus is due to the fact that we took $p^0 = -1$ for the Lagrange multiplier associated with the cost). This notion is well known in classical optimization theory, in relation with primal and dual problems. It is proved in [43, Theorem 3] that the strong duality property implies the dissipativity property, with the storage function $S(x) = \langle \bar{p}_x, x \rangle$.

Note that strong duality is in some sense an infinitesimal version of the dissipativity inequality, at least when the storage function is continuously differentiable (see also [24, 26]). Note also that the infinitesimal form of the (non-strict) dissipativity inequality (12), with $\alpha = 0$, is the Hamilton-Jacobi inequality

$$H_1(x, \nabla S(x), u) \leq -f^0(\bar{x}, \bar{u}),$$

where H_1 is the maximized normal Hamiltonian (indeed, divide by $t_1 - t_0 > 0$ and take the limit $t_1 - t_0 \rightarrow 0$). The existence of C^1 solutions is therefore related to the so-called weak KAM theory (see [19]). In this context, the singleton $\{(\bar{x}, \bar{u})\}$ is the Aubry set and $f^0(\bar{x}, \bar{u})$ is the Mañé critical value.

Remark 1. Under (v)-(vi)-(vii), Assumptions (i)-(ii)-(iii)-(iv) are automatically satisfied in the linear-quadratic case where

$$f(x, u) = \bar{A}_1 x + \bar{B}_1 u, \quad f^0(x, u) = \frac{1}{2}(x - x_d)^\top \bar{W}(x - x_d) + \frac{1}{2}(u - u_d)^\top \bar{U}(u - u_d)$$

for some $x_d \in \mathbb{R}^n$ and $u_d \in \mathbb{R}^m$, and $\Omega = \mathbb{R}^m$.

Remark 2. Theorem 1 does not appear as such in the literature. Compared with [45] where the main result is established under the assumptions of local nature only (i.e., (v)-(vi)-(vii)), we have added here assumptions of global nature (i.e., (i)-(ii)-(iii)-(iv)): compactness and uniqueness of minimizers, strict dissipativity, global controllability to and from the turnpike set, and then we obtain a *global* turnpike property.

When the minimizers are not unique or when the system is not dissipative, we obtain only a *local* turnpike property, in the sense that the turnpike estimate (13) can only be expected to be satisfied in a neighborhood of an extremal steady-state $(\bar{x}, \bar{p}_x, \bar{u})$ (see [45]). This is so, because the dynamics is nonlinear. In dynamical systems theory, when an equilibrium is not degenerate, the solutions of the nonlinear system resemble the solutions of the linearized system at the equilibrium, only locally around the equilibrium. The above turnpike theorem is of this nature.

When the static problem has several local minimizers, then one has local turnpike properties. When the static problem has, for instance, two global minimizers, then there is a region of the state space in which globally optimal solutions of the optimal control problems have a turnpike property corresponding to the first global minimizer, and another region in which the turnpike property corresponds to the second global minimizer. We refer to Appendix A for a discussion on these topics and for some numerical simulations.

The turnpike result of [45] has been extended to the infinite-dimensional setting in [43, 44] and in [23, 29]. One can find many turnpike results in the literature (see [13, 24, 27, 28, 35, 36, 49, 50], and see historical references in [22, 45]), but the specificity of the above result is that the turnpike property is established as well for the adjoint vector: as explained in [45], this is particularly important in view of deriving appropriate initializations in numerical computation methods (direct methods or shooting methods, see [42]).

Remark 3. In the optimal control problem (1)-(2)-(3)-(4), we have put some control constraints $u(t) \in \Omega$. This is a (slight) novelty with respect to [45] where there is no control constraint. We say the novelty is slight, because we however assume that $\bar{u} \in \overset{\circ}{\Omega}$, so that, in the long middle part of the time interval, the optimal control, which is close to \bar{u} , does not saturate the constraints. Bang arcs can only occur at the beginning and at the end of the time frame.

We can as well add some state constraints: $x(t) \in M$ for every $t \in [0, T]$, where M is a subset of \mathbb{R}^n . This does not lead to significant changes, provided that, in Assumption (ii), we assume not only that $\bar{u} \in \overset{\circ}{\Omega}$ but also that $\bar{x} \in \overset{\circ}{M}$. In presence of state constraints, the extremal system (5) may change: along a boundary arc, there is an additional adjoint vector $\mu_x^T(\cdot)$, standing for the Lagrange multiplier of the state constraint, which may live in a measure space (see, e.g., [16]). Except the additional assumption $\bar{x} \in \overset{\circ}{M}$, the rest of Theorem 1 is unchanged. Indeed, the optimal trajectory remains exponentially close to the optimal steady-state, on an interval $[\varepsilon, T - \varepsilon]$, hence $x^T(\cdot) \in \overset{\circ}{M}$ along this interval and thus $\mu_x^T(\cdot) = 0$ there along. This is only at the beginning and at the end that the multiplier $\mu_x^T(\cdot)$ may be nontrivial.

In presence of control and/or state constraints, if $\bar{u} \in \partial\Omega$ and/or $\bar{x} \in \partial M$, i.e., if the optimal steady-state saturates the constraints, it is not clear how to derive an exponential turnpike property by exploiting the Pontryagin extremal system, because our approach strongly relies on a linearization of this first-order optimality system at the optimal steady-state and requires smoothness properties of the extremal control. In such cases, the approach by dissipativity, which is softer, is more appropriate but leads to weaker turnpike results, like the “measure-turnpike property” (see [24, 27, 43]).

Until now, we have discussed a turnpike phenomenon around an *equilibrium point* of the dynamics. In the next section, we state a turnpike theorem for systems having some coordinates that evolve in a monotone way and thus have no equilibrium point. Theorem 1 (and its proof) will thus be a particular case of Theorem 2.

1.2 A new result: linear turnpike phenomenon

1.2.1 Framework

We keep the notations introduced in Section 1.1, and we add to the control system (1)-(3)-(4) p additional differential equations, for some $p \in \mathbb{N}^*$, as follows. Given any $T > 0$, we consider the general optimal control problem in \mathbb{R}^{n+p} in fixed final time T :

$$\dot{x}(t) = f(x(t), u(t)) \tag{14}$$

$$\dot{y}(t) = g(x(t), u(t)) \tag{15}$$

$$x(0) \in M_0, \quad x(T) \in M_1, \quad y(0) = y_0, \quad y(T) = y_1^T \tag{16}$$

$$u(t) \in \Omega \tag{17}$$

$$\min \int_0^T f^0(x(t), u(t)) dt \tag{18}$$

where $y_0, y_1^T \in \mathbb{R}^p$ are fixed and $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is of class C^2 (the rest of the assumptions is unchanged, with respect to Section 1.1). We assume that $m \geq p$.

Compared with (1)-(2)-(3)-(4), we have added the p differential equations $\dot{y}(t) = g(x(t), u(t))$, where g does not depend on the new coordinate $y \in \mathbb{R}^p$. The instantaneous cost function f^0 also, still does not depend on y . This is important to derive the result hereafter.

As in the previous section, the final time T is fixed in the optimal control problem (14)-(15)-(16)-(17)-(18) and the linear turnpike property will describe the behavior of optimal solutions when T is large.

In (16), the data M_0 , M_1 and y_0 do not depend on T , but the final condition y_1^T depends on T : we have thus put a superscript T in order to underline this important dependence. We have indeed in mind to consider cases where $\|y_1^T\|$ is large as T is large: in such cases, we expect that the components of $y(t)$ should evolve in a monotone way, i.e., $g(x(t), u(t)) \neq 0$ along the optimal trajectory: we have then no equilibrium in the y components.

We are going to derive a turnpike result in which the “turnpike set” consists of a *partial equilibrium*, i.e., the turnpike is a path $t \mapsto (\bar{x}, \bar{y}(t), \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$ where $f(\bar{x}, \bar{u}) = 0$.

Remark 4. Following [20, 21], it is worth noting that such problems (14)-(15)-(17)-(18) frequently appear in applications, for instance in mechanical problems where, denoting by $x(t)$ a position variable, by $v(t) = \dot{x}(t)$ the speed variable, one has a system of the form

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = f(v(t), u(t)).$$

The result that we are going to derive allows one to obtain a turnpike phenomenon along a trajectory $x(t) = \bar{v}t$, where $\bar{v} \in \mathbb{R}^n$ is such that $f(\bar{v}, \bar{u}) = 0$. Here, (\bar{v}, \bar{u}) is an equilibrium of f , but $(\bar{v}t, \bar{v}, \bar{u})$ is not an equilibrium of the complete dynamics. In [20, 21], this phenomenon is called a “velocity turnpike”. Because of the estimate that we are going to establish in Theorem 2 hereafter, we rather speak of a “linear turnpike” phenomenon.

Note that, in (14)-(15)-(16)-(17)-(18), we have fixed the initial and final conditions on $y(t)$. Indeed, otherwise, if either $y(0)$ or $y(T)$ is let free, then there is nothing new, because f and f^0 do not depend on y and then (14)-(15)-(16)-(17)-(18) is then equivalent to (1)-(2)-(3)-(4), with the coordinate $y(t)$ evolving without having any impact on the optimal control problem (1)-(2)-(3)-(4). But here, we impose $y(0) = y_0$ and $y(T) = y_1^T$, which creates additional constraints with respect to (1)-(2)-(3)-(4), and then the turnpike result stated in Theorem 1 cannot be applied.

1.2.2 A motivating example

In order to shed light on what can be expected, let us consider the very simple example in \mathbb{R}^2 :

$$\begin{aligned} \dot{x}(t) &= u(t), & x(0) &= 1, & x(T) &= 2 \\ \dot{y}(t) &= x(t), & y(0) &= 0, & y(T) &= y_1^T \\ \min \int_0^T & (x(t)^2 + u(t)^2) dt \end{aligned} \tag{19}$$

for which the optimal trajectory can be computed explicitly. Here, $y_1^T \in \mathbb{R}$ is a smooth function of T that is sublinear at infinity. For instance, one can take $y_1^T = \alpha T$ or $y_1^T = \alpha T \sin(T)$ for some $\alpha \in \mathbb{R}$. We also assume that $\Omega = \mathbb{R}$, i.e., there is no control constraint.

In a first step, if we ignore the dynamics in y in (19), then we are in the framework of Theorem 1. The static optimal control problem consists of minimizing $x^2 + u^2$ under the constraint $u = 0$, which gives the optimal solution $\bar{x} = 0$, $\bar{u} = 0$ with the Lagrange multiplier $\bar{p}_x = 0$ (which is normal; here, we take $p^0 = -1/2$). By Theorem 1, we obtain $|x^T(t)| + |p_x^T(t)| + |u^T(t)| \leq C(e^{-\nu t} + e^{-\nu(T-t)})$ for every $t \in [0, T]$. In other words, roughly, $x(t) \simeq 0$ and $u(t) \simeq 0$, except around $t = 0$ and $t = T$.

Now, let us take into account the additional dynamics in y with the terminal conditions on y . We are not anymore in the framework of Theorem 1, because there is no term in $y(t)^2$ in the cost and therefore the matrix W is not positive definite. We define the “turnpike-static” optimal control problem (depending on T) as

$$\min\{x^2 + u^2 \mid (x, u) \in \mathbb{R} \times \mathbb{R}, \quad u = 0, \quad Tx = y_1^T\} \tag{20}$$

i.e., with respect to the previous static problem, we add the terminal constraints coming from the dynamics in y : the unique solution of the Cauchy problem $\dot{y}(t) = x$ (with x constant), $y(0) = 0$, must satisfy $y(T) = y_1^T$.

It is easy to check that the (unique) optimal solution of (20) is $\bar{x}^T = \frac{y_1^T}{T}$, $\bar{u}^T = 0$, with Lagrange multipliers $\bar{p}_x^T = 0$ associated with the constraint $u = 0$, and $\bar{p}_y^T = \frac{y_1^T}{T}$ associated with the constraint $x = \frac{y_1^T}{T}$ (which are normal; here, we take $p^0 = -1/2$). The corresponding trajectory in y is $\bar{y}^T(t) = \frac{y_1^T}{T}t$ (and we have $\bar{y}^T(T) = y_1^T$ as expected). For this example, we expect that, if $T > 0$ is large, then $x^T(t) \simeq 0$, $u^T(t) \simeq 0$ and $y^T(t) \simeq \bar{y}^T(t)$ for $t \in [0, T]$, where the order of this approximation is to be determined, what we do hereafter by explicit calculations.

Note that, now, the solution of the “turnpike-static” optimal control problem, denoted with upper bars, depends on T .

Let us apply the Pontryagin maximum principle to (19). The Hamiltonian is $H = p_x u + p_y x - \frac{1}{2}(x^2 + u^2)$ (there is no abnormal minimizer). Then $u^T(t) = p_x^T(t)$ and the extremal system is

$$\dot{x}^T(t) = p_x^T(t), \quad \dot{y}^T(t) = x^T(t), \quad \dot{p}_x^T(t) = x^T(t) - p_y^T$$

with p_y^T constant. We have then

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x^T(t) - p_y^T \\ p_x^T(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^T(t) - p_y^T \\ p_x^T(t) \end{pmatrix} \\ \frac{d}{dt} y^T(t) &= x^T(t) \end{aligned}$$

In particular, the dynamics in $(x^T - p_y^T, p_x^T)$ is hyperbolic: setting $v^T(t) = x^T(t) - p_y^T + p_x^T(t)$ and $w^T(t) = x^T(t) - p_y^T - p_x^T(t)$, we have

$$\dot{v}^T(t) = v^T(t), \quad \dot{w}^T(t) = -w^T(t)$$

with terminal conditions $v^T(0) + w^T(0) = 2 - 2p_y^T$, $v^T(T) + w^T(T) = 4 - 2p_y^T$. The solutions of this differential system are given by the familiar phase portrait around a saddle point, and we infer that

$$|v^T(t)| + |w^T(t)| = O(e^{-t} + e^{-(T-t)}) + O(|p_y^T|(e^{-t} + e^{-(T-t)}))$$

for every $t \in [0, T]$, where the $O(\cdot)$ is uniform with respect to T , and thus

$$|x^T(t) - p_y^T| + |p_x^T(t)| = O(e^{-t} + e^{-(T-t)}) + O(|p_y^T|(e^{-t} + e^{-(T-t)})).$$

In particular,

$$x^T(t) = p_y^T + O(e^{-t} + e^{-(T-t)}) + O(|p_y^T|(e^{-t} + e^{-(T-t)}))$$

and, integrating $\dot{y}^T(t) = x^T(t)$ we obtain $y^T(t) = p_y^T t + O(1) + O(|p_y^T|)$ for every $t \in [0, T]$. Since $y^T(T) = y_1^T$, we obtain that

$$p_y^T = \frac{y_1^T}{T} + O\left(\frac{1}{T}\right) + O\left(\frac{|p_y^T|}{T}\right),$$

and since y_1^T is sublinear at infinity, we conclude that

$$p_y^T = \bar{p}_y^T + O\left(\frac{1}{T}\right).$$

Therefore

$$x^T(t) = \frac{y_1^T}{T} + \mathcal{O}\left(\frac{1}{T} + e^{-t} + e^{-(T-t)}\right) = \bar{x}^T + \mathcal{O}\left(\frac{1}{T} + e^{-t} + e^{-(T-t)}\right)$$

and

$$y^T(t) = \frac{y_1^T}{T}t + \mathcal{O}(1) = \bar{y}^T(t) + \mathcal{O}(1)$$

for every $t \in [0, T]$. In turn, we have also $p_x^T(t) = \mathcal{O}(e^{-t} + e^{-(T-t)})$.

We conclude that there exists $C > 0$ (independent of T) such that

$$\boxed{\begin{aligned} |x^T(t) - \bar{x}^T| &\leq C \left(\frac{1}{T} + e^{-t} + e^{-(T-t)}\right) \\ |u^T(t) - \bar{u}^T| + |p_x^T(t) - \bar{p}_x^T| &\leq C \left(e^{-t} + e^{-(T-t)}\right) \\ |y^T(t) - \bar{y}^T(t)| &\leq C \\ |p_y^T - \bar{p}_y^T| &\leq \frac{C}{T} \end{aligned}} \quad (21)$$

for every $t \in [0, T]$ (with $\bar{p}_x^T = 0$ and $\bar{p}_y^T = \frac{y_1^T}{T}$). Comparing with Theorem 1, we see on this example that we do not have an exponential turnpike phenomenon for the component $x^T(t)$: the above inequality is weaker; it says that, except around $t = 0$ and $t = T$, $x^T(t)$ is bounded by $\frac{1}{T}$ (which is small as T is large), while the component $y^T(t)$ remains in a bounded neighborhood of $\bar{y}^T(t) = \frac{y_1^T}{T}t$.

It is interesting to note that, in this example, if we take $y_1^T = T \sin(T)$ (as mentioned at the beginning) then $\bar{x}^T = \sin(T)$ and thus \bar{x}^T oscillates as T varies and has no limit as $T \rightarrow +\infty$. Despite of that, we have the estimates (21).

In what follows, we are going to generalize the above example.

1.2.3 Main result

Pontryagin maximum principle. As in Section 1.1, we assume that there exists a unique optimal solution $(x^T(\cdot), y^T(\cdot), u^T(\cdot))$ of (14)-(15)-(16)-(17)-(18). Compared with (1)-(2)-(3)-(4), the Hamiltonian of (14)-(15)-(16)-(17)-(18) is now

$$H(x, p_x, p_y, p^0, u) = \langle p_x, f(x, u) \rangle + \langle p_y, g(x, u) \rangle + p^0 f^0(x, u).$$

Note that H does not depend on the variable y . Assuming as well that, for every $T \geq T_0$, the optimal solution has a unique extremal lift, which is not abnormal (so that we take $p^0 = -1$), the application of the Pontryagin maximum principle leads to the existence of an absolutely continuous mapping $p_x^T(\cdot) : [0, T] \rightarrow \mathbb{R}^n$ and of a constant $p_y^T \in \mathbb{R}^p$ such that

$$\begin{aligned} \dot{x}^T(t) &= \frac{\partial H}{\partial p_x}(x^T(t), p_x^T(t), p_y^T, -1, u^T(t)) = f(x^T(t), u^T(t)) \\ \dot{y}^T(t) &= \frac{\partial H}{\partial p_y}(x^T(t), p_x^T(t), p_y^T, -1, u^T(t)) = g(x^T(t), u^T(t)) \\ \dot{p}_x^T(t) &= -\frac{\partial H}{\partial x}(x^T(t), p_x^T(t), p_y^T, -1, u^T(t)) \\ &= -\frac{\partial f}{\partial x}(x^T(t), u^T(t))^\top p_x^T(t) - \frac{\partial g}{\partial x}(x^T(t), u^T(t))^\top p_y^T + \frac{\partial f^0}{\partial x}(x^T(t), u^T(t))^\top \end{aligned} \quad (22)$$

(p_y^T is constant because $\frac{\partial H}{\partial y} = 0$) and

$$u^T(t) = \operatorname{argmax}_{v \in \Omega} H(x^T(t), p_x^T(t), p_y^T, -1, v) \quad (23)$$

for almost every $t \in [0, T]$.

Note that (23) gives

$$\begin{aligned} & \frac{\partial H}{\partial u}(x^T(t), p_x^T(t), p_y^T, -1, u^T(t)) \\ &= \frac{\partial f}{\partial u}(x^T(t), u^T(t))^\top p_x^T(t) + \frac{\partial g}{\partial u}(x^T(t), u^T(t))^\top p_y^T - \frac{\partial f^0}{\partial x}(x^T(t), u^T(t))^\top = 0 \end{aligned} \quad (24)$$

at every Lebesgue time $t \in [0, T]$ at which $u^T(\cdot)$ is approximately continuous and $u^T(t) \in \hat{\Omega}$.

“Turnpike-static” optimal control problem. With respect to (8), the “turnpike-static” optimal control problem (depending on T) is now the following one:

$$\begin{aligned} \min \left\{ f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, \quad f(x, u) = 0, \right. \\ \left. \dot{y}(t) = g(x, u), \quad y(0) = y_0, \quad y(T) = y_1^T \right\} \end{aligned} \quad (25)$$

where we have added the constraint coming from the dynamics and terminal conditions in y . In contrast to the static optimal control problem (8), which does not depend on T , here, we have a *family* of optimization problems indexed by T . We call it “turnpike-static” because (x, u) is still searched as an equilibrium point, while the component $y(\cdot)$ is time-varying.

We assume that, for every $T \geq T_0$, there exists a unique optimal solution $(\bar{x}^T, \bar{y}^T(\cdot), \bar{u}^T)$ of (25) (depending on T), with $u^T \in \hat{\Omega}$, and that it has a unique extremal (Lagrange multiplier) lift, which is not abnormal. Integrating the differential equation in (25), noting that (x, u) is constant, the turnpike-static optimal control problem (25) is equivalent to

$$\min \left\{ f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, \quad f(x, u) = 0, \quad g(x, u) = \frac{y_1^T - y_0}{T} \right\} \quad (26)$$

which is a classical minimization problem under $n + p$ equality constraints (note that its optimal solution (\bar{x}^T, \bar{u}^T) is assumed to be such that $\bar{u}^T \in \hat{\Omega}$ for every $T \geq T_0$). The Lagrange multiplier rule implies (since the optimal solution is moreover assumed not to be abnormal) that $df^0(\bar{x}^T, \bar{u}^T)$ can be expressed linearly in function of $df(\bar{x}^T, \bar{u}^T)$ and $dg(\bar{x}^T, \bar{u}^T)$. In the next lemma, we write these classical first-order optimality conditions in an equivalent Hamiltonian form, which seems more complicated but is exactly devised to be comparable to (22).

Lemma 1. *There exist $\bar{p}_x^T \in \mathbb{R}^n$ and $\bar{p}_y^T \in \mathbb{R}^p$ such that, for every $t \in [0, T]$,*

$$\begin{aligned} 0 &= \frac{\partial H}{\partial p_x}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = f(\bar{x}^T, \bar{u}^T) \\ \dot{y}(t) &= \frac{\partial H}{\partial p_y}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = g(\bar{x}^T, \bar{u}^T), \quad \bar{y}^T(0) = y_0, \quad \bar{y}^T(T) = y_1^T \\ 0 &= -\frac{\partial H}{\partial x}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = -\frac{\partial f}{\partial x}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_x^T - \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_y^T + \frac{\partial f^0}{\partial x}(\bar{x}^T, \bar{u}^T) \\ 0 &= -\frac{\partial H}{\partial y}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) \end{aligned}$$

and

$$\frac{\partial H}{\partial u}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T) = \frac{\partial f}{\partial u}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_x^T + \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T)^\top \bar{p}_y^T - \frac{\partial f^0}{\partial u}(\bar{x}^T, \bar{u}^T) = 0.$$

This lemma follows by applying the Lagrange multiplier rule to the optimization problem (26) (see also Lemma 2 in Section 2). The vector \bar{p}_x^T is the Lagrange multiplier associated with the constraint $f(x, u) = 0$, while the vector \bar{p}_y^T is the Lagrange multiplier associated with the constraint $g(x, u) = \frac{y_1^T - y_0}{T}$.

We observe that

$$\bar{y}^T(t) = y_0 + tg(\bar{x}^T, \bar{u}^T) \quad \forall t \in [0, T] \quad (27)$$

and therefore

$$y_0 + Tg(\bar{x}^T, \bar{u}^T) = y_1^T. \quad (28)$$

When $g(\bar{x}^T, \bar{u}^T) \neq 0$, this is only possible if the norm of $y_1^T - y_0$ is large as T is large. In the proof of the turnpike theorem hereafter, we are going to consider T as a parameter, tending to $+\infty$. This is why we have underlined the dependence of y_1^T with respect to T : when T becomes larger, it may be required that $y_1^T - y_0$ becomes larger as well (otherwise, the controllability problem may have no solution).

Linear turnpike theorem. We use the same notations as before (note anyway that, now, the matrices $A_1^T, \bar{A}_1^T, \bar{B}_1^T, \bar{U}^T$ and \bar{W}^T depend on T because (\bar{x}^T, \bar{u}^T) depends on T), and additionally, we set

$$\begin{aligned} \bar{A}_2^T &= \bar{H}_{p_y x} = \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T), & \bar{B}_2^T &= \bar{H}_{p_y u} = \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T), & A_2^T &= \bar{A}_2^T + \bar{B}_2^T (\bar{U}^T)^{-1} \bar{H}_{ux}, \\ A^T &= \begin{pmatrix} \bar{A}_1^T & 0 \\ \bar{A}_2^T & 0 \end{pmatrix}, & \bar{A}^T &= \begin{pmatrix} \bar{A}_1^T & 0 \\ \bar{A}_2^T & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(\bar{x}^T, \bar{u}^T) & 0 \\ \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T) & 0 \end{pmatrix}, & \bar{B}^T &= \begin{pmatrix} \bar{B}_1^T \\ \bar{B}_2^T \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u}(\bar{x}^T, \bar{u}^T) \\ \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T) \end{pmatrix}. \end{aligned}$$

Here, A^T and \bar{A}^T are square matrices of size $n + p$ and \bar{B}^T is a matrix of size $(n + p) \times m$. In our main result hereafter, we make all assumptions done in Theorem 1 and we adapt them in an obvious way to the present context where the turnpike set is not anymore the singleton $\{\bar{x}\}$, but is now the path $t \mapsto (\bar{x}^T, \bar{y}^T(t))$ that we call the *turnpike trajectory*. For instance, the controllability assumption (iv) now stipulates that there exists a trajectory steering in finite time some point of the initial set M_0 to some point of the turnpike trajectory, and that any point of the turnpike trajectory can be steered in finite time to the final set M_1 .

We say that the family of optimal control problems (14)-(15)-(17)-(18), indexed by $T > 0$, is *partially x -strictly dissipative* at (\bar{x}^T, \bar{u}^T) with respect to the supply rate function (11), if the control system (14), in the x variable, is strictly dissipative with respect to the supply rate function (11). In other words, here, we do not require any dissipativity with respect to y .

Theorem 2. *There exist $\delta > 0$, $K > 0$ and $T_0 > 0$ such that, for every $T \geq T_0$:*

– *Assumptions of global nature:*

- (i) *There exists a unique optimal tuple $(x^T(\cdot), y^T(\cdot), p_x^T(\cdot), p_y^T(\cdot), u^T(\cdot))$ solution of (14)-(15)-(16)-(17)-(18)-(22)-(23)-(6) (assumed to be normal); moreover, $\|x^T(t)\| + \|u^T(t)\| \leq K$ for every $t \in [0, T]$.*
- (ii) *There exists a unique optimal tuple $(\bar{x}^T, \bar{y}^T(\cdot), \bar{p}_x^T, \bar{p}_y^T, \bar{u}^T)$ solution of (25) (with $\bar{y}^T(\cdot)$ given by (27)) and of the first-order optimality system stated in Lemma 1, assumed to be normal; moreover, $\|\bar{x}^T\| + \|\bar{u}^T\| \leq K$ and $\bar{u}^T \in \Omega_1$ where Ω_1 is a closed subset of $\hat{\Omega}$ not depending on T .*

(iii) The family of optimal control problems (14)-(15)-(17)-(18), indexed by $T > 0$, is partially x -strictly dissipative at (\bar{x}^T, \bar{u}^T) with respect to the supply rate function $w(x, u) = f^0(x, u) - f^0(\bar{x}^T, \bar{u}^T)$, with a storage function S^T (satisfying (12), with α not depending on T) that is bounded on M_0 and M_1 uniformly with respect to T .

(iv) There exists a trajectory $(\tilde{x}^T(\cdot), \tilde{y}^T(\cdot), \tilde{u}^T(\cdot))$ solution of (14)-(15) on $[0, T]$, such that $\|\tilde{x}^T(t)\| + \|\tilde{u}^T(t)\| \leq K$ for every $t \in [0, T]$ and $\tilde{x}^T(t) = \bar{x}^T$ and $\tilde{u}^T(t) = \bar{u}^T$ for every $t \in [t_1^T, T - t_2^T]$, for some $t_1^T, t_2^T \in [0, T_0]$.

– Assumptions of local nature:

(v) $\bar{U}^T = -\bar{H}_{uu} = -\frac{\partial^2 \bar{H}}{\partial u^2}(\bar{x}^T, \bar{p}_x^T, \bar{p}_y^T, -1, \bar{u}^T)$ is a positive definite symmetric matrix, uniformly with respect to T , i.e., $\bar{U}^T \geq \delta I_m$ for every $T \geq T_0$.

(vi) $\bar{W}^T = -\bar{H}_{xx} - \bar{H}_{xu}(\bar{U}^T)^{-1}\bar{H}_{ux}$ is a positive definite symmetric matrix, uniformly with respect to T , i.e., $\bar{W}^T \geq \delta I_n$ for every $T \geq T_0$.

(vii) The pair (\bar{A}^T, \bar{B}^T) (equivalently, the pair (A^T, \bar{B}^T)) satisfies the Kalman condition in a uniform way with respect to T , i.e., denoting $\bar{K}^T = (\bar{B}^T, \bar{A}^T \bar{B}^T, \dots, (\bar{A}^T)^{n+p-1} \bar{B}^T)$ the Kalman matrix, there exists $\delta > 0$ such that $\det(K^T (K^T)^\top) \geq \delta$ for every $T \geq T_0$.

In addition, we assume that there exists $C_1 > 0$ such that:

(viii) $\|y_1^T\| \leq C_1 T$ for every $T \geq T_0$.

(ix) There exists $C_2 > C_1$ such that, for every $\theta \in \mathbb{R}^p$ satisfying $\|\theta\| \leq C_2$, the set $\{(x, u) \in \mathbb{R}^n \times \Omega \mid f(x, u) = 0, \quad g(x, u) = \theta\}$ is a (C^2) submanifold of $\mathbb{R}^n \times \mathbb{R}^m$.

Then, there exist $C > 0$ and $\nu > 0$ such that if $T \geq T_0$ then

$$\boxed{\begin{aligned} \|x^T(t) - \bar{x}^T\| + \|p_x^T(t) - \bar{p}_x^T\| + \|u^T(t) - \bar{u}^T\| &\leq C \left(\frac{1}{T} + e^{-\nu t} + e^{-\nu(T-t)} \right) \\ \|y^T(t) - \bar{y}^T(t)\| &\leq C, \quad \|p_y^T - \bar{p}_y^T\| \leq \frac{C}{T} \end{aligned}} \quad (29)$$

for every $t \in [0, T]$.

The first estimate in (29) states that, except around $t = 0$ and $t = T$, the discrepancies $x^T(t) - \bar{x}^T$, $p_x^T(t) - \bar{p}_x^T$ and $u^T(t) - \bar{u}^T$ are bounded above by $\frac{1}{T}$, which is small as T is large, but not exponentially small: it is weaker than (13), and we speak of a *linear turnpike estimate*. The second estimate in (29) gives a bound (uniform with respect to T) on the discrepancy $y^T(t) - \bar{y}^T(t)$ along the whole interval $[0, T]$: it says that $y^T(t)$ remains at a uniform (wrt T) distance of $\bar{y}^T(t) = y_0 + tg(\bar{x}^T, \bar{u}^T)$ as $t \in [0, T]$. Finally, the third estimate in (29) says that $p_y^T - \bar{p}_y^T$ is (linearly) small as T is large.

On specific examples, it may happen that some of the components of the extremal triple $(x^T(\cdot), p_x^T(\cdot), u^T(\cdot))$ enjoy the exponential turnpike property, i.e., an estimate that is stronger than (29), without the term $\frac{1}{T}$ at the right-hand side. This is the case in the example (19): one can see in (21) that the component x^T satisfies the linear turnpike estimate (29), while the components p_x^T and u^T satisfy the stronger exponential turnpike estimate (13).

Remark 5. Under (v)-(vi)-(vii)-(viii), Assumptions (i)-(ii)-(iii)-(iv) and (ix) are automatically satisfied in the linear-quadratic (LQ) case

$$\begin{aligned} \dot{x}(t) &= \bar{A}_1^T x(t) + \bar{B}_1^T u(t), & x(0) &\in M_0, & x(T) &\in M_1 \\ \dot{y}(t) &= \bar{A}_2^T x(t) + \bar{B}_2^T u(t), & y(0) &= y_0, & y(T) &= y_1^T \\ \min \int_0^T & ((x(t) - x_d)^\top \bar{W}^T (x(t) - x_d) + (u(t) - u_d)^\top \bar{U}^T (u(t) - u_d)) dt \end{aligned} \quad (30)$$

for some $x_d \in \mathbb{R}^n$ and $u_d \in \mathbb{R}^m$, with $\Omega = \mathbb{R}^m$. Actually, (ix) follows from (vii), because, using the Hautus test, (vii) implies that the $(n+p) \times (n+m)$ matrix $(\bar{A}^T \ \bar{B}^T)$ is surjective.

Remark 6. As said earlier, to simplify the exposition we have assumed that, for every $T \geq T_0$, the optimal control problem (14)-(15)-(16)-(17)-(18) has a unique solution, whose extremal lift is unique and normal, and that the static optimal control problem (26) has a unique solution, whose Lagrange multiplier is unique and normal. When the optimal solution is not unique, or when the extremal lift (still assumed to be normal) is not unique, the statement of Theorem 2 must be adapted as in [45]: we obtain in this case a result that is satisfied *locally* around the extremal triple.

Remark 7. As in Remark 3, we can add state constraints in the optimal control problem (14)-(15)-(16)-(17)-(18). This does not change the statement of Theorem 2, provided that, in Assumption (ii), it is also assumed that $(\bar{x}^T, \bar{y}^T(\cdot))$ lies in a closed subset of \dot{M} not depending on T .

Remark 8. Theorem 1 (and its proof) is a particular case of Theorem 2, by ignoring the variables y and p_y (i.e., formally, $p = 0$), and the term $\frac{1}{T}$ in the turnpike estimate (29).

Theorem 2 is proved in Section 2. In Section 3 we give several optimal control problems that illustrate our main result and we provide numerical simulations showing evidence of the linear turnpike phenomenon.

2 Proof of Theorem 2

The proof of Theorem 2 is quite long and involved in some parts. Let us describe shortly the strategy.

First, we prove in Section 2.1, Lemma 2, that the family $(\bar{x}^T, \frac{y_1^T}{T}, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)_{T \geq T_0}$, solution of the turnpike-static optimal control problem (25), is of class C^1 and remains bounded as $T \rightarrow +\infty$. Then, throughout the section, we consider an arbitrary closure point $(\bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ of that bounded family and an arbitrary sequence $T_k \rightarrow +\infty$ of positive real numbers such that $(\bar{x}^{T_k}, \frac{y_1^{T_k}}{T_k}, \bar{u}^{T_k}, \bar{p}_x^{T_k}, \bar{p}_y^{T_k})$ converges to $(\bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ as $k \rightarrow +\infty$.

In Section 2.2, we study the auxiliary optimal control problem $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$, consisting of the same control system and cost functional as in (14)-(15)-(16)-(17)-(18), but on a fixed time interval $[t_0, t_1]$ and with fixed terminal conditions $x(t_0) = x_0, y(t_0) = t_0 \eta_0, x(t_1) = x_1, y(t_1) = t_1 \eta_1$. Noting in Lemma 5 that the trivial trajectory $t \mapsto (x(t), y(t), u(t)) = (\bar{x}^\infty, t \bar{d}^\infty, \bar{u}^\infty)$ is an optimal solution of $\mathcal{P}(t_0, t_1, \bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$, with normal extremal lift $(\bar{x}^\infty, t \bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty, -1, \bar{u}^\infty)$, we then establish in Proposition 1 (the main result of the section) that, for any x_0, x_1 near \bar{x}^∞ and for any η_0, η_1 near \bar{d}^∞ , the optimal control problem $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$ has a unique locally optimal solution, with a normal extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y, -1, u(\cdot))$ that is close to $(\bar{x}^\infty, t \bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty, -1, \bar{u}^\infty)$, and moreover their discrepancy satisfies linear turnpike estimates that prefigure the estimates (29) to be established in Theorem 2. In some sense, in that section, we pass from static to dynamic.

The proof of Proposition 1 itself is quite long and is decomposed in several steps. Since its strategy is described in Section 2.2, we do not reproduce it here. We just mention that the proof relies on sensitivity analysis and conjugate point theory (Section 2.2.1), on the analysis of a dynamical system near an hyperbolic singular point (Section 2.2.2), and on the careful analysis of the shooting problem. The most delicate point is to obtain a neighborhood that does not depend on t_0, t_1 for $t_1 - t_0$ large enough: this is done in Section 2.2.5 by solving the shooting problem while exploiting and bootstrapping the estimates obtained until that step.

In Section 2.3, we show how to pass from local to global by exploiting the dissipativity property. We first show in Lemma 12 that the optimal trajectory $t \mapsto (x^{T_k}(t), \frac{y^{T_k}(t)}{\max(t, 1)})$, starting at $t = 0$ (resp., starting back in time at $t = T_k$), enters a neighborhood of $(\bar{x}^\infty, \bar{d}^\infty)$ within a time that converges to $+\infty$ as $k \rightarrow +\infty$ but can be chosen as a $o(T_k)$. Thanks to Proposition 1 and by local uniqueness, we first derive some estimates that can be bootstrapped to finally show that there exists a finite entering time. The proof of Theorem 2 can then be concluded.

2.1 Preliminaries on the turnpike-static optimal control problem

We consider the turnpike-static optimal control problem (25), written in the form of the optimization problem (26). Recall that, by Assumption (ii), there exists a unique minimizer (\bar{x}^T, \bar{u}^T) of (26) and that $\bar{p}_x^T \in \mathbb{R}^n$ and $\bar{p}_y^T \in \mathbb{R}^p$ are the (normal) Lagrange multipliers respectively associated to the two constraints (see Lemma 1). For every $T \geq T_0$, we have (with $df = (\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u})$)

$$\begin{aligned} df(\bar{x}^T, \bar{u}^T)^\top \bar{p}_x^T + dg(\bar{x}^T, \bar{u}^T)^\top \bar{p}_y^T - df^0(\bar{x}^T, \bar{u}^T)^\top &= 0, \\ f(\bar{x}^T, \bar{u}^T) &= 0, \quad g(\bar{x}^T, \bar{u}^T) = \frac{y_1^T - y_0}{T}. \end{aligned} \tag{31}$$

Lemma 2. *For every $T \geq T_0$, the mapping $T \mapsto (\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)$ is of class C^1 and*

$$(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T) = O(1) \tag{32}$$

as $T \rightarrow +\infty$.

Proof. Let us first prove the C^1 regularity property. The optimality system (31) is written as

$$F(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T) = \begin{pmatrix} df(\bar{x}^T, \bar{u}^T)^\top \bar{p}_x^T + dg(\bar{x}^T, \bar{u}^T)^\top \bar{p}_y^T - df^0(\bar{x}^T, \bar{u}^T)^\top \\ f(\bar{x}^T, \bar{u}^T) \\ g(\bar{x}^T, \bar{u}^T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{y_1^T - y_0}{T} \end{pmatrix}$$

which is a system of $2n + m + p$ equations with the $2n + m + p$ unknowns (x, u, p_x, p_y) , where $F : \mathbb{R}^{2n+m+p} \rightarrow \mathbb{R}^{2n+m+p}$ is a C^1 mapping. The Jacobian of F at the point $(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)$, which is the usual sensitivity matrix in optimization, is

$$dF(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_{y_1}^T, \bar{p}_{y_2}^T) = \left(\begin{array}{cc|cc} \bar{H}_{xx} & \bar{H}_{xu} & \bar{H}_{xp_x} & \bar{H}_{xp_y} \\ \bar{H}_{ux} & \bar{H}_{uu} & \bar{H}_{up_x} & \bar{H}_{up_y} \\ \hline \bar{H}_{p_x x} & \bar{H}_{p_x u} & 0 & 0 \\ \bar{H}_{p_y x} & \bar{H}_{p_y u} & 0 & 0 \end{array} \right) = \left(\begin{array}{c|c} E_1 & E_2^\top \\ \hline E_2 & 0 \end{array} \right).$$

Noting that

$$E_2 = \begin{pmatrix} \frac{\partial f}{\partial x}(\bar{x}^T, \bar{u}^T) & \frac{\partial f}{\partial u}(\bar{x}^T, \bar{u}^T) \\ \frac{\partial g}{\partial x}(\bar{x}^T, \bar{u}^T) & \frac{\partial g}{\partial u}(\bar{x}^T, \bar{u}^T) \end{pmatrix} = (\bar{A}^T \quad \bar{B}^T),$$

Using the Hautus test, Assumption (vii), which says in particular that the pair (\bar{A}^T, \bar{B}^T) satisfies the Kalman assumption, implies that the $(n+p) \times (n+p+m)$ matrix E_2 is surjective.

Besides, by the Schur complement lemma (see, e.g., [10, Appendix A.5.5]), the symmetric matrix E_1 is negative definite, because by Assumption (v) the matrix $\bar{H}_{uu} = -\bar{U}^T$ is negative definite and by Assumption (vi) the matrix $\bar{H}_{xx} - \bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_{ux} = -\bar{W}^T$ (Schur complement) is negative definite.

Now, since E_1 is symmetric negative definite and E_2 is surjective, we conclude that the sensitivity matrix $dF(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)$ is invertible.² By Assumption (viii), we have $\frac{y_1^T - y_0}{T} = O(1)$ as $T \rightarrow +\infty$. Hence, we have to solve the nonlinear system of equations $F(\bar{x}^T, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T) = O(1)$. The C^1 regularity property follows by applying the implicit function theorem.

Let us now establish (32). We already know that $(\bar{x}^T, \bar{u}^T) = O(1)$ as $T \rightarrow +\infty$ by the K -boundedness property in Assumption (ii). Hence, it remains to prove that \bar{p}_x^T and \bar{p}_y^T are uniformly bounded as $T \rightarrow +\infty$. By contradiction, let us assume that there exists a sequence $T_k \rightarrow +\infty$ such that $\|\bar{p}_x^{T_k}\| + \|\bar{p}_y^{T_k}\| \rightarrow +\infty$ as $k \rightarrow +\infty$.

First, let us show how to pass to the limit in (31). Since the sequences $\bar{x}^{T_k}, \bar{u}^{T_k}, \bar{p}_x^{T_k}/(\|\bar{p}_x^{T_k}\| + \|\bar{p}_y^{T_k}\|), \bar{p}_y^{T_k}/(\|\bar{p}_x^{T_k}\| + \|\bar{p}_y^{T_k}\|)$ and y_1^T/T are bounded (the latter, by Assumption (viii)), taking a subsequence if necessary, we have

$$\bar{x}^{T_k} \rightarrow \bar{x}^\infty, \quad \bar{u}^{T_k} \rightarrow \bar{u}^\infty, \quad \frac{\bar{p}_x^{T_k}}{\|\bar{p}_x^{T_k}\| + \|\bar{p}_y^{T_k}\|} \rightarrow \psi_x^\infty, \quad \frac{\bar{p}_y^{T_k}}{\|\bar{p}_x^{T_k}\| + \|\bar{p}_y^{T_k}\|} \rightarrow \psi_y^\infty, \quad \frac{y_1^{T_k}}{T_k} \rightarrow \bar{d}^\infty$$

as $k \rightarrow +\infty$, for some $\bar{x}^\infty \in \mathbb{R}^n, \bar{u}^\infty \in \mathbb{R}^m, (\psi_x^\infty, \psi_y^\infty) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$ (the nontriviality of the latter being crucial in the reasoning) and $\bar{d}^\infty \in \mathbb{R}^p$ satisfying $\|\bar{d}^\infty\| \leq C_1$. Moreover, by Assumption (ii), $\bar{u}^{T_k} \in \Omega_1$ where $\Omega_1 \subset \mathring{\Omega}$ is closed, hence $\bar{u}^\infty \in \Omega_1$ and thus $\bar{u}^\infty \in \mathring{\Omega}$. Considering (31) with $T = T_k$ and dividing the first line by $\|\bar{p}_x^{T_k}\| + \|\bar{p}_y^{T_k}\|$, which converges to $+\infty$, and then taking the limit $k \rightarrow +\infty$ yields

$$\begin{aligned} df(\bar{x}^\infty, \bar{u}^\infty)^\top \psi_x^\infty + dg(\bar{x}^\infty, \bar{u}^\infty)^\top \psi_y^\infty &= 0, \\ f(\bar{x}^\infty, \bar{u}^\infty) &= 0, \quad g(\bar{x}^\infty, \bar{u}^\infty) = \bar{d}^\infty. \end{aligned} \tag{33}$$

Second, let us prove that $(\bar{x}^\infty, \bar{u}^\infty)$ is an optimal solution of the optimization problem

$$\min \{ f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, \quad f(x, u) = 0, \quad g(x, u) = \bar{d}^\infty \}. \tag{34}$$

By definition, we have

$$f^0(\bar{x}^{T_k}, \bar{u}^{T_k}) \leq f^0(x, u) \quad \forall (x, u) \mid f(x, u) = 0 \quad \text{and} \quad g(x, u) = \frac{y_1^{T_k} - y_0}{T_k} \quad \forall k \in \mathbb{N}.$$

The fact follows by passing to the limit and by using the fact that, by Assumption (ix), the set $\{(x, u) \in \mathbb{R}^n \times \Omega \mid f(x, u) = 0, \quad g(x, u) = \bar{d}^\infty\}$ is a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$: indeed, the latter assumption entails the fact that, at every point of that set, the mapping $(df(x, u), dg(x, u))$ is surjective, which allows us to use the implicit function theorem (more precisely, the tubular neighborhood theorem).

Let us now conclude by raising a contradiction. By Assumption (ix), the optimization problem (34) is qualified and has no abnormal minimizer. But we have proved that $(\bar{x}^\infty, \bar{u}^\infty)$ is a minimizer of (34), which has a nontrivial abnormal lift by (33) (recall that $(\psi_x^\infty, \psi_y^\infty) \neq (0, 0)$). This is a contradiction. \square

²Indeed, let $(Y_1, Y_2) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+p}$ be such that $dF \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., $E_1 Y_1 + E_2^\top Y_2 = 0$ and $E_2 Y_1 = 0$.

Then $0 = Y_1^\top E_1 Y_1 + Y_1^\top E_2^\top Y_2 = Y_1^\top E_1 Y_1$ because $Y_1^\top E_2^\top = 0$, and thus $Y_1 = 0$ because E_1 is negative definite. Then, $E_2^\top Y_2 = 0$ implies $Y_2 = 0$ because E_2^\top is injective.

Remark 9. In the LQ case (30), the above proof is simpler and does not require to use the implicit function theorem.

Remark 10. As noticed in the above proof, by Assumption (viii), the family $(\frac{y_1^T}{T})_{T \geq T_0}$ is bounded. Therefore, by Lemma 2, the family $(\bar{x}^T, \frac{y_1^T}{T}, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)_{T \geq T_0}$ is bounded. It may however fail to have a limit as $T \rightarrow +\infty$. For instance, in the motivating example (19), if we choose $y_1^T = T \sin(T)$ then we have $\bar{x}^T = \bar{p}_y^T = \sin(T)$, which do not have any limit as $T \rightarrow +\infty$.

Hereafter, throughout all the proof (which is quite long) of Theorem 2, we fix a limit point (closure point) of that family and a converging subsequence $(T_k)_{k \in \mathbb{N}^*}$. This gives a “limit turnpike-static” optimal control problem.

Limit turnpike-static optimal control problem. By Lemma 2 and Assumption (viii), the family $(\bar{x}^T, \frac{y_1^T}{T}, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)_{T \geq T_0}$ is bounded as $T \rightarrow +\infty$. Let $(\bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ and let $(T_k)_{k \in \mathbb{N}^*}$ be a sequence of positive real numbers converging to $+\infty$ such that

$$\bar{x}^{T_k} \rightarrow \bar{x}^\infty, \quad \frac{y_1^{T_k}}{T_k} \rightarrow \bar{d}^\infty, \quad \bar{u}^{T_k} \rightarrow \bar{u}^\infty, \quad \bar{p}_x^{T_k} \rightarrow \bar{p}_x^\infty, \quad \bar{p}_y^{T_k} \rightarrow \bar{p}_y^\infty \quad (35)$$

as $k \rightarrow +\infty$. Note that $\|\bar{d}^\infty\| \leq C_1$ and $\bar{u}^\infty \in \Omega_1$ (recall that Ω_1 is a closed subset of Ω). We infer from (31) that

$$\begin{aligned} df(\bar{x}^\infty, \bar{u}^\infty)^\top \bar{p}_x^\infty + dg(\bar{x}^\infty, \bar{u}^\infty)^\top \bar{p}_y^\infty - df^0(\bar{x}^\infty, \bar{u}^\infty)^\top &= 0, \\ f(\bar{x}^\infty, \bar{u}^\infty) &= 0, \quad g(\bar{x}^\infty, \bar{u}^\infty) = \bar{d}^\infty. \end{aligned} \quad (36)$$

Lemma 3. *The pair $(\bar{x}^\infty, \bar{u}^\infty)$ is an optimal solution of the so-called “limit turnpike-static” problem*

$$\min \{ f^0(x, u) \mid (x, u) \in \mathbb{R}^n \times \Omega, \quad f(x, u) = 0, \quad g(x, u) = \bar{d}^\infty \} \quad (37)$$

and its unique Lagrange multiplier is normal and is $(\bar{p}_x^\infty, \bar{p}_y^\infty)$.

Proof. We set $N^\infty = \{(x, u) \in \mathbb{R}^n \times \Omega \mid f(x, u) = 0, \quad g(x, u) = \bar{d}^\infty\}$ and $N^T = \{(x, u) \in \mathbb{R}^n \times \Omega \mid f(x, u) = 0, \quad g(x, u) = \frac{y_1^T - y_0}{T}\}$. Let $(x, u) \in N^\infty$. For every k , by Assumption (ix) and by the tubular neighborhood theorem, let $(x^k, u^k) \in N^{T_k}$ such that $(x^k, u^k) \rightarrow (x, u)$ as $k \rightarrow +\infty$. We have $f^0(\bar{x}^{T_k}, \bar{u}^{T_k}) \leq f^0(x^k, u^k)$ because $(\bar{x}^{T_k}, \bar{u}^{T_k})$ minimizes f^0 on N^{T_k} , and then, taking the limit, we obtain that $f^0(\bar{x}^\infty, \bar{u}^\infty) \leq f^0(x, u)$. Hence, $(\bar{x}^\infty, \bar{u}^\infty)$ is an optimal solution of (37).

By Assumption (ix), $(\bar{x}^\infty, \bar{u}^\infty)$ has a unique extremal lift, which is normal. By (36), the Lagrange multiplier must be $(\bar{p}_x^\infty, \bar{p}_y^\infty)$. \square

We denote by $A^\infty, \bar{A}^\infty, \bar{B}^\infty, \bar{U}^\infty$ and \bar{W}^∞ the limits, respectively, of $A^{T_k}, \bar{A}^{T_k}, \bar{B}^{T_k}, \bar{U}^{T_k}$ and \bar{W}^{T_k} as $k \rightarrow +\infty$. By taking the limit $T_k \rightarrow +\infty$ in Assumptions (v), (vi) and (vii), we have $\bar{U}^\infty \geq \delta I_m, \bar{W}^\infty \geq \delta I_n$, and the pair $(\bar{A}^\infty, \bar{B}^\infty)$ (equivalently, (A^∞, B^∞)) satisfies the Kalman condition.

Lemma 4. *The convergence estimates in (35) are in $\frac{1}{T_k}$, i.e., there exists $C > 0$ such that $\|\bar{x}^{T_k} - \bar{x}^\infty\| \leq \frac{C}{T_k}$, and the same for the others, for every $k \in \mathbb{N}^*$.*

Proof. The proof is similar to the first part of the proof of Lemma 2: it suffices to replace the optimality system (31) with the limit one (36). Then all arguments work exactly in the same way (also, replace $\frac{y_1^T - y_0}{T}$ with the limit \bar{d}^∞), since the limit pair $(\bar{A}^\infty, \bar{B}^\infty)$ satisfies the Kalman assumption, \bar{W}^∞ is positive definite \square

2.2 Auxiliary optimal control problem

Given any $t_0, t_1 \in [0, +\infty)$ such that $t_0 < t_1$, any $x_0, x_1 \in \mathbb{R}^n$ and any $\eta \in \mathbb{R}^p$, we consider the auxiliary optimal control problem

$$\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1) \begin{cases} \dot{x}(t) = f(x(t), u(t)), & x(t_0) = x_0, \quad x(t_1) = x_1 \\ \dot{y}(t) = g(x(t), u(t)), & y(t_0) = t_0 \eta_0, \quad y(t_1) = t_1 \eta_1 \\ u(t) \in \Omega \\ \min \int_{t_0}^{t_1} f^0(x(t), u(t)) dt \end{cases}$$

According to the Pontryagin maximum principle, if $(x(\cdot), y(\cdot), u(\cdot))$ is a locally optimal solution of $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$ having at least one normal extremal lift, then there exist $p_x(\cdot) : [t_0, t_1] \rightarrow \mathbb{R}^n$ absolutely continuous and $p_y \in \mathbb{R}^p$ such that

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ \dot{y}(t) &= g(x(t), u(t)) \\ \dot{p}_x(t) &= -\frac{\partial f}{\partial x}(x(t), u(t))^\top p_x(t) - \frac{\partial g}{\partial x}(x(t), u(t))^\top p_y + \frac{\partial f^0}{\partial x}(x(t), u(t))^\top \end{aligned} \quad (38)$$

where $u(t)$ is solution of

$$\frac{\partial f}{\partial u}(x(t), u(t))^\top p_x(t) + \frac{\partial g}{\partial u}(x(t), u(t))^\top p_y - \frac{\partial f^0}{\partial x}(x(t), u(t))^\top = 0 \quad (39)$$

almost everywhere on $[t_0, t_1]$.

Lemma 5. *Taking $x_0 = x_1 = \bar{x}^\infty$ and $\eta_0 = \eta_1 = \bar{d}^\infty$, for any $t_0, t_1 \in [0, +\infty)$ such that $t_0 < t_1$, the trajectory $t \mapsto (x(t), y(t), u(t)) = (\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$ is an optimal solution of the optimal control problem $\mathcal{P}(t_0, t_1, \bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$. Moreover, this optimal solution has a unique extremal lift which is normal, the adjoint vectors corresponding to x and y being constant, equal to $\bar{p}_x^\infty, \bar{p}_y^\infty$ respectively.*

Proof. By Assumption (iii), the partial x -dissipativity inequality (12) implies that, for any $x(\cdot)$ solution of $\dot{x}(t) = f(x(t), u(t))$ on $[t_0, t_1]$ such that $x(t_0) = x(t_1)$, since $S^{T_k}(x(t_0)) = S^{T_k}(x(t_1))$ we get that $\int_{t_0}^{t_1} f^0(\bar{x}^{T_k}, \bar{u}^{T_k}) dt \leq \int_{t_0}^{t_1} f^0(x(t), u(t)) dt$, and letting k tend to $+\infty$ we obtain that $\int_{t_0}^{t_1} f^0(\bar{x}^\infty, \bar{u}^\infty) dt \leq \int_{t_0}^{t_1} f^0(x(t), u(t)) dt$. Therefore $(x(t), y(t), u(t)) = (\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$ is an optimal solution of $\mathcal{P}(t_0, t_1, \bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$. The rest of the statement follows from (36). \square

Remark 11. Lemma 5 shows that $(x(t), y(t), u(t)) = (\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$ is not only solution of the limit turnpike-static problem, but also of the dynamical optimal control problem $\mathcal{P}(t_0, t_1, \bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$. This stronger fact is due to the dissipativity assumption. This is an important step because here, in some sense, we pass from static to dynamic.

Note that $t \mapsto (\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$ is a globally optimal solution of $\mathcal{P}(t_0, t_1, \bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$. In the next lemma, we show that the property that the trajectory $t \mapsto (\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$ is a locally optimal solution having a normal extremal lift is robust near \bar{x}^∞ and \bar{d}^∞ , for any $0 \leq t_0 < t_1$.

The main result of that section is the following.

Proposition 1. *There exist $\varepsilon_0 > 0$, $T_0 > 0$, $C > 0$ and $\nu > 0$ such that, for any $t_0, t_1 \in [0, +\infty)$ such that $t_1 - t_0 \geq T_0$, for any $\varepsilon \in (0, \varepsilon_0)$, for any $x_0, x_1 \in B(\bar{x}^\infty, \varepsilon)$ and any $\eta_0, \eta_1 \in B(\bar{d}^\infty, \varepsilon)$:*

(i) the optimal control problem $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$ has a locally optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ (for the L^∞ topology on u) that has a normal extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y, -1, u(\cdot))$, with u being C^1 ;

moreover, this optimal solution is locally unique in the sense that there does not exist any other control $\tilde{u} \in L^\infty([t_0, t_1], \mathcal{U})$ whose corresponding trajectory would also be a locally optimal solution of $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$;

(ii) $t \mapsto (x(t), \frac{y(t)}{\max(1,t)}, p_x(t), p_y, u(t))$ converges to $(\bar{x}^\infty, \bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty, \bar{u}^\infty)$ in C^0 topology on $[t_0, t_1]$ as $\varepsilon \rightarrow 0$;

(we divide $y(t)$ by $\max(1, t)$ to avoid division by 0)

(iii) we have the estimates

$$\begin{aligned} & \|x(t) - \bar{x}^\infty\| + \|p_x(t) - \bar{p}_x^\infty\| + \|u(t) - \bar{u}^\infty\| \\ & \leq C\varepsilon \left(e^{-\nu(t-t_0)} + e^{-\nu(t_1-t)} \right) + \frac{C}{t_1 - t_0} (\varepsilon + \|y(t_1) - y(t_0) - (t_1 - t_0)\bar{d}^\infty\|), \\ & \|y(t) - t\bar{d}^\infty\| \leq C (\varepsilon + \|y(t_0) - t_0\bar{d}^\infty\| + \|y(t_1) - y(t_0) - (t_1 - t_0)\bar{d}^\infty\|), \\ & \|p_y - \bar{p}_y^\infty\| \leq \frac{C}{t_1 - t_0} (\varepsilon + \|y(t_1) - y(t_0) - (t_1 - t_0)\bar{d}^\infty\|), \end{aligned} \tag{40}$$

for every $t \in [t_0, t_1]$.

The rest of the section is devoted to proving Proposition 1. The proof is quite lengthy. We use sensitivity analysis, conjugate point theory, analysis of a dynamical system near a hyperbolic singular point, and a number of bootstrap arguments aiming at improving, step by step, the estimates. The successive bootstrap arguments ultimately lead to (40). A major difficulty in Proposition 1 is to prove the existence of ε_0 not depending on t_0, t_1 for $t_1 - t_0$ large enough.

More precisely, in Section 2.2.1, we establish Lemma 6, which gives (i) (except the local uniqueness property) and (ii) of Proposition 1 but with ε_0 depending on t_0 and t_1 . The proof of the existence of ε_0 not depending on t_0, t_1 will be done further, in Section 2.2.5: it requires other results and several bootstrap arguments.

In-between, in Section 2.2.2, we study the extremal system (38)-(39) locally around the trivial trajectory $t \mapsto (\bar{x}^\infty, t\bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$. We will call “linearized system”, the linearization of the extremal system (38)-(39) along this trivial trajectory. The main result in Section 2.2.2 is Lemma 7, which gives (iii), i.e., the linear turnpike estimates (40).

Sections 2.2.3 and 2.2.4 are devoted to proving two technical lemmas, used in the proof of Lemma 7.

Finally, in Section 2.2.5, we conclude the proof of Proposition 1, by bootstrapping Lemmas 6 and 7 and finally establish the existence of ε_0 that is uniform with respect to t_0 and t_1 whenever $t_1 - t_0$ is large enough, and obtaining the local uniqueness property.

2.2.1 Sensitivity analysis

In this section, we establish the following lemma, which implies (i) (except the local uniqueness property) and (ii) of Proposition 1 but, for the moment, with ε_0 depending on t_0 and t_1 .

Lemma 6. *There exist $T_0 > 0$ such that, for any $t_0, t_1 \in [0, +\infty)$ satisfying $t_1 - t_0 \geq T_0$, there exist $\varepsilon_0 > 0$ (depending on t_0 and t_1) such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x_0, x_1 \in B(\bar{x}^\infty, \varepsilon)$ and $\eta_0, \eta_1 \in B(\bar{d}^\infty, \varepsilon)$, the optimal control problem $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$ has a locally optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ that has a normal extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y, -1, u(\cdot))$, with u being C^1 .*

Moreover, $t \mapsto (x(t), \frac{y(t)}{\max(1,t)}, p_x(t), p_y, u(t))$ converges to $(\bar{x}^\infty, \bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty, \bar{u}^\infty)$ in C^0 topology on $[t_0, t_1]$ as $\varepsilon \rightarrow 0$ converges to $(\bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$.

Proof. The proof goes by classical sensitivity analysis arguments. We denote by $E : [0, +\infty)^2 \times \mathbb{R}^n \times \mathbb{R}^p \times L^\infty([t_0, t_1], \Omega) \rightarrow \mathbb{R}^{n+p}$ the end-point mapping, defined by $E(t_0, t_1, x_0, \eta_0, u) = (x(t_1), y(t_1))$ where $(x(\cdot), y(\cdot))$ is the solution on $[t_0, t_1]$ of $\dot{x}(t) = f(x(t), u(t))$, $\dot{y}(t) = g(x(t), u(t))$ corresponding to the control u and such that $x(t_0) = x_0$ and $y(t_0) = t_0 d^\infty$, and we denote by $C(t_0, t_1, x_0, \eta_0, u) = \int_{t_0}^{t_1} f^0(x(t), u(t)) dt$ the associated cost. The optimal control problem $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$ is then equivalent to minimize $C(t_0, t_1, x_0, \eta_0, u)$ over all possible controls $u \in L^\infty([t_0, t_1], \Omega)$ such that $E(t_0, t_1, x_0, \eta_0, u) = (x_1, t_1 \eta_1)$.

For $0 < t_0 < t_1$ fixed, if $(x(\cdot), y(\cdot), u(\cdot))$ is an optimal solution of $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$ on $[t_0, t_1]$, such that u takes its values in $\Omega_1 \subsetneq \Omega$ and is not abnormal, then according to the Lagrange multiplier rule there must exist $\psi \in \mathbb{R}^{n+p}$ such that

$$\frac{\partial C}{\partial u}(t_0, t_1, x_0, \eta_0, u) + \langle \psi, \frac{\partial E}{\partial u}(t_0, t_1, x_0, \eta_0, u) \rangle = 0,$$

$$\text{i.e., } \frac{\partial L}{\partial u}(t_0, t_1, x_0, \eta_0, x_1, \eta_1, u, \psi) = 0, \quad (41)$$

(this equality is in the dual of $L^1([t_0, t_1], \mathbb{R}^m)$ that is identified with $L^\infty([t_0, t_1], \mathbb{R}^m)$) where

$$L(t_0, t_1, x_0, \eta_0, x_1, \eta_1, u, \psi) = C(t_0, t_1, x_0, \eta_0, u) + \langle \psi, E(t_0, t_1, x_0, \eta_0, u) - (x_1, t_1 \eta_1) \rangle$$

is the Lagrangian of the optimization problem. Therefore

$$G(t_0, t_1, x_0, \eta_0, x_1, \eta_1, u, \psi) = \begin{pmatrix} \frac{\partial L}{\partial u}(t_0, t_1, x_0, \eta_0, x_1, \eta_1, u, \psi) \\ E(t_0, t_1, x_0, \eta_0, u) - (x_1, t_1 \eta_1) \end{pmatrix} = 0$$

where $G : [0, +\infty)^2 \times (\mathbb{R}^{n+p})^2 \times L^\infty([t_0, t_1], \mathbb{R}^m) \times \mathbb{R}^{n+p} \rightarrow L^\infty([t_0, t_1], \mathbb{R}^m) \times \mathbb{R}^{n+p}$ is a C^1 mapping. The first-order optimality condition (41) is the preliminary equation ultimately leading to the “weak” Pontryagin maximum principle (see [40, 41, 42]); this means that, equivalently, there exist $p_x(\cdot) : [0, t_1] \rightarrow \mathbb{R}^n$ and $p_y \in \mathbb{R}^p$ such that the extremal equations (38) and (39) are satisfied almost everywhere on $[t_0, t_1]$. Moreover, we have $-\psi = (p_x(t_1), p_y)$, i.e., the Lagrange multiplier $-\psi$ coincides with the adjoint vector at the final time. Note, since u takes its values in Ω_1 , we indeed have $\frac{\partial H}{\partial u} = 0$ along the extremal (which is (39)), and when $(x(t), p_x(t), p_y, u(t))$ is close enough to $(\bar{x}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty, \bar{u}^\infty)$ (what we are going to ensure next), thanks to Assumption (v) one can solve this equation by the implicit function theorem and get u as a C^1 function of $(x(t), p_x(t), p_y)$.

Note that we have not proved yet that $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$ has an optimal solution. The equations (38)-(39) constitute the first-order optimality system for a locally optimal solution, if it exists and if it has a normal extremal lift. Hereafter, we are going to prove that, locally near the trivial solution $t \mapsto (\bar{x}^\infty, t \bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty, \bar{u}^\infty)$ of (38)-(39), for all $0 \leq t_0 < t_1$ there exists a solution of the extremal system (38)-(39) such that $x(t_0) = x_0$, $x(t_1) = x_1$, $y(t_0) = t_0 \eta_0$ and $y(t_1) = t_1 \eta_1$, with u of class C^1 as above, and that such a solution is indeed a locally optimal solution of $\mathcal{P}(t_0, t_1, x_0, x_1, \eta_0, \eta_1)$. This is done, hereafter, by sensitivity analysis and by conjugate point theory.

By Lemma 5, given any $0 \leq t_0 < t_1$ we have

$$G(t_0, t_1, \bar{x}^\infty, \bar{d}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, (\bar{p}_x^\infty, \bar{p}_y^\infty)) = 0.$$

Now, for any $0 \leq t_0 < t_1$ fixed and any $(x_0, x_1, \eta_0, \eta_1)$ sufficiently close to $(\bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$, we can solve with respect to (u, ψ) the equation $G(t_0, t_1, x_0, \eta_0, x_1, \eta_1, u, \psi) = 0$: this is possible thanks to the implicit function theorem, because the sensitivity matrix (of operators)

$$\frac{\partial G}{\partial(u, \psi)} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^2} & \frac{\partial E^*}{\partial u} \\ \frac{\partial E}{\partial u} & 0 \end{pmatrix}$$

is boundedly invertible at the point $(t_0, t_1, \bar{x}^\infty, \bar{d}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, (\bar{p}_x^\infty, \bar{p}_y^\infty))$ (the argument here is quite similar to the one used in the proof of Lemma 2, which is not a surprise in view of Remark 11: we pass from static to dynamic). This is so, because of the two following facts:

- The Fréchet differential

$$\frac{\partial E}{\partial u}(t_0, t_1, \bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty) : L^\infty([t_0, t_1], \mathbb{R}^m) \rightarrow \mathbb{R}^{n+p}$$

is surjective: indeed, equivalently (see, e.g., [9, 41]), the linearized control system at $(\bar{x}^\infty, \bar{u}^\infty)$ is controllable, because the pair $(A^\infty, \bar{B}^\infty)$ (defined at the end of Section 2.1) satisfies the Kalman condition.

- The quadratic form

$$Q_{t_0, t_1} = \frac{\partial^2 L}{\partial u^2}(t_0, t_1, \bar{x}^\infty, \bar{d}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, (\bar{p}_x^\infty, \bar{p}_y^\infty))|_{\ker \frac{\partial E}{\partial u}(t_0, t_1, \bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty)}$$

is positive definite for any $0 \leq t_0 < t_1$ as a consequence of conjugate point theory (see [6, 8, 9, 42]): indeed, the strong Legendre condition $\bar{U}^\infty \geq \delta I_m$ (see the end of Section 2.1) implies that $Q_{t_0, t_1} > 0$ for any $0 \leq t_0 < t_1$. Note that, usually, the strong Legendre assumption would imply $Q_{t_0, t_1} > 0$ only for $t_1 - t_0 > 0$ small enough (see [6, Proposition 20.2]), but here positive-definiteness is valid for any $t_1 - t_0 > 0$, or equivalently, the first conjugate time associated with the trivial trajectory $(x(t), y(t), u(t)) = (\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$ is equal to $+\infty$. To be more precise, using [6, 8, 9], the matrix of the variational system along this reference trajectory is the constant matrix³

$$\begin{pmatrix} A^\infty & \bar{B}^\infty (\bar{U}^\infty)^{-1} (\bar{B}^\infty)^\top \\ \widetilde{W} & -(A^\infty)^\top \end{pmatrix} \quad \text{with} \quad \widetilde{W} = \begin{pmatrix} \bar{W}^\infty & 0 \\ 0 & 0 \end{pmatrix}.$$

The variational system coincides with the optimality system of the LQ problem consisting of minimizing the cost functional

$$\int_{t_0}^{t_1} (\delta x(t))^\top \bar{W}^\infty \delta x(t) + (\delta u(t))^\top \bar{U}^\infty \delta u(t) dt$$

for the control system $\delta \dot{x}(t) = A^\infty \delta x(t) + \bar{B}^\infty \delta u(t)$, $\delta x(t_0) = 0$, $\delta x(t_1) = 0$. According to the classical conjugate point theory, for t_0 fixed, the first conjugate time t_c is the infimum of times $t_1 > t_0$ such that the minimal cost is 0 if $t_0 < t_1 < t_c$ and is $-\infty$ if $t_0 < t_c < t_1$. Here, we have $\bar{U}^\infty > 0$ and $\bar{W}^\infty > 0$ (see the end of Section 2.1) and thus the minimal cost is 0 for every $t_1 > t_0$, which implies that $t_c = +\infty$.

Hence, by the implicit function theorem, for any $0 \leq t_0 < t_1$, for any $(x_0, x_1, \eta_0, \eta_1)$ close enough to $(\bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$, there exist (u, ψ) close to $(\bar{u}^\infty, (\bar{p}_x^\infty, \bar{p}_y^\infty))$ such that $G(t_0, t_1, x_0, \eta_0, x_1, \eta_1, u, \psi) = 0$, with, moreover, $u \in L^1([t_0, t_1], \Omega_1)$. This means that the trajectory $(x(\cdot), y(\cdot), u(\cdot))$ has a normal extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y, -1, u(\cdot))$ satisfying the extremal equations (38)-(39), and is such that $x(0) = x_0$, $y(0) = t_0 \eta_0$, $x(t_1) = x_1$ and $y(t_1) = t_1 \eta_1$. In turn, since Q_{t_0, t_1} is positive definite, it follows from the conjugate point theory (see, e.g., [6, Theorem 21.3]) that $(x(\cdot), y(\cdot), u(\cdot))$ is locally optimal on $[0, t_1]$ (for the L^∞ topology on u). Moreover, $t \mapsto (x(t), \frac{y(t)}{\max(1, t)}, p_x(t), p_y, u(t))$ converges to $(\bar{x}^\infty, \bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty, \bar{u}^\infty)$ in C^0 topology when $(x_0, x_1, \eta_0, \eta_1) \rightarrow (\bar{x}^\infty, \bar{x}^\infty, \bar{d}^\infty, \bar{d}^\infty)$. \square

³Actually, this matrix is exactly the matrix of the linearized system (43) studied further.

2.2.2 Local linear turnpike estimates

In this section, we study the extremal system (38)-(39) locally near the trivial solution $t \mapsto (\bar{x}^\infty, t\bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ of that system. Throughout the section, we set

$$\begin{aligned} x(t) &= \bar{x}^\infty + \delta x(t), & y(t) &= t\bar{d}^\infty + \delta y(t), \\ p_x(t) &= \bar{p}_x^\infty + \delta p_x(t), & p_y &= \bar{p}_y^\infty + \delta p_y, & u(t) &= \bar{u}^\infty + \delta u(t), \end{aligned} \quad (42)$$

and we anticipate that $(\delta x, \delta y, \delta p_x, \delta p_y, \delta u)$ are small, uniformly on $[t_0, t_1]$; in particular, δu is small enough so that, thanks to Assumption (v) (passed to the limit), one can solve (39) by the implicit function theorem and get u as a C^1 function of $(x(t), p_x(t), p_y)$.

In this section, we establish linear turnpike estimates similar to those stated in the theorem for the system (38)-(39), close enough to $(\bar{x}^\infty, t\bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$, by analyzing the linearized system.

Plugging the variables (42) in (39) and then in (38), and performing Taylor expansions at the first order, we obtain first

$$\delta u(t) = (\bar{U}^\infty)^{-1} (\bar{H}_{ux}^\infty \delta x(t) + (\bar{B}_1^\infty)^\top \delta p_x(t) + (\bar{B}_2^\infty)^\top \delta p_y) + O(\|\delta x(t)\|^2, \|\delta p_x(t)\|^2, \|\delta p_y\|^2)$$

(note that, for instance, $2\|\delta x(t)\|\|\delta p_x(t)\| \leq \|\delta x(t)\|^2 + \|\delta p_x(t)\|^2$ by the Young inequality), and then, recalling that $\delta p_y^T \in \mathbb{R}^p$ is constant,

$$\begin{aligned} \delta \dot{x}(t) &= A_1^\infty \delta x(t) + \bar{B}_1^\infty (\bar{U}^\infty)^{-1} (\bar{B}_1^\infty)^\top \delta p_x(t) + \bar{B}_1^\infty (\bar{U}^\infty)^{-1} (\bar{B}_2^\infty)^\top \delta p_y + O(\star) \\ \delta \dot{y}(t) &= A_2^\infty \delta x(t) + \bar{B}_2^\infty (\bar{U}^\infty)^{-1} (\bar{B}_1^\infty)^\top \delta p_x(t) + \bar{B}_2^\infty (\bar{U}^\infty)^{-1} (\bar{B}_2^\infty)^\top \delta p_y + O(\star) \\ \delta \dot{p}_x(t) &= \bar{W}^\infty \delta x(t) - (A_1^\infty)^\top \delta p_x(t) - (A_2^\infty)^\top \delta p_y + O(\star) \\ \delta \dot{p}_y(t) &= 0 \end{aligned} \quad (43)$$

where $O(\star) = O(\|\delta x(t)\|^2, \|\delta p_x(t)\|^2, \|\delta p_y\|^2)$, which can also be written, in a matrix form, as

$$\dot{Z}(t) = \begin{pmatrix} A^\infty & \bar{B}^\infty (\bar{U}^\infty)^{-1} (\bar{B}^\infty)^\top \\ \widetilde{W} & -(A^\infty)^\top \end{pmatrix} Z(t) + O(\|\delta x(t)\|^2, \|\delta p_x(t)\|^2, \|\delta p_y\|^2) \quad (44)$$

with

$$Z(t) = \begin{pmatrix} \delta x(t) \\ \delta y(t) \\ \delta p_x(t) \\ \delta p_y \end{pmatrix}, \quad A^\infty = \begin{pmatrix} A_1^\infty & 0 \\ A_2^\infty & 0 \end{pmatrix}, \quad \widetilde{W} = \begin{pmatrix} \bar{W}^\infty & 0 \\ 0 & 0 \end{pmatrix},$$

where A^∞ , \bar{B}^∞ and \bar{W}^∞ have been defined at the end of Section 2.1. Here and in what follows, the “big-O” notation $O(\|\delta x(t)\|^2, \|\delta p_x(t)\|^2, \|\delta p_y\|^2)$ stands for a term satisfying: there exist $C > 0$ and $\alpha > 0$ such that, given any $t \in \mathbb{R}$ satisfying $\max(\|\delta x(t)\|, \|\delta p_x(t)\|, \|\delta p_y\|) \leq \alpha$, we have

$$\|O(\|\delta x(t)\|^2, \|\delta p_x(t)\|^2, \|\delta p_y\|^2)\| \leq C \max(\|\delta x(t)\|^2, \|\delta p_x(t)\|^2, \|\delta p_y\|^2). \quad (45)$$

The “little-o” notation $o(\delta x(t), \delta p_x(t), \delta p_y)$ stands for a term satisfying: for every $\varepsilon > 0$ there exists $\eta > 0$ such that, given any $t \in \mathbb{R}$ satisfying $\max(\|\delta x(t)\|, \|\delta p_x(t)\|, \|\delta p_y\|) \leq \eta$, we have

$$\|o(\delta x(t), \delta p_x(t), \delta p_y)\| \leq \varepsilon \max(\|\delta x(t)\|, \|\delta p_x(t)\|, \|\delta p_y\|).$$

Of course, a term $O(\|\delta x(t)\|^2, \|\delta p_x(t)\|^2, \|\delta p_y\|^2)$ is *a fortiori* a $o(\delta x(t), \delta p_x(t), \delta p_y)$.

It is important to note that, in the system (43) (equivalently, (44)), the remainder terms do not involve $\delta y(t)$.

The “linearized system”, i.e., the linearization along $t \mapsto (\bar{x}^\infty, t\bar{d}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ of the system (38)-(39) is the system (44) without the remainder terms. But, in what follows, we do not ignore those remainder terms: they have to be handled carefully.

To this aim, we write the system (43) in the following form, more amenable to its analysis. Setting

$$z(t) = \begin{pmatrix} \delta x(t) \\ \delta p_x(t) \end{pmatrix}$$

and defining the matrices

$$M = \begin{pmatrix} A_1^\infty & \bar{B}_1^\infty (\bar{U}^\infty)^{-1} (\bar{B}_1^\infty)^\top \\ \bar{W}^\infty & -(A_1^\infty)^\top \end{pmatrix}, \quad V = \begin{pmatrix} -\bar{B}_1^\infty (\bar{U}^\infty)^{-1} (\bar{B}_2^\infty)^\top \\ (A_2^\infty)^\top \end{pmatrix}, \quad (46)$$

we write the system (43) as

$$\dot{z}(t) = Mz(t) - V\delta p_y + O(\|z(t)\|^2, \|\delta p_y\|^2) \quad (47)$$

$$\delta \dot{y}(t) = Lz(t) + \bar{B}_2^\infty (\bar{U}^\infty)^{-1} (\bar{B}_2^\infty)^\top \delta p_y + O(\|z(t)\|^2, \|\delta p_y\|^2) \quad (48)$$

where

$$L = (A_2^\infty \quad \bar{B}_2^\infty (\bar{U}^\infty)^{-1} (\bar{B}_1^\infty)^\top).$$

Note that, when $\bar{H}_{ux} = 0$, the system (47)-(48), without the remainder terms, corresponds exactly to the extremal system associated with the linear-quadratic optimal control problem (30).

We have the following result.

Lemma 7. *There exist $\varepsilon_1 > 0$, $C > 0$ and $\nu > 0$ such that, for all $0 \leq t_0 < t_1$, for every solution $(z(\cdot), \delta y(\cdot), \delta p_y)$ of (47)-(48) satisfying*

$$\gamma_{t_0, t_1} = \max(\|z(t_0)\|, \|z(t_1)\|, \|\delta p_y\|) \leq \varepsilon_1, \quad (49)$$

we have

$$\begin{aligned} \|z(t)\| &\leq C\gamma_{t_0, t_1} \left(e^{-\nu(t-t_0)} + e^{-\nu(t_1-t)} \right) + \frac{C}{t_1 - t_0} (\gamma_{t_0, t_1} + \|\delta y(t_1) - \delta y(t_0)\|), \\ \|\delta y(t)\| &\leq C(\gamma_{t_0, t_1} + \|\delta y(t_0)\| + \|\delta y(t_1) - \delta y(t_0)\|), \\ \|\delta p_y\| &\leq \frac{C}{t_1 - t_0} (\gamma_{t_0, t_1} + \|\delta y(t_1) - \delta y(t_0)\|), \end{aligned} \quad (50)$$

for every $t \in [t_0, t_1]$.

Lemma 7 can be seen as a *bootstrap* result: as soon as a solution $(\delta x(\cdot), \delta y(\cdot), \delta p_x(\cdot), \delta p_y)$ of (43) is bounded at its extremities in δx and δp_x with a sufficiently small bound, then, automatically, this solution enjoys the estimates (50) on $[t_0, t_1]$, which are similar to the linear turnpike estimates stated in Theorem 2. Note that the constants ε_1 , C and ν do not depend on t_0, t_1 .

Proof. We have the following spectral property for the matrix M :

The matrix M is hyperbolic, i.e., all (complex) eigenvalues of M have a nonzero real part.

Let us establish this property. The fact that a Hamiltonian matrix is hyperbolic under a Kalman assumption is classical (see, e.g., [1, 32]). The proof that we give hereafter is the same as in [45] (which, itself, was borrowed from [7, 47]). We recall it not only for completeness but also because the precise changes of variables that are performed below are going to be used, further, in an instrumental way.

Let E_- (resp., E_+) be the minimal symmetric negative definite (resp., maximal symmetric positive definite) matrix solution of the algebraic Riccati equation

$$XA_1^\infty + (A_1^\infty)^\top X + X\bar{B}_1^\infty(\bar{U}^\infty)^{-1}(\bar{B}_1^\infty)^\top X - \bar{W}^\infty = 0.$$

Setting

$$P = \begin{pmatrix} I_n & I_n \\ E_- & E_+ \end{pmatrix}, \quad (51)$$

the matrix P is invertible and we have (note that $E_+ - E_-$ is invertible)

$$P^{-1} = \begin{pmatrix} E_-^{-1}E_+(E_+ - E_-)^{-1}E_- & -(E_+ - E_-)^{-1} \\ -(E_+ - E_-)^{-1}E_- & (E_+ - E_-)^{-1} \end{pmatrix} \quad (52)$$

and

$$P^{-1}MP = \begin{pmatrix} M_- & 0 \\ 0 & M_+ \end{pmatrix}$$

where

$$M_- = A_1^\infty + \bar{B}^\infty(\bar{U}^\infty)^{-1}(\bar{B}^\infty)^\top E_-, \quad M_+ = A_1^\infty + \bar{B}^\infty(\bar{U}^\infty)^{-1}(\bar{B}^\infty)^\top E_+.$$

Moreover, subtracting the Riccati equations satisfied by E_+ and E_- , we have

$$(E_+ - E_-)M_+ + M_-^\top(E_+ - E_-) = 0,$$

hence the eigenvalues of M_+ are the negative of those of M_- , which have negative real parts by the well known Riccati theory (see [1]). Here, we have used the facts that the pair $(A_1^\infty, \bar{B}_1^\infty)$ satisfies the Kalman assumption (this is inferred from Assumption (vii) and by taking the limit) and that the matrices \bar{W}^∞ and \bar{U}^∞ are symmetric positive definite (by taking the limit in Assumptions (v) and (vi)). This proves the claimed spectral property of M .

Since M is hyperbolic, first of all, it is invertible and then (47) can be written as

$$\frac{d}{dt}(z(t) - M^{-1}V\delta p_y) = M(z(t) - M^{-1}V\delta p_y) + O(\|z(t)\|^2, \|\delta p_y\|^2).$$

Second, by the proof of Lemma 7, setting

$$z(t) - M^{-1}V\delta p_y = \begin{pmatrix} \delta x(t) \\ \delta p_x(t) \end{pmatrix} - M^{-1}V\delta p_y = P \begin{pmatrix} v_-(t) \\ v_+(t) \end{pmatrix} = \begin{pmatrix} v_-(t) + v_+(t) \\ E_-v_-(t) + E_+v_+(t) \end{pmatrix}, \quad (53)$$

where P is defined by (51), and equivalently, thanks to (52),

$$\begin{pmatrix} v_-(t) \\ v_+(t) \end{pmatrix} = \begin{pmatrix} E_-^{-1}E_+(E_+ - E_-)^{-1}E_- \delta x(t) - (E_+ - E_-)^{-1} \delta p_x(t) \\ -(E_+ - E_-)^{-1}E_- \delta x(t) + (E_+ - E_-)^{-1} \delta p_x(t) \end{pmatrix} - P^{-1}M^{-1}V\delta p_y, \quad (54)$$

(we write these formulas explicitly because we will use them in an instrumental way further) we have

$$\begin{aligned} \dot{v}_-(t) &= M_-v_-(t) + O(\|v_-(t)\|^2, \|v_+(t)\|^2, \|\delta p_y\|^2), \\ \dot{v}_+(t) &= M_+v_+(t) + O(\|v_-(t)\|^2, \|v_+(t)\|^2, \|\delta p_y\|^2). \end{aligned} \quad (55)$$

We have the following result.

Lemma 8. *There exist $\varepsilon_1, C > 0$ and $\nu > 0$ such that, for all $0 \leq t_0 \leq t_1$, for every solution $(v_-(\cdot), v_+(\cdot), \delta p_y)$ of (55) on $[t_0, t_1]$ satisfying*

$$\|v_-(t_0)\| \leq \varepsilon_1, \quad \|v_-(t_1)\| \leq \varepsilon_1, \quad \|v_+(t_0)\| \leq \varepsilon_1, \quad \|v_+(t_1)\| \leq \varepsilon_1, \quad \|\delta p_y\| \leq \varepsilon_1, \quad (56)$$

we have

$$\|v_-(t)\| \leq C\|v_-(t_0)\| e^{-\nu(t-t_0)} + o(\|v_+(t_1)\| e^{-\nu(t_1-t)}) + o(\delta p_y) \quad (57)$$

$$\|v_+(t)\| \leq C\|v_+(t_1)\| e^{-\nu(t_1-t)} + o(\|v_-(t_0)\| e^{-\nu(t-t_0)}) + o(\delta p_y) \quad (58)$$

for every $t \in [t_0, t_1]$.

Lemma 8 is proved in Section 2.2.3 hereafter. Let us admit it temporarily and finish the proof of Lemma 7.

Using (53), multiplying if necessary ε_1 and C by a positive constant only depending on P, M, E_- and E_+ , we infer from the assumption (49) done in Lemma 7 that (56) is satisfied and thus also (57) and (58) by Lemma 8, and therefore

$$\|z(t) - M^{-1}V\delta p_y\| \leq C\gamma_{t_0, t_1} \left(e^{-\nu(t-t_0)} + e^{-\nu(t_1-t)} \right) + o(\delta p_y) \quad \forall t \in [t_0, t_1]. \quad (59)$$

It remains to estimate the norm of δp_y . Writing (48) in the form

$$\delta \dot{y}(t) = L(z(t) - M^{-1}V\delta p_y) + R\delta p_y + o(z(t), \delta p_y) \quad (60)$$

where R is the square matrix of size p defined by

$$R = LM^{-1}V + \bar{B}_2^\infty (U^\infty)^{-1} (\bar{B}_2^\infty)^\top, \quad (61)$$

integrating in time, using that $\int_{t_0}^{t_1} e^{-\nu t} dt \leq \frac{1}{\nu}$, we infer (taking a larger constant C if necessary), using (59), that

$$\|\delta y(t) - \delta y(t_0) - (t - t_0)R\delta p_y\| \leq C\gamma_{t_0, t_1} + o((t - t_0)\delta p_y) \quad \forall t \in [t_0, t_1] \quad (62)$$

and thus

$$\|R\delta p_y\| \leq \frac{1}{t_1 - t_0} (C\gamma_{t_0, t_1} + \|\delta y(t_1) - \delta y(t_0)\|) + o(\delta p_y). \quad (63)$$

By Lemma 9, which is established in Section 2.2.4, the matrix R is symmetric positive definite, and thus is invertible. Taking a larger constant C if necessary, we infer from (63) and from Lemma 9 the estimate for δp_y given in (50). We finally infer from (59) and (62) the estimates for $z(t)$ and $\delta y(t)$ in (50). \square

Remark 12. In (14)-(15)-(17)-(18), the dynamics in y is written as $\dot{y}(t) = h(x(t), y(t), u(t))$ with $h(x, y, u) = g(x, u)$: the mapping h does not depend on y . Our strategy of proof would not apply to a general mapping h , depending on y . Indeed, otherwise, when linearizing $\dot{y}(t) = h(x(t), y(t), u(t))$ along the path $(\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$, we would get $\delta \dot{y}(t) = A_2^\infty \delta x(t) + \bar{B}_2^\infty \delta u(t) + Q(t)\delta y(t)$ where $Q(t) = \frac{\partial h}{\partial y}(\bar{x}^\infty, t\bar{d}^\infty, \bar{u}^\infty)$ depends on t and thus the matrix M defined by (46) would as well depend on t (anyway, only through the part in Q). The argument of hyperbolicity, then, cannot be applied in general, in particular we think that it may fail whenever the eigenvectors of $Q(t)$ are oscillating too fast. Nevertheless, we think that the technique of proof may be extended to the case where $t \mapsto Q(t)$ is slowly-varying, as in the context of control or stabilization by quasi-static deformation (see [17] and references therein). We leave this issue open.

2.2.3 Proof of Lemma 8

Similar estimates have been given in [45, Section 3.2, (34)], but without the remainder term in δp_y . The proof uses the method of successive approximations, following the classical Banach fixed point argument, like in the proof of the stable manifold theorem done in [33, Chapter 2.7, proof of the theorem page 109].

We infer from (55) that

$$\begin{aligned} v_-(t) &= e^{(t-t_0)M_-} v_-(t_0) + \int_{t_0}^t e^{(t-s)M_-} r_-(v_-(s), v_+(s), \delta p_y) ds \\ v_+(t) &= e^{(t-t_1)M_+} v_+(t_1) - \int_t^{t_1} e^{(t-s)M_+} r_+(v_-(s), v_+(s), \delta p_y) ds \end{aligned}$$

where $r_{\pm}(v_-, v_+, \delta p_y) = O(\|v_-\|^2, \|v_+\|^2, \|\delta p_y\|^2)$. Since M_- and $-M_+$ are Hurwitz, there exist $C, \nu > 0$ such that $\|e^{\tau M_-}\| \leq C e^{-2\nu\tau}$ and $\|e^{-\tau M_+}\| \leq C e^{-2\nu\tau}$ for every $\tau \geq 0$. Taking a larger constant C if necessary, using the definition (45), there exists $\alpha_0 > 0$ such that, for every $\alpha \in (0, \alpha_0)$, under the *a priori* assumption that, for every $t \in [t_0, t_1]$, $\max(\|v_-(t)\|, \|v_+(t)\|, \|\delta p_y\|) \leq \alpha$, we have

$$\begin{aligned} \|v_-(t)\| &\leq C e^{-2\nu(t-t_0)} \|v_-(t_0)\| + \alpha \frac{C}{2\nu} \|\delta p_y\| + \alpha C \int_{t_0}^t e^{-2\nu(t-s)} (\|v_-(s)\| + \|v_+(s)\|) ds \\ \|v_+(t)\| &\leq C e^{-2\nu(t_1-t)} \|v_+(t_1)\| + \alpha \frac{C}{2\nu} \|\delta p_y\| + \alpha C \int_t^{t_1} e^{-2\nu(s-t)} (\|v_-(s)\| + \|v_+(s)\|) ds \end{aligned} \quad (64)$$

(we have used that $\int_0^{+\infty} e^{-2\nu\tau} d\tau = \frac{1}{2\nu}$). The reader who is familiar with the proof of the stable manifold theorem can then conclude straightforwardly, by plugging the rough estimates $\|v_-(t)\| \lesssim C e^{-\nu(t-t_0)} \|v_-(t_0)\|$ and $\|v_+(t)\| \lesssim C e^{-\nu(t_1-t)}$ into the integrals and, since the argument goes by fixed point, we know in advance that it is valid to derive in this way the estimates (57) and (58).

Hereafter, as required by one of the referees, we give the complete argument. Using that $e^{-2\nu s} \leq e^{-\nu s}$ for every $s \geq 0$ (like in the proof of the stable manifold theorem done in [33, Chapter 2.7, page 109]), it is important to let a ‘‘margin’’ in order to avoid a resonance phenomenon), we infer from (64) that

$$y \leq a + \alpha K y + \alpha K z \quad (65)$$

$$z \leq b + \alpha L y + \alpha L z \quad (66)$$

where

$$\begin{aligned} y(t) &= \|v_-(t)\|, & z(t) &= \|v_+(t)\|, \\ a(t) &= C e^{-\nu(t-t_0)} y(t_0) + \alpha \frac{C}{2\nu} \|\delta p_y\|, & b(t) &= C e^{-\nu(t_1-t)} z(t_1) + \alpha \frac{C}{2\nu} \|\delta p_y\|, \end{aligned}$$

and where the linear operators $K, L : C^0([t_0, t_1]) \rightarrow C^0([t_0, t_1])$ are defined by

$$(Kf)(t) = C \int_{t_0}^t e^{-2\nu(t-s)} f(s) ds, \quad (Lf)(t) = C \int_t^{t_1} e^{-2\nu(s-t)} f(s) ds \quad \forall f \in C^0([t_0, t_1]).$$

The operators K and L are bounded on $C^0([t_0, t_1])$ (we have $\|K\| \leq \frac{C}{2\nu}$ and the same for L) and moreover are positively monotone in the following sense: given any $f_1, f_2 \in C^0([t_0, t_1])$,

$$0 \leq f_1 \leq f_2 \quad \Rightarrow \quad 0 \leq K f_1 \leq K f_2 \quad \text{and} \quad 0 \leq L f_1 \leq L f_2.$$

Taking α_0 smaller if necessary so that $0 < \alpha_0 < \frac{2\nu}{C}$, the operators $\text{id} - \alpha K$ and $\text{id} - \alpha L$ are boundedly invertible, and the operators $(\text{id} - \alpha K)^{-1}$ and $(\text{id} - \alpha L)^{-1}$ are positively monotone. Indeed,

$$(\text{id} - \alpha K)^{-1} = \sum_{j=0}^{+\infty} \alpha^j K^j$$

is a converging sum of positively monotone operators, hence is positively monotone (and the same is true for $(\text{id} - \alpha L)^{-1}$).

By (66), we have $(\text{id} - \alpha L)z \leq b + \alpha Ly$ and thus, by positive monotonicity, $z \leq (\text{id} - \alpha L)^{-1}(b + \alpha Ly)$. Plugging this inequality into (65), using that $\text{id} + (\text{id} - \alpha L)^{-1}\alpha L = (\text{id} - \alpha L)^{-1}$, we get

$$y \leq a + \alpha K(\text{id} - \alpha L)^{-1}b + \alpha K(\text{id} - \alpha L)^{-1}y$$

and thus $(\text{id} - U_\alpha)y \leq a + U_\alpha b$ where $U_\alpha = \alpha K(\text{id} - \alpha L)^{-1}$. The linear operator U_α is bounded on $C^0([t_0, t_1])$, and $\|U_\alpha\| \leq \frac{\alpha C}{2\nu - \alpha C}$. Taking α_0 smaller if necessary so that $0 < \alpha_0 < \frac{\nu}{C}$, $\text{id} - U_\alpha$ is boundedly invertible and

$$(\text{id} - U_\alpha)^{-1} = \sum_{j=0}^{+\infty} \alpha^j K^j \left(\sum_{k=0}^{+\infty} \alpha^k L^k \right)^j$$

is a converging sum, with positive coefficients, of positively monotone operators, hence is positively monotone. Therefore

$$y \leq (\text{id} - U_\alpha)^{-1}a + (\text{id} - U_\alpha)^{-1}U_\alpha b = a + (\text{id} - U_\alpha)^{-1}U_\alpha(a + b)$$

where we have used that $(\text{id} - U_\alpha)^{-1} = \text{id} + (\text{id} - U_\alpha)^{-1}U_\alpha$. Noting that $(\text{id} - \alpha K(\text{id} - \alpha L)^{-1})^{-1}$ and αK commute (this is straightforwardly checked by a series expansion), we even have

$$(\text{id} - U_\alpha)^{-1}U_\alpha = \alpha K(\text{id} - \alpha(K + L))^{-1}.$$

The same can be done for z . We finally have

$$\begin{aligned} y &\leq a + \alpha K(\text{id} - \alpha(K + L))^{-1}(a + b) \\ z &\leq b + \alpha L(\text{id} - \alpha(K + L))^{-1}(a + b) \end{aligned}$$

Since $(\text{id} - \alpha(K + L))^{-1} = \sum_{j=0}^{+\infty} \alpha^j (K + L)^j$, let us estimate the power iterates of $\alpha(K + L)$ applied to $e^{-\nu(t-t_0)}$ and $e^{-\nu(t_1-t)}$. We compute

$$\begin{aligned} Ke^{-\nu(t-t_0)} &\leq \frac{C}{\nu} e^{-\nu(t-t_0)}, & Le^{-\nu(t-t_0)} &\leq \frac{C}{3\nu} e^{-\nu(t-t_0)}, \\ Ke^{-\nu(t_1-t)} &\leq \frac{C}{3\nu} e^{-\nu(t_1-t)}, & Le^{-\nu(t_1-t)} &\leq \frac{C}{\nu} e^{-\nu(t_1-t)}, \end{aligned}$$

hence $\alpha^j (K + L)^j e^{-\nu(t-t_0)} \leq (\frac{2C\alpha}{\nu})^j e^{-\nu(t-t_0)}$, and the same for $e^{-\nu(t_1-t)}$, and thus, taking the (convergent) sum and changing slightly constants if necessary, we finally infer (taking α_0 small enough) that, given any $\alpha \in (0, \alpha_0)$,

$$\begin{aligned} y(t) &\leq Ce^{-\nu(t-t_0)}y(t_0) + 2\alpha \frac{C}{\nu} (\|\delta p_y\| + e^{-\nu(t-t_0)}y(t_0) + e^{-\nu(t_1-t)}y(t_1)) \\ z(t) &\leq Ce^{-\nu(t_1-t)}z(t_1) + 2\alpha \frac{C}{\nu} (\|\delta p_y\| + e^{-\nu(t-t_0)}y(t_0) + e^{-\nu(t_1-t)}y(t_1)) \end{aligned} \tag{67}$$

for every $t \in [t_0, t_1]$, under the *a priori* assumption that $\max(y(t), z(t), \|\delta p_y\|) \leq \alpha$ for every $t \in [t_0, t_1]$. Finally, we infer from (67) that, if $\max(y(t_0), y(t_1), z(t_0), z(t_1), \|\delta p_y\|) \leq \alpha/(C + 6\frac{C}{\nu}\alpha_0)$, then actually $\max(y(t), z(t), \|\delta p_y\|) \leq \alpha$ for every $t \in [t_0, t_1]$. Since this has been done for every $\alpha \in (0, \alpha_0)$, the lemma follows.

2.2.4 Proof of Lemma 9

Lemma 9. *The matrix R defined by (61) is symmetric positive definite, and thus is invertible.*

This section is devoted to proving Lemma 9. Throughout this section, we do not write the superscript ∞ in all matrices, to facilitate the reading. Let us first express the matrix R , defined by (61), in a more explicit way. Recall that

$$R = (A_2 \bar{B}_2 \bar{U}^{-1} \bar{B}_1^\top) M^{-1} \begin{pmatrix} -\bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top \\ A_2^\top \end{pmatrix} + \bar{B}_2 \bar{U}^{-1} \bar{B}_2^\top \quad \text{with} \quad M = \begin{pmatrix} A_1 & \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top \\ \bar{W} & -A_1^\top \end{pmatrix}.$$

We set

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M^{-1} \begin{pmatrix} -\bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top \\ A_2^\top \end{pmatrix}$$

so that

$$\begin{aligned} A_1 C_1 + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top C_2 &= -\bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top \\ \bar{W} C_1 - A_1^\top C_2 &= A_2^\top \end{aligned} \tag{68}$$

Since \bar{W} is invertible, we infer from the second equation of (68) that $C_1 = \bar{W}^{-1} A_1^\top C_2 + \bar{W}^{-1} A_2^\top$, and plugging this expression into the first equation of (68) gives

$$(A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top) C_2 = -A_1 \bar{W}^{-1} A_2^\top - \bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top \tag{69}$$

Lemma 10. *The matrix $A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top$ is invertible.*

Proof. Let $\xi \in \mathbb{R}^n$ be such that $(A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top) \xi = 0$. Multiplying to the left by ξ^\top , we obtain $\|\bar{W}^{-1/2} A_1^\top \xi\|^2 + \|\bar{U}^{-1/2} \bar{B}_1^\top \xi\|^2 = 0$ and hence $\xi \in \ker(A_1^\top) \cap \ker(\bar{B}_1^\top)$. As in the proof of Lemma 7, since the pair (A_1, \bar{B}_1) satisfies the Kalman condition, the latter intersection is $\{0\}$. The claim follows. \square

We infer from (69) and from Lemma 10 that

$$C_2 = - (A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top)^{-1} (A_1 \bar{W}^{-1} A_2^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top)$$

and thus

$$C_1 = -\bar{W}^{-1} A_1^\top (A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top)^{-1} (A_1 \bar{W}^{-1} A_2^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top) + \bar{W}^{-1} A_2^\top.$$

We conclude that

$$\begin{aligned} R &= - (A_1 \bar{W}^{-1} A_2^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top)^\top (A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top)^{-1} (A_1 \bar{W}^{-1} A_2^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_2^\top) \\ &\quad + A_2 \bar{W}^{-1} A_2^\top + \bar{B}_2 \bar{U}^{-1} \bar{B}_2^\top \end{aligned}$$

(note that R is symmetric) that we can write as the sum of two symmetric matrices

$$\begin{aligned} R &= A_2 \bar{W}^{-1/2} \left(-\bar{W}^{-1/2} A_1^\top (A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top)^{-1} A_1 \bar{W}^{-1/2} + I_n \right) \bar{W}^{-1/2} A_2^\top \\ &\quad + \bar{B}_2 \bar{U}^{-1/2} \left(-\bar{U}^{-1/2} \bar{B}_1^\top (A_1 \bar{W}^{-1} A_1^\top + \bar{B}_1 \bar{U}^{-1} \bar{B}_1^\top)^{-1} \bar{B}_1 \bar{U}^{-1/2} + I_m \right) \bar{U}^{-1/2} \bar{B}_2^\top \\ &= \tilde{A}_2 \left(-\tilde{A}_1^\top \left(\tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top \right)^{-1} \tilde{A}_1 + I_n \right) \tilde{A}_2^\top + \tilde{B}_2 \left(-\tilde{B}_1^\top \left(\tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top \right)^{-1} \tilde{B}_1 + I_m \right) \tilde{B}_2^\top \end{aligned}$$

where we have set $\tilde{A}_i = A_i \bar{W}^{-1/2}$ and $\tilde{B}_i = \bar{B}_i \bar{U}^{-1/2}$ for $i = 1, 2$, and we are going to prove hereafter that the above two matrices are symmetric positive definite, so that R is itself symmetric positive definite.

We have the following general result.

Lemma 11. *Let A be an arbitrary real-valued square matrix of size n , and let B be an arbitrary real-valued matrix of size $n \times m$, where n and m are arbitrary nonzero integers. If $\ker(A^\top) \cap \ker(B^\top) = \{0\}$ then*

$$B^\top (AA^\top + BB^\top)^{-1} B \preceq I_m \quad (70)$$

meaning that the symmetric matrix $I_m - B^\top (AA^\top + BB^\top)^{-1} B$ is positive semi-definite.

Proof. Since $\ker(A^\top) \cap \ker(B^\top) = \{0\}$, the matrix $AA^\top + BB^\top$ is invertible (same argument as in Lemma 10).

When B is invertible, the result is obvious: starting from

$$AA^\top + BB^\top \succeq BB^\top,$$

taking the inverse and multiplying to the left by B^\top and to the right by B , we obtain (70).

When B is not invertible, we follow anyway the above reasoning, adding εI_m for $\varepsilon > 0$ to recover an appropriate invertibility property: starting from

$$AA^\top + BB^\top + \varepsilon I_m \succeq BB^\top + \varepsilon I_m,$$

taking the inverse and multiplying to the left by B^\top and to the right by B , we obtain

$$B^\top (AA^\top + BB^\top + \varepsilon I_m)^{-1} B \preceq B^\top (BB^\top + \varepsilon I_m)^{-1} B.$$

Now, we use the general fact that

$$B^\top (BB^\top + \varepsilon I_m)^{-1} B \xrightarrow{\varepsilon \rightarrow 0} B^\# B$$

which is a consequence of the Tikhonov regularization in the Moore-Penrose pseudo-inverse theory (see [39]). Here, $B^\#$ is the Moore-Penrose pseudo-inverse of B . We also note the general fact that $B^\# B \preceq I_m$, because $I_m - B^\# B$ is an orthogonal projection.

Then, taking the limit $\varepsilon \rightarrow 0$ gives (70). \square

Noting that $\ker(\tilde{A}_1^\top) \cap \ker(\tilde{B}_1^\top) = \{0\}$ (this comes again from the Kalman condition on the pair (A_1, \tilde{B}_1)), we infer from Lemma 11 that

$$\tilde{A}_1^\top \left(\tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top \right)^{-1} \tilde{A}_1 \preceq I_n \quad \text{and} \quad \tilde{B}_1^\top \left(\tilde{A}_1 \tilde{A}_1^\top + \tilde{B}_1 \tilde{B}_1^\top \right)^{-1} \tilde{B}_1 \preceq I_m.$$

Therefore $R \succeq 0$, i.e., R is positive semi-definite. This concludes the proof of Lemma 9.

2.2.5 Uniform neighborhoods: end of the proof of Proposition 1

To finish the proof of Proposition 1, it remains to prove that one can choose $\varepsilon_0 > 0$ that does not depend on t_0, t_1 whenever $t_1 - t_0$ is large enough. This fact will follow by inspecting more finely (and bootstrapping again) the study performed in Section 2.2.2.

Before starting the proof, we note that it suffices to establish Proposition 1 for $t_0 = 0$ and $\eta_0 = 0$. Indeed, if Proposition 1 is true for $t_0 = 0$ and $\eta_0 = 0$, then we infer the general case by shifting time and by considering $y(t) - y(t_0 \eta_0)$ instead of $y(t)$. Hence, in what follows, we assume without loss of generality that $t_0 = 0$ and that $y(0) = 0$, i.e., $\eta_0 = 0$.

Given $t_1 > 0$ arbitrarily fixed, we have proved in Lemma 6 that there exists $\varepsilon_0 > 0$ depending on t_1 such that, for any $\varepsilon \in (0, \varepsilon_0)$, for any $x_0, x_1 \in B(\bar{x}^\infty, \varepsilon)$ and any $\eta_1 \in B(\bar{d}^\infty, \varepsilon)$, the problem $\mathcal{P}(0, t_1, x_0, x_1, 0, \eta_1)$ has a locally optimal solution $(x(\cdot), y(\cdot), p_x(\cdot), p_y(\cdot), u(\cdot))$ on $[0, t_1]$ satisfying

(38)-(39). We set $\delta x(t) = x(t) - \bar{x}^\infty$, $\delta y(t) = y(t) - t\bar{d}^\infty$, $\delta p_x(t) = p_x(t) - \bar{p}_x^\infty$, $\delta p_y = p_y - \bar{p}_y^\infty$ as in (42).

Having in mind the *shooting method*, given any $t_1 > 0$, any $\delta x(0)$ such that $\|\delta x(0)\| \leq \varepsilon_0$ (recall that $\delta y(0) = 0$), we consider the mapping $\Phi_{t_1, \delta x(0)} : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n \times \mathbb{R}^p$ defined (in a neighborhood of $(0, 0)$) by

$$\Phi_{t_1, \delta x(0)}(\delta p_x(0), \delta p_y) = \left(\delta x(t_1), \frac{\delta y(t_1)}{t_1} \right)$$

where $(\delta x(\cdot), \delta y(\cdot), \delta p_x(\cdot), \delta p_y)$ is solution of (43). The mapping $\Phi_{t_1, \delta x(0)}$ is interpreted as the *exponential mapping* (see [8]) associated with the Hamiltonian system (38)-(39), and it follows from the conjugate point theory that t_1 is not a conjugate time for the initial point $(\delta x(0), \delta y(0) = 0)$ if and only if the mapping $\Phi_{t_1, \delta x(0)}$ is a local diffeomorphism (see [8, Proposition 2.9] or see [6, Theorem 21.1] in a more general context).

Here, the *shooting problem* that we are going to solve is the following: we will prove that there exists $\varepsilon_0 > 0$ (to be chosen small enough) and $T_0 > 0$ (to be chosen large enough) such that, given any $t_1 > T_0$, given any $x_0, x_1 \in B(\bar{x}^\infty, \varepsilon_0)$ and given any $y_1^{t_1} \in \mathbb{R}^p$ such that $\frac{y_1^{t_1}}{t_1} \in B(\bar{d}^\infty, \varepsilon_0)$, there exist $\delta p_x(0) \in \mathbb{R}^n$ and $\delta p_y \in \mathbb{R}^p$ such that

$$\Phi_{t_1, \delta x(0)}(\delta p_x(0), \delta p_y) = \left(x_1 - \bar{x}^\infty, \frac{y_1^{t_1}}{t_1} - \bar{d}^\infty \right) \quad (71)$$

(this is the shooting problem), i.e., such that the solution $(\delta x(\cdot), \delta y(\cdot), \delta p_x(\cdot), \delta p_y)$ of (43), of initial condition $(\delta x(0) = x_0 - \bar{x}^\infty, \delta y(0) = 0, \delta p_x(0), \delta p_y)$, satisfies $\delta x(t_1) = x_1 - \bar{x}^\infty$ and $\delta y(t_1) = y_1^{t_1} - t_1 \bar{d}^\infty$, and moreover we want to prove that $\Phi_{t_1, \delta x(0)}$ is a local diffeomorphism (and this, we insist, whatever $t_1 > 0$ large enough may be).

Let us consider again and follows the steps of the proof of Lemma 7. Assuming that $t_1 \geq T_0$, setting $M^{-1}V = (H_1 \ H_2)^\top$ and $P^{-1}M^{-1}V = (H_3 \ H_4)^\top$, it follows from (53) that

$$\delta x(t_1) = v_-(t_1) + v_+(t_1) + H_1 \delta p_y, \quad (72)$$

and from (54) that

$$v_-(0) = E_-^{-1}E_+(E_+ - E_-)^{-1}E_- \delta x(0) - (E_+ - E_-)^{-1} \delta p_x(0) + H_3 \delta p_y \quad (73)$$

$$v_+(0) = -(E_+ - E_-)^{-1}E_- \delta x(0) + (E_+ - E_-)^{-1} \delta p_x(0) + H_4 \delta p_y \quad (74)$$

Now, using (57) at $t = t_1$ and (73), we infer that

$$v_-(t_1) = O(\delta x(0), \delta p_x(0), \delta p_y) e^{-\nu t_1} + o(v_+(t_1)) + o(\delta p_y),$$

and combining with (72) we obtain

$$\begin{aligned} v_+(t_1) &= \delta x(t_1) - v_-(t_1) - H_1 \delta p_y \\ &= \delta x(t_1) - H_1 \delta p_y + O(\delta x(0), \delta p_x(0), \delta p_y) e^{-\nu t_1} + o(\delta x(t_1), \delta p_y). \end{aligned} \quad (75)$$

Since $\delta x(0) = O(1)$ and since we want to solve $\delta x(t_1) = x_1 - \bar{x}^\infty = O(1)$, using (58) at $t = 0$, (73) and (75), we must have

$$v_+(0) = O(\delta x(t_1), \delta p_y) e^{-\nu t_1} + o(\delta x(0), \delta p_x(0), \delta p_y),$$

and thus using (74), finally, we find that

$$\delta p_x(0) = E_- \delta x(0) - (E_+ - E_-) H_4 \delta p_y + O(\delta x(t_1), \delta p_y) e^{-\nu t_1} + o(\delta x(0), \delta p_y). \quad (76)$$

Finally, let us consider the last equation of the shooting problem (71) to be solved, $\delta y(t_1) = y_1^{t_1} - t_1 \bar{d}^\infty$. Using (53), we rewrite (60) as

$$\delta y(t) = L \begin{pmatrix} v_-(t) + v_+(t) \\ E_- v_-(t) + E_+ v_+(t) \end{pmatrix} + R \delta p_y + o(z(t), \delta p_y). \quad (77)$$

It follows from (57), (58), (73), (75) and (76) that

$$v_-(t) = O(\delta x(0), \delta p_y, e^{-\nu t_1}) e^{-\nu t} + o(\delta x(t_1), \delta p_y, e^{-\nu t_1}) e^{-\nu(t_1-t)} + o(\delta p_y), \quad (78)$$

$$v_+(t) = o(\delta x(0), \delta p_y, e^{-\nu t_1}) e^{-\nu t} + O(\delta x(t_1), \delta p_y, e^{-\nu t_1}) e^{-\nu(t_1-t)} + o(\delta p_y). \quad (79)$$

Now, we infer from (77), (78) and (79) that

$$\delta y(t) = R \delta p_y + O(\delta x(0), \delta x(t_1), \delta p_y, e^{-\nu t_1})(e^{-\nu t} + e^{-\nu(t_1-t)}) + o(\delta p_y)$$

and, integrating in time, using that $\delta y(0) = 0$ and $\int_0^{t_1} e^{-\nu t} dt \leq \frac{1}{\nu}$, noting that $O(\delta p_y) = o(t_1 \delta p_y)$ for t_1 large enough, we finally obtain

$$\delta y(t_1) = t_1 R \delta p_y + o(t_1 \delta p_y) + O(\delta x(0), \delta x(t_1), e^{-\nu t_1}).$$

Since we want to solve $\delta y(t_1) = y_1^{t_1} - t_1 \bar{d}^\infty$, and since R is invertible by Lemma 9, we must have

$$\delta p_y = R^{-1} \left(\frac{y_1^{t_1}}{t_1} - \bar{d}^\infty \right) + o \left(\frac{y_1^{t_1}}{t_1} - \bar{d}^\infty \right) + \frac{1}{t_1} O(\delta x(0), \delta x(t_1), e^{-\nu t_1}) \quad (80)$$

and, plugging (80) into (76) we obtain

$$\delta p_x(0) = E_- \delta x(0) - (E_+ - E_-) H_4 R^{-1} \left(\frac{y_1^{t_1}}{t_1} - \bar{d}^\infty \right) + o \left(\delta x(0), \delta x(t_1), \frac{y_1^{t_1}}{t_1} - \bar{d}^\infty, e^{-\nu t_1} \right). \quad (81)$$

All in all, the shooting problem (71) is equivalent to the equations (80) and (81), which can be solved for an ε_0 small enough that is independent of t_1 , and whose solving shows that the mapping $\Phi_{t_1, \delta, x(0)}$, which maps $(\delta p_x(0), \delta p_y)$ to $(\delta x(t_1), \frac{\delta y(t_1)}{t_1})$, is a local diffeomorphism, whatever $t_1 > 0$ large enough may be. In turn, this gives the local uniqueness property stated in (i). This concludes the proof of the proposition.

Remark 13. When $\frac{y_1^{t_1}}{t_1} = \bar{d}^\infty + o(1)$ as $t_1 \rightarrow +\infty$, passing to the limit $t_1 \rightarrow +\infty$ in (81) gives $\delta p_x(0) = E_- \delta x(0)$, which is the classical relation in algebraic Riccati theory (see, e.g., [41, Proof of Theorem 4.4.5]).

2.3 Exploiting dissipativity

Recall that we have considered a sequence $(T_k)_{k \in \mathbb{N}^*}$ of positive real numbers converging to $+\infty$ such that (35) holds, i.e., $(\bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ is a closure point of the family $(\bar{x}^T, \frac{y_1^T}{T}, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)_{T \geq T_0}$. Considering the optimal solution $(x^{T_k}(\cdot), y^{T_k}(\cdot), u^{T_k}(\cdot))$ of (14)-(15)-(16)-(17)-(18), let us now exploit the strict dissipativity property to derive the following result.

Lemma 12. *Let $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ be a function such that $\varphi(k) \leq T_k$ for every $k \in \mathbb{N}^*$ and $\varphi(k) \rightarrow +\infty$ as $k \rightarrow +\infty$. Considering a subsequence of $(T_k)_{k \in \mathbb{N}^*}$ if necessary, there exists a measurable subset $C_\varphi \subset [0, 1]$ of full Lebesgue measure such that, for every $s \in C_\varphi$,*

$$x^{T_k}(\varphi(k)s) \rightarrow \bar{x}^\infty, \quad u^{T_k}(\varphi(k)s) \rightarrow \bar{u}^\infty, \quad \frac{y^{T_k}(\varphi(k)s)}{\varphi(k)} \rightarrow s \bar{d}^\infty, \quad (82)$$

$$x^{T_k}(T_k - \varphi(k)s) \rightarrow \bar{x}^\infty, \quad u^{T_k}(T_k - \varphi(k)s) \rightarrow \bar{u}^\infty, \quad \frac{y^{T_k}(T_k - \varphi(k)s)}{T_k - \varphi(k)s} \rightarrow s \bar{d}^\infty, \quad (83)$$

as $k \rightarrow +\infty$.

Of course, this lemma can be applied with $\varphi(k) = T_k$, but as we are going to see it will be of interest to apply it with functions φ such that $\varphi(k) = o(T_k)$. A difficulty will arise from the fact that the full Lebesgue measure set C_φ depends on φ .

Proof. The partial x -strict dissipativity inequality (12) given by Assumption (iii), applied to $(x^{T_k}(\cdot), u^{T_k}(\cdot))$ on the time interval $[0, \varphi(k)]$, implies that

$$f^0(\bar{x}^{T_k}, \bar{u}^{T_k}) \leq \frac{1}{\varphi(k)} \int_0^{\varphi(k)} f^0(x^{T_k}(t), u^{T_k}(t)) dt + \frac{S^{T_k}(x^{T_k}(0)) - S^{T_k}(x^{T_k}(\varphi(k)))}{\varphi(k)} - \frac{1}{\varphi(k)} \int_0^{\varphi(k)} \alpha(\|x^{T_k}(t) - \bar{x}^{T_k}\|, \|u^{T_k}(t) - \bar{u}^{T_k}\|) dt. \quad (84)$$

Let us establish (82). To establish (83), the method is the same by replacing the time interval $[0, \varphi(k)]$ with $[T_k - \varphi(k), T_k]$ in (84).

Let us prove that $\frac{1}{\varphi(k)} \int_0^{\varphi(k)} \alpha(\|x^{T_k}(t) - \bar{x}^{T_k}\|, \|u^{T_k}(t) - \bar{u}^{T_k}\|) dt \rightarrow 0$ as $k \rightarrow +\infty$. By contradiction, assume that there exists $\eta > 0$ and a (sub)sequence of integers $k \rightarrow +\infty$ such that

$$\frac{1}{\varphi(k)} \int_0^{\varphi(k)} \alpha(\|x^{T_k}(t) - \bar{x}^{T_k}\|, \|u^{T_k}(t) - \bar{u}^{T_k}\|) dt \geq \eta \quad (85)$$

for every k large enough. The cost on the time interval $[0, \varphi(k)]$ of the trajectory $(\tilde{x}^{T_k}(\cdot), \tilde{y}^{T_k}(\cdot))$ solution of (14)-(15) given by Assumption (iv) is (by splitting the time interval $[0, \varphi(k)]$ and thus the integral with respect to the bounded times $t_1^{T_k}, t_2^{T_k}$ given by that assumption)

$$\frac{1}{\varphi(k)} \int_0^{\varphi(k)} f^0(\tilde{x}^{T_k}(t), \tilde{u}^{T_k}(t)) dt = f^0(\bar{x}^{T_k}, \bar{u}^{T_k}) + o(1)$$

as $k \rightarrow +\infty$. Hence, for k large enough, using (84) and (85), we have obtained a trajectory solution of (14)-(15) whose cost is (strictly) less than the cost of the optimal trajectory $(x^{T_k}(\cdot), y^{T_k}(\cdot), u^{T_k}(\cdot))$: this is a contradiction.

Using the change of variable $t = \varphi(k)s$, we have therefore proved that

$$\lim_{k \rightarrow +\infty} \int_0^1 \alpha(\|x^{T_k}(\varphi(k)s) - \bar{x}^{T_k}\|, \|u^{T_k}(\varphi(k)s) - \bar{u}^{T_k}\|) ds = 0.$$

By the converse of the Lebesgue dominated theorem, taking a subsequence if necessary, for almost every $s \in [0, 1]$, $x^{T_k}(\varphi(k)s) - \bar{x}^{T_k} \rightarrow 0$ and $u^{T_k}(\varphi(k)s) - \bar{u}^{T_k} \rightarrow 0$ as $k \rightarrow +\infty$, and thus $x^{T_k}(\varphi(k)s) \rightarrow \bar{x}^\infty$ and $u^{T_k}(\varphi(k)s) \rightarrow \bar{u}^\infty$ as $k \rightarrow +\infty$. It remains to prove that $\frac{y^{T_k}(\varphi(k)s)}{\varphi(k)} \rightarrow s\bar{d}^\infty$. We set, for almost every $s \in [0, 1]$,

$$\hat{x}_k(s) = x^{T_k}(\varphi(k)s), \quad \hat{y}_k(s) = y^{T_k}(\varphi(k)s), \quad \hat{u}_k(s) = u^{T_k}(\varphi(k)s).$$

We already know that $(\hat{x}_k(\cdot), \hat{u}_k(\cdot))$ converges almost everywhere to $(\bar{x}^\infty, \bar{u}^\infty)$. Moreover, we have

$$\frac{1}{\varphi(k)} \frac{d}{ds} \hat{y}_k(s) = g(\hat{x}_k(s), \hat{u}_k(s)), \quad \frac{1}{\varphi(k)} \hat{y}_k(0) = \frac{y_0}{\varphi(k)},$$

and thus $\frac{1}{\varphi(k)} \hat{y}_k(s) \rightarrow \bar{y}(s)$ almost everywhere on $[0, 1]$ with $\bar{y}(s) = s g(\bar{x}^\infty, \bar{u}^\infty)$ and $g(\bar{x}^\infty, \bar{u}^\infty) = \bar{d}^\infty$. The lemma is proved. \square

First improvement by bootstrap. Lemma 12 alone does not imply that, for instance, $t \mapsto x^{T_k}(t)$ converges to \bar{x}^∞ uniformly on $[\varepsilon, T - \varepsilon]$ for any $\varepsilon > 0$ fixed (one could a priori have “sliding bumps”). This much stronger convergence property is however true and is obtained by bootstrapping again with the estimates (40) given by Proposition 1. The main restriction in Lemma 12 comes from the full Lebesgue measure subset C_φ depending on φ ; we start with the following lemma, showing that we can take $s = 1$ in the convergence of $x^{T_k}(\cdot)$, independently of φ .

Lemma 13. *Let $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ be a function such that $\varphi(k) \leq T_k/2$ for every $k \in \mathbb{N}^*$ and $\varphi(k) \rightarrow +\infty$ as $k \rightarrow +\infty$. Considering a subsequence of $(T_k)_{k \in \mathbb{N}^*}$ if necessary, we have*

$$\begin{aligned} x^{T_k}(\varphi(k)) &\rightarrow \bar{x}^\infty, & x^{T_k}(T_k - \varphi(k)) &\rightarrow \bar{x}^\infty, \\ p_x^{T_k}(\varphi(k)) &\rightarrow \bar{p}_x^\infty, & p_x^{T_k}(T_k - \varphi(k)) &\rightarrow \bar{p}_x^\infty, \end{aligned}$$

as $k \rightarrow +\infty$.

Proof. By Lemma 12, there exist $s_0, s_1 \in (0, 1/4)$ such that

$$\begin{aligned} x^{T_k}(\varphi(k)s_0) &\rightarrow \bar{x}^\infty, & \frac{y^{T_k}(\varphi(k)s_0)}{\varphi(k)s_0} &\rightarrow \bar{d}^\infty, \\ x^{T_k}(T_k - \varphi(k)s_1) &\rightarrow \bar{x}^\infty, & \frac{y^{T_k}(T_k - \varphi(k)s_1)}{T_k - \varphi(k)s_1} &\rightarrow \bar{d}^\infty \end{aligned}$$

as $k \rightarrow +\infty$. Given any $k \in \mathbb{N}^*$, setting

$$\begin{aligned} t_0^k &= \varphi(k)s_0, & t_1^k &= T_k - \varphi(k)s_1, \\ x_0^k &= x^{T_k}(t_0), & x_1^k &= x^{T_k}(t_1), & \eta_0^k &= \frac{y^{T_k}(t_0)}{t_0}, & \eta_1^k &= \frac{y^{T_k}(t_1)}{t_1}, \end{aligned}$$

by the above convergence properties, for k large enough we have $x_0^k \simeq \bar{x}^\infty$, $x_1^k \simeq \bar{x}^\infty$, $\eta_0^k \simeq \bar{d}^\infty$ and $\eta_1^k \simeq \bar{d}^\infty$, and we also note that $t_1^k - t_0^k = T_k - \varphi(k)(s_0 + s_1) \rightarrow +\infty$ as $k \rightarrow +\infty$. Hence, for k large enough we have $x_0^k, x_1^k \in B(\bar{x}^\infty, \varepsilon_0)$ and $\eta_0^k, \eta_1^k \in B(\bar{d}^\infty, \varepsilon_0)$, with the notations of Proposition 1: this is at this step that we use in a crucial way the fact, established in Proposition 1, that ε_0 does not depend on t_0, t_1 whenever $t_1 - t_0$ is large enough.

Being the restriction of an optimal solution, by the dynamic programming principle the triple $(x^{T_k}(\cdot), y^{T_k}(\cdot), u^{T_k}(\cdot))$ restricted to the time interval $[t_0^k, t_1^k]$ must therefore be the unique locally optimal solution of the auxiliary optimal control problem $\mathcal{P}(t_0^k, t_1^k, x_0^k, x_1^k, \eta_0^k, \eta_1^k)$ considered in Section 2.2.

It follows from Proposition 1 that, on the time interval $[t_0^k, t_1^k] = [\varphi(k)s_0, T_k - \varphi(k)s_1]$, the optimal solution $(x^{T_k}(\cdot), y^{T_k}(\cdot), u^{T_k}(\cdot))$ has a normal extremal lift and that $u^{T_k}(\cdot)$ is C^1 . But, by Assumption (i), $(x^{T_k}(\cdot), y^{T_k}(\cdot), u^{T_k}(\cdot))$ has a unique extremal lift, which is moreover normal, with adjoint vectors $p_x^{T_k}(\cdot)$ and $p_y^{T_k}(\cdot)$. Hence, these adjoint vectors must coincide with those given by Proposition 1. Therefore, still by Proposition 1, if k is large enough then we have also $p_x^{T_k}(\varphi(k)s_0) \simeq \bar{p}_x^\infty$, $p_x^{T_k}(T_k - \varphi(k)s_1) \simeq \bar{p}_x^\infty$ and $p_y^{T_k} \simeq \bar{p}_y^\infty$. Note that the latter fact, on the adjoint vectors, is far from obvious and follows from the quite difficult Proposition 1, adequately combined with the previous analysis.

At this step, we have therefore obtained that, for every k large enough,

$$\begin{aligned} \|x^{T_k}(\varphi(k)s_0) - \bar{x}^\infty\| &\leq \varepsilon_0, & \|x^{T_k}(T_k - \varphi(k)s_1) - \bar{x}^\infty\| &\leq \varepsilon_0, \\ \|p_x^{T_k}(\varphi(k)s_0) - \bar{p}_x^\infty\| &\leq \varepsilon_0, & \|p_x^{T_k}(T_k - \varphi(k)s_1) - \bar{p}_x^\infty\| &\leq \varepsilon_0. \end{aligned}$$

Hence, applying Proposition 1 on the time interval $[\varphi(k)s_0, T_k - \varphi(k)s_1]$, we infer from (40) (here, we bootstrap again) that

$$\begin{aligned} \|x^{T_k}(t) - \bar{x}^\infty\| + \|p_x^{T_k}(t) - \bar{p}_x^\infty\| &\leq C\varepsilon_0 \left(e^{-\nu(t-\varphi(k)s_0)} + e^{-\nu(T_k-\varphi(k)s_1-t)} \right) \\ &+ \frac{C}{T_k - \varphi(k)(s_0 + s_1)} \left(\varepsilon_0 + \|y^{T_k}(T_k - \varphi(k)s_1) - y^{T_k}(\varphi(k)s_0) - (T_k - \varphi(k)(s_0 + s_1))\bar{d}^\infty\| \right) \end{aligned}$$

for every $t \in [\varphi(k)s_0, T_k - \varphi(k)s_1]$. The conclusion stated in the lemma now easily follows. \square

Of course, at the end of the above proof, we could already infer a stronger conclusion than that stated in Lemma 13. But before that, we are first going to establish the following result, providing an additional improvement.

Lemma 14. *For every $\varepsilon \in (0, 1)$ there exist $\tau_0, \tau_1 > 0$ and $k_0 \in \mathbb{N}^*$ such that*

$$\begin{aligned} \|x^{T_k}(\tau_0) - \bar{x}^\infty\| &< \varepsilon, & \|x^{T_k}(T_k - \tau_1) - \bar{x}^\infty\| &< \varepsilon, \\ \|p_x^{T_k}(\tau_0) - \bar{p}_x^\infty\| &< \varepsilon, & \|p_x^{T_k}(T_k - \tau_1) - \bar{p}_x^\infty\| &< \varepsilon, \end{aligned}$$

for every $k \geq k_0$.

Proof. Let us prove the existence of τ_0 (and only for the x component), the property for τ_1 being similar. By contradiction, assume that there exists $\varepsilon > 0$ such that, for every $\tau > 0$, for every $k \in \mathbb{N}^*$, there exists $j(k) \geq k$ such that $\|x^{T_{j(k)}}(\tau) - \bar{x}^\infty\| \geq \varepsilon$. Applying this property to $\tau = k$, we obtain that, for every $k \in \mathbb{N}^*$, there exists $j(k) \geq k$ such that $\|x^{T_{j(k)}}(k) - \bar{x}^\infty\| \geq \varepsilon$. Without loss of generality, we assume that $k \mapsto j(k)$ is increasing. Then, there exists a function $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that $\varphi(j(k)) = k$ for every $k \in \mathbb{N}^*$. By Lemma 13, we have $x^{T_{j(k)}}(k) = x^{T_{j(k)}}(\varphi(j(k))) \rightarrow \bar{x}^\infty$. This is a contradiction. \square

Second (and final) improvement by bootstrap. With respect to Lemma 13, in Lemma 14 we have obtained that the trajectory $(x^{T_k}(\cdot), p^{T_k}(\cdot))$, starting at $t = 0$ (resp., starting backward in time at $t = T_k$), enters an ε_0 -neighborhood of $(\bar{x}^\infty, \bar{p}_x^\infty)$ in finite time τ_0 (resp., τ_1). This is instrumental to conclude, what we now do.

Applying again Proposition 1, but now, on the time interval $[\tau_0, T_k - \tau_1]$, we infer from (40) (again a bootstrap) that, for k large enough,

$$\begin{aligned} \|x^{T_k}(t) - \bar{x}^\infty\| + \|p_x^{T_k}(t) - \bar{p}_x^\infty\| &\leq C\varepsilon_0 \left(e^{-\nu(t-\tau_0)} + e^{-\nu(T_k-\tau_1-t)} \right) \\ &+ \frac{C}{T_k - \tau_0 - \tau_1} \left(\varepsilon_0 + \|y^{T_k}(T_k - \tau_1) - y^{T_k}(\tau_0) - (T_k - \tau_0 - \tau_1)\bar{d}^\infty\| \right), \\ \|y^{T_k}(t) - t\bar{d}^\infty\| &\leq C \left(\varepsilon_0 + \|y^{T_k}(\tau_0) - \tau_0\bar{d}^\infty\| + \|y^{T_k}(T_k - \tau_1) - y^{T_k}(\tau_0) - (T_k - \tau_0 - \tau_1)\bar{d}^\infty\| \right), \\ \|p_y^{T_k} - \bar{p}_y^\infty\| &\leq \frac{C}{T_k - \tau_0 - \tau_1} \left(\varepsilon_0 + \|y^{T_k}(T_k - \tau_1) - y^{T_k}(\tau_0) - (T_k - \tau_0 - \tau_1)\bar{d}^\infty\| \right), \end{aligned}$$

for every $t \in [\tau_0, T_k - \tau_1]$. Since $y^{T_k}(0) = y_0$ and $\dot{y}^{T_k}(t) = g(x^{T_k}(t), u^{T_k}(t))$ is bounded (see Assumption (i)), the sequence $(y^{T_k}(\tau_0) - \tau_0\bar{d}^\infty)_{k \in \mathbb{N}^*}$ is bounded. Besides, since

$$y^{T_k}(T_k - \tau_1) = y^{T_k}(T_k) - \int_{T_k - \tau_1}^{T_k} g(x^{T_k}(t), u^{T_k}(t)) dt,$$

since $y^{T_k}(T_k) = y_1^{T_k}$ and since $y_1^{T_k} = y_0 + T_k \bar{d}^\infty$ (see (28)), we infer that the sequence $(y^{T_k}(T_k - \tau_1) - y^{T_k}(\tau_0) - (T_k - \tau_0 - \tau_1) \bar{d}^\infty)_{k \in \mathbb{N}^*}$ is bounded. Therefore, taking a larger constant C if necessary, we obtain

$$\begin{aligned} \|x^{T_k}(t) - \bar{x}^\infty\| + \|p_x^{T_k}(t) - \bar{p}_x^\infty\| &\leq C \left(\frac{1}{T_k} + e^{-\nu t} + e^{-\nu(T_k-t)} \right) \\ \|y^{T_k}(t) - t \bar{d}^\infty\| \leq C, \quad \|p_y^{T_k} - \bar{p}_y^\infty\| &\leq \frac{C}{T_k} \quad \forall t \in [0, T_k]. \end{aligned} \quad (86)$$

Conclusion. Let us summarize what we have done. Given any closure point $(\bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ of the bounded family $(\bar{x}^T, \frac{y_1^T}{T}, \bar{u}^T, \bar{p}_x^T, \bar{p}_y^T)_{T \geq T_0}$ and any sequence $T_k \rightarrow +\infty$ of positive real numbers such that $(\bar{x}^{T_k}, \frac{y_1^{T_k}}{T_k}, \bar{u}^{T_k}, \bar{p}_x^{T_k}, \bar{p}_y^{T_k})$ converges to $(\bar{x}^\infty, \bar{d}^\infty, \bar{u}^\infty, \bar{p}_x^\infty, \bar{p}_y^\infty)$ (see (35)), we have established the estimates (86), and thus, using Lemma 4 and the triangular inequality, and taking a larger C if necessary,

$$\begin{aligned} \|x^{T_k}(t) - \bar{x}^{T_k}\| + \|p_x^{T_k}(t) - \bar{p}_x^{T_k}\| &\leq C \left(\frac{1}{T_k} + e^{-\nu t} + e^{-\nu(T_k-t)} \right) \\ \|y^{T_k}(t) - t \frac{y_1^{T_k}}{T_k}\| \leq C, \quad \|p_y^{T_k} - \bar{p}_y^{T_k}\| &\leq \frac{C}{T_k} \quad \forall t \in [0, T_k]. \end{aligned}$$

It follows that

$$\begin{aligned} \|x^T(t) - \bar{x}^T\| + \|p_x^T(t) - \bar{p}_x^T\| &\leq C \left(\frac{1}{T} + e^{-\nu t} + e^{-\nu(T-t)} \right) \\ \|y^T(t) - t \frac{y_1^T}{T}\| \leq C, \quad \|p_y^T - \bar{p}_y^T\| &\leq \frac{C}{T} \quad \forall t \in [0, T]. \end{aligned}$$

Indeed, here, we use the following obvious fact⁴: let h_1 and h_2 be two bounded functions on \mathbb{R} such that, for every sequence $x_k \rightarrow +\infty$ for which $h_2(x_k)$ converges, we have $h_1(x_k) - h_2(x_k) \rightarrow 0$ as $k \rightarrow +\infty$; then $h_1(x) - h_2(x) \rightarrow 0$ as $x \rightarrow +\infty$.

This gives (29) and concludes the proof of Theorem 2.

3 Examples

In this section, we illustrate the linear turnpike phenomenon on several practical examples, with numerical simulations. Hereafter, we compute numerically the optimal solutions either with a direct method, using `AMPL` (see [25]) combined with `IpOpt` (see [46]), following a full discretization of the problem, or with the variant of the shooting method introduced in [45]. In both cases, we initialize the numerical method at the turnpike, thus guaranteeing its convergence.

3.1 The Zermelo problem

The Zermelo problem is a famous optimal control problem, often used as a simple model or exercise to illustrate theory or practice of optimal control (see [41]). In this problem, we consider a boat

⁴It suffices to prove that 0 is the only possible closure point of the (bounded) function $h_1 - h_2$ at $+\infty$. Let $a \in \mathbb{R}$ be such a closure point: there exists a sequence $x_k \rightarrow +\infty$ such that $h_1(x_k) - h_2(x_k) \rightarrow a$. Since the sequence $(h_2(x_k))_{k \in \mathbb{N}}$ is bounded, there exists a subsequence such that $h_2(x_{k_p}) \rightarrow 0$. Hence, by assumption, $h_1(x_{k_p}) - h_2(x_{k_p}) \rightarrow 0$, and thus $a = 0$.

moving with constant speed along a river of constant width ℓ , in which there is a current $c(x)$. The movement of the center of mass of the boat is governed by the control system

$$\begin{aligned} \dot{x}(t) &= v(t) \sin u(t), & x(0) &= 0, \\ \dot{y}(t) &= v(t) \cos u(t) + c(x(t)), & y(0) &= 0, \end{aligned}$$

where the controls are the modulus of the speed $v(t) \in [0, v_{\max}]$ and the angle $u(t) \in \mathbb{R}$ of the axis of the boat with respect to the axis $(0y)$ (see Figure 2).

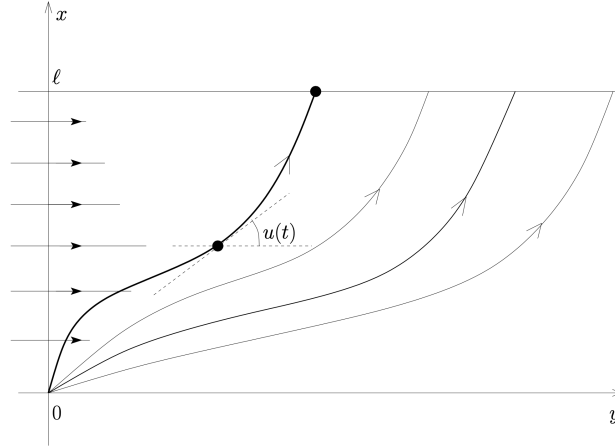


Figure 1: Zermelo problem.

Many variants of terminal conditions and of cost functionals can be considered. Here, in order to illustrate the linear turnpike phenomenon, given some $T > 0$ fixed (to be chosen large enough), we consider the optimal control problem of steering in time T this control system from the initial point $(0, 0)$ to the final point $(\ell, 5T)$ (on the opposite river), by minimizing the cost functional $\int_0^T v(t)^2 dt$.

We take $L = 2$, $v_{\max} = 1.1$ and $c(x) = 3 + x(L - x)$. The current in the river is so strong that we always have $\dot{y}(t) > 0$, whatever the control may be; hence $y(\cdot)$ is bound to be increasing.

The turnpike-static optimal control problem consists of minimizing v^2 under the constraints $v \sin u = 0$ and $T(v \cos u + c(y)) = 5T$. It has the unique solution $\bar{x}^T = 1$, $\bar{y}^T(t) = 5t$, $\bar{x}^T = 1$, $\bar{u}^T = 0$, $\bar{v}^T = 1$, $\bar{p}_x^T = 0$, $\bar{p}_y^T = 2$.

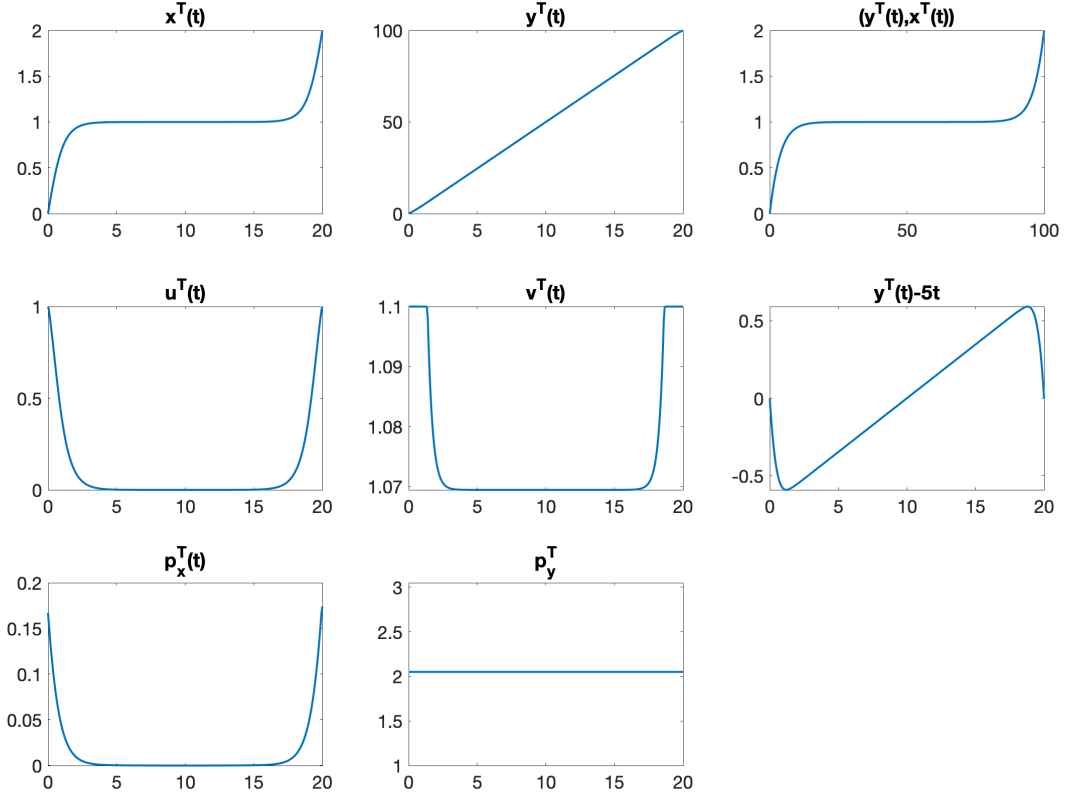


Figure 2: Numerical results for the Zermelo problem

The numerical results for $T = 20$ are reported on Figure 2. We observe, as expected, that $|y^T(t) - \bar{y}^T(t)|$ is bounded by a constant. Besides, we observe that, in the long middle part of the time interval, $(x^T(t), u^T(t), p_x^T(\cdot))$ is exponentially close to $(\bar{x}^T, \bar{u}^T, \bar{p}_x^T)$ while $(v^T(t), p_y^T)$ is linearly close to (\bar{v}^T, \bar{p}_y^T) .

Note that the optimal control $v^T(\cdot)$ has two bang arcs along which $v^T(t) = v_{\max} = 1.1$: a short one at the beginning, and another short one at the end.

3.2 Optimal control model of a runner

Consider the runner model developed in [2, 3, 4, 5], originating from [31]:

$$\begin{aligned}
 \dot{x}(t) &= v(t) & x(0) &= 0, & x(t_f) &= d \\
 \dot{v}(t) &= -\frac{v(t)}{\tau} + f(t) & v(0) &= v^0 \\
 \dot{e}(t) &= \sigma - f(t)v(t) & e(0) &= e^0, & e(t_f) &= 0, & e(t) &\geq 0 \\
 \dot{f}(t) &= \gamma(u(t)(F_{\max} - f(t)) - f(t)) & 0 &\leq f(t) \leq F_{\max}
 \end{aligned} \tag{87}$$

where $d > 0$ is the prescribed distance to run, $e^0 > 0$ is the initial energy, $\tau > 0$ is the friction coefficient related to the runner's economy, $\sigma > 0$ is a constant standing for the energetic equivalent of the oxygen uptake VO_2 , $F_{\max} > 0$ is a threshold upper bound for the force $f(t)$, $\gamma > 0$ is the time constant of motor activation and $u(t) \in [-M, M]$ is the neural drive which is the control, where $M > 0$ is some control bound. Here, $x(t)$ is the distance travelled at time t by the runner, $v(t)$ the velocity, $e(t)$ the anaerobic energy, and $f(t)$ the propulsive force per unit mass. Actually, the typical values of the initial energy e_0 are such that the energy variable $e(t)$ is decreasing. The minimization criterion is

$$\min \left(t_f + \frac{\alpha}{2} \int_0^{t_f} u(t)^2 dt \right) \quad (88)$$

where $\alpha > 0$ is a fixed constant and the final time t_f is free.

Although, in the optimal control problem (87)-(88), t_f , $v(t_f)$, $f(0)$ and $f(t_f)$ are let free, the problem can be reparametrized by the distance s , with the change of variable $t'(s) = \frac{dt}{ds} = \frac{1}{v(s)}$. In terms of s , the optimal control problem is

$$\begin{aligned} v'(s) &= -\frac{1}{\tau} + \frac{f(s)}{v(s)} & v(0) &= v^0 \\ e'(s) &= \frac{\sigma}{v(s)} - f(s) & e(0) &= e^0, \quad e(d) = 0, \quad e(s) \geq 0 \\ f'(s) &= \frac{\gamma}{v(s)} (u(s)(F_{\max} - f(s)) - f(s)) & 0 &\leq f(s) \leq F_{\max} \\ \min \int_0^d &\frac{1}{v(s)} \left(1 + \frac{\alpha}{2} u(s)^2 \right) ds \end{aligned}$$

and now fits in the general framework developed in this paper.

Solving the "static" problem leads to $\bar{e}(t) = e_0 + (\sigma - \bar{f}\bar{v})t$ (we re-express it in function of t) where

$$\bar{v} = \frac{\tau}{2d} \left(e_0 + \sqrt{e_0^2 + 4\frac{\sigma d^2}{\tau}} \right), \quad \bar{f} = \frac{\bar{v}}{\tau} = \frac{1}{2d} \left(e_0 + \sqrt{e_0^2 + 4\frac{\sigma d^2}{\tau}} \right), \quad \bar{u} = \frac{\bar{f}}{F_{\max} - \bar{f}}.$$

For the numerical simulations, we take $d = 1500$, $\tau = 0.932$, $\sigma = 22$, $\alpha = 10^{-5}$, $F_{\max} = 8$, $\gamma = 0.0025$, $v_0 = 3$, $e_0 = 4651$. We obtain $\bar{v} = 6.2$, $\bar{f} = 6.65$, $\bar{u} = 4.92$. The distance run (1500 meters) is large enough so that we observe the linear turnpike phenomenon. The numerical results are reported on Figure 3.

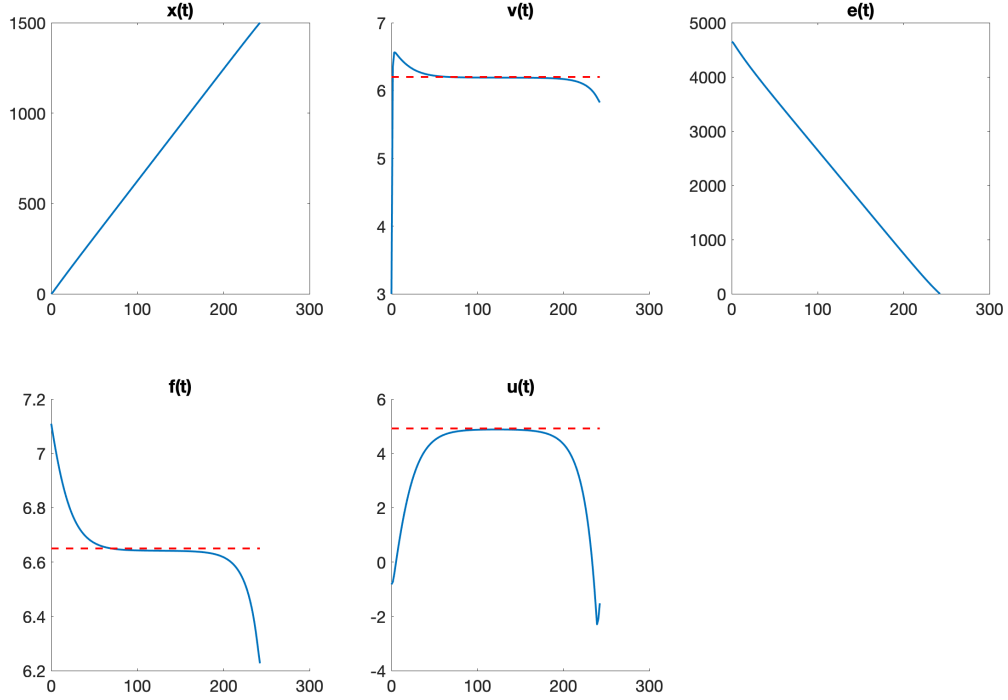


Figure 3: Numerical results for the runner problem

This runner optimal control problem has actually been the initial point for the present paper, and the author warmly thanks Amandine Aftalion for having raised such an interesting problem. The linear turnpike phenomenon has been recently exploited in [5].

A Appendix

This appendix is devoted to illustrating the comments done in Remark 2. We take a very simple example and we give numerical simulations showing the competition between two global turnpikes, or, at the level of the initialization of numerical methods, between local and global turnpikes.

We consider the one-dimensional optimal control problem

$$\begin{aligned} \dot{x} &= -3x + 3x^3 + u, & x(0) &= x_0, & x(T) &= x_f \text{ or free} \\ \min \int_0^T & ((x(t) - x_d)^2 + (u(t) - u_d)^2) dt \end{aligned} \quad (89)$$

The corresponding static problem is

$$\min_{(x,u) \mid -3x+3x^3+u=0} ((x - x_d)^2 + (u - u_d)^2). \quad (90)$$

A.1 Competition between two (global) turnpikes

We take $x_d = 1$ and $u_d = 3.47197$. The choice of u_d is done so that the static problem (90) has two global minima, at $\bar{x}_1 = -1.347372066$ and $\bar{x}_2 = 0.5939615956$, see Figure 4.

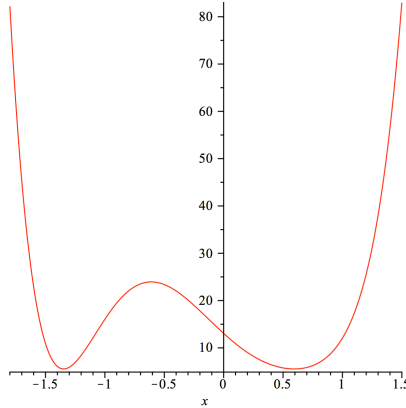


Figure 4: Plot of the function $x \mapsto (x - 1)^2 + (3x - 3x^3 - 3.47197)^2$

On Figure 5, in dashed blue, we have computed the optimal trajectory with $x_0 = -5$, $x_f = -1$, $T = 10$: we observe a turnpike phenomenon around \bar{x}_1 . In solid red, we have computed the optimal trajectory with $x_0 = 2$, $x_f = 1$, $T = 10$: we observe a turnpike phenomenon around \bar{x}_2 .

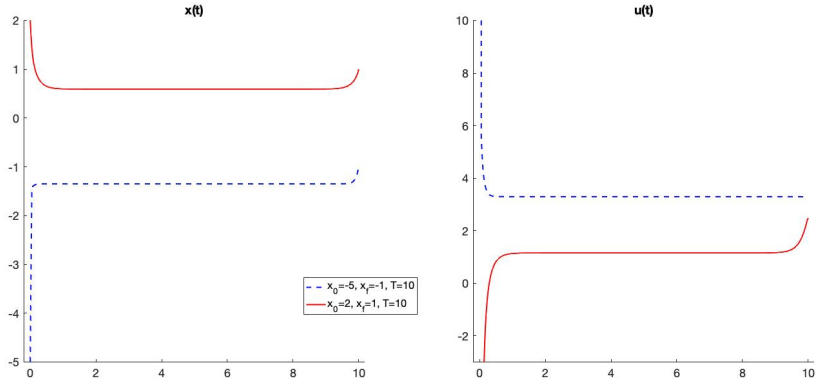


Figure 5: Global turnpikes around \bar{x}_1 and \bar{x}_2 .

The fact that the turnpike phenomenon is either around \bar{x}_1 or around \bar{x}_2 depends on the terminal conditions. For instance, if x_0 and x_f are close to \bar{x}_1 (resp., \bar{x}_2) then the optimal trajectory will make a turnpike around \bar{x}_1 (resp., \bar{x}_2). But when the terminal conditions are farther, it is not clear to predict the behavior of the optimal trajectory.

Since the minima are global, we expect to observe a competition between both turnpikes, depending on the terminal conditions (see [37]). Let us provide some numerical simulations illustrating this competition. To facilitate the understanding, we consider the problem with x_f free. It is interesting to note that the numerical result strongly depends on the initialization of the

numerical method. Here, to compute numerically the optimal solutions of (89), we use AMPL (see [25]) combined with IpOpt (see [46]): the trajectory and the controls are discretized (the control is piecewise constant and the trajectory is piecewise linear, on a given subdivision that is chosen fine enough), and we initialize the trajectory with the same constant value over all the subdivision. Then according to the value of this initialization, we can make emerge such or such turnpike, and all solutions are anyway optimal.

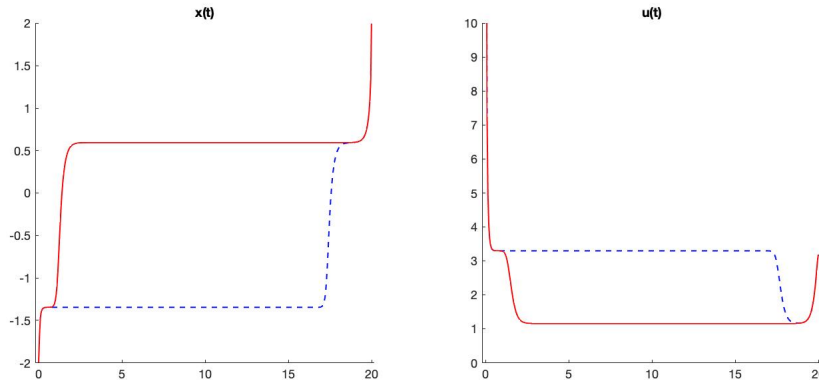


Figure 6: $x_0 = -2$, x_f free, $T = 20$.

On Figure 6, we have taken $T = 20$, $x_0 = -2$ (and x_f is let free). In dashed blue, we have initialized the trajectory to the constant trajectory \bar{x}_1 , and we then obtain an optimal trajectory which stays essentially near $\bar{x}_1 = -1.347372066$, with a kind of “hesitation” towards $\bar{x}_2 = 0.5939615956$ near the end. In solid red, we have initialized the trajectory to the constant trajectory \bar{x}_2 , and we obtain a trajectory staying essentially near \bar{x}_2 , with a kind of “hesitation” towards \bar{x}_1 near the beginning. We stress that the two solutions are optimal: both have a cost $C \simeq 6.2822$. We could make emerge other similar trajectories, which “hesitate” between the two turnpikes. All of them are optimal, or, at least, “quasi-optimal” (there is a small error due to switches from one turnpike to the other).

A.2 Local versus global turnpike

We take $x_d = 1$ and $u_d = 1$. The choice of u_d is now such that the static problem (90) has a unique global solution $\bar{x} = 0.781538640850898$, see Figure 7. But it has also a local solution $\bar{x}_{loc} = -1.10551208794920$.

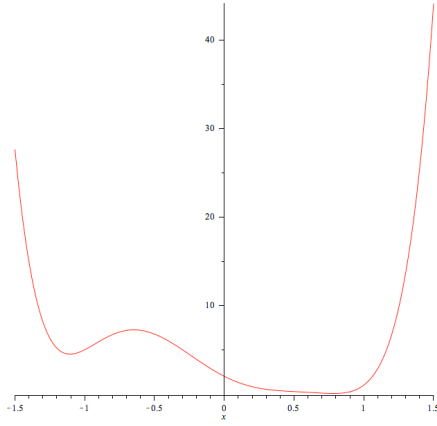


Figure 7: Plot of the function $x \mapsto (x - 1)^2 + (3x - 3x^3 - 1)^2$

The global minimum \bar{x} is a global turnpike, while the local minimum \bar{x}_{loc} is a local turnpike. In the numerical simulations, when performing either an optimization or a Newton method (shooting), we compute local solutions. Hence, we must expect that the numerical results depend on the initialization. To check global optimality, we have to compare the costs.

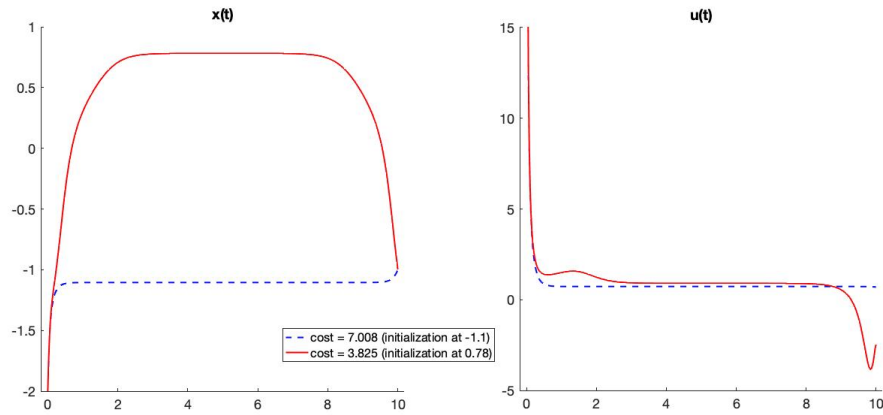


Figure 8: $x_0 = -2$, $x_f = -1$, $T = 10$.

On Figure 8, in dashed blue, we have initialized the code with the constant trajectory \bar{x}_{loc} . We observe a turnpike around $\bar{x}_{loc} = -1.10551208794920$. The cost is $C \simeq 7.008$. But this is a local turnpike only. This trajectory that we obtain is only locally optimal, and is not globally optimal. In solid red, we have initialized the code with the constant trajectory \bar{x} . We observe a turnpike around $\bar{x} = 0.781538640850898$. The cost is $C \simeq 3.825$ and is lower than the one of the previous one. Here, we have actually computed the globally optimal trajectory. This is the global turnpike.

It is also interesting to see what happens if we take T much smaller. Let us take $T = 2$. On Figure 9, in dashed blue, we have initialized the code with the constant trajectory \bar{x} . We observe a trend to the turnpike around $\bar{x} = 0.781538640850898$. The cost is $C \simeq 17.792$. But this trajectory, now, is not globally optimal. In solid red, we have initialized the code with the constant trajectory

\bar{x}_{loc} . We observe a turnpike around $\bar{x}_{loc} = -1.347372066$. The cost is $C \simeq 16.322$. It is the globally optimal trajectory.

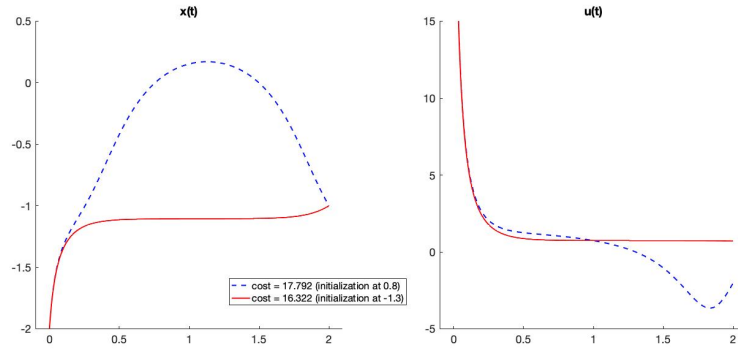


Figure 9: $x_0 = -2, x_f = -1, T = 2$.

We can search a time $2 < T_c < 10$ for which both previous initializations give equivalent turnpikes around \bar{x}_{loc} and \bar{x} (with same cost). This is done hereafter. What is important is that, in large time T , we have indeed the global turnpike around \bar{x} .

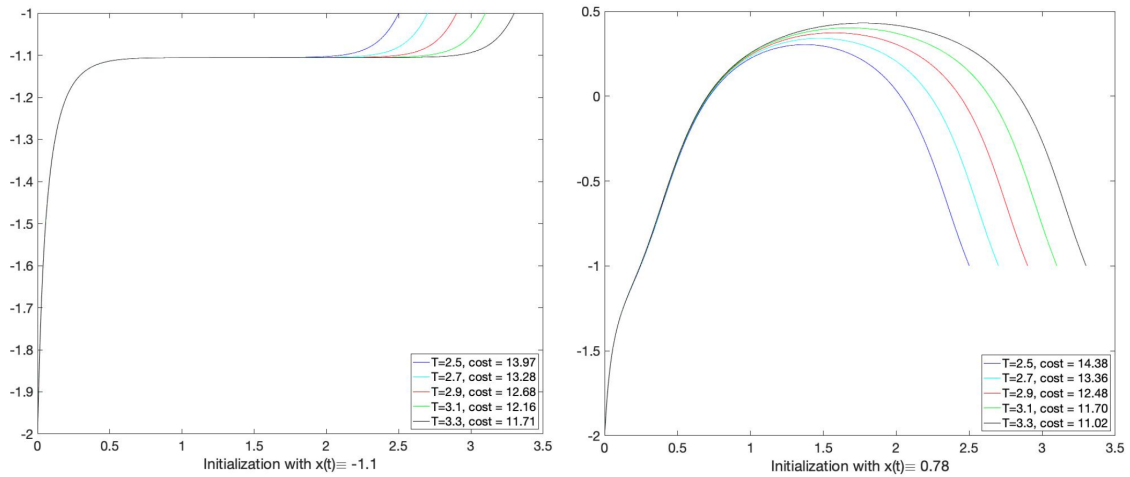


Figure 10: $x_0 = -2, x_f = -1$

On Figure 10, at the left, the code is initialized with $x(t) \equiv -1.1$, in order to promote the turnpike around $\bar{x}_{loc} = -1.347372066$. At the right, the code is initialized with $x(t) \equiv 0.78$, in order to promote the turnpike around $\bar{x} = 0.781538640850898$. We have computed trajectories for the following successive values of T : 2.5, 2.7, 2.9, 3.1, 3.3. We observe that the blue and cyan trajectories at the top are optimal (see the value of their cost); and that the red, green and black trajectories at the bottom are optimal. The bifurcation occurs around $T = 2.9$.

Finally, on Figure 11, we represent the global optimal trajectory. For $T \lesssim 2.9$, the global optimal trajectory makes a turnpike around $\bar{x}_{loc} = -1.347372066$, which is a local minimizer of

the optimal static problem. For $T \geq 2.9$ we have a bifurcation and the global optimal trajectory makes a turnpike around $\bar{x} = 0.781538640850898$, which is the global minimizer of the optimal static problem.

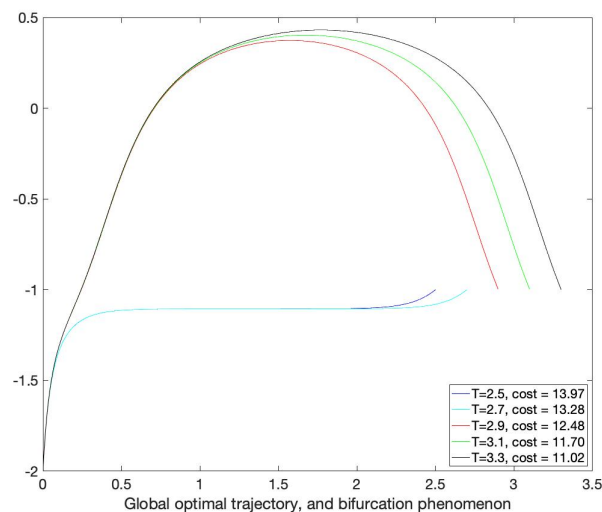


Figure 11: $x_0 = -2$, $x_f = -1$

References

- [1] H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, *Matrix Riccati equations*, Systems & Control: Foundations & Applications. Birkhäuser Verlag, Basel, 2003. xx+572 pp. ISBN: 3-7643-0085-X
- [2] A. Aftalion, *How to run 100 meters*, SIAM J. Applied Math. **77** (2017), no. 4, 1320–1334.
- [3] A. Aftalion, J.-F. Bonnans, *Optimization of running strategies based on anaerobic energy and variations of velocity*, SIAM J. Applied Math. **74** (2014), no. 5, 1615–1636.
- [4] A. Aftalion, E. Trélat, *How to build a new athletic track to break records*, R. Soc. Open Sci. **7** (2020), April, 200007, 10 pp.
- [5] A. Aftalion, E. Trélat, *Pace and motor control optimization for a runner*, J. Math. Biol. **83** (2021), no. 1, Paper No. 9.
- [6] A. Agrachev, Y. Sachkov, *Control theory from the geometric viewpoint*, Encyclopaedia of Mathematical Sciences, 87. Control Theory and Optimization, II. Springer-Verlag, Berlin, 2004. xiv+412 pp.
- [7] B.D.O. Anderson, P.V. Kokotovic, *Optimal control problems over large time intervals*, Automatica J. IFAC **23** (1987), no. 3, 355–363.
- [8] B. Bonnard, J.-B. Caillaud, E. Trélat, *Second order optimality conditions in the smooth case and applications in optimal control*, ESAIM Control Optim. Calc. Var. **13** (2007), no. 2, 207–236.
- [9] B. Bonnard, M. Chyba, *Singular trajectories and their role in control theory*, Mathématiques & Applications (Berlin) [Mathematics & Applications], 40. Springer-Verlag, Berlin, 2003. xvi+357 pp.
- [10] S. Boyd, L. Vandenberghe, *Convex optimization*, Cambridge University Press, Cambridge, 2004. xiv+716 pp.
- [11] B. Brogliato, R. Lozano, B. Maschke, O. Egeland. *Dissipative Systems Analysis and Control*, Second Edition, Springer-Verlag London, Ltd., London, 2007.

- [12] P. Cannarsa, C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston, Inc., Boston, MA, 2004. xiv+304 pp.
- [13] D.A. Carlson, A.B. Haurie, A. Leizarowitz, *Infinite horizon optimal control* Springer-Verlag, Berlin, 1991, Deterministic and stochastic systems, Second revised and enlarged edition of the 1987 original.
- [14] Y. Chitour, F. Jean, E. Trélat, *Genericity results for singular curves*, J. Differential Geom. **73** (2006), no. 1, 45–73.
- [15] Y. Chitour, F. Jean, E. Trélat, *Singular trajectories of control-affine systems*, SIAM J. Control Optim. **47** (2008), no. 2, 1078–1095.
- [16] F.H. Clarke, *Functional analysis, calculus of variations and optimal control*, Graduate Texts in Mathematics, 264. Springer, London, 2013, xiv+591 pp.
- [17] J.-M. Coron, E. Trélat, *Global steady-state controllability of one-dimensional semilinear heat equations*, SIAM J. Control Optim. **43** (2004), no. 2, 549–569.
- [18] T. Damm, L. Grüne, M. Stieler, K. Worthmann. *An exponential turnpike theorem for dissipative discrete time optimal control problems*, SIAM J. Control Optim. **52** (2014), 1935–1957.
- [19] A. Fathi, *Weak KAM theorem in Lagrangian dynamics*, Cambridge Studies in Advanced Mathematics, Hardcover, 2016.
- [20] T. Faulwasser, K. Flasskamp, S. Ober-Blöbaum, K. Worthmann, *Towards velocity turnpikes in optimal control of mechanical systems*, IFAC-PapersOnLine, 52(16):490–495. In Proc. 11th IFAC Symposium on Nonlinear Control Systems, NOLCOS 2019.
- [21] T. Faulwasser, K. Flasskamp, S. Ober-Blöbaum, K. Worthmann, *A Dissipativity Characterization of Velocity Turnpikes in Optimal Control Problems for Mechanical Systems*, IFAC-PapersOnLine, 54(9), 2021, 624–629. In 24th International Symposium on Mathematical Theory of Networks and Systems MTNS 2020: Cambridge, United Kingdom.
- [22] T. Faulwasser, L. Grüne, *Turnpike properties in optimal control*, Handbook of Numerical Analysis, Volume 23, 2022, pp. 367–400.
- [23] T. Faulwasser, L. Grüne, J.P. Humaloja, M. Schaller, *The interval turnpike property for adjoints*, Pure and Applied Functional Analysis, in press, arXiv:2005.12120.
- [24] T. Faulwasser, M. Korda, C.N. Jones, D. Bonvin, *On turnpike and dissipativity properties of continuous-time optimal control problems*, Automatica J. IFAC **81** (2017), 297–304.
- [25] R. Fourer, D.M. Gay, B.W. Kernighan, *AMPL: A modeling language for mathematical programming*, Duxbury Press, Second edition (2002), 540 pages.
- [26] L. Grüne, *Economic receding horizon control without terminal constraints*, Automatica **49** (2013), 725–734.
- [27] L. Grüne, M.A. Müller, *On the relation between strict dissipativity and turnpike properties*, Systems Control Lett. **90** (2016), 45–53.
- [28] L. Grüne, R. Guglielmi, *Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems*, SIAM J. Control Optim. **56** (2018), no. 2, 1282–1302.
- [29] L. Grüne, M. Schaller, A. Schiela, *Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations*, Journal of Differential Equations **268** (2020), no. 12, 7311–7341.
- [30] V. Jurdjevic, *Geometric control theory*, Cambridge Studies in Advanced Mathematics, 52. Cambridge University Press, Cambridge, 1997. xviii+492 pp.
- [31] J.B. Keller, *Optimal velocity in a race*, American Mathematical Monthly (1974), 474–480.
- [32] V. Kučera, *A contribution to matrix quadratic equations*, IEEE Trans. Automat. Control **17** (1972), no. 3, 344–347.
- [33] L. Perko, *Differential equations and dynamical systems*, Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001. xiv+553 pp.

- [34] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The mathematical theory of optimal processes*, Translated from the Russian by K. N. Trirogoff; edited by L.W. Neustadt. Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.
- [35] A. Porretta, E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim. **51** (2013), no. 6, 4242–4273.
- [36] A. Rapaport, P. Cartigny, *Turnpike theorems by a value function approach* ESAIM Control Optim. Calc. Var. **10** (2004), no. 1, 123–141.
- [37] A. Rapaport, P. Cartigny, *Competition between most rapid approach paths: necessary and sufficient conditions*, J. Optim. Theory Appl. **124** (2005), no. 1, 1–27.
- [38] E.D. Sontag, *Mathematical control theory, deterministic finite-dimensional systems*, Texts in Applied Mathematics, Vol. 6, Springer-Verlag, New York, second edition, 1998.
- [39] A. N. Tikhonov, V. Y. Arsenin, *Solutions of ill-posed problems*, V. H. Winston & Sons, Washington, D.C.: John Wiley & Sons, New York-Toronto, Ont.-London, 1977. Translated from the Russian, Preface by translation editor Fritz John, Scripta Series in Mathematics.
- [40] E. Trélat, *Some properties of the value function and its level sets for affine control systems with quadratic cost*, J. Dynam. Control Systems **6** (2000), no. 4, 511–541.
- [41] E. Trélat, *Contrôle optimal, théorie & applications*, Mathématiques Concrètes, Vuibert, Paris, 2005.
- [42] E. Trélat, *Optimal control and applications to aerospace: some results and challenges*, J. Optim. Theory Appl. **154** (2012), no. 3, 713–758.
- [43] E. Trélat, C. Zhang, *Integral and measure-turnpike properties for infinite-dimensional optimal control systems*, Math. Control Signals Systems **30** (2018), no. 1, Art. 3, 34.
- [44] E. Trélat, C. Zhang, E. Zuazua, *Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces*, SIAM J. Control Optim. **56** (2018), no. 2, 1222–1252.
- [45] E. Trélat, E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, Journal of Differential Equations **258** (2015), no. 1, 81–114.
- [46] A. Wächter, L.T. Biegler, *On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming*, Mathematical Programming **106** (2006), 25–57.
- [47] R.R. Wilde, P.V. Kokotovic, *A dichotomy in linear control theory*, IEEE Trans. Automat. Control **17** (1972), no. 3, 382–383.
- [48] J.C. Willems, *Dissipative dynamical systems, Part I: General theory*, Arch. Ration. Mech. Anal. **45** (1972), 321–351.
- [49] A.J. Zaslavski, *Turnpike properties in the calculus of variations and optimal control*, Nonconvex Optimization and its Applications, Vol. 80, Springer, New York, 2006.
- [50] A.J. Zaslavski, *Turnpike theory of continuous-time linear optimal control problems*, Springer Optimization and Its Applications, Vol. 104, Springer, Cham, 2015.